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2-Perfect m-cycle systems

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Abstract

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The spectrum for 2-perfect m-cycle systems of K_n has been considered by several authors in the case when $m \leq 7$. In this paper we essentially solve the problem for 2-perfect m-cycle systems of K_n in the case where m is prime and $2m + 1$ is a prime power. In particular we settle the problem for $m = 11$ and 13 except for two or one possible exceptions respectively. The problem for $m = 9$ is also considered.

1. Introduction

An *m-cycle* is a cycle of length m, which we denote by $(v_0, \ldots, v_{m-1})$; so $v_i v_{i+1}$ is an edge for $0 \leq i < m$, reducing the subscripts modulo m. An *m-cycle system of K_n* is an ordered pair (V, C) where V is the vertex set of K_n and C is a set of m-cycles which induce a partition of the edge set of K_n; so the m-cycles in C form an edge-disjoint decomposition of K_n. If c is an m-cycle then define $c(i)$ to be the graph formed by joining vertices that are distance i apart in c.

If m is odd then a *2-perfect m-cycle system of K_n* is an m-cycle system of K_n, (V, C) with the additional property that $(V, \{c(2) \mid c \in C\})$ is also an m-cycle system of K_n. So if an m-cycle system (V, C) is 2-perfect then each pair of vertices is distance 2 apart in exactly one m-cycle in C. Throughout this paper we assume that $m = 2t + 1$ is an *odd* integer.

Associated with any 2-perfect m-cycle system of K_n, (V, C) there is a natural binary operation $\cdot$ defined by $x \cdot y = z$ if and only if $(x, y, z, v_3, \ldots, v_{m-1}) \in C$. It is not hard to see that the binary operation defined in this way actually produces a quasigroup. In fact, this quasigroup will also satisfy the following identities:

$$x^2 = x \qquad \text{for all } x \in V,$$

$$(yx)x = y \qquad \text{for all } x, y \in V,$$

and some further identities depending on the length of the cycles. For example, one further identity can be obtained by recognizing that there are two directions that one can proceed around a given m-cycle, so the products defined must be consistent regardless of which direction is chosen. This further identity becomes complicated as m gets large but we illustrate it for some small values of m:

when $m = 3$ it is $xy = yx$ for all x, y in V;

when $m = 5$ it is $x(yx) = y(xy)$ for all x, y in V; and

when $m = 7$ it is $(xy)(y(xy)) = (yx)(x(yx))$ for all x, y in V.

In fact, when $m \in \{3, 5, 7\}$ there are no further identities since it has been shown that [6] 2-perfect m-cycle systems produce quasigroups satisfying the three identities mentioned, but the converse is also true; that is, to any quasigroup satisfying the three given identities there naturally corresponds a 2-perfect m-cycle system. For further information concerning connections between 2-perfect m-cycle systems of K_n and related quasigroups, see [6].

Perhaps the most natural problem concerning 2-perfect m-cycle systems of K_n is the *spectrum problem*: for which values of n do there exist 2-perfect m-cycle systems of K_n? Recent progress in this area has essentially settled the spectrum problem for 2-perfect m-cycle systems of K_n when $m \in \{5, 7\}$ (of course for $m = 3$, every 3-cycle system of K_n is 2-perfect): the spectrum for 2-perfect 5-cycle systems of K_n (also called Steiner pentagon systems) is all $n \equiv 1$ or $5 \pmod{10}$ except for $n = 15$ [8], and the spectrum for 2-perfect 7-cycle systems of K_n is all $n \equiv 1$ or $7 \pmod{14}$ except possibly for $n \in \{21, 85\}$ [9]. In this paper we consider the spectrum problem for 2-perfect m-cycle systems of K_n: we actually solve the problem except possibly for two or one values of n when $m = 11$ or 13 respectively, and except for 3 possible values in the case where m is a prime and $2m + 1$ is a prime power. We also obtain some results when $m = 9$.

Clearly some necessary conditions for the existence of just an m-cycle system of K_n are:

(N1) if $n > 1$ then $n \geq m$;

(N2) n is odd (each vertex has even degree); and

(N3) $2m$ divides $n(n-1)$ (since m divides the total number of edges).

It is likely that these conditions are sufficient for the existence of a 2-perfect m-cycle system of K_n except for some small values of n (for example, there is no 2-perfect 5-cycle system of K_{15} (since no $(15, 5, 2)$ BIBD exists)). However, even showing that these conditions are sufficient for the existence of an m-cycle system has yet to be proved [3, 10].

It is worth remarking here that it is not necessary to restrict one's attention to the case where m is odd, nor just to 2-perfect m-cycle systems of K_n. For $1 \leq i \leq m/2$ we can define an *i-perfect m-cycle system of K_n*, (V, C) to be an m-cycle system of K_n with the additional property that $(V, \{c(i) \mid c \in C\})$ is an x-cycle system of K_n for some integer x. So for example, a 2-perfect 6-cycle system of K_n is a 6-cycle system of K_n, (V, C) with the additional property that $(V, \{c(2) \mid c \in C\})$ is a 3-cycle system of K_n (or if you prefer, a Steiner triple system). The spectrum for 2-perfect 6-cycle systems of K_n has been shown to be all $n \equiv 1$ or $9 \pmod{12}$ except for $n = 9$ and possibly for $n \in \{45, 57\}$ [7]. Notice also that if (V, C) is a 2-perfect 7-cycle system of K_n then $(V, \{c(2) \mid c \in C\})$ is a 3-perfect 7-cycle system of K_n. Similarly, 3-perfect 7-cycle systems of K_n produce 2-perfect 7-cycle systems of K_n, so their spectra are identical (see [10] for this and related problems). We point out one last generalization: one might look for m-cycle systems that are i-perfect for more than one value of i. Indeed, when m is odd a *Steiner m-cycle system of K_n* is defined to be an m-cycle system of K_n that is i-perfect for $1 \leq i \leq m/2$ (hence the name Steiner pentagon system, mentioned earlier). Determining the spectrum of Steiner m-cycle systems is a very difficult problem.

Finally, we remark that m-cycle systems have been used in the design of experiments for use in serology. In this application, the cycles are permitted to be degenerate, in which case they are called *neighbor designs*. However avoiding degeneracy or having the additional structure of being 2-perfect are properties of some use in this application [4, 5].

Throughout the rest of this paper, $m = 2t + 1$ is an odd integer. Let $Z_x = \{0, 1, \ldots, x-1\}$.

2. Preliminary results

We begin with some well-known constructions of Steiner m-cycle systems.

Lemma 2.1. *If m is a prime then there exists a Steiner m-cycle system of K_m.*

Proof. A Steiner m-cycle system (Z_m, C) is defined by

$$C = \{(0, x, 2x, \ldots, -x) \mid 1 \leq x < m/2\},$$

where, of course, each component of each m-cycle is reduced modulo m. □

Lemma 2.2 [2]. *If $n \equiv 1 \pmod{2m}$ is a prime power then there exists a Steiner m-cycle system of K_n.*

Proof. Let β be a primitive element of GF(n). Then a Steiner m-cycle system of K_n is formed by the cycles

$$(\beta^i + \alpha, \beta^{i+(n-1)/m} + \alpha, \beta^{i+2(n-1)/m} + \alpha, \ldots, \beta^{i+(m-1)(n-1)/m} + \alpha)$$

for $0 \leq i < (n-1)/2m$ and for each field element α. □

The following 2-perfect 9-cycle system will be used in Section 4.

Lemma 2.3. *There exists a 2-perfect 9-cycle system of K_{27}.*

Proof. Define a 2-perfect 9-cycle system $(\{\infty\} \cup (Z_2 \times Z_{13}), C)$ as follows:

$$C\{(\infty, (0,0), (0,3), (0,1), (1,4), (0,4), (1,8), (1,2), (1,0)) + (0,i),$$
$$((0,0), (0,1), (0,5), (0,12), (1,0), (1,1), (0,4), (0,9), (1,7)) + (0,i),$$
$$((0,0), (1,2), (1,7), (1,4), (0,8), (1,3), (1,12), (0,6), (1,5)) + (0,i) \mid$$
$$0 \leqslant i \leqslant 12\}$$

where $c + (0, i)$ is the m-cycle formed by adding $(0, i)$ to each vertex in c, reducing the second component modulo 13 and defining $\infty + (0, i) = \infty$. □

The constructions developed in Section 3 make use of commutative quasigroups with holes of size 2, self-orthogonal quasigroups and self-orthogonal quasigroups with holes of size 2.

Let $H = \{\{0, 1\}, \{2, 3\}, \ldots, \{2s-2, 2s-1\}\}$. The 2-element subsets in H are called *holes*. Let $(Z_{2s}, \cdot)$ be a commutative quasigroup with the property that, for each hole $h \in H$, $(h, \cdot)$ is a subquasigroup; such a quasigroup is called a *commutative quasigroup with holes* H and of *order* $2s$ (it is common in other settings to leave the subquasigroup products undefined). A commutative quasigroup with holes $H = \{\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}\}$ is shown in Fig. 1.

Lemma 2.4 [1]. *There exists a commutative quasigroup with holes H and of order $2s$ for all $s \geqslant 3$.*

A pair of quasigroups with the same holes H, say $(Q, \cdot_1)$ and $(Q, \cdot_2)$ are *orthogonal* provided that when the partial latin squares obtained by deleting the symbols in the cells in $h \times h$, $h \in H$ are superimposed, the resulting set of ordered pairs is precisely

$$(Q \times Q) \setminus \{(x, y) \mid \{x, y\} \in H\};$$

if, in addition, these partial latin squares are transposes of each other then $(Q, \cdot_1)$ is a *self-orthogonal quasigroup with holes H*.

0	1	7	4	3	6	5	2
1	0	5	6	7	2	3	4
7	5	2	3	6	1	4	0
4	6	3	2	0	7	1	5
3	7	6	0	4	5	2	1
6	2	1	7	5	4	0	3
5	3	4	1	2	0	6	7
2	4	0	5	1	3	7	6

Fig. 1

Lemma 2.5 [11]. *There exists a self-orthogonal quasigroup of order $2s+1$ for all s except $s=1$. There exists a self-orthogonal quasigroup with holes H and of order $2s$ for all $s \geqslant 4$.*

3. Constructing 2-perfect m-cycle systems

Recall that $m = 2t+1$ is an odd integer. Define an *m-sequence* to be a sequence $(a_0, \ldots, a_{2t})$ of $t+1$ distinct elements of Z_m satisfying

(1) $a_{t-z} = a_{t+z}$ for $1 \leqslant z \leqslant t$,

(2) $\{a_z - a_{z-1} \mid z \in Z_m\} = Z_m$, and

(3) $\{a_z - a_{z-2} \mid z \in Z_m\} = Z_m$,

where, of course, both the subscript $x-y$ and the difference $a_x - a_y$ are reduced modulo m.

Example 3.1. (0, 2, 3, 7, 4, 7, 3, 2, 0) is a 9-sequence. (3, 5, 9, 4, 1, 0, 1, 4, 9, 5, 3) is an 11-sequence.

Lemma 3.2. *If m is a prime then there exists an m-sequence.*

Proof. Define an m-sequence $(a_0, \ldots, a_{2t})$ by $a_i \in Z_m$ and $a_{t-x} = a_{t+x} \equiv x^2 \pmod m$ for $0 \leqslant x \leqslant t$. $\square$

We now define an m-cycle $c(x, y, \cdot, \circ)$ where $x, y \in Z_m$ and $(Z_m, \cdot)$ and $(Z_m, \circ)$ are particular quasigroups. Let

$$c(x, y, \cdot, \circ) = ((a_0, b_0), (a_1, b_1), \ldots, (a_{2t}, b_{2t}))$$

where $(a_0, a_1, \ldots, a_{2t})$ is an m-sequence and where for $k \geqslant 0$,

$$b_t = x \circ y,$$
$$b_{t+1+3k} = b_{t-2-3k} = x,$$
$$b_{t+2+3k} = b_{t-1-3k} = y,$$
$$b_{t+3k} = x \cdot y, \text{ and}$$
$$b_{t-3k} = y \cdot x$$

except that if $m \equiv 1$ or $5 \pmod 6$ then we redefine $b_0 = y$, $b_1 = x \cdot y$, $b_{2t-1} = y \cdot x$ and $b_{2t} = x$.

Using the 9-sequence $(0, 2, 3, 7, 4, 7, 3, 2, 0)$ from Example 3.1, when $m = 9$, $c(x, y, \cdot, \circ)$ can be depicted as in Fig. 2, and using the 11-sequence (3, 5, 9, 4, 1, 0, 1, 4, 9, 5, 3), when $m = 11$, $c(x, y, \cdot, \circ)$ can be depicted as in Fig. 3, so hopefully these pictures will help the reader.

The two important properties that $c(x, y, \cdot, \circ)$ possesses are that for $1 \leqslant z \leqslant t$,

(i) if $b_{t-z} = x, y, x \cdot y$ or $y \cdot x$ then b_{t+z} is $y, x, y \cdot x$ or $x \cdot y$ respectively, and

(ii) $b_{t-z} \neq b_{t-z-1}$ and $b_{t-z} \neq b_{t-z-2}$.

Clearly property (i) implies that $c(x, y, \cdot, \circ) = c(y, x, \cdot, \circ)$.

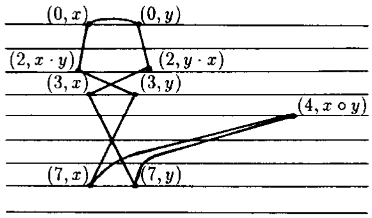

Fig. 2

Theorem 3.3. *If there exists an m-sequence and a 2-perfect m-cycle system of K_m then there exists a 2-perfect m-cycle system of K_n for all $n \equiv m \pmod{2m}$ except possibly $n = 3m$.*

Proof. Let $n = (2s+1)m$ where $s \neq 1$. Let $(Z_{2s+1}, \cdot)$ be a self-orthogonal quasigroup (see Lemma 2.5) and let $(Z_{2s+1}, \circ)$ be a commutative idempotent quasigroup. Let $(a_0, \ldots, a_{2t})$ be an m-sequence. Define a 2-perfect m-cycle system of K_n, $(Z_m \times Z_{2s+1}, C)$ as follows:

(i) for each $x \in Z_{2s+1}$ let $(Z_m \times \{x\}, C_x)$ be a 2-perfect m-cycle system of K_m and let $C_x \subseteq C$; and

(ii) for each $x, y \in Z_{2s+1}$, $x \neq y$ and for each $j \in Z_m$ let $c(x, y, \cdot, \circ) + (j, 0)$ be in C (where, of course, $c(x, y, \cdot, \circ) + (j, 0)$ is formed by adding $(j, 0)$ to each vertex of $c(x, y, \cdot, \circ)$, reducing the first component modulo m).

To see that this defines an m-cycle system, consider the edge $\{(k, u), (l, v)\}$. If $u = v$ then this edge occurs in an m-cycle in C_u. Suppose that $u \neq v$. By property (2) of m-sequences, there exists a $w \in Z_m$ so that $a_w - a_{w-1} \equiv k - l \pmod{m}$. Replacing $w = t - z$ and using property (1) of sequences shows that

$$k - l \equiv a_{t-z} - a_{t-z-1} = a_{t+z} - a_{t+z+1} \pmod{m}.$$

Therefore there exists a j so that $k = a_{t-z} + j$ and $l = a_{t-z-1} + j$. Now, from property (i) of $c(x, y, \cdot, \circ)$ If $b_{t-z} = x, y, x \cdot y$ or $y \cdot x$ then b_{t+z} is $y, x, y \cdot x$ or $x \cdot y$ respectively, and similarly for b_{t-z-1} and b_{t+z+1}. Also from property (ii) of $c(x, y, \cdot, \circ) b_{t-z} \neq b_{t+z}$ and $b_{t-z-1} \neq b_{t+z+1}$. So we have several cases to consider.

If $b_{t-z} = x$ (or y) and $b_{t-z-1} = y$ (or x) then $\{(k, u), (l, v)\}$ is in $c(k, l, \cdot, \circ) + (j, 0)$. If $b_{t-z} = x$ and if $b_{t-z-1} = x \cdot y$ (or $y \cdot x$) then let $a \in Z_{2s+1}$ be the unique

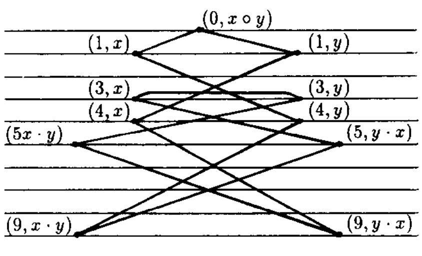

Fig. 3

element so that $u \cdot a = v$ (or $a \cdot u = v$); $\{(k, u), (l, v)\}$ is in $c(u, a, \cdot, \circ) + (j, 0)$. If $b_{t-z} = x \cdot y$ and $b_{t-z-1} = y \cdot x$ then let $a, b \in Z_{2s+1}$ be the unique elements satisfying $a \cdot b = u$ and $b \cdot a = v$ (this uses the self-orthogonality of $(Z_{2s+1}, \cdot)$); then $\{(k, u), (l, v)\}$ is in $c(a, b, \cdot, \circ) + (j, 0)$. If $b_{t-z} = x \circ y$ and $b_{t-z-1} = x$ (so $z = 0$), then let $a \in Z_{2s+1}$ be the unique element so that $v \circ a = a \circ v = u$; then $\{(k, u), (l, v)\}$ is in $c(a, v, \cdot, \circ)$. The remaining cases follow in the same way as the cases considered so far.

To see that this defines a 2-perfect m-cycle system, one can apply the same argument as we just used except that property (3) of m-sequences is used instead of property (2), and then a_{t-z-2}, a_{t+z+2}, b_{t-z-2} and b_{t+z+2} are considered instead of a_{t-z-1}, a_{t+z+1}, b_{t-z-1} and b_{t+z+1}. □

Theorem 3.4. *If there exists an m-sequence and a 2-perfect m-cycle system of* K_{2m+1} then there exists a 2-perfect m-cycle system for all $n \equiv 1 \pmod{2m}$ *except possibly* $n \in \{4m+1, 6m+1\}$.

Proof. Let $n = 2sm + 1$ where $s \notin \{2, 3\}$. Let $(Z_{2s}, \cdot)$ be a self-orthogonal quasigroup with holes H (see Lemma 2.5) and let $(Z_{2s}, \circ)$ be a commutative quasigroup with holes H (see Lemma 2.4). Let $(a_0, \ldots, a_{2t})$ be an m-sequence. Define a 2-perfect m-cycle system of K_n, $(\{\infty\} \cup (Z_m \times Z_{2s}), C)$ as follows:

(i) for each $h \in H$ let $(\{\infty\} \cup (Z_m \times h), C_h)$ be a 2-perfect m-cycle system of K_{2m+1} and let $C_h \subseteq C$; and

(ii) for each $x, y \in Z_{2s}$ that belong to different holes of H, and for each $j \in Z_m$ let $c(x, y, \cdot, \circ) + (j, 0)$ be in C.

The proof that this indeed defines a 2-perfect m-cycle system of K_n is almost identical to the proof of Theorem 3.3. □

Corollary 3.5. *If m is a prime and if* $2m+1$ *is a prime power then the spectrum for 2-perfect m-cycle systems is* $n \equiv 1$ *or* $m \pmod{2m}$ *except possibly for* $n \in \{3m, 4m+1, 6m+1\}$.

Proof. By Lemmas 2.1 and 2.2 there exist 2-perfect m-cycle system of K_m and of K_{2m+1}. By Lemma 3.2 there exists an m-sequence. Therefore, by Theorems 3.3 and 3.4 there exist 2-perfect m-cycle system of K_n for all $n \equiv 1$ or $m \pmod{2m}$ except possibly for $n \in \{3m, 4m+1, 6m+1\}$. The result follows since the necessary conditions (N1), (N2) and (N3) require that $n \equiv 1$ or $m \pmod{2m}$. □

Corollary 3.6. *The spectrum for 2-perfect 11-cycle systems of* K_n *is* $n \equiv 1$ *or* $11 \pmod{22}$ *except possibly for* $n \in \{33, 45\}$. *The spectrum for 2-perfect 13-cycle systems of* K_n *is* $n \equiv 1$ *or* $13 \pmod{26}$ *except possibly for* $n = 39$.

Proof. By Lemma 3.2 there exist an 11-sequence and a 13-sequence. By Lemmas 2.1 and 2.2 there exist 2-perfect 11-cycle systems of K_{11}, K_{23} and K_{67} and there exist 2-perfect 13-cycle system of K_{13}, K_{27}, K_{53} and K_{79}. Therefore the result follows from Theorems 3.3 and 3.4. □

4. Conclusions

The careful reader might at first wonder why the case when $m = 9$ has not been considered, especially since Example 3.1 exhibits a 9-sequence. The answer is that there is no 2-perfect 9-cycle system of K_9. One method of settling the spectrum problem for 2-perfect 9-cycle systems of K_n would simply (!) be to find a 2-perfect 9-cycle system of $K_{27} - K_9$; that is, of the graph on 27 vertices in which all pairs of vertices are adjacent except for pairs occurring in a distinguished set of 9 vertices (here we are using the obvious generalization to 2-perfect m-cycle systems of graphs other than K_n). Together with the 2-perfect 9-cycle system of K_{27} exhibited in Lemma 2.3, this would completely solve the spectrum problem for 2-perfect 9-cycle systems.

Using this more general notation of 2-perfect m-cycle systems of a graph G, one could regard the cycles $c(x, y, \cdot, \circ)$ in Theorems 3.3 and 3.4 as forming 2-perfect m-cycle systems of K_m^{2s+1} and of K_{2m}^s respectively (K_x^y is the complete y-partite graph with x vertices in each part). As when considering the spectrum problem for m-cycle systems [3], results concerning 2-perfect m-cycle systems of $K_x - K_y$ can be used in conjunction with such decompositions of K_m^{2s+1} and of K_{2m}^s to great effect.

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