Journal of Combinatorial Theory, Series A 119 (2012) 1789–1797

Contents lists available at SciVerse ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Races among products

Alexander Berkovich, Keith Grizzell

Department of Mathematics, University of Florida, Gainesville, FL 32611-8105, United States

A R T I C L E   I N F O

Article history:
Received 15 December 2011
Available online 13 June 2012

Keywords:
q-Series
Generating functions
Partition inequalities
Injections
Eliezer Ehrenpreis' question

A B S T R A C T

In this paper we revisit a 1987 question of Rabbi Ehrenpreis. Among many things, we provide an elementary injective proof that

$$P_1(L, y, n) \geqslant P_2(L, y, n)$$

for any $L, n > 0$ and any odd $y > 1$. Here, $P_1(L, y, n)$ denotes the number of partitions of n into parts congruent to 1, $y + 2$, or $2y \pmod{2y + 2}$ with the largest part not exceeding $(2y + 2)L - 2$ and $P_2(L, y, n)$ denotes the number of partitions of n into parts congruent to 2, y, or $2y + 1 \pmod{2y + 2}$ with the largest part not exceeding $(2y + 2)L - 1$.

1. Introduction

The celebrated Rogers–Ramanujan identities [9] are given analytically as follows:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q, q^4; q^5)_\infty} \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2, q^3; q^5)_\infty}. \tag{1.2}$$

Here we are using the following standard notations:

$$(a;q)_L = \begin{cases} 1 & \text{if } L = 0, \\ \prod_{j=0}^{L-1}(1 - aq^j) & \text{if } L > 0, \end{cases}$$

E-mail addresses: alexb@ufl.edu (A. Berkovich), grizzell@ufl.edu (K. Grizzell).

http://dx.doi.org/10.1016/j.jcta.2012.06.003

$$(a_1, a_2, \dots, a_n; q)_L = (a_1; q)_L (a_2; q)_L \cdots (a_n; q)_L,$$

$$(a; q)_\infty = \lim_{L \to \infty} (a; q)_L.$$

Subtracting (1.2) from (1.1) we have

$$\sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q)_{n-1}} = \frac{1}{(q, q^4; q^5)_\infty} - \frac{1}{(q^2, q^3; q^5)_\infty}, \tag{1.3}$$

from which it is obvious that the coefficients in the q-series expansion of the difference of the two products in (1.3) are all non-negative. In other words, for all $n > 0$ we have

$$p_1(n) \geqslant p_2(n), \tag{1.4}$$

where $p_r(n)$ denotes the number of partitions of n into parts congruent to $\pm r \pmod 5$.

At the 1987 A.M.S. Institute on Theta Functions, Rabbi Ehrenpreis asked if one can prove (1.4) without resorting to the Rogers–Ramanujan identities. In 1999, Kadell [8] provided an affirmative answer to this question by constructing an injection of partitions counted by $p_2(n)$ into partitions counted by $p_1(n)$. In 2005, Berkovich and Garvan [5] constructed an injective proof for an infinite family of partition function inequalities related to finite products, thus giving us the following theorem.

Theorem 1.1. *Suppose $L > 0$, and $1 < r < m - 1$. Then the coefficients in the q-series expansion of the difference of the two finite products*

$$\frac{1}{(q, q^{m-1}; q^m)_L} - \frac{1}{(q^r, q^{m-r}; q^m)_L}$$

are all non-negative if and only if $r \nmid (m - r)$ and $(m - r) \nmid r$.

We note that (1.4) is an immediate corollary of this theorem with $m = 5$, $r = 2$ and $L \to \infty$.

In 2011, Andrews [4] used a clever combination of injective and anti-telescoping techniques to establish the following remarkable theorem.

Theorem 1.2. *For $L > 0$, the q-series expansion of*

$$\frac{1}{(q, q^5, q^6; q^8)_L} - \frac{1}{(q^2, q^3, q^7; q^8)_L}$$

has non-negative coefficients.

The main object of the present manuscript is the following new theorem.

Theorem 1.3. *For any $L > 0$ and any odd $y > 1$, the q-series expansion of*

$$\frac{1}{(q, q^{y+2}, q^{2y}; q^{2y+2})_L} - \frac{1}{(q^2, q^y, q^{2y+1}; q^{2y+2})_L} = \sum_{n=1}^{\infty} a(L, y, n) q^n \tag{1.5}$$

has non-negative coefficients. Furthermore, the coefficient $a(L, y, n)$ is 0 if and only if either $n \in \{2, 4, 6, \dots, y + 1\} \cup \{y\}$ or $(L, y, n) = (1, 3, 9)$.

We note that the products on the left of (1.5) can be interpreted as

$$\frac{1}{(q, q^{y+2}, q^{2y}; q^{2y+2})_L} = \sum_{n=1}^{\infty} P_1(L, y, n) q^n \tag{1.6}$$

and

$$\frac{1}{(q^2, q^y, q^{2y+1}; q^{2y+2})_L} = \sum_{n=1}^{\infty} P_2(L, y, n) q^n, \tag{1.7}$$

where $P_1(L, y, n)$ denotes the number of partitions of n into parts $\equiv 1, y+2, 2y \pmod{2(y+1)}$ with the largest part not exceeding $2(y+1)L-2$ and $P_2(L, y, n)$ denotes the number of partitions of n into parts $\equiv 2, y, 2y+1 \pmod{2(y+1)}$ with the largest part not exceeding $2(y+1)L-1$.

In the next section, we examine a norm-preserving injection of partitions counted by $P_2(L, y, n)$ into partitions counted by $P_1(L, y, n)$, where the norm of a partition π — denoted by $|\pi|$ — is the sum of its parts. In Section 3, we give a proof of Theorem 1.3. In Section 4, we look at a generalization of Theorem 1.3. In Section 5, we conclude with a brief discussion of our plans for future work in this area.

2. The injection

Let s_t denote $s + (t-1)(2y+2)$ for any positive integers s and t, and let $\nu(s_t)$ denote the number of occurrences of s_t in a given partition. Then we may write any partition π_1 counted by $P_1(L, y, n)$ as

$$\pi_1 = \left\langle 1_1^{\nu(1_1)}, (y+2)_1^{\nu((y+2)_1)}, (2y)_1^{\nu((2y)_1)}, \ldots, 1_L^{\nu(1_L)}, (y+2)_L^{\nu((y+2)_L)}, (2y)_L^{\nu((2y)_L)} \right\rangle,$$

where

$$|\pi_1| = \sum_{k=1}^{L} \big(1_k \cdot \nu(1_k) + (y+2)_k \cdot \nu\big((y+2)_k\big) + (2y)_k \cdot \nu\big((2y)_k\big)\big) = n; \tag{2.1}$$

and similarly any partition π_2 counted by $P_2(L, y, n)$ may be written as

$$\pi_2 = \left\langle 2_1^{\nu(2_1)}, y_1^{\nu(y_1)}, (2y+1)_1^{\nu((2y+1)_1)}, \ldots, 2_L^{\nu(2_L)}, y_L^{\nu(y_L)}, (2y+1)_L^{\nu((2y+1)_L)} \right\rangle,$$

where

$$|\pi_2| = \sum_{k=1}^{L} \big(2_k \cdot \nu(2_k) + y_k \cdot \nu(y_k) + (2y+1)_k \cdot \nu\big((2y+1)_k\big)\big) = n. \tag{2.2}$$

Here we are following the convention as in [3] whereby the exponents represent the frequencies of the parts.

Let $Q(s_t)$ and $R(s_t)$ denote the quotient and remainder, respectively, upon dividing $\nu(s_t)$ by 2, taking $0 \leqslant R(s_t) \leqslant 1$. Our injection then maps a partition π_2 to a partition π_1 as follows:

$$\nu\big((2y)_k\big) = \begin{cases} Q\left((2y+1)_{k/2}\right) & \text{if } k \text{ is even,} \\ Q\left(y_{(k+1)/2}\right) & \text{if } k \text{ is odd,} \end{cases} \tag{2.3a}$$

$$\nu\big((y+2)_k\big) = \begin{cases} \nu((2y+1)_k) & \text{if } L/2 < k \leqslant L, \\ 2\nu(2_{2k}) + R((2y+1)_k) & \text{if } 1 \leqslant k \leqslant L/2, \end{cases} \tag{2.3b}$$

$$\nu(1_k) = \begin{cases} \nu(y_k) & \text{if } (L+1)/2 < k \leqslant L, \\ 2\nu(2_{2k-1}) + R(y_k) & \text{if } 1 < k \leqslant (L+1)/2, \\ R(y_1) + 2\nu(2_1) + (y-1)(A+B+C+D) & \text{if } k = 1, \end{cases} \tag{2.3c}$$

where

$$A = \sum_{1 \leqslant b \leqslant L/2} R\big((2y+1)_b\big), \tag{2.3d}$$

$$B = \sum_{L/2 < b \leqslant L} \nu\big((2y+1)_b\big), \tag{2.3e}$$

$$C = \sum_{1 \leqslant b \leqslant (L+1)/2} R(y_b), \tag{2.3f}$$

$$D = \sum_{(L+1)/2 < b \leqslant L} \nu(y_b). \tag{2.3g}$$

For example, we have the following mappings $\pi_2 \mapsto \pi_1$ as part of our injection:

$$\begin{array}{ll}
\langle (2y+1)_k^{2i} \rangle \mapsto \langle (2y)_{2k}^i \rangle & \text{if } 1 \leqslant k \leqslant L/2, \\
\langle (2y+1)_k^{2i+1} \rangle \mapsto \langle 1_1^{y-1}, (y+2)_k, (2y)_{2k}^i \rangle & \text{if } 1 \leqslant k \leqslant L/2, \\
\langle (2y+1)_k \rangle \mapsto \langle 1_1^{y-1}, (y+2)_k \rangle & \text{if } L/2 < k \leqslant L, \\
\langle y_k^{2i} \rangle \mapsto \langle (2y)_{2k-1}^i \rangle & \text{if } 1 \leqslant k \leqslant (L+1)/2, \\
\langle y_k^{2i+1} \rangle \mapsto \langle 1_1^{y-1}, 1_k, (2y)_{2k-1}^i \rangle & \text{if } 2 \leqslant k \leqslant (L+1)/2, \\
\langle y_k^{2i+1} \rangle \mapsto \langle 1_1^{y}, (2y)_1^i \rangle & \text{if } k = 1, \\
\langle y_k \rangle \mapsto \langle 1_1^{y-1}, 1_k \rangle & \text{if } (L+1)/2 < k \leqslant L, \\
\langle 2_k \rangle \mapsto \langle 1_{(k+1)/2}^2 \rangle & \text{if } k \text{ is odd}, \\
\langle 2_k \rangle \mapsto \langle (y+2)_{k/2}^2 \rangle & \text{if } k \text{ is even}.
\end{array}$$

From the rules (2.3a)–(2.3g), it is a relatively straightforward (though perhaps slightly tedious) matter to verify that for any partition π_2 counted by $P_2(L,y,n)$, the corresponding image partition π_1 will be one that is counted by $P_1(L,y,n)$; i.e. if π_2 maps to π_1, then $|\pi_1| = |\pi_2|$. To show that this mapping is injective, we give the inverse map:

$$\nu\big((2y+1)_k\big) = \begin{cases} 2\nu((2y)_{2k}) + R((y+2)_k) & \text{if } 1 \leqslant k \leqslant L/2, \\ \nu((y+2)_k) & \text{if } L/2 < k \leqslant L, \end{cases} \tag{2.4a}$$

$$\nu(y_k) = \begin{cases} 2\nu((2y)_{2k-1}) + R(1_k) & \text{if } 1 \leqslant k \leqslant (L+1)/2, \\ \nu(1_k) & \text{if } (L+1)/2 < k \leqslant L, \end{cases} \tag{2.4b}$$

$$\nu(2_k) = \frac{1}{2} \cdot \begin{cases} \nu((y+2)_{k/2}) - R((y+2)_{k/2}) & \text{if } k \text{ is even}, \\ \nu(1_{(k+1)/2}) - R(1_{(k+1)/2}) & \text{if } k > 1, \text{ odd}, \\ \nu(1_1) - R(1_1) - (y-1)(W+X+Y+Z) & \text{if } k = 1, \end{cases} \tag{2.4c}$$

where

$$W = \sum_{1 \leqslant b \leqslant L/2} R\big((y+2)_b\big), \tag{2.4d}$$

$$X = \sum_{L/2 < b \leqslant L} \nu\big((y+2)_b\big), \tag{2.4e}$$

$$Y = \sum_{1 \leqslant b \leqslant (L+1)/2} R(1_b), \tag{2.4f}$$

$$Z = \sum_{(L+1)/2 < b \leqslant L} \nu(1_b). \tag{2.4g}$$

We note that the only possibly negative quantity exhibited in either the forward map or the inverse map is the partition statistic μ, which takes a partition π_1 counted by $P_1(L,y,n)$ and maps it to

$$\mu(\pi_1) = \nu(1_1) - R(1_1) - (y-1)(W+X+Y+Z), \tag{2.5}$$

i.e. the numerator of the expression given for $\nu(2_1)$ in (2.4c). This is a useful statistic since we must have $\mu(\pi_1) \geqslant 0$ iff π_1, counted by $P_1(L, y, n)$, is the image of some π_2 counted by $P_2(L, y, n)$. (Note that the value of μ is automatically even.)

For example, if $L = 2$, $y = 3$, and $n = 14$, then we have the following partitions π_1 counted by $P_1(2, 3, 14)$ and π_2 counted by $P_2(2, 3, 14)$, where $\pi_2 \mapsto \pi_1$ if they are on the same row, as well as the corresponding value of the statistic μ.

π_2	π_1	$\mu(\pi_1)$
$\langle 3_1, 3_2\rangle$	$\langle 1_1^5, 1_2\rangle$	0
$\langle 2_1^2, 2_2\rangle$	$\langle 1_1^4, 5_1^2\rangle$	4
$\langle 7_1^2\rangle$	$\langle 6_2\rangle$	0
$\langle 2_1^2, 3_1, 7_1\rangle$	$\langle 1_1^9, 5_1\rangle$	4
$\langle 2_1, 3_1^4\rangle$	$\langle 1_1^2, 6_1^2\rangle$	2
$\langle 2_1^4, 3_1^2\rangle$	$\langle 1_1^8, 6_1\rangle$	8
$\langle 2_1^7\rangle$	$\langle 1_1^{14}\rangle$	14
	$\langle 1_1, 5_2\rangle$	−4
	$\langle 5_1, 1_2\rangle$	−4
	$\langle 1_1^3, 5_1, 6_1\rangle$	−2

3. Proof of Theorem 1.3

First, we note that the injection given in Section 2 proves the first part of Theorem 1.3 by virtue of the partition interpretations given by (1.6) and (1.7). What remains to be shown is the last statement in the theorem: *the coefficient* $a(L, y, n)$ *is* 0 *if and only if either* $n \in \{2, 4, 6, \ldots, y+1\} \cup \{y\}$ *or* $(L, y, n) = (1, 3, 9)$.

If $n \in \{2, 4, 6, \ldots, y+1\}$, then $P_1(L, y, n) = P_2(L, y, n) = 1$: here P_1 counts $\langle 1^n\rangle$ and P_2 counts $\langle 2^{n/2}\rangle$. If $n = y$, then $P_1(L, y, n) = P_2(L, y, n) = 1$: here P_1 counts $\langle 1^y\rangle$ and P_2 counts $\langle y\rangle$. If $(L, y, n) = (1, 3, 9)$, then $P_1(1, 3, 9) = P_2(1, 3, 9) = 3$: here P_1 counts $\langle 1^9\rangle$, $\langle 1^4, 5\rangle$, and $\langle 1^3, 6\rangle$; and P_2 counts $\langle 2^3, 3\rangle$, $\langle 2, 7\rangle$, and $\langle 3^3\rangle$. Thus, if either $n \in \{2, 4, 6, \ldots, y+1\} \cup \{y\}$ or $(L, y, n) = (1, 3, 9)$, we have $a(L, y, n) = 0$.

To show the reverse implication we will show that the inverse is true: if $n \notin \{2, 4, 6, \ldots, y+1\} \cup \{y\}$ and $(L, y, n) \neq (1, 3, 9)$, we have $a(L, y, n) > 0$. To accomplish this, we use the partition statistic μ defined previously and we exhibit partitions π_1 counted by $P_1(L, y, n)$ that have $\mu(\pi_1) < 0$, and hence are not mapped to under the injection. We now consider the following three cases (and several subcases), where throughout we assume that y is odd.

Case 1: $y > 3$ *and* $L \geqslant 1$. Here we will examine six subcases.

Subcase 1a. If $0 < b < 2y-1$, with b odd, then $\langle 1^b, (y+2), (2y)^m\rangle$ cannot be in the image since we would have $\mu = b - 1 - (y-1)(1+1) = b - (2y-1) < 0$. This implies that the inequality is strict for all even $n > y+1$ except possibly when $n \equiv y+1 \pmod{2y}$.

Subcase 1b. If $L = 1$ then $\langle 1^{y-3}, (y+2)^2, (2y)^m\rangle$ cannot be in the image since we would have $\mu = y - 3 - (y-1)(2) = -y - 1 < 0$. This implies that the inequality is strict for all even $n > y+1$ with $n \equiv y+1 \pmod{2y}$ when $L = 1$.

Subcase 1c. If $L > 1$ then $\langle 1^{y-2}, 1_2, (2y)^m\rangle$ cannot be in the image since we would have $\mu = y - 2 - 1 - (y-1)(2) = -y - 1 < 0$. This implies that the inequality is strict for all even $n > y+1$ with $n \equiv y+1 \pmod{2y}$ when $L > 1$.

Note that Subcases 1a, 1b, and 1c together show that the inequality is strict for any even $n > y+1$.

Subcase 1d. If $0 \leqslant b < y-1$, with b even, then $\langle 1^b, (y+2), (2y)^m\rangle$ cannot be in the image since we would have $\mu = b - (y-1)(1) = b - (y-1) < 0$. This implies that the inequality is strict for all odd $n > y$ with $n \equiv y+2, y+4, y+6, \ldots,$ or $2y-1 \pmod{2y}$.

Subcase 1e. If $0 < b < y$, with b odd, then $\langle 1^b, (2y)^m\rangle$ cannot be in the image since we would have $\mu = b - 1 - (y-1)(1) = b - y < 0$. This implies that the inequality is strict for all odd $n > 0$ with $n \equiv 1, 3, 5, \ldots,$ or $y-2 \pmod{2y}$.

Subcase 1f. The partition $\langle 1^{y-4}, (y+2)^2, (2y)^m \rangle$ cannot be in the image since we would have $\mu = (y-4) - 1 - (y-1)(1) = -4 < 0$ if $L > 1$ and $\mu = (y-4) - 1 - (y-1)(1+2) = -2y-2 < 0$ if $L = 1$. This implies that the inequality is strict for all odd $n > y$ with $n \equiv y \pmod{2y}$.

From Subcases 1a–1f we may conclude that the inequality is strict when $y > 3$, $L \geqslant 1$, and $n \notin \{2, 4, \ldots, y+1\} \cup \{y\}$.

Case 2: $y = 3$ *and* $L > 1$. In this case the partition $\langle 1_2, 6^m \rangle$ cannot be in the image since we would have $\mu = -2 < 0$. This implies that the inequality is strict for all odd $n > 3$ with $n \equiv 3 \pmod 6$. Together with Subcases 1a, 1c, 1d, and 1e (all with $y = 3$), this shows that if $y = 3$ and $L > 1$, then the inequality is strict when $n \notin \{2, 3, 4\}$.

Case 3: $y = 3$ *and* $L = 1$. In this case the partition $\langle 5^3, 6^m \rangle$ cannot be in the image since we would have $\mu = -6 < 0$. This implies that the inequality is strict for all odd $n > 9$ with $n \equiv 3 \pmod 6$. Together with Subcases 1a, 1b, 1d, and 1e (all with $y = 3$), this shows that if $y = 3$ and $L = 1$, then the inequality is strict when $n \notin \{2, 3, 4, 9\}$.

Cases 1, 2, and 3 together show that the inequality is strict except for the following possibilities:

- $y > 3$, $L \geqslant 1$, and $n \in \{2, 4, \ldots, y+1\} \cup \{y\}$;
- $y = 3$, $L > 1$, and $n \in \{2, 3, 4\}$;
- $y = 3$, $L = 1$, and $n \in \{2, 3, 4, 9\}$.

However, we have already shown that the inequality is an equality at these points; thus the theorem is proven.

4. A further generalization

In Theorem 1.3 we may replace $y+2$ with any integer x, provided $1 < x \leqslant y+2$, and still have a perfectly viable inequality; thus, the following generalization.

Theorem 4.1. *For any $L > 0$, any odd $y > 1$, and any x with $1 < x \leqslant y+2$, the q-series expansion of*

$$\frac{1}{(q, q^x, q^{2y}; q^{2y+2})_L} - \frac{1}{(q^2, q^y, q^{2y+1}; q^{2y+2})_L} = \sum_{n=1}^{\infty} a(L, y, n, x) q^n \tag{4.1}$$

has non-negative coefficients. Furthermore, the coefficient $a(L, y, n, x)$ is 0 if and only if one of the following three conditions holds:

(1) *$n < x$ and n is even.*
(2) *$n = y$ and $y < x$.*
(3) *$n = 9$ and $(L, y, n, x) = (1, 3, 9, 5)$.*

We note that the products on the left of (4.1) can be interpreted as

$$\frac{1}{(q, q^x, q^{2y}; q^{2y+2})_L} = \sum_{n=1}^{\infty} P_1'(L, y, n, x) q^n \tag{4.2}$$

and

$$\frac{1}{(q^2, q^y, q^{2y+1}; q^{2y+2})_L} = \sum_{n=1}^{\infty} P_2(L, y, n) q^n, \tag{4.3}$$

where $P_1'(L, y, n, x)$ denotes the number of partitions of n into parts $\equiv 1, x, 2y \pmod{2(y+1)}$ with the largest part not exceeding $2(y+1)L - 2$ and, as before, $P_2(L, y, n)$ denotes the number of partitions of n into parts $\equiv 2, y, 2y+1 \pmod{2(y+1)}$ with the largest part not exceeding $2(y+1)L - 1$.

Proof of Theorem 4.1. We will prove the first part of the theorem by producing a norm-preserving injection from partitions π_1 counted by $P_1(L, y, n)$ to partitions π_1' counted by $P_1'(L, y, n, x)$ and then relying on the fact that composition of injections is injective. Using ν' to distinguish counting parts of π_1' from counting parts of π_1, we take

$$\nu'\big((2y)_k\big) = \nu\big((2y)_k\big), \tag{4.4a}$$

$$\nu'(x_k) = \nu\big((y+2)_k\big), \tag{4.4b}$$

$$\nu'(1_k) = \begin{cases} \nu(1_k) & \text{if } k > 1, \\ \nu(1_1) + (y+2-x)\sum_{1 \leqslant b \leqslant L} \nu((y+2)_b) & \text{if } k = 1. \end{cases} \tag{4.4c}$$

The inverse map is immediate:

$$\nu\big((2y)_k\big) = \nu'\big((2y)_k\big), \tag{4.5a}$$

$$\nu(x_k) = \nu'\big((y+2)_k\big), \tag{4.5b}$$

$$\nu(1_k) = \begin{cases} \nu'(1_k) & \text{if } k > 1, \\ \nu'(1_1) - (y+2-x)\sum_{1 \leqslant b \leqslant L} \nu'(x_b) & \text{if } k = 1. \end{cases} \tag{4.5c}$$

It is then very straightforward to show that this injection is norm-preserving. Thus, we have

$$P_1'(L, y, n, x) - P_1(L, y, n) \geqslant 0, \tag{4.6}$$

and when we compose the injection given by (2.3a)–(2.3g) with the one presented above, we obtain a mapping of partitions

$$\pi_2 \mapsto \pi_1 \mapsto \pi_1' \tag{4.7}$$

which is an injection that maps $\pi_2 \mapsto \pi_1'$. Thus, we have

$$P_1'(L, y, n, x) \geqslant P_1(L, y, n) \geqslant P_2(L, y, n). \tag{4.8}$$

For the second part of the theorem, we note that it is straightforward to verify that for any n prescribed by conditions (1)–(3), one does, in fact, obtain $a(L, y, n, x) = 0$. Also, if $P_1(L, y, n) > P_2(L, y, n)$, then $P_1'(L, y, n, x) > P_2(L, y, n)$. So if n is even and $x \leqslant n < y + 2$, then $P_1'(L, y, n, x) \geqslant 2$, counting at least $\langle 1^n \rangle$ and $\langle 1^{n-x}, x \rangle$, whereas $P_2(L, y, n) = 1$, counting only $\langle 2^{n/2} \rangle$; hence condition (1). Now if $n = y$ and $x \leqslant y$ then $P_1'(L, y, n, x) \geqslant 2$, counting at least $\langle 1^n \rangle$ and $\langle 1^{n-x}, x \rangle$, whereas $P_2(L, y, n) = 1$, counting only $\langle y \rangle$; hence condition (2). Finally, $(L, y, n, x) = (1, 3, 9, 5)$ is the same as $(L, y, n) = (1, 3, 9)$ in Theorem 1.3; hence condition (3). □

In 1971, Andrews [1] used a simple inductive technique to prove the following theorem.

Theorem 4.2. *Let $S = \{a_i\}_{i=1}^{\infty}$ and $T = \{b_i\}_{i=1}^{\infty}$ be two strictly increasing sequences of positive integers such that $b_1 = 1$ and $a_i \geqslant b_i$ for all i. Let $\rho(S; n)$ (resp. $\rho(T; n)$) denote the number of partitions of n into parts taken from S (resp. T). Then*

$$\rho(T; n) \geqslant \rho(S; n)$$

for all n.

We note that this theorem provides an alternate proof of the partition inequality (4.6), as well as the subset of cases $2 \leqslant x \leqslant y$ in Theorem 4.1. Observe, however, that the cases when $x = y + 2$ and when $x = y + 1$ in Theorem 4.1 are not covered by Andrews' theorem but are covered by our new Theorems 1.3 and 4.1. In addition, Theorems 1.3 and 4.1 also provide explicit conditions for when the inequality is strict.

5. Concluding remarks

We plan to study more general partition inequalities in a later paper, including cases with higher modulus, cases where y is even, and cases with more than three residues. Experimental evidence leads us to the following conjecture for three residues, which we are actively pursuing.

Conjecture 5.1. *For any $L > 0$, any $z > 1$, any $y > z$, any x with $y < x \leqslant y + z$, and any $m \geqslant yz + 2$, the q-series expansion of*

$$\frac{1}{(q, q^x, q^{yz}; q^m)_L} - \frac{1}{(q^z, q^y, q^{yz+1}; q^m)_L} = \sum_{n=1}^{\infty} a(L, y, n, x, z, m) q^n \tag{5.1}$$

has only non-negative coefficients iff z does not divide y; if z divides y then there are finitely many negative coefficients.

In particular, we plan to prove the following.

Proposal 5.2. *For any $L > 0$ and any even $y > 2$, the q-series expansion of*

$$\frac{1}{(q, q^{y+2}, q^{2y}; q^{2y+2})_L} - \frac{1}{(q^2, q^y, q^{2y+1}; q^{2y+2})_L} = \sum_{n=1}^{\infty} a(L, y, n) q^n \tag{5.2}$$

has non-negative coefficients except for $a(L, y, y) = -1$.

Note that Conjecture 5.1 with $z = 2$, $m = 2y + 2$, and y odd is part of Theorem 4.1, and that Proposal 5.2 is the natural companion to Theorem 1.3. Also note that, as before, Theorem 4.2 would clearly establish the corresponding result to Conjecture 5.1 when $x \leqslant y$, leaving the more difficult cases when $x > y$ to be addressed.

Finally, we would like to point out that the problems discussed in this paper belong to a broad class of positivity problems in q-series and partitions. These problems often are very deceptive because they are so easy to state but so painfully hard to solve. As an example, consider the famous Borwein problem:

> Let $B_e(L, n)$ (resp. $B_o(L, n)$) denote the number of partitions of n into an even (resp. odd) number of distinct non-multiples of 3 with each part less than $3L$. Prove that for all positive integers L and n, $B_e(L, n) - B_o(L, n)$ is non-negative if n is a multiple of 3 and non-positive otherwise.

Further background on this conjecture may be found in [2,6,7,10,11].

Acknowledgments

We are grateful to George Andrews for bringing our attention to [1], and to Krishna Alladi and Frank Garvan for their interest.

References

[1] George E. Andrews, On a partition problem of H.L. Alder, Pacific J. Math. 36 (1971) 279–284.
[2] George E. Andrews, On a conjecture of Peter Borwein, in: Symbolic Computation in Combinatorics Δ_1, Ithaca, NY, 1993, J. Symbolic Comput. 20 (5–6) (1995) 487–501.
[3] George E. Andrews, The Theory of Partitions, Cambridge Math. Lib., Cambridge University Press, Cambridge, 1998, reprint of the 1976 original.
[4] George E. Andrews, Differences of partition functions: The anti-telescoping method, 2011, preprint; in: H. Farkas, R. Gunning, M. Knopp, B.A. Taylor (Eds.), Memorial Volume for Leon Ehrenpreis, in: Dev. Math., vol. 28, in press.
[5] Alexander Berkovich, Frank G. Garvan, Dissecting the Stanley partition function, J. Combin. Theory Ser. A 112 (2) (2005) 277–291.

[6] Alexander Berkovich, S. Ole Warnaar, Positivity preserving transformations for q-binomial coefficients, Trans. Amer. Math. Soc. 357 (6) (2005) 2291–2351 (electronic).
[7] David M. Bressoud, The Borwein conjecture and partitions with prescribed hook differences, in: The Foata Festschrift, Electron. J. Combin. 3 (2) (1996), Research Paper 4, approx. 14 pp. (electronic).
[8] Kevin W.J. Kadell, An injection for the Ehrenpreis Rogers–Ramanujan problem, J. Combin. Theory Ser. A 86 (2) (1999) 390–394.
[9] L.J. Rogers, S. Ramanujan, Proof of certain identities in combinatory analysis, Proc. Cambridge Philos. Soc. 19 (1919) 211–216.
[10] S. Ole Warnaar, The generalized Borwein conjecture. I. The Burge transform, in: q-Series with Applications to Combinatorics, Number Theory and Physics, Urbana, IL, 2000, in: Contemp. Math., vol. 291, Amer. Math. Soc., Providence, RI, 2001, pp. 243–267.
[11] S. Ole Warnaar, The generalized Borwein conjecture. II. Refined q-trinomial coefficients, Discrete Math. 272 (2–3) (2003) 215–258.