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Completeness theory for the product of finite partial algebras

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Abstract

A general completeness criterion for the finite product $\prod \mathbb{P}(k_i)$ of full partial clones $\mathbb{P}(k_i)$ (composition-closed subsets of partial operations) defined on finite sets $E(k_i)$ ($|E(k_i)| \geqslant 2$, $i = 1,\ldots,n$, $n \geqslant 2$) is considered and a Galois connection between the lattice of subclones of $\prod \mathbb{P}(k_i)$, called partial n-clones, and the lattice of subalgebras of multiple-base invariant relation algebra, with operations of a restricted quantifier free calculus, is established. This is used to obtain the full description of all maximal partial n-clones via multiple-base invariant relations and, thus, to solve the general completeness problem in $\prod \mathbb{P}(k_i)$.

Keywords: Product of partial clones; Galois connection; Maximal partial clone; Completeness problem

1. Introduction and basic definitions

Let $k \geqslant 2$ be an integer and $E(k) = \{0, 1, \ldots, k-1\}$. For an integer $m \geqslant 1$ an m-ary partial operation f on $E(k)$ (an m-ary partial function of k-valued logic) is a one-to-one map from a subset $D_f = \mathrm{Dom}(f)$ of $E^m(k)$ (called the domain of f) into $E(k)$, $f : D_f \to E(k)$. Denote $P^m(k)$ the set of all partial m-ary operations on $E(k)$ including the empty operation p_m having an empty domain. Set $P(k) = \bigcup_{m\geqslant 1} P^m(k)$.

The notion of a composition of partial operations from $P(k)$ is formally equivalent to the operations of iterative Post algebra $\mathbb{P}(k) = \langle P(k); \zeta, \tau, \Delta, *, e_1^2\rangle$ (see [11]), where $e_1^2(x_1, x_2) = x_1$ is a binary selector (projection) and for any $n > 1$ and $f \in P^n(k)$ we

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have

$$(\zeta f)(x_1,x_2,x_3,\dots,x_n) = f(x_2,x_3,\dots,x_n,x_1),$$

$$(\tau f)(x_1,x_2,x_3,\dots,x_n) = f(x_2,x_1,x_3,\dots,x_n),$$

$$(\Delta f)(x_1,x_2,x_3,\dots,x_n) = f(x_1,x_1,x_3,\dots,x_{n-1}),$$

where the left sides of identities are defined whenever the right sides are defined. For $n=1$ we put $\zeta f = \tau f = \Delta f = f$.

Next for $f \in P^n(k)$ and $g \in P^m(k)$ $(n,m \geqslant 1)$ we set

$$(f * g)(x_1,\dots,x_{m+n-1}) = f(g(x_1,\dots,x_m),x_{m+1},\dots,x_{m+n-1}),$$

where again the left side is defined whenever the right side is defined.

In universal algebra terminology $\mathbb{P}(k)$ is called the *full partial clone* [7] and each subalgebra of it is called a *partial clone* on $E(k)$. A set S of partial operations is *complete* in $\mathbb{P}(k)$ when it is a generating set in $P(k)$ with respect to operations of the iterative Post algebra (or, equivalently, with respect to any compositions of partial operations). A general completeness criterion establishes the necessary and sufficient conditions for a given set $S \subset P(k)$ to be complete. Since $\mathbb{P}(k)$ is finitely generated this criterion is known (see, e.g., [2] or [4]) to be based on the knowledge of the full list of all maximal subalgebras of $\mathbb{P}(k)$ or *maximal partial clones* on $E(k)$ $(k \geqslant 2)$.

For $k=2$ this problem was introduced and solved by Freivald [3,4] who listed all 8 maximal partial clones on $E(2)$. The case $k \geqslant 3$ was considered in [15], where the list of maximal partial clones on $E(3)$ was presented (3 clones were inadvertently omitted, see [6,20]),[2] and the Slupecki-type criterion for $k \geqslant 3$ was given (completeness with all unary partial operations), as well as some series of maximal partial clones on $E(k)$, $k \geqslant 4$, were found. The full description of all maximal partial clones on $E(k)$, $k \geqslant 4$, was provided independently by Lo Czukai [9,10] (see also comments on these results in [20]), Haddad and Rosenberg [5,7] and the author [20]. All of the variants of a final solution were grounded on the fact [15] that, with one exception, each maximal partial clone is determined by a relation of arity less or equal k defined on the same set $E(k)$, $k \geqslant 4$.

Remark. In the case of an infinite base set E the general completeness criterion cannot be formulated entirely in terms of maximal partial clones (see, e.g., [16,24]), although the knowledge of these clones is still of a great importance. We'll mention only three results in this field: (1) Slupecki-type criterion for local completeness in $P(E)$ [17]; (2) the full description of all maximal local partial clones [22]; (3) the full description of maximal partial clones which can be determined by a finite arity relation on E [24].

[2] The list of all 58 maximal partial clones on $E(3)$ was also presented in the thesis: D. Lau, "Eingenschaften gewisser abgeschlossener Klassen in Postschen Algebren", University Rostock, GDR, 1977.

In this paper we consider the completeness problem for vectors of partial operations defined on finite sets. For integers $k_1,\dots,k_n$ greater than 1 and $m \geqslant 1$ consider the set:

$$A(m) = P^m(k_1) \times \cdots \times P^m(k_n) \tag{1}$$

of all n-vectors ($n \geqslant 2$) of partial m-ary operations defined on the sets $E(k_1),\dots,E(k_n)$ resp. Denote $\mathbf{e}_1^2 = \langle e_1^2(x,y),\dots,e_1^2(x,y)\rangle \in A(2)$ the n-vector produced from the projection $e_1^2(x,y) = x$. We introduce the arity-calibrated product of full partial clones as follows:

$$\prod \mathbb{P}(k_i) := \prod_{i=1}^{n} \mathbb{P}(k_i) = \mathbb{P}(k_1) \times \cdots \times \mathbb{P}(k_n)$$

$$= \left\langle \bigcup_{m \geqslant 1} A(m); \zeta, \tau, \Delta, *, \mathbf{e}_1^2 \right\rangle, \tag{2}$$

where the operations ζ, τ, Δ, and $*$ are applied coordinatewise.

So if $\mathbf{f} = \langle f_1,\dots,f_n\rangle \in A(m)$ and $\mathbf{g} = \langle g_1,\dots,g_n\rangle \in A(s)$ $(m,s \geqslant 1)$, then $\mathbf{f} * \mathbf{g} = \langle f_1 * g_1,\dots,f_n * g_n\rangle$ and $\varepsilon\mathbf{f} = \langle \varepsilon f_1,\dots,\varepsilon f_n\rangle$, where $\varepsilon \in \{\zeta,\tau,\Delta\}$. The n-vector $\mathbf{e}_1^2$ is a constant operation. This formalism represents all compositions of n-vectors of partial algebraic operations. The product $\prod \mathbb{P}(k_i)$ is called the *full partial n-clone*. Any its subalgebra is called a *partial n-clone*, which is exactly a subdirect product of n partial clones defined on the sets $E(k_i)$ $(i=1,\dots,n)$. Next a partial n-clone is called *maximal* if there is no partial n-clone, other than the full n-clone, covering it.

Similarly to its factors $\mathbb{P}(k_i)$ $(i=1,\dots,n)$ the full partial n-clone $\prod \mathbb{P}(k_i)$ is finitely generated (e.g. it is easy to verify that $A(2)$ is a finite generating set in it). Hence, from the common algebraic results (see [2]) it follows that each proper partial n-clone is contained in a maximal partial n-clone and, therefore, a set S is complete in $\prod \mathbb{P}(k_i)$ if and only if it is not contained in any maximal partial n-clone. So the description of all maximal partial n-clones (dual atoms in the lattice of all partial n-clones) provides the solution of the general completeness problem in $\prod \mathbb{P}(k_i)$.

We will explore the properties of the lattice of partial n-clones via multiple-base invariant relations defined on the same base sets $E(k_i)$ $(i=1,\dots,n)$, similar to the case of products of the full clones of everywhere defined operations $\mathbb{Q}(k_1)\times\cdots\times\mathbb{Q}(k_n)$ (see e.g., [14,18,19,21]), where $\mathbb{Q}(k) = \langle Q(k);\ \zeta,\tau,\Delta,*,e_1^2\rangle$ is the full clone of algebraic operations and $Q(k)$ is the set of all everywhere defined operations on $E(k)$ $(k \geqslant 2)$.

We will follow a traditional way (see [1,14,16]) in providing the relational description of dual atoms in the lattice of partial n-clones. First we establish a Galois connection between the lattice of partial n-clones closed under all restrictions of their elements and the lattice of multiple-base relations sets closed under the formation of $(\&, =_{1,\dots,} =_n)$-formulas of the restricted quantifier free first order calculus. Then we prove that each maximal partial n-clone, with n exceptions, is determined by a multiple-base relation, which is minimal under the expressibility by these formulas. Next starting with the Slupecki criterion we find all those multiple-base relations for the general case $\mathbb{P}(k_1) \times \cdots \times \mathbb{P}(k_n)$ using predicative descriptions and also combinatorial considerations as well as for the case $\mathbb{P}(2) \times \cdots \times \mathbb{P}(2)$ which requires only predicative descriptions of relations. The short version of these results, without proofs, was published in [26].

2. Multiple-base relations

We consider multiple-base relations on n base sets $E(k_1),\dots,E(k_n)$ $(n \geqslant 1)$, each of them corresponds to its own *sort* of variables from the set $I=\{1,\dots,n\}$. In what follows we denote x^i or y^i variables of ith sort in both function and relation taking on values from $E(k_i)$ $(i=1,\dots,n)$. Let $m_1,\dots,m_n$ be nonnegative integers. A *multiple-base* relation $R(x_1^1,\dots,x_{m_1}^1,x_1^2,\dots,x_{m_2}^2,\dots,x_1^n,\dots,x_{m_n}^n)$ of arity $(m_1,\dots,m_n)$ is a relation with m_i coordinates from the set $E(k_i)$, where $m_i \geqslant 0$ $(i=1,\dots,n)$. In case $m_j>0$, while $m_i=0$ for all $1\leqslant i\leqslant n,\ i\neq j$, we identify this relation with an ordinary single-base relation on the set $E(k_j)$. The set $J(R)$ of all indices j for which $m_j>0$ is called *type* of R, $J(R)\subseteq I$.

Example 2.1. Let $n=3$ and $k_i=2$ $(i=1,2,3)$. Then $R\equiv(x_1^1=x_2^1)\,\&\,(x_1^2=x_2^2)$, where & is a conjunction of multi-sorted predicates, is a multiple-base relation of arity $(2,2,0)$ and type $J(R)=\{1,2\}$. Notice that in order to present R as a set of $(2,2)$-tuples one has to distinguish each base set from the others. Namely, one way is to put semicolon to separate coordinates of different sorts. So we have $R=\{(0,0;0,0),(0,0;1,1),(1,1;0,0),(1,1;1,1)\}$. Another way [14] is to assume that all $E(k_i)$ $(i=1,\dots,n)$ are distinct pairwise disjoint sets (this assumption in no way affects further results). So we may rewrite $R=\{(0,0,a,a),(1,1,a,a),(0,0,b,b),(1,1,b,b)\}$, where $E(k_1)=\{0,1\},E(k_2)=\{a,b\}$. In the sequel we will use (whenever it is possible) different letters for variables from different sorts, so we may put in our case $R(x_1,x_2,y_1,y_2)\equiv(x_1=x_2)\,\&\,(y_1=y_2)$.

Definition 2.1. A vector of partial operations $\mathbf{f}=\langle f_1(x_1,\dots,x_m),\ f_2(y_1,\dots,y_m),\dots,f_n(z_1,\dots,z_m)\rangle$ $(m\geqslant 1)$ preserves a multiple-base relation $R(x_1,\dots,x_k,y_1,\dots,y_p,\dots,z_1,\dots,z_s)$ of arity $(k,p,\dots,s)$ if

$$\begin{aligned}
&R(x_{11},\dots,x_{1k},y_{11},\dots,y_{1p},\dots,z_{11},\dots,z_{1s})\&\cdots\\
&\quad\&R(x_{m1},\dots,x_{mk},y_{m1},\dots,y_{mp},\dots,z_{m1},\dots,z_{ms})\\
&\quad\&f_1(x_{11},\dots,x_{m1})=x_1\&\cdots\\
&\quad\&f_1(x_{1k},\dots,x_{mk})=x_k\&f_2(y_{11},\dots,y_{m1})=y_1\&\cdots\\
&\quad\&f_2(y_{1p},\dots,y_{mp})=y_p\&\cdots\\
&\quad\&f_n(z_{11},\dots,z_{m1})=z_1\&\cdots\\
&\quad\&f_n(z_{1s},\dots,z_{ms})=z_s\to R(x_1,\dots,x_k,y_1,\dots,y_p,\dots,z_1,\dots,z_s)
\end{aligned}\tag{3}$$

holds for all values of all sorts of variables $x,y,\dots,z$ involved.

Notice that a predicate $f(x_1,\dots,x_m)=x(f\in P^m(k))$ is valid in (3) whenever $f(x_1,\dots,x_m)$ is defined and equals x. Hence each $\mathbf{f}$ that contains a void (empty) function as its coordinate preserves any relation R. Denote $F=\bigcup_{m\geqslant 1}\{\langle f_1,\dots,f_n\rangle\in A(m)$: $\exists i\in\{1,\dots,n\}\ \ f_i=p_m\}$ the set of all vector-functions having at least one empty coordinate.

Definition 2.1 can be interpreted in terms of constructing of all possible $m \times (k + p + \cdots + s)$ matrices over the sets $E(k_1), \ldots, E(k_n)$ with rows that are tuples from R and then applying $\mathbf{f}$ coordinatewise to these matrices according to each sort of variables. Namely, f_1 is applying to k coordinates of the 1st sort, $\ldots, f_n$ is applying to s coordinates of the nth sort. Finally, if the result of each application of $\mathbf{f}$ to any matrix constructed above (while existed) is also a tuple of R, then $\mathbf{f}$ preserves R.

For everywhere defined vector-operations from $\mathbb{Q}(k_1) \times \cdots \times \mathbb{Q}(k_n)$, the expression (1) coincides with the definition given in [14,19]. If $n = 1$, then we obtain partial operations and relations on $E(k)$, $k \geqslant 2$ (see, e.g., [16]). And, finally, for $f \in Q(k)$ we get the conventional definition of an algebraic operation preserving a relation on the same set $E(k)$.

Let $\mathrm{Pol}(R) = \{\mathbf{f} \in \prod \mathbb{P}(k_i)$: $\mathbf{f}$ preserves $R\}$ and $\mathrm{Pol}^t(R) = \{\mathbf{f} \in \prod \mathbb{Q}(k_i)$: preserves $R\}$. Clearly $\mathrm{Pol}(R)(\mathrm{Pol}^t(R))$ is a partial n-clone (n-clone, respectively) and $F \subset \mathrm{Pol}(R)$. Set $\mathrm{Pol}(\mathfrak{R}) = \cap_{R \in \mathfrak{R}} \mathrm{Pol}(R)$ for any set $\mathfrak{R}$ of multiple-base relations.

Example 2.2. Let R be the relation of Example 2.1. Then it is easy to verify that $\mathrm{Pol}(R) = \prod \mathbb{P}(k_i)$ and also $\mathrm{Pol}^t\ (R) = \prod \mathbb{Q}(k_i)$ for any $n \geqslant 2$.

Let $\mathbf{f}, \mathbf{g} \in A(m)$ ($m \geqslant 1$) be such that $\mathrm{Dom}(g_i) \subseteq \mathrm{Dom}(f_i)$ and $g_i = f_i|\mathrm{Dom}(g_i)$ ($i = 1, \ldots, n$). We call $\mathbf{g}$ a *restriction* of $\mathbf{f}$ and in turn $\mathbf{f}$ is called an *extension* of $\mathbf{g}$. Clearly if $\mathbf{f}$ preserves R, then $\mathbf{g}$ also preserves R and so each partial n-clone $\mathrm{Pol}(\mathfrak{R})$ is restriction-closed. The converse is also true.

Proposition 2.1. *Any partial n-clone can be presented in the form* $\mathrm{Pol}(\mathfrak{R})$ *if and only if it is restriction-closed and also contains F.*

Proof. Let $\mathbf{A}$ be a restriction-closed partial n-clone and $F \subset \mathbf{A}$. Similar to the case $n = 1$ (see [16]) we introduce *m-graphs* of $\mathbf{A}$ ($m = 1, 2, \ldots$) as follows: for each set $D \subseteq E^m(k_1) \cup \cdots \cup E^m(k_n)$, $D \neq \emptyset$, $1 \leqslant |D| \leqslant k_1^m + \cdots + k_n^m$, which is considered as m multiple-base tuples $r_1, \ldots, r_m$ of the same arity $(s_1, \ldots, s_n)$ $(0 \leqslant s_i \leqslant k_i^m,\ i = 1, \ldots, n)$ and presented as a $m \times (s_1 + \cdots + s_n)$ matrix $D = [r_1, \ldots, r_{m.}]$ over $E(k_1), \ldots, E(k_{n.})$, we define the relation of arity $(s_1, \ldots, s_n)$:

$$G_m(\mathbf{A}, D) = \{r\text{: } \mathbf{f}(r_1, \ldots, r_m) = r \text{ for some } \mathbf{f} \in \mathbf{A} \text{ of arity } m \geqslant 1\}, \tag{4}$$

where $\mathbf{f}(r_1, \ldots, r_m)$ is a $(s_1, \ldots, s_n)$-tuple obtained by column-wise application of $\mathbf{f}$ to $[r_1, \ldots, r_m]$.

Notice that in this case we have $D \subseteq \mathrm{Dom}(f_1) \cup \cdots \cup \mathrm{Dom}(f_n)$. Then we introduce the set of relations $\mathbf{G} = \{G_m(\mathbf{A}, D)$: for all non-void subsets D and $m \geqslant 1\}$. Next we prove:

$$\mathbf{A} = \mathrm{Pol}(\mathbf{G}). \tag{5}$$

It is easy to verify that $\mathbf{A}$ preserves each relation (4) and so we have $\mathbf{A} \subseteq \mathrm{Pol}(\mathbf{G})$. Now assume that there exists $\mathbf{f} \in \mathrm{Pol}(\mathbf{G}) \backslash \mathbf{A}$ of arity $m \geqslant 1$. Consider $G_m(\mathbf{A}, D)$, where

$D = \mathrm{Dom}(f_1) \cup \cdots \cup \mathrm{Dom}(f_n)$. Then by (4) we have $\mathbf{f}(r_1,\ldots,r_m) = r \notin G_m(\mathbf{A},D)$ (otherwise $\mathbf{f}\in\mathbf{A}$). Hence $\mathbf{f}$ does not preserve this relation. On the other hand, $\mathbf{f}$ preserves each relation from $\mathbf{G}$. This contradiction proves (5). □

For any nonempty system $\mathbf{A}$ of partial n-operations let $\mathrm{Inv}(\mathbf{A})$ be the set of all multiple-base relations that are preserved by each element of $\mathbf{A}$: $\mathrm{Inv}(\mathbf{A}) = \{R\colon \mathbf{A} \subseteq \mathrm{Pol}(R)\}$. The functors Pol and Inv establish the *Galois connection* (see, e.g., [1]) between the sets of partial n-operations and multiple-base relations. The sets having the form $\mathrm{Pol}(\mathfrak{R})$ and $\mathrm{Inv}(\mathbf{A})$ are called *Galois-closed* and consequently $\mathrm{Pol}(\mathrm{Inv}(\mathbf{A}))$ $(\mathrm{Inv}(\mathrm{Pol}(\mathfrak{R}))$ is called the *Galois closure* on sets of partial n-operations (sets of n-base relations, respectively).

Notice that Proposition 2.1 gives us the description of Galois-closed sets on the side of partial n-operations. In order to produce similar description on another side we consider some operations on n-base relations. Let $=_i$ be the equality relation on $E(k_i)$ $(i=1,\ldots,n)$. We introduce $(\&,=_1,\ldots,=_n)$*-formulas* of the restricted multi-sorted first order calculus over the set of relations $\mathfrak{R}$ which are constructed by the operation & from $=_i$ $(i=1,\ldots,n)$ and the symbols of relations from $\mathfrak{R}$ with arbitrary permutations and identifications of variables. Operations π_i $(i=1,\ldots,n)$, peculiar to the case of partial n-operations, are used to obtain relations of the smaller type. Namely, if R can be presented in the form $(x^i = x^i)\,\&\,R'$, then $\pi_i(R) = R'$, otherwise $\pi_i(R) = R$ $(i = 1,\ldots,n)$.

Example 2.3. If $\mathfrak{R}$ is the empty set, then applying & -formulas to $=_i$ $(i = 1,\ldots,n)$ we obtain *multiple-base diagonals* [14], which can be presented in the form $D = D_1\,\&\,\cdots\,\&\,D_n$, where each D_i is a single-base diagonal on $E(k_i)$ constructed by a & -formula from $=_i$ $(i = 1,\ldots,n)$. Denote $\mathbf{D}$ the set of all n-base diagonals including empty relations. It is easy to check that $\mathrm{Pol}(D) = \prod \mathbb{P}(k_i)$ and also $\mathrm{Pol}^t(D) = \prod \mathbb{Q}(k_i)$ for any $D\in\mathbf{D}$ $(n \geqslant 2)$.

Clearly $\mathrm{Pol}(R) = \mathrm{Pol}(\pi_i R)$ $(i = 1,\ldots,n)$, and if a relation Q is constructed by some $(\&,=_1,\ldots,=_n)$-formula from $\mathfrak{R}$, then $\mathrm{Pol}(\mathfrak{R}) \subseteq \mathrm{Pol}(Q)$. Applying antimonotone property of the functor Inv we obtain $\mathrm{Inv}(\mathrm{Pol}(\mathfrak{R})) \supseteq \mathrm{Inv}(\mathrm{Pol}(Q))$, which with $Q\in\mathrm{Inv}(\mathrm{Pol}((Q))$, gives us $Q\in\mathrm{Inv}(\mathrm{Pol}(\mathfrak{R}))$. Thus, we proved the property:

Any set of the form $\mathrm{Inv}(\mathbf{A})$, $\mathbf{A} \subseteq \prod \mathbb{P}(k_i)$, is closed under application of $(\&,=_1,\ldots,=_n)$-formulas and also operations π_i $(i = 1,\ldots,n)$.

The converse is also true, and in this way we obtain the characteristics of Galois-closed sets of multiple-base relations.

Theorem 2.1. *Any system of n-base relations has the form* $\mathrm{Inv}(\mathbf{A})$, $\mathbf{A} \subseteq \prod \mathbb{P}(k_i)$, *if and only if it is closed under formation of* $(\&,=_1,\ldots,=_n)$*-formulas and application of* π_i $(i = 1,\ldots,n)$.

Proof. $(\Rightarrow)$ See the property from the above.

$(\Leftarrow)$ Without the loss of generality we consider $n = 2$. The common case can be obtained by using the same technique.

Lemma 2.1. *Let* $\mathfrak{R}$ *be a set of* 2*-base relations which is closed under formation of* $(\&, =_1, =_2)$*-formulas and* π_i $(i=1,2)$. *Then for every* $R \in \mathrm{Inv}(\mathrm{Pol}(\mathfrak{R}))$ *we have* $R \in \mathfrak{R}$.

Proof. Clearly $\mathbf{D} \subseteq \mathfrak{R}$. Let $R(x_1,\dots,x_s,\ y_1,\dots,y_m)$, $R \in \mathrm{Inv}(\mathrm{Pol}(\mathfrak{R}))$, be a 2-base non-diagonal relation of arity (s,m), $s,m \geqslant 1$. Consider the set $N=\{Q_1,\dots,Q_t\}$ of all 2-base relations Q_i from $\mathfrak{R}$ such that $R \subseteq Q_i$ $(i=1,\dots,t)$ (inclusion of 2-base relations as sets of (s,m)-tuples). It is obvious that this set is non-void, since it contains at least the full relation of arity (s,m). Then we construct the relation T of arity (s,m):

$$T(x_1,\dots,x_s,y_1,\dots,y_m) \equiv \&_{i=1}^{t} Q_i(x_1,\dots,x_s,y_1,\dots,y_m). \tag{6}$$

Since T itself is constructed by a $(\&, =_1, =_2)$-formula we have $T \in \mathfrak{R}$ and, therefore, $R \subseteq T$. Our goal is to show that $R \equiv T$ which proves the lemma.

Let $R \subset T$ (strict inclusion) and $R=\{r_1,\dots,r_n\}$ be presented as a set of n (s,m)-tuples, $n=|R| \geqslant 1$. Choose an (s,m)-tuple $r \in T \backslash R$. Then we define a 2-mapping $\mathbf{f}=\langle f_1, f_2\rangle$ of arity n: $\mathrm{Dom}(\mathbf{f})=[r_1,\dots,r_n]=\{\langle r_1(i),\dots,r_n(i)\rangle\colon i=1,\dots,s+m\}$ and $\mathbf{f}(r_1,\dots,r_n)=\langle f_1(r_1(1),\dots,r_n(1)),\dots,f_1(r_1(s),\dots,r_n(s)),\quad f_2(r_1(s+1),\dots,r_n(s+1)),\dots, f_2(r_1(s+m),\dots,r_n(s+m)\rangle = \langle r(1),\dots,r(s+m)\rangle = r$.

Since R is a non-full relation $\mathbf{f}$ is not everywhere defined. In addition, $\mathbf{f}$ is a partial 2-operation, i.e., both components f_1 and f_2 are one-to-one partial operations. In other words, for every equal columns $\langle r_1(i),\dots,r_n(i)\rangle$ and $\langle r_1(j),\dots,r_n(j)\rangle$ from $\mathrm{Dom}(\mathbf{f})$ we have $r(i)=r(j)$ $(1 \leqslant i,j \leqslant s$ or $s+1 \leqslant i,j \leqslant s+m)$. It is true because in this case $R \subset D$, where $D(x_1,\dots,x_s,y_1,\dots,y_m) \equiv (x_i=x_j)$ $(1 \leqslant i,j \leqslant s)$ is a 2-base diagonal of arity (s,m), and hence D is involved in formula (6) which gives us $T \subseteq D$ and $r \in D$.

Next we need three facts about $\mathbf{f}$.

Fact 1. $\mathbf{f} \notin \mathrm{Pol}(R)$ ($\mathbf{f}$ *does not preserve* $\mathrm{Pol}(R)$).

Holds straightforward from the definition of $\mathbf{f}$.

Fact 2. $\mathbf{f} \notin \mathrm{Pol}(\mathfrak{R})$.

Since $R \in \mathrm{Inv}(\mathrm{Pol}(\mathfrak{R}))$ we obtain $\mathrm{Pol}(\mathfrak{R}) \subseteq \mathrm{Pol}(R)$ by using antimonotone property of the functor Pol (see, e.g., [1]). Then we apply Fact 1.

Fact 3. *There exists such relation* $Q \in \mathfrak{R}$ *that* $\mathbf{f}$ *does not preserve* Q.

Follows straight from the Fact 2.

First let Q be a 2-base relation of arity (p,t) $(p,t \geqslant 1)$. Then from the Fact 3 there exist n 2-base (p,t)-tuples $q_1,\dots,q_n \in Q$ such that $\mathbf{f}(q_1,\dots,q_n)=q \notin Q$. In addition, since $\mathrm{Dom}(\mathbf{f})=[r_1,\dots,r_n]$ we have $[q_1,\dots,q_n] \subseteq [r_1,\dots,r_n]$ (inclusion as sets of n-tuples $[q_1,\dots,q_n]=\{\langle q_1(i),\dots,q_n(i)\rangle\colon i=1,\dots,p+t\}$ and $[r_1,\dots,r_n]=\{\langle r_1(j),\dots,r_n(j)\rangle\colon j=1,\dots,s+m\}$). Notice that by identification of equal coordinates in Q one can reduce its arity to $p \leqslant s$ and $t \leqslant m$ still satisfying the Fact 3.

We introduce two everywhere defined one-to-one mappings $\varphi\colon\{1,\dots,p\} \to \{1,\dots,s\}$, $i \mapsto \varphi i$, and $\psi\colon\{1,\dots,t\} \to \{1,\dots,m\}$, $j \mapsto \psi j$, between the numbers of n-tuples from

$[q_1,\ldots,q_n]$ and $[r_1,\ldots,r_n]$:

$$\langle q_1(i),\ldots,q_n(i)\rangle = \langle r_1(\varphi i),\ldots,r_n(\varphi i)\rangle \quad \text{for all } n\text{-tuples on } E(k_1),$$

$$\langle q_1(j),\ldots,q_n(j)\rangle = \langle r_1(\psi j),\ldots,r_n(\psi j)\rangle \quad \text{for all } n\text{-tuples on } E(k_2). \tag{7}$$

Now we define the relation S of arity (s,m) as follows:

$$S(x_1,\ldots,x_s,y_1,\ldots,y_m) \equiv Q(x_{\varphi 1},\ldots,x_{\varphi p},y_{\psi 1},\ldots,y_{\psi t}), \tag{8}$$

where all coordinates, other than explicitly shown on the right side, are free or complete.

Next we establish several properties of S:

(i) $R \subseteq S$.

According to (7) we have $r_1,\ldots,r_n \in S$ and so $R=\{r_1,\ldots,r_n\} \subseteq S$.

(ii) $T \subseteq S$.

Since S is constructed via &-formula from $Q \in \mathfrak{R}$ we get that S is involved in the formula (6) and so $T \subseteq S$.

(iii) $r \in S$

Follows straight from (ii).

Since f_1 and f_2 are one-to-one operations we obtain from (7) that:

$$\langle q(i)\rangle = \langle r(\varphi i)\rangle \quad \text{for elements from } E(k_1),$$

$$\langle q(j)\rangle = \langle r(\psi j)\rangle \quad \text{for elements from } E(k_2). \tag{9}$$

Next we define two mappings $\alpha:\{1,\ldots,s\} \to \{1,\ldots,p\}$, $i \mapsto \alpha$, and $\beta:\{1,\ldots,m\} \to \{1,\ldots,t\}$, $j \mapsto \beta j$ such that: $\alpha i{=}j$, when $\varphi j{=}i$ and $\alpha i{=}1$ otherwise ($i{=}1,\ldots,s$); $\beta i{=}j$, when $\psi j=i$, and $\beta i=1$ otherwise ($i=1,\ldots,m$).

Finally, from the formula (8) we obtain:

$$Q(x_1,\ldots,x_p,y_1,\ldots,y_t) \equiv S(x_{\alpha 1},\ldots,x_{\alpha s},y_{\beta 1},\ldots,y_{\beta m}). \tag{10}$$

Moreover, from (iii) ($r \in S$) and (9) we obtain that in formula (10) $q \in Q$ that contradicts our previous assumptions.

In the case, when Q is single-sorted, we use only one mapping $\varphi:\{1,\ldots,p\} \to \{1,\ldots,s\}$ and obtain $S(x_1,\ldots,x_s,y_1,\ldots,y_m) \equiv Q(x_{\varphi 1},\ldots,x_{\varphi p})$ with the converse identification (instead of (10)): $\pi_2 S(x_{\alpha 1},\ldots,x_{\alpha s},y,\ldots,y) \equiv Q(x_1,\ldots,x_p)$.

So there is no $r \in T \backslash R$ and $R \equiv T$, which proves the lemma.

Applying Lemma 2.1 we get $\mathfrak{R} = \mathrm{Inv}(\mathrm{Pol}(\mathfrak{R}))$ for any set $\mathfrak{R}$ closed under formation of $(\&,=_1,\ldots,=_n)$-formulas and application of π_i ($i=1,\ldots,n$). This proves the theorem. □

If $\mathbf{A} = \mathrm{Pol}(R)$, then we say that R *determines* partial n-clone $\mathbf{A}$. Using Galois connection properties we obtain that in this case R is a generating relation for the set $\mathrm{Inv}(\mathbf{A})$ with respect to operations mentioned in Theorem 2.1.

Corollary 2.1. *A relation R determines $\prod \mathbb{P}(k_i)$ if and only if R is a multiple-base diagonal.*

Let $\Phi(k) = Q(k) \cup \{p_m\colon m \geqslant 1\}$ be a partial clone on $E(k)$, $k \geqslant 2$, consisting of all everywhere defined and empty operations. It is known [4] that $\Phi(k)$ is a maximal partial clone (moreover, in [24] this result was extended to an infinite base set E). Consider n $(n \geqslant 2)$ partial n-clones:

$$\Phi_1 = (\Phi(k_1) \times P(k_2) \times \cdots \times P(k_n)) \cup F,$$

$$\Phi_2 = (P(k_1) \times \Phi(k_2) \times \cdots \times P(k_n)) \cup F, \ldots,$$

$$\Phi_n = (P(k_1) \times P(k_2) \times \cdots \times \Phi(k_n)) \cup F. \tag{11}$$

Proposition 2.2. *Φ_i $(i = 1, \ldots, n)$ are the only maximal partial n-clones containing the n-clone $\prod \mathbb{Q}(k_i)$.*

Proof. Consider n-clone $\mathrm{Sel} = \mathrm{Sel}(k_1) \times \mathrm{Sel}(k_2) \times \cdots \times \mathrm{Sel}(k_n)$, which is the direct arity-calibrated product of n clones of all projections (selectors) $\mathrm{Sel}(k_i)$ on $E(k_i)$ $(i = 1, \ldots, n)$. In what follows, we will use the fact, which is based on the properties of Sel.

Fact. *If $\mathbf{A}$ is a partial n-clone with $\mathrm{Sel} \subset \mathbf{A}$, then $\mathbf{A} \backslash F$ can be presented in the form $A_1 \times A_2 \times \cdots \times A_n$ of an arity-calibrated direct product of n partial clones A_i on $E(k_i)$ $(i = 1, \ldots, n)$.*

First it is easy to prove maximality of each Φ_i using that $\Phi(k_i)$ is maximal in $P(k_i)$ $(i = 1, \ldots, n)$. Next from $\mathrm{Sel} \subset \prod \mathbb{Q}(k_i)$ we get that each maximal partial n-clone containing $\prod \mathbb{Q}(k_i)$ can be presented as a direct product. This proves the second part of the proposition. $\square$

Denote $[A]$ the partial n-clone generated by a set of n-operations A.

Corollary 2.2. *$\prod \mathbb{P}(k_i)$ is generated by the set $A(2)$.*

Proof. Since all binary n-selectors $\mathrm{Sel}^{(2)}$ are contained in $A(2)$ and also $\mathrm{Sel}^{(2)}$ generates Sel the partial n-clone $[A(2)]$ generated by $A(2)$ is presented as a direct product. Next we apply the result that the set of all partial binary operations generates $P(k_i)$ $(i = 1, \ldots, n)$ (see [4]). $\square$

Hence from common algebraic results (see, e.g., [2]) it follows that each proper partial n-clone is contained in a maximal partial n-clone and, therefore, a set of partial n-operations is complete in $\prod \mathbb{P}(k_i)$ if and only if it is not contained in any maximal partial n-clone $(n \geqslant 2)$.

Theorem 2.2. *Each maximal partial n-clone, with the exception of Φ_i $(i = 1, \ldots, n)$, is determined by a multiple-base relation that is minimal under the expressibility by & -formulas and distinct from a multiple-base diagonal.*

Proof. Without the loss of generality consider $n = 2$. Let $\mathbf{A}$ be a maximal partial 2-clone, other than Φ_i $(i = 1, 2)$. Then applying Proposition 2.2 we obtain $\mathbf{B} = \mathbf{A} \cap Q(k_1) \times Q(k_2) \subset Q(k_1) \times Q(k_2)$, where $\mathbf{B}$ is a proper 2-clone. Next for binary operations we have $\mathbf{B}^{(2)} = \mathbf{A}^{(2)} \cap Q^{(2)}(k_1) \times Q^{(2)}(k_2)$. Clearly $\mathbf{B}^{(2)}$ is included properly in $Q^{(2)}(k_1) \times Q^{(2)}(k_2)$, otherwise $[\mathbf{B}^{(2)}] = [Q^{(2)}(k_1) \times Q^{(2)}(k_2)] = Q(k_1) \times Q(k_2)$, a contradiction to Proposition 2.2.

Consider a 2-graph $G_2(\mathbf{B})$ of the n-clone $\mathbf{B}$. We choose the set $D = E^2(k_1) \cup E^{(2)}(k_2)$, $|D| = k_1^2 + k_2^2$, where $D = [r_1, r_2]$ consists of two 2-base tuples r_1 and r_2 of arity (k_1^2, k_2^2) over $E(k_1)$ and $E(k_2)$. Next we define the relation $G_2(\mathbf{B})$ of arity (k_1^2, k_2^2) as follows:

$$G_2(\mathbf{B}) = \{r\colon\ \mathbf{f}(r_1, r_2) = r \text{ for some } \mathbf{f} \in \mathbf{B}^{(2)}\}.$$

Clearly $G_2(\mathbf{B})$ is a non-full relation hence it is not a 2-base diagonal as well (the $2 \times (k_1^2 + k_2^2)$ matrix $[r_1, r_2]$ does not have equal columns and so no non-full diagonal contains $G_2(\mathbf{B})$). Finally, it is easy to verify, applying the maximality of $\mathbf{A}$, that $\mathbf{A} = \mathrm{Pol}(G_2(\mathbf{B}))$.

Hence we proved that each maximal partial n-clone, other than Φ_i $(i = 1, \ldots, n)$, is determined by a multiple-base relation (in the common case of arity $(k_1^2, \ldots, k_n^2)$). Now from Proposition 2.1 we get that maximal partial n-clones of this type are precisely maximal restriction-closed partial n-clones. So applying properties of the Galois connection we obtain that $G_2(\mathbf{B}))$ is a generating relation with the minimal expressibility property in the atom $\mathrm{Inv}(\mathbf{A})$ of the lattice of Galois-closed sets of multiple-base relations, i.e., every non-diagonal Q, $Q \in \mathrm{Inv}(\mathbf{A})$, can be obtained from $G_2(\mathbf{B})$ by using operations of the Galois closure on the set of relations and, conversely, $G_2(\mathbf{B})$ is constructed from Q via the same operations. Notice that $G_2(\mathbf{B})$ has no equal or fictitious (dummy) coordinates. Moreover, if we also consider Q without equal or fictitious coordinates, then Q can be obtained from $G_2(\mathbf{B})$ via a & -formula and vice versa. □

In the sequel, we call relations without equal or fictitious coordinates satisfying Theorem 2.2 *minimal*. Straight from the definition of minimal relations we obtain the corollary which enables us to incorporate minimal m-base relations into n-base relations, i.e., partial m-clones into partial n-clones $(1 \leqslant m \leqslant n)$.

Corollary 2.3. *Every minimal relation over the type J, $|J| \geqslant 1$, is also minimal over any type I, $J \subset I$.*

3. Slupecki-type criterion

In order to find the exact estimates of minimal relations arities we will establish a Slupecki-type criterion, i.e., a completeness criterion for systems of partial n-operations, containing the set $\Omega(k_1, \ldots, k_n) = P^{(1)}(k_1) \times P^{(1)}(k_2) \times \cdots \times P^{(1)}(k_n)$ of all unary partial n-operations.

Namely, we will find all maximal partial n-clones containing $\Omega(k_1, \ldots, k_n)$, called *Slupecki* partial n-clones, via n-base relations determining them. Notice that $\Omega(k_1, \ldots, k_n)$ is a direct product of n semigroups $\Omega(k_i)$ $(i = 1, \ldots, n)$ of all partial unary operations defined on n base sets. At the same time we may also consider $\Omega(k_1,\dots,k_n)$ as a partial n-clone by applying n-selectors [14] (or constant operation $\mathbf{e}_1^2$) to it.

We will describe the structure of $\mathrm{Inv}(\mathbf{A})$ in the case of unary partial n-operations (for $n=1$ see [16]).

Proposition 3.1. *Let* $\mathbf{A}$ *be a restriction-closed partial n-clone. Then* $\mathbf{A}$ *is a subsemigroup of* $\Omega(k_1,\dots,k_n)$ (*consists of only unary partial n-operations*, *n-selectors and* F) *if and only if* $\mathrm{Inv}(\mathbf{A})$ *is closed under application of any disjunction of relations.*

The proof basically follows the case $n=1$ (see, e.g., [1]).

Corollary 3.1. *The set* $\mathrm{Inv}(\Omega(k_1,\dots,k_n))$ *consists of any disjunction of n-base diagonals.*

Denote $\mathfrak{I}$ the set consisted of any disjunction of n-base diagonals ($n \geqslant 1$). Applying Proposition 2.1 we get the corollary.

Corollary 3.2. *Each restriction-closed partial n-clone*, *containing* $\Omega(k_1,\dots,k_n)$, *is determined by a set of relations from* $\mathfrak{I}$.

Then applying Theorem 2.2 we obtain the following corollary.

Corollary 3.3. *Each Slupecki partial n-clone is determined by a minimal relation from the set* $\mathfrak{I}$.

Now it suffices to find all minimal relations in the set $\mathfrak{I}\backslash\mathbf{D}$, which determine distinct partial n-clones.

Definition 3.1. A non-diagonal n-base relation S ($n \geqslant 2$) is called irreducible if by applying to S intersections with permutations of coordinates, identifications of coordinates of the same sort and also π_i ($i=1,\dots,n$) one cannot obtain a non-diagonal relation of either less arity, or less type, or less number of tuples.

For any $(h_1,\dots,h_n)$-tuple $r(h_1,\dots,h_n \geqslant 1)$ denote $\varepsilon(r)$ the equivalence relation on numbers of coordinates induced by equal coordinates in r, e.g., for a $(2,2)$-tuple $r=(0,0;1,1)$ we have $\varepsilon(r)=\{(1,2),(3,4)\}$ and $\varepsilon(r)=\varepsilon(D)$, where $D \equiv x_1=x_2 \,\&\, y_1=y_2$ is a 2-base diagonal corresponding to $\varepsilon(r)$. If r has no equal coordinates, then $\varepsilon(r)$ is the trivial equivalence which represents the full n-base diagonal of arity $(h_1,\dots,h_n)$.

Lemma 3.1. *Let* S *be an irreducible n-base relation. Then for every* $r \in S$ *such that* $\varepsilon(r)$ *is non-trivial we have* $D \subset S$, *where* $\varepsilon(D)=\varepsilon(r)$ *and* $D \in \mathbf{D}$.

Proof. Assume $D \not\subset S$ for some $r \in S$ and $\varepsilon(D)=\varepsilon(r)$. Then applying to S identifications of coordinates according to all blocks of $\varepsilon(D)$ we obtain a non-diagonal relation which contradicts the fact that S is irreducible. □

Set $T(h) \equiv \bigvee_{1\leqslant i<j\leqslant h} (x_i = x_j), \quad h \geqslant 2.$

Proposition 3.2. *Each irreducible relation S, $S\in\mathfrak{I}\backslash\mathbf{D}$, of arity $(h_1,\ldots,h_n)$ and type $\{1,\ldots,n\}(2\leqslant h_1\leqslant k_1,\ldots,2\leqslant h_n\leqslant k_n,\ n\geqslant 2)$ is presented as a disjunction $T(h_1)\vee\cdots\vee T(h_n)$ of n single-base relations defined on sets $E(k_1),\ldots,E(k_n)$, respectively.*

Proof. We consider the proof for the case $n=2$. The same idea is applicable to $n\geqslant 2$.

Let $S(x_1,\ldots,x_n,y_1,\ldots,y_m)$ $(n,m\geqslant 2)$ be a 2-base irreducible relation of arity (n,m) (if $n=1$, then using π_1 we obtain a single-base non-diagonal relation). So S can be presented in the form:

$$S \equiv \bigvee_{i=1}^{t} D_1^i(x_1,\ldots,x_n) \,\&\, D_2^i(y_1,\ldots,y_m), \tag{12}$$

where D_1^i are diagonals of the 1st sort and D_2^i are diagonals of the 2nd sort $(i=1,\ldots,t)$.

Now consider the relation $D(y_1,\ldots,y_m)\equiv\pi_1 S(x,\ldots,x,y_1,\ldots,y_m)$, which is a diagonal due to Definition 2.1. If D is a non-full diagonal, then from (12) we get $D_2^i\subseteq D$ $(i=1,\ldots,t)$. Hence $S\subset\bigvee_{1\leqslant i\leqslant t} D_2^i\subseteq D$ and so $S\cap D=S$ (here D has n fictitious variables of the 1st sort). Then by Lemma 3.1 $D\subseteq S$ and $D=S$. Contradiction.

So D is the full diagonal and, therefore, there exists $a\in E(k_1)$ such that $(a,\ldots,a;\ b_1,\ldots,b_m)\in S$, where $(b_1,\ldots,b_m)$ are all possible m-tuples from $E^m(k_2)$. Then applying Lemma 3.1 we have $x_1=\cdots=x_n\subset S$. Similarly we obtain $y_1=\cdots=y_m\subset S$. Hence we proved that S can be presented in a form of separated single-base disjunctive components:

$$S \equiv R_1(x_1,\ldots,x_n)\vee T(x_1,\ldots,x_n,y_1,\ldots,y_m)\vee R_2(y_1,\ldots,y_m), \tag{13}$$

where R_1 and R_2 are non-full single-base diagonals and $R_1,R_2,T\in\mathfrak{I}$.

In addition, we choose R_1 and R_2 as the greatest single-base disjunctive components, i.e., if a single-base diagonal $D^1\subset S$ $(D^2\subset S)$, then $D^1\subseteq R_1$ $(D^2\subseteq R_2$, respectively). At the same time, we assume that T does not contain any single-base diagonals with fictitious coordinates.

Fact 1. *Relations R_1 and R_2 in the expression* (13) *are totally symmetric*, *i.e.*, *stable under any permutations of coordinates.*

Proof. Let α be a permutation of n variables in R_1: $R_1^\alpha(x_1,\ldots,x_n)\equiv R_1(x_{\alpha 1},\ldots,x_{\alpha n})$. Then from (13) we get $S^\alpha\equiv R_1^\alpha(x_1,\ldots,x_n)\vee T(x_{\alpha 1},\ldots,x_{\alpha n},y_1,\ldots,y_m)\vee R_2(y_1,\ldots,y_m)$. Hence by using properties of operations & and $\vee$ we have

$$S\,\&\,S^\alpha \equiv R_1^\alpha(x_1,\ldots,x_n)\,\&\,R_1(x_1,\ldots,x_n)\vee T_1(x_1,\ldots,x_n,y_1,\ldots,y_m)$$
$$\vee R_2(y_1,\ldots,y_m),$$

where $T_1\equiv R_1\,\&\,T^\alpha\vee T\,\&\,T^\alpha\vee R_1^\alpha\,\&\,T$ is a 2-base relation from $\mathfrak{I}$.

Since $x_1 = \cdots = x_n \subseteq R_1^\alpha \,\&\, R_1$, the relation $S \,\&\, S^\alpha$ is not a diagonal (it has single-base disjunctive components for each sort of variables). Hence from Definition 3.1 $S \,\&\, S^\alpha \equiv S$ and so $R \equiv R_1^\alpha \,\&\, R_1$, which implies $R_1^\alpha \equiv R_1$. □

Now consider S in formula (13) in two different cases.

Case $n = 2$ $(m = 2)$: Here it is easy to verify that $T(x_1, x_2, y_1, \ldots, y_m) \subset x_1 = x_2$ and so S can be presented in the form: $S \equiv x_1 = x_2 \vee R_2(y_1, \ldots, y_m)$. If $m = 2$, then $R_2 \equiv y_1 = y_2$ and so $S \equiv x_1 = x_2 \vee y_1 = y_2$.

Let $m \geqslant 3$. Then the relation $S(x_1, x_2, y_1, y_1, y_3, \ldots, y_m)$ is the full 2-base diagonal of arity $(2, m-1)$ (because of the disjunctive component $x_1 = x_2$ it cannot be a non-full diagonal). Hence applying Lemma 3.1 we get $y_1 = y_2 \subset S$ and also $y_1 = y_2 \subset R_2$ (greatest disjunctive component property). Next from the Fact 1 we conclude that $R_2 \equiv T(m) \equiv \bigvee_{1 \leqslant i < j \leqslant m} (x_i = x_j)$. Note that here we have $3 \leqslant m \leqslant k_2$, since for $m > k_2$ $T(m)$ is the full relation and S is also full relation. Finally, we get $S \equiv x_1 = x_2 \vee T(m)$, $2 \leqslant m \leqslant k_2$. (For the case $m = 2$ we have $S \equiv T(n) \vee x_1 = x_2$, $2 \leqslant n \leqslant k_1$).

Case $n, m > 2$: Consider the relation $S' \equiv S(x_1, \ldots, x_n, y_1, y_1, y_3, \ldots, y_m)$ which is a 2-base diagonal of arity $(n, m-1)$ (see Definition 3.1). If S' is a non-full diagonal, then this contradicts the inclusion $R_1(x_1, \ldots, x_n) \subset S'$ which follows straight from (13). So S' is the full diagonal and applying Lemma 3.1 we obtain $y_1 = y_2 \subset S$ and from the greatest disjunctive component property we have $y_1 = y_2 \subset R_2$. Then from the Fact 1 we get $R_2 \equiv T(m)$, $2 \leqslant m \leqslant k_2$. Next by repeating the same steps we obtain $R_1 \equiv T(n)$, $2 \leqslant n \leqslant k_1$. Finally, $S \equiv T(n) \vee T(m)$, $n, m \geqslant 2$. □

It is obvious that every maximal partial n-clone, with n exceptions, can be determined by an irreducible relation. Hence from Corollary 3.3 and Proposition 3.2 we get corollary.

Corollary 3.4. *Each Slupecki partial n-clone, that is a subdirect product of m* $(2 \leqslant m \leqslant n)$ *factors, is determined by a relation which is contained among the relations having the form*: $T(h_1) \vee \cdots \vee T(h_m)$ $(2 \leqslant h_1 \leqslant k_1, \ldots, 2 \leqslant h_m \leqslant k_m)$, *where* $T(h_i)$ *has the type* $\{i\}$ $(1 \leqslant i \leqslant m)$, *or by a relation obtained from them by a permutation of numbers of base sets.*

The converse is also true.

Proposition 3.3. *Each relation* $T(h_1) \vee \cdots \vee T(h_m)$ $(2 \leqslant h_1 \leqslant k_1, \ldots, 2 \leqslant h_m \leqslant k_m,\ 2 \leqslant m \leqslant n)$ *is a minimal m-base relation.*

The proof for the general case will be presented in the next section (Proposition 4.2).

Recall that all k Slupecki partial clones on $E(k)$, $k \geqslant 3$, were described in [15] by k invariant relations: $H_1 \equiv x_1 = x_2 \,\&\, x_3 = x_4 \vee x_1 = x_3 \,\&\, x_2 = x_4$, $H_2 \equiv x_1 = x_2 \,\&\, x_3 = x_4 \vee x_1 = x_3 \,\&\, x_2 = x_4 \vee x_1 = x_4 \,\&\, x_2 = x_3$, $T(h)$ $(h = 3, \ldots, k)$. If $k = 2$, then there exist 2 Slupecki partial clones $\mathrm{Pol}(H_1)$ and $\mathrm{Pol}(H_2)$ (see [4]). We define the set G_i of k_i single-base relations of type $\{i\}$ on $E(k_i)$ $(i = 1, \ldots, n)$ as follows: if $k_i \geqslant 3$, $G_i = \{H_1, H_2, T(h)\ (3 \leqslant h \leqslant k_i)\}$ and if $k_i = 2$, then $G_i = \{H_1, H_2\}$ $(i = 1, \ldots, n)$.

Finally, from the results of this section we obtain the theorem.

Theorem 3.1. *Each Slupecki partial n-clone* ($n \geqslant 2$) *is defined by a relation such that*

(1) $R \in G_1 \cup \cdots \cup G_n$ *or*
(2) R *is represented as a disjunction* $R_1 \vee \cdots \vee R_n$, *where each* R_i ($i=1,\ldots,n$) *is either one of* $T(h)$ ($2 \leqslant h \leqslant k_i$) *with the type* $J(R_i)=\{i\}$, *or empty and, moreover, at least two of disjunctive components* R_i *are nonempty.*

Similarly to the case $n=1$ [15] each maximal restriction-closed partial n-clone $\mathbf{A}$, except for Slupecki partial n-clones, is determined by its 1-graph $G_1(\mathbf{A})$ (the graph of all unary n-operations $\mathbf{A} \cap \Omega(k_1,\ldots,k_n)$) which has an arity $(k_1,\ldots,k_n)$.

Corollary 3.5. *Each maximal partial n-clone, other than* Φ_i (i=$1,\ldots,n$), *is determined by a minimal multiple-base relation of arity* $(k_1,\ldots,k_n)$.

Slupecki criterion for $\mathbb{Q}(k_1) \times \cdots \times \mathbb{Q}(k_n)$. We will apply results of this section to the description of all Slupecki n-clones ($n \geqslant 2$) (maximal n-clones including all unary n-operations). Similar to partial n-clones by establishing analogues of Proposition 3.1 and Corollary 3.1 we obtain the fact: every non-full n-clone $\mathbf{B}$, which contains all unary n-operations, has the form $\mathbf{B} = \mathrm{Pol}^t(\mathfrak{R})$, where $\mathfrak{R} \subseteq \mathfrak{I}$ and $\mathfrak{R} \cap (\mathfrak{I} \backslash \mathbf{D}) \neq \emptyset$. Hence we get the following proposition.

Proposition 3.4. *Each Slupecki n-clone* ($n \geqslant 2$) *is determined by a non-diagonal relation* R, $R \in \mathfrak{I}$, *such that* $\mathrm{Pol}^t(R)$ *is a maximal element by inclusion among all n-clones of the form* $\mathrm{Pol}^t(S)$, $S \in \mathfrak{I} \backslash \mathbf{D}$.

Next it suffices to investigate only irreducible relations described in Proposition 3.2, because if S is reduced by intersections and identifications to irreducible R, then it is obvious that $\mathrm{Pol}^t(S) \subseteq \mathrm{Pol}^t(R)$. Further we will need the lemma.

Lemma 3.2. *An n-operation* $\mathbf{f} = \langle f_1,\ldots,f_n \rangle$, $\mathbf{f} \notin \mathrm{Sel}$, *belongs to* $\mathrm{Pol}^t(T(h_1) \vee \cdots \vee T(h_m))$ ($2 \leqslant h_1 \leqslant k_1,\ldots,2 \leqslant h_m \leqslant k_m$, $2 \leqslant m \leqslant n$) *if and only if there exists* i, $1 \leqslant i \leqslant n$, *such that the range of* f_i *is less or equal* $h_i - 1$ ($2 \leqslant h_i \leqslant k_i$).

The proof of this lemma is based on the case $n=1$ (Slupecki criterion for k-valued logic, see, e.g., [8]). Recall that Slupecki n-clones determined by single-base relations for each type $\{i\}$ are: the Slupecki clone $\mathrm{Pol}^t(T(k_i))$, when $k_i \geqslant 3$, or the clone of all linear Boolean functions [13] having the form $\mathrm{Pol}^t(H_2)$, when $k_i=2$ ($i=1,\ldots,n$).

Applying Lemma 3.2 we get $\mathrm{Pol}^t(T(h_1) \vee \cdots \vee T(h_m)) \subset \mathrm{Pol}^t(T(t_1) \vee \cdots \vee T(t_m))$, where $h_1 \leqslant t_1 \leqslant k_1,\ldots,h_m \leqslant t_m \leqslant k_m$, and there is at least one strict inequality ($2 \leqslant m \leqslant n$). So all maximal elements by inclusion satisfying Proposition 3.4 are exactly $\mathrm{Pol}^t(T(k_1) \vee \cdots \vee T(k_m))$ and the ones obtained from them by permutations of the numbers of base sets. Finally, we obtain the description of all Slupecki n-clones (see also [12,19,25]).

Theorem 3.2. *There are exactly* $2^n - 1$ *Slupecki n-clones that are determined by multiple-base relations having the form*

(a) $R \equiv R_1 \vee \cdots \vee R_n$, *where* $R_i \in \{ø, T(k_i)\}$, $J(R_i) = \{i\}$, *and at least two* R_i *are nonempty*;
(b) *single-sorted relations of the type* $\{i\}$ $(i = 1, \ldots, n)$, namely, $R \equiv T(k_i)$, *when* $k_i \geqslant 3$, *or* $R \equiv H_2$, *when* $k_i = 2$.

We call an n-operation $\mathbf{f} = \langle f_1, \ldots, f_n \rangle$ *essential over type* $\{i\}$ $(1 \leqslant i \leqslant n)$, if either f_i is essential (has the full range k_i and is a non-selector), when $k_i \geqslant 3$, or f_i is a non-linear Boolean function, when $k_i = 2$. Then $\mathbf{f}$ is *essential over type* $J = \{i_1, \ldots, i_m\}$, $J \subseteq I$, $2 \leqslant |J| \leqslant n$, if for every $i \in J$ f_i has the full range and $\langle f_{i_1}, \ldots f_{i_m} \rangle$ is not equal to an m-vector of unary partial operations $\langle \alpha_{i_1}, \ldots, \alpha_{i_m} \rangle \in \Omega(k_{i_1}) \times \cdots \times \Omega(k_{i_m})$ (up to fictitious coordinates). Next $\mathbf{f}$ is *essential* if for every $i \in \{1, \ldots, n\}$ either f_i has the full range and is a non-selector, when $k_i \geqslant 3$, or f_i is a non-linear Boolean function, when $k_i = 2$. Finally, we obtain Slupecki criterion for n-clones (see also [23]).

Proposition 3.5. *A set B of n-operations is complete in* $\mathbb{Q}(k_1) \times \cdots \times \mathbb{Q}(k_n)$ *with all unary n-operations if and only if for every type J*, $J \subseteq \{1, \ldots, n\}, 1 \leqslant |J| \leqslant n$ $(n \geqslant 2)$, *there exists an n-operation* $\mathbf{f} \in B$ *which is essential over J*.

Corollary 3.6. *An n-operation* $\mathbf{f}$ *is complete in* $\mathbb{Q}(k_1) \times \cdots \times \mathbb{Q}(k_n)$ *with all unary n-operations if and only if* $\mathbf{f}$ *is essential.*

Corollary 3.7. *Each maximal n-clone is determined by a multiple-base relation of arity* $(k_1, \ldots, k_n)$ (*with the exception of a single-base relation* H_2 *of arity* 4 *on* $E(2)$).

4. Maximal partial *n*-clones

In what follows we explore irreducible relations of arity less or equal $(k_1, \ldots, k_n)$ which do not belong to $\mathfrak{I}$. Without loss of generality we consider n-base relations of arity $\langle h_1, \ldots, h_m, 1, \ldots, 1, 0, \ldots, 0 \rangle$, where $0 \leqslant m \leqslant n$ and $2 \leqslant h_i \leqslant k_i$ $(i = 1, \ldots, m)$ (one can pass to the general case by changing numbers of base sets). We also need definitions extending case $n = 1$.

1. A multiple-base relation R is called *areflexive* if it contains no tuples with equal coordinates, i.e., $R \cap (T(h_1) \vee \cdots \vee T(h_m)) = \emptyset$. Denote $\mathbf{R}$ the set of all areflexive relations.

2. A multiple-base relation H is called *totally symmetric*, when it is stable under each permutation of coordinates of the same ith sort $(1 \leqslant i \leqslant m)$ and *totally reflexive*, when $T(h_1) \vee \cdots \vee T(h_m) \subseteq H$. Denote $\mathbf{H}$ the set of all totally reflexive and totally symmetric non-full relations (for $n = 1$ see [27]).

Example 4.1. Let $E(k_1) = k \geqslant 3$, $E(k_2) = 2$ and a 2-base relation of arity $(h, 1)$, $2 \leqslant h \leqslant k$, be defined as follows: $H(x_1, \ldots, x_h, y) \equiv \{(x_1, \ldots, x_h, y)$: $(x_1, \ldots, x_h) \in T(h)$ or

$(x_1+\cdots+x_h)=0 \pmod k) \,\&\, y=1\} \equiv T(h)(x_1,\dots,x_h) \vee \langle\langle(x_1+\cdots+x_h)=0\rangle\rangle \,\&\, y=1$. Then $H\in\mathbf{H}$.

3. For every non-single type J, $2\leqslant |J|\leqslant n$, the set $\mathbf{K}$ consists of all nonempty, non-full relations of arity $(1,\dots,1)$ and type J.

Note that if a relation T is obtained by a & -formula from irreducible Q of arity $\langle h_1,\dots,h_m,1,\dots,1,0,\dots,0\rangle$, then by identification of coordinates of types $s>m$ in T we also get a non-diagonal relation S of arity $\langle s_1,\dots,s_m,1,\dots,1,0,\dots,0\rangle$ and $s_i\geqslant h_i$ $(i=1,\dots,m)$. Now we will consider a special presentation of S by a & -formula from Q. Without loss of generality Q has arity $(h_1,\dots,h_m)$ and $S-(s_1,\dots,s_m)$ respectively ($h_i\leqslant s_i$, $i=1,\dots,m$). Then we introduce an *index m-base relation* M of arity $(h_1,\dots,h_m)$ on base sets $E(s_1),\dots,E(s_m)$. An index relation M represents any S constructed by a & -formula from Q:

$$S(x_0,\dots,x_{s_1-1},y_0,\dots,y_{s_2-1},\dots,z_0,\dots,z_{s_m-1})\equiv \&_{r\in M}Q^r, \tag{14}$$

where $r=(r(1,1),\dots,r(1,h_1);\ r(2,1),\dots,r(m,h_m))\in M$ is a $(h_1,\dots,h_m)$-tuple over $E(s_1),\dots,E(s_m)$ and $Q^r\equiv Q(x_{r(1,1)},\dots,y_{r(2,1)},\dots,z_{r(m,1)},\dots)$.

Next if Q, in turn, can be obtained by a & -formula from S, then clearly it can be done by using intersections with identifications and permutations of coordinates. So we get.

Lemma 4.1. *Any irreducible multiple-base relation Q of arity $\langle h_1,\dots,h_m,1,\dots,1,0,\dots,0\rangle$ is minimal if and only if from every non-diagonal relation S of arity $\langle s_1,\dots,s_m,1,\dots,1,0,\dots,0\rangle$ constructed by the formula* (14) *one can obtain Q using intersections with identifications and permutations of coordinates in S.*

Proposition 4.1. *Each $Q\in\mathbf{K}$ is a minimal relation.*

Proof. Let S be constructed from $Q\in\mathbf{K}$ by the formula (14). We consider an identification Δ of coordinates in S as follows: for all $r\in M$ $r(1,1)\to 1,\dots,r(m,1)\to 1$. Hence we get $\Delta S\equiv Q$. Then we apply Lemma 4.1. □

Proposition 4.2. *Each $Q\in\mathbf{H}$ is a minimal relation.*

Proof. Clearly that in this case if S in the formula (14) is not a diagonal, then there exists a point $q\in M$ with all pairwise distinct coordinates of the same sort. Consider identification Δ of coordinates in $S: q(i,j)\to q(i,j)$ $(i=1,\dots,m;\ j=1,\dots,h_i)$ and $r(i,j)\to q(i,j)$ for any $r\in M\backslash\{q\}$. Hence we have $\Delta S\equiv Q$. Next see Lemma 4.1. □

Note that all minimal relations from Proposition 3.3 are included into the set $\mathbf{H}$. So the above proof also covers that case.

Lemma 4.2. *For each irreducible non-single sort relation Q of arity less or equal $(k_1,\dots,k_n)$ we have either*:

(1) *Q belongs to* $\mathbf{K}$ $(Q\in\mathbf{K})$;
(2) *Q belongs to* $\mathbf{H}$ $(Q\in\mathbf{H})$;

(3) *Q is areflexive* ($Q \in \mathbf{R}$);
(4) *Q has the form $R \vee D$, where $R \in \mathbf{R}$ and D is a multiple-base non-full diagonal of the same arity as R.*

Proof. Let Q be an irreducible relation of arity $\langle h_1, \ldots, h_m, 1, \ldots, 1, 0, \ldots, 0\rangle$ $(m \geqslant 1)$ and $Q \notin \mathbf{K}$. Then either Q is areflexive or $Q \cap D \neq \emptyset$, where D is a multiple-base non-full diagonal. Applying Lemma 3.1 we obtain that $Q \equiv R \vee S$ or $Q \equiv S$, where $R \in \mathbf{R}$ and $S \in \mathfrak{I}$. If S is a diagonal, then $Q \equiv R \vee D$ (case 4). Next if S is a non-diagonal, then according to Proposition 3.2, S has the form $T(h_1) \vee \cdots \vee T(h_m)$ and, moreover, R admits all permutations, since Q is irreducible. Hence $Q \in \mathbf{H}$. □

Now it suffices to clear cases (3) and (4) in the previous lemma. In what follows Q will be of arity $\langle h_1, \ldots, h_m, 1, \ldots, 1, 0, \ldots, 0\rangle$ with s $(s \geqslant 2)$ non-void sorts of coordinates $(m \leqslant s \leqslant n)$.

Lemma 4.3. *An s-base irreducible relation of the form $Q \equiv R$ $(R \in \mathbf{R})$ or $Q \equiv R \vee D$ $(R \in \mathbf{R}, D \in \mathbf{D})$ is minimal if and only if every relation $T \equiv \&_{r \in M} Q^r$ $(M \subseteq R)$ of arity $\langle k_1, \ldots, k_m, k_{m+1}, \ldots, k_s, 0, \ldots, 0\rangle$ can be reduced by some identification of coordinates to Q.*

Proof. First it easy to verify that Pol(Q) is not included in any Slupecki partial n-clone, i.e., using any & -formula one cannot obtain from Q an s-base relation of the form $T(h_1) \vee \cdots \vee T(h_t)$ $(2 \leqslant t \leqslant s)$. The proof of this fact is similar to the case $n = 1$ (see [22]). Hence from the results of the previous section each maximal partial n-clone $\mathbf{A}$, such that Pol(Q) $\subseteq \mathbf{A}$, satisfy the condition $\Omega(k_1, \ldots, k_n) \not\subset \mathbf{A}$. Moreover, there exists such $\mathbf{A}$ that it is determined by a non-diagonal s-base relation. To construct such relation consider 1-graph of any $\mathbf{A} = \mathbf{A}' \times \mathbb{P}(k_{s+1}) \times \cdots \times \mathbb{P}(k_n)$, where $\mathbf{A}'$ is a subdirect product of s factors $\mathbb{P}(k_i)$ $(1 \leqslant i \leqslant s)$. Namely, we have a relation $G_1(\mathbf{A}) = \{\mathbf{f}\, p\colon \mathbf{f} \in A \cap \Omega(k_1, \ldots, k_n)\}$ of arity $(k_1, \ldots, k_s)$, where $p = \langle \mathbf{E}(k_1,); \ \ldots; \mathbf{E}(k_s)\rangle$ is a $(k_1, \ldots, k_s)$-tuple, $\mathbf{E}(k_i) = (0, 1, \ldots, k_i - 1)$ $(1 \leqslant i \leqslant s)$. From the fact that $\mathbf{A}$ is maximal we get $\mathbf{A} = \mathrm{Pol}(G_1(\mathbf{A}))$. Hence $G_1(\mathbf{A}) \in \mathrm{Inv}(\mathrm{Pol}(Q))$ and so $G_1(\mathbf{A})$ can be obtained by a & -formula from Q. Therefore, grounding on Lemma 4.1 it is sufficient to consider in the formula (14) only relations $T \equiv \&_{r \in M} Q^r$ of arity $(k_1, \ldots, k_s)$, which contain the point p. It is easy to prove two facts about such relations:

(1) if $M \subseteq R$, then $p \in T$;
(2) if there exists $r \in M$ and $r \notin R$, then $p \notin T$.

So we may consider only index relations M, $M \subseteq R$. Moreover, since Q is irreducible each identification of T to arity $\langle h_1, \ldots, h_m, 1, \ldots, 1, 0, \ldots, 0\rangle$ is either Q or a diagonal. □

Example 4.2. Consider 2-base irreducible relation $R = \{(0, 1, a), (1, 0, b)\}$ of arity (2,1) on the sets $E(k_1) = \{0, 1\}$ and $E(k_2) = \{a, b\}$. Then by Lemma 4.3 we need to investigate only three relations containing the point p: $T_1(x_0, x_1, y_0, y_1) \equiv R(x_0, x_1, y_0)$,

$T_2(x_0,x_1,y_0,y_1) \equiv R(x_1,x_0,y_1)$ and $T_3(x_0,x_1,y_0,y_1) \equiv R(x_0,x_1,y_0) \& R(x_1,x_0,y_1)$, where $T_3 = \{(0,1,a,b),(1,0,b,a)\}$. So there is no identification of T_3 to arity (2,1) other then empty. Hence applying Lemma 4.3 we obtain that R is not a minimal relation. At the same time, a single-base projection of R on the type $\{1\}$ $R' = \{(0,1),(1,0)\}$ is a minimal relation [4].

Let $G(R)$ be a *symmetry group* of R, i.e., $G(R)$ is a subgroup of the product $S(h_1) \times \cdots \times S(h_m)$ $(m \geqslant 1)$ of the symmetric permutation groups on numbers of coordinates of each sort i, $1 \leqslant i \leqslant m$, for which R contains at least two coordinates, such that for each $\alpha \in G(R)$ the resulting relation $R^{\alpha}(x_1,\dots,y_1,\dots,z_1,\dots) \equiv R(x_{\alpha 1},\dots,y_{\alpha 1},\dots,z_{\alpha 1},\dots)$ under application of α to the numbers of its coordinates equals R and for each $\beta \notin G(R)$ we have $R^{\beta} \neq R$. We call R *normal* [20] if for each $\beta \notin G(R)$ we have $R \cap R^{\beta} = \emptyset$. It is obvious, that areflexive R is normal if and only if it is irreducible, e.g., for R from Example 4.2 we have $R(x_0,x_1,y_0) \& R(x_1,x_0,y_1) = \emptyset$ and so R is a normal relation. Also notice that in this case $G(R) = \{e\}$ is the identity group.

Denote $\mathrm{Orb}(G(R))$ the $(h_1,\dots,h_m)$-orbit of the group $G(R)$ (a generalization of the notion of the h-orbit of a permutation group) that consists of the images of all vector-permutations $\alpha = \langle \alpha_1,\dots,\alpha_m \rangle \in G(R)$ applied to the $(h_1,\dots,h_m)$-tuple $p = \langle \mathbf{E}(h_1);\dots;\ \mathbf{E}(h_m) \rangle$, i.e., $\mathrm{Orb}(G(R)) = \{(\alpha_1\mathbf{E}(h_1);\dots;\ \alpha_m\mathbf{E}(h_m))\colon \langle \alpha_1,\dots,\alpha_m \rangle \in G(R)\}$, where $\alpha_i\mathbf{E}(h_i) = (\alpha_i 0,\dots,\alpha_i(h_i-1))$, $1 \leqslant i \leqslant m$. Hence $\mathrm{Orb}(G(R))$ is an m-base relation of arity $(h_1,\dots,h_m)$ and type $J = \{1,\dots,m\}$.

Let $\Psi_i : E(k_i) \to E(h_i)$ $(2 \leqslant h_i \leqslant k_i,\ i = 1,\dots,m)$ be epimorphisms (one-to-one onto mappings) and $\Psi = \langle \Psi_1,\dots,\Psi_m \rangle$ be the corresponding vector-epimorphism. Also denote ΨR the m-base relation defined on the sets $E(h_1),\dots,E(h_m)$ that is obtained from the restriction of R on the coordinates of type $J = \{1,\dots,m\}$ (each sort of J contains at least two coordinates in R) by application Ψ to all its points, while Ψ_i is applied to coordinates of sort i $(i = 1,\dots,m)$. For example, let $R(x_1,x_2,y_1,y_2,z) = \{(0,1;0,1;0),(1,0;1,0;1)\}$ be a relation of arity $(2,2,1)$ over three two-element base sets $E(2) = \{0,1\}$. Then for any 2-epimorphism $\Psi = \langle \Psi_1,\Psi_2 \rangle (\Psi_1 : E(2) \to E(2), \Psi_2 : E(2) \to E(2))$ we have a $(2,2)$-relation $\Psi R = \{(\Psi_1 x_1, \Psi_1 x_2, \Psi_2 y_1, \Psi_2 y_2)\colon (x_1,x_2,y_1,y_2) \in R\}$. Notice that in this case $G(R)$ is the identity group and so $\mathrm{Orb}(G(R)) = \{(0,1;0,1)\}$.

Proposition 4.3. *Each areflexive s-base relation R of arity* $\langle h_1,\dots,h_m,1,\dots,1,0,\dots,0 \rangle$, *where* $1 \leqslant m \leqslant s \leqslant n$, $2 \leqslant h_i \leqslant k_i$ $(i = 1,\dots,m)$, $s \geqslant 2$, *is minimal if and only if*:

(1) *R is normal* (*sufficient condition for arity* $(k_1,\dots,k_s)$);
(2) *there exists a vector-epimorphism* $\Psi = \langle \Psi_1,\dots,\Psi_m \rangle$ *such that* $\Psi R = \mathrm{Orb}(G(R))$.

Proof. Straight from the Lemma 4.3 we get that R of arity $(k_1,\dots,k_s)$ is minimal if and only if it is normal (the case $n = 1$ see in [20]). Now consider the common case.

First we show that the part (1) of this proposition is the necessary condition for a relation to be minimal (this condition is absent in the results [9,10] for $n = 1$). Indeed, if $R \cap R^{\alpha} = R'$, $\emptyset \neq R' \subset R$, for some vector-permutation α, then we have

$\mathrm{Pol}(R) \subset \mathrm{Pol}(R')$, since one cannot obtain R via & -formula from areflexive R' of the same arity, which is included in R.

Then it is obvious that any identification of T in Lemma 4.3 to a relation of arity $\langle h_1,\dots,h_m,1,\dots,1,0,\dots,0\rangle$ corresponds to application of some vector-epimorphism Ψ to the index relation M, $M \subseteq R$, provided that all variables of each sort i, $m \leqslant i \leqslant s$, are identified with a single variable of the same sort.

Next since R is a normal relation each identification of $T \equiv \&_{r\in M} R^r$ $(M \subseteq R)$ in Lemma 4.3 to the arity $\langle h_1,\dots,h_m,1,\dots,1,0,\dots,0\rangle$ is either R or empty. Hence if there exists Ψ such that $T^{\Psi} \equiv \&_{r\in \Psi R} R^r = R$, then the same Ψ, while applied to any non-void $M \subseteq R$, gives us $T^{\Psi} \equiv \&_{r\in \Psi M} R^r = R$. So for the case $Q \equiv R$ it is sufficient to consider in Lemma 4.3 only relations of the form $T \equiv \&_{r\in R} R^r$.

It is easy to verify that $\Psi R \subseteq \mathrm{Orb}(G(R))$ implies $T^{\Psi} = R$. Moreover, in this case we have $\Psi R = \mathrm{Orb}(G(R))$, since Ψ is a vector-epimorphism and R is normal. Next if there exists Ψ such that $T^{\Psi} \equiv R$ and $p \notin \Psi R$, then we can find such vector-permutation $\alpha = \langle \alpha_1,\dots,\alpha_m\rangle$ on $E(h_1)\times\cdots\times E(h_m)$, $\alpha \notin G(R)$, that for $\alpha\Psi = \langle \alpha\Psi_1,\dots,\alpha\Psi_m\rangle$ we have $p \in (\alpha\Psi)R$ and also $T^{\alpha\Psi} \equiv R$.

The class of relations established in the previous proposition, including those obtained by arbitrary permutations of numbers of base sets, is denoted by $\mathbf{R}_1$ (similarly to $n=1$ [20]).

Consider incomplete s-base relations of arity $\langle h_1,\dots,h_m,1,\dots,1,0,\dots,0\rangle, 1 \leqslant m \leqslant s \leqslant n, s \geqslant 2, 2 \leqslant h_i \leqslant k_i$ $(i=1,\dots,m)$, having the form $Q \equiv R \vee D_1 \& \cdots \& D_m$, where R is non-empty areflexive relation of the same arity as Q, D_i is a single-base diagonal of arity h_i and sort i $(1 \leqslant i \leqslant m)$. Let $G(D_i)$ be the symmetry group of D_i, i.e., the group of all permutations of coordinates preserving the equivalence relation $\varepsilon(D_i)$ on the set of numbers of coordinates $E(h_i)$ induced by equal, non-dummy coordinates in D_i $(i=1,\dots,m)$. Denote, $D_i(h_i)$ the diagonal on $E(h_i)$ induced by the same equivalence relation: $\varepsilon(D_i) \equiv \varepsilon(D_i(h_i))$.

Proposition 4.4. *An s-base incomplete relation $Q \equiv R \vee D_1 \& \cdots \& D_m$ of arity $\langle h_1,\dots,h_m,1,\dots,1,0,\dots,0\rangle$, $1 \leqslant m \leqslant s \leqslant n$, $s \geqslant 2$, $2 \leqslant h_i \leqslant k_i$ $(i=1,\dots,m)$, is minimal if and only if*:

(1) *R is normal and $G(R) \subseteq G(D_1)\times\cdots\times G(D_m)$ (a sufficient condition for arity $(k_1,\dots,k_s)$)*;

(2) *For each non-empty subrelation $M \subseteq R$ there exists a vector-epimorphism $\Psi = \langle \Psi_1,\dots,\Psi_m\rangle$ such that $\Psi M \subseteq \mathrm{Orb}(G(R)) \cup D(h_1)\times\cdots\times D(h_m)$ and $\Psi M \cap \mathrm{Orb}(G(R)) \neq \emptyset$.*

Proof. Part 1. Clearly the condition of Part 1 is equivalent to the fact that Q is irreducible. Next similarly to Proposition 4.3 one can show that this condition is the necessary for Q to be minimal. From Lemma 4.3 we obtain that it is also a sufficient condition for the arity $(k_1,\dots,k_s)$.

Part 2. Notice that each identification of T in Lemma 4.3 gives us either Q (up to permutations of coordinates of the same sort) or a diagonal (since Q is irreducible) and it is equivalent to application of some vector-epimorphism Ψ to the index relation M. Next if $r \in M$ and $\Psi r \in D(h_1) \times \cdots \times D(h_m)$, then $Q^{\Psi r}$ is a full diagonal, which does not affect the result of identification. But if $\Psi r \in D \backslash D(h_1) \times \cdots \times D(h_m)$, where D is an m-base incomplete diagonal, then $D^{\Psi r}$ is an incomplete diagonal itself and so $T^{\Psi} \neq Q$. Therefore, any reflexive part of ΨM leading to $T^{\Psi} = Q$ is included in $D(h_1) \times \cdots \times D(h_m)$.

It is obvious that the requirements of Part 2 imply the minimality of an irreducible relation Q. On the other side, if there exists Ψ such that $T^{\Psi} = Q$ and $p \notin \Psi R$, then by using some vector-permutation α on $E(h_1) \times \cdots \times E(h_m)$, $\alpha \notin G(R)$, one can prove (similar to Proposition 4.3) that $\alpha\Psi$ satisfies conditions of Part 2 and we have $T^{\alpha\Psi} = Q$. $\square$

Denote $\mathbf{R}_2$ the class of relations established in Proposition 4.4 including the ones obtained by permutations of numbers of base sets. Let $\mathbf{B}(k_i)$ be the set of single-base relations of sort i determining all maximal partial clones on $E(k_i)$, except $\Phi(k_i)$ $(i = 1, \ldots, n)$ (see [20] or [7] and also [3]). Set $\mathbf{B} = \mathbf{B}(k_1) \cup \cdots \cup \mathbf{B}(k_n)$.

Finally, summarizing the results of the three sections we obtain the theorem.

Theorem 4.1. *Every maximal partial n-clone* $(n \geqslant 2)$, *except* Φ_i $(i = 1, \ldots, n)$, *is determined by a relation from classes* $\mathbf{K}$, $\mathbf{H}$, $\mathbf{R}_1$, $\mathbf{R}_2$ *and* $\mathbf{B}$.

Corollary. *A system of partial n-operations S is complete in* $\prod \mathbb{P}(k_i)$ *if and only if*:

(1) *for every* i $(1 \leqslant i \leqslant n)$ *the set* $(S \backslash F)^i$ *of all restrictions* $S \backslash F$ *on its ith coordinate is complete in* $\mathbb{P}(k_i)$;
(2) *for every relation from classes* $\mathbf{K}$, $\mathbf{H}$, $\mathbf{R}_1$, *and* $\mathbf{R}_2$ *the set S contains a partial n-operation not preserving it*.

Remark. Note that the elements of F play the same role as empty operations for the case $n=1$. So, if we consider completeness criteria for $\prod \mathbb{P}(k_i)$, then elements of F are not supposed to be produced by compositions of partial n-operations from a complete set.

Note: (1) all relations from the above listed classes determine distinct partial n-clones unless they could be transposed to one another by some permutation of coordinates; (2) minimal relations have the minimum arity (comparing coordinatewise) among all relations determining the same maximal partial n-clone (for $n = 1$ see [22]).

5. Completeness in $\mathbb{P}(2) \times \cdots \times \mathbb{P}(2)$

We apply the previous results to vectors of partial Boolean functions (partial Boolean n-operations). In this case the description of maximal n-clones has a special simplified form that avoids the usage of epimorphic images. We introduce all these classes of minimal relations defined on n base sets E(2)={0,1}.

(1) Class **K** is the set of all nonempty, incomplete relations of arity $(1,\dots,1)$ having an arbitrary non-single type $J \subseteq \{1,\dots,n\}$, $2 \leqslant |J| \leqslant n$, which cannot be reduced by π_i $(i \in J)$ to relations of smaller type.

Next according to Corollary 3.5 we may consider only relations of arity $(2,\dots,2, 1,\dots,1,0,\dots,0)$ with the first m sorts having arity 2 and the next $s-m$ sorts having arity 1 $(1 \leqslant m \leqslant s \leqslant n,\ s \geqslant 2)$. Let $H \in \mathbf{H}$ be a relation of the above arity. By Lemma 3.1 being totally reflexive in the case $k_1 = 2$ means $x_1 = x_2 \subset H$. Moreover, if there exist two points $(0,1;q)$, $(1,0;q) \in H$, where q is a tuple over the type $\{2,\dots,s\}$, then clearly that together with $x_1 = x_2 \subset H$ we get $x_1 = x_2 \vee q \subseteq H$. Hence $H \equiv x_1 = x_2 \vee H'$, where H' has the type $\{2,\dots,s\}$.

(2) Class **H** consists of all relations having the form (as well as ones obtained from them by permutations of base sets):

$$x_1^1 = x_2^1 \vee \cdots \vee x_1^m = x_2^m \vee K(x^{m+1},\dots,x^s), \tag{15}$$

where either $K \in \mathbf{K}$ of type $\{m+1,\dots,s\}$, when $s \geqslant m+2$, or $K \in \{x^s = 0, x^s = 1\}$, when $s = m+1$, or K is void, when $s = m$.

(3) Note that in the Boolean case a vector-epimorphism Ψ from the Proposition 4.3 becomes a vector-isomorphism. Hence here each minimal $R \in \mathbf{R}_1$ consists of only one block (orbit) of its group $G(R)$ which in this case is a subgroup of the direct product $S_2 \times \cdots \times S_2$ of m symmetric groups $S_2 = \{e,\alpha\}$ on $E(2): \alpha: 0 \to 1,\ 1 \to 0$ and $\alpha^2 = e$. Notice that $G(R)$ consists of vectors $\langle \alpha_1,\dots,\alpha_m \rangle$, where either $\alpha_i = \alpha$ or $\alpha_i = e$ $(i = 1,\dots,m)$. Next if $G(R) = S_2 \times A$, where A is a group over the type $\{2,\dots,m\}$, then $R \equiv (x_1^1 \neq x_2^1) \mathbin{\&} T$, where T has the type $\{2,\dots,m\}$. Moreover, if we have $\alpha_i = e$ for all elements $\alpha \in G(R)$, then $R \equiv (x_1^i = 0 \mathbin{\&} x_2^i = 1) \mathbin{\&} T$, where T has the type $\{2,\dots,m\} \backslash \{i\}$.

So for a group $G(R)$ which is the direct product of S_2 and the unit group $\{e\}$, i.e., $G(R) = S_2(1) \times \cdots \times S_2(m)$, where $S_2(i) \in \{\{e\}, S_2\}$ $(i = 1,\dots,m)$, we have the presentation of the corresponding $R \in \mathbf{R}_1$ (up to arbitrary permutations of base sets):

$$R \equiv R_1 \mathbin{\&} \cdots \mathbin{\&} R_m \mathbin{\&} K, \tag{16}$$

where $R \in \{x_1^i = 0 \mathbin{\&} x_2^i = 1, x_1^i \neq x_2^i\}$ $(i = 1,\dots,m)$; $K \in \mathbf{K}$ is of the type $\{m+1,\dots,s\}$, when $s \geqslant m+2$, or $K \in \{x^s = 0, x^s = 1\}$, when $s = m+1$, or K is the full relation of the type $\{1,\dots,m\}$, when $s = m$.

Now consider the common case $G(R) = S_2[t] \times S_2(t+1) \times \cdots \times S_2(m)$, where $S_2[t]$ is a subdirect product of t groups S_2 $(2 \leqslant t \leqslant m)$. Let $\mathrm{Orb}(S_2[t]) = \{(\alpha_1 0, \alpha_1 1; \dots; \alpha_t 0, \alpha_t 1): \langle \alpha_1,\dots,\alpha_t \rangle \in S_2[t]\}$ be the $(2,\dots,2)$-orbit of this group.

Hence $\mathbf{R}_1$ consists of all relations defined in (16) and also relations having the form (including those obtained by permutations of numbers of base sets):

$$\mathrm{Orb}(S_2[t]) \mathbin{\&} R, \tag{17}$$

where either R is a relation from (16) over the type $\{t+1,\dots,s\}$, when $t < m$, or $R \in \mathbf{K}$ over the type $\{m+1,\dots,s\}$, when $t = m < s$ and $s \geqslant m+2$, or $R \in \{x^s = 0,\ x^s = 1\}$, when $s = m+1$ and $t = m$, or R is the full relation over the type $\{1,\dots,m\}$, when $t = m = s$.

Example. Let $t=3, m=5, s=7, S_2[3]=\{\langle e,e,e\rangle,\langle\alpha,e,\alpha\rangle,\langle e,\alpha,e\rangle,\langle\alpha,\alpha,\alpha\rangle\}$ and $G(R)=S_2[3]\times S_2\times\{e\}$ is the symmetry group of R. We have $\mathrm{Orb}(S_2[3])=\{(0,1;0,1;0,1),(0,1;1,0;0,1),(1,0;0,1;1,0),(1,0;1,0;1,0)\}$. Next we construct relations $R\in\mathbf{R}_1$:

$$R\equiv \mathrm{Orb}(S_2[3])\,\&\,x_1^4\neq x_2^4\,\&\,x_1^5=0\,\&\,x_2^5=1\,\&\,K,$$

where $K\in\mathbf{K}$ has the type $\{6,7\}$.

(4) From Proposition 4.4 we get that each $Q\in\mathbf{R}_2$ is obtained from $R\in\mathbf{R}_1$ using disjunction with an incomplete m-base diagonal of the same arity:

$$Q\equiv R\vee D_1\,\&\cdots\&\,D_m, \tag{18}$$

where $R\in\mathbf{R}_1$, D_i is a single-base diagonal of the sort i $(i=1,\ldots,m)$ and at least one of D_i is the equality relation $(1\leqslant m\leqslant n)$.

In total each $R\in\mathbf{R}_1$ produces 2^m-1 different relations $Q\in\mathbf{R}_2$.

(5) Recall that we have 7 single-base minimal relations (see [3]) over the type i $(i=1,\ldots,n)$: $x=0, x=1, x_1\neq x_2, x_1\leqslant x_2, x_1=0\,\&\,x_2=1, H_1\equiv x=y\,\&\,u=z\vee x=u\,\&\,y=z$ and $H_2\equiv x=y\,\&\,u=z\vee x=u\,\&\,y=z\vee x=z\,\&\,y=u$. The 8th maximal partial Boolean clone $\Phi(2)$, consisting of $Q(2)$ and empty operations, produces the maximal partial n-clone Φ_i $(i=1,\ldots,n)$.

Thus, we obtained the following theorem.

Theorem 5.1. *A system of partial Boolean n-operations S is complete in* $\mathbb{P}(2)\times\cdots\times\mathbb{P}(2)$ *if and only if*:

(1) *each coordinate set* $(S\backslash F)^i$ $(i=1,\ldots,n)$ *is complete in* $\mathbb{P}(2)$;
(2) *for each relation from the classes* (1)–(4) *S contains a partial n-operation not preserving it.*

Recall that all maximal n-clones of $\mathbb{Q}(2)\times\cdots\times\mathbb{Q}(2)$ were described in [19] by the following relations (another approach see in [28]):

(a) single-base relations determining all 5 maximal clones on $E(2)$ (see [13]): $x=0, x=1, x_1\neq x_2, x_1\leqslant x_2$, and H_2 of the sort $\{i\}$ $(i=1,\ldots,n)$;
(b) 2-base relations $x=0\,\&\,y=0\vee x=1\,\&\,y=1, x=0\,\&\,y=1\vee x=1\,\&\,y=0$ for all pairs of different sorts from $\{1,\ldots,n\}$;
(c) s-base relations $(1\leqslant h\leqslant s\leqslant n, s\geqslant 2)$ of the form (including the ones obtained by permutations of numbers of base sets):

$$x_1^1=x_2^1\vee\cdots\vee x_1^h=x_2^h\vee R_{h+1}\vee\cdots\vee R_s,$$

where $R_i\in\{x^i=0,\ x^i=1\}$ $(i=h+1,\ldots,s)$.

Clearly class (a) in included in (5) and relations from (b) and (c) are contained in $\mathbf{K}\cup\mathbf{H}$.

Corollary 5.1. *Each relation from classes* (a), (b) *and* (c) *determining maximal n-clone of Boolean functions also determines maximal partial n-clone of partial Boolean functions.*

Case $\mathbb{P}(2) \times \mathbb{P}(2)$:

Applying the results of this section we describe all 67 maximal partial 2-clones of Boolean operations, i.e., all maximal iterative Post subalgebras in the system of all pairs of partial Boolean functions.

(1) Considering class **K** we get 10 minimal double-base relations: $x=a\,\&\,y=b, x=a \vee y=b$, where $a,b \in \{0,1\}$, $x=0\,\&\,y=0 \vee x=1\,\&\,y=1$, $x=0\,\&\,y=1 \vee x=1\,\&\,y=0$.

(2) Class **H** contributes 5 relations: $x_1=x_2 \vee y_1=y_2$, $x=0 \vee y_1=y_2$, $x=1 \vee y_1=y_2$, $x_1=x_2 \vee y=0$, $x_1=x_2 \vee y=1$.

(3) Classes $\mathbf{R}_1$ and $\mathbf{R}_2$ give 20 relations of arity $(2,2)$: $R_1 \equiv x_1=0\,\&\,x_2=1\,\&\,y_1=0\,\&\,y_2=1$, $R_2 \equiv x_1=0\,\&\,x_2=1\,\&\,y_1 \neq y_2, R_3 \equiv x_1 \neq x_2\,\&\,y_1=0\,\&\,y_2=1$, $R_4 \equiv x_1 \neq x_2\,\&\,y_1 \neq y_2$, $R_5 \equiv x_1=0\,\&\,x_2=1\,\&\,y_1=0\,\&=y_2=1 \vee x_1=1\,\&\,x_2=0\,\&\,y_1=1\,\&=y_2=0$. And also $R_i \vee D$ $(i=1,\ldots,5)$, where $D \in \{x_1=x_2,\ y_1=y_2,\ x_1=x_2\,\&\,y_1=y_2\}$.

(4) There are also 16 relations from $\mathbf{R}_1 \cup \mathbf{R}_2$ of arity (2,1) and (1,2): $Q_1 \equiv x_1=0\,\&\,y_1=0\,\&=y_2=1, Q_2 \equiv x_2=1\,\&\,y_1=0\,\&=y_2=1, Q_3 \equiv x=0\,\&\,y_1 \neq y_2$, $Q_4 \equiv x=1\,\&\,y_1 \neq y_2, Q_i \vee y_1=y_2$ $(i=1,\ldots,4)$-yields 8 relations. Interchanging x and y we obtain 8 relations of arity (2,1).

(5) Add 16 partial 2-clones of the form $(A \times P(2)) \cup F$ and $(P(2) \times A) \cup F$, where A is maximal partial clone on $E(2)$ (see [4]).

Finally, we get in total 67 maximal partial 2-clones of Boolean functions.

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