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Separation of internal and interaction dynamics for NLS-described wave packets with different carrier waves

Martina Chirilus-Bruckner [a], Christopher Chong [a], Guido Schneider [a,*], Hannes Uecker [b]

[a] Institut für Analysis, Dynamik und Modellierung, Universität Stuttgart, Pfaffenwaldring 57, D-70569 Stuttgart, Germany
[b] Institut für Mathematik, Universität Oldenburg, D-26111 Oldenburg, Germany

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ABSTRACT

We give a detailed analysis of the interaction of two NLS-described wave packets with different carrier waves for a nonlinear wave equation. By separating the internal dynamics of each wave packet from the dynamics caused by the interaction we prove that there is almost no interaction of such wave packets. We also prove the validity of a formula for the envelope shift caused by the interaction of the wave packets.

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1. Introduction

We consider the nonlinear wave equation

$$\partial_t^2 u = \partial_x^2 u - u + u^3 \tag{1}$$

with $t \geqslant 0$, $x \in \mathbb{R}$, and $u = u(x,t) \in \mathbb{R}$. For this equation the ansatz

$$u(x,t) = \varepsilon A(X,T)\mathrm{e}^{\mathrm{i}(kx-\omega t)} + \text{c.c.} + \mathcal{O}\big(\varepsilon^2\big), \quad X = \varepsilon(x-ct), \ T = \varepsilon^2 t, \tag{2}$$

* Corresponding author.
E-mail address: guido.schneider@mathematik.uni-stuttgart.de (G. Schneider).

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where $k, \omega \in \mathbb{R}$ satisfy the linear dispersion relation $\omega^2 = k^2 + 1$, where $c = \mathrm{d}\omega/\mathrm{d}k = k/\omega$ is the linear group velocity, and where $0 < \varepsilon \ll 1$ is a small perturbation parameter, leads to the nonlinear Schrödinger (NLS) equation

$$2\mathrm{i}\omega\partial_T A = \big(1 - c^2\big)\partial_X^2 A + 3|A|^2 A$$

describing slow modulations in time and space of the underlying carrier wave $\mathrm{e}^{\mathrm{i}(kx-\omega t)}$. This procedure is common in nonlinear optics and allows one to reduce the dimension of the problem in numerical simulations by a factor up to 10^5, cf. [1]. In modern fiber optics, however, not only a single carrier wave, but a number of different carrier waves is used, cf. [5].

In the particular case of two different carrier waves, i.e., $k_A \neq k_B$, the ansatz is given by

$$u(x,t) = \varepsilon A\big(\varepsilon(x - c_A t), \varepsilon^2 t\big)\mathrm{e}^{\mathrm{i}(k_A x - \omega_A t)} + \varepsilon B\big(\varepsilon(x - c_B t), \varepsilon^2 t\big)\mathrm{e}^{\mathrm{i}(k_B x - \omega_B t)} + \text{c.c.} + \mathcal{O}\big(\varepsilon^2\big),$$

leading to a system of *coupled* NLS equations

$$2\mathrm{i}\omega_A\partial_T A = \big(1 - c_A^2\big)\partial_{X_A}^2 A + 3A|A|^2 + 6A|B|^2,$$
$$2\mathrm{i}\omega_B\partial_T B = \big(1 - c_B^2\big)\partial_{X_B}^2 B + 3B|B|^2 + 6B|A|^2.$$

Since $X_A = \varepsilon(x - c_A t) = \varepsilon(x - c_B t) - \varepsilon(c_A - c_B)t = X_B - \frac{c_A - c_B}{\varepsilon}T$ and since the group velocities $c_A \neq c_B$ of the wave packets are different, this system has still the multiple scale character of the original problem. However, the interaction of localized wave packets will only happen on a very short time scale, such that asymptotically the interaction terms

$$6A(X_A, T)\big|B(X_B, T)\big|^2 = 6A(X_A, T)\left|B\left(X_A - \frac{(c_B - c_A)}{\varepsilon}T, T\right)\right|^2$$

and

$$6B(X_B, T)\big|A(X_A, T)\big|^2 = 6B(X_B, T)\left|A\left(X_B - \frac{(c_A - c_B)}{\varepsilon}T, T\right)\right|^2$$

are negligible. As a consequence, in lowest order we have a system of *uncoupled* NLS equations

$$2\mathrm{i}\omega_A\partial_T A = \big(1 - c_A^2\big)\partial_{X_A}^2 A + 3A|A|^2,$$
$$2\mathrm{i}\omega_B\partial_T B = \big(1 - c_B^2\big)\partial_{X_B}^2 B + 3B|B|^2,$$

or, in other words, each band is described independently by a single NLS equation.

In applications the neglection of the coupling terms is a common procedure, cf. [1]. There exist a number of mathematical papers [2,3,9] which validate this procedure rigorously. Our research is dedicated to an improvement of existing estimates for wave interaction aiming towards applications in optical communication lines which use wavelength division multiplexing technologies, cf. [5].

In our previous work [3] we presented improved bounds for two waves modulated by *NLS* 1-*solitons* (in the following called *well-prepared pulses*, see Fig. 1). Here, we further extend our results to waves whose envelopes are *general localized profiles* evolving according to the NLS equation (in the following called *non-well-prepared pulses*, see Fig. 1). We show for these general wave packets that the interaction leads to an $\mathcal{O}(\varepsilon)$-phase shift of the carrier wave and to an $\mathcal{O}(\varepsilon)$-shift of the envelope. Thus, we improve the bound for the possible envelope shift caused by the interaction of general localized NLS-described wave packets from $\mathcal{O}(1)$, cf. [9], to $\mathcal{O}(\varepsilon)$ and generalize the $\mathcal{O}(\varepsilon)$-bound for the interaction of wave packets with NLS 1-solitons as envelope to general NLS-described wave packets. Moreover, we prove the validity of a formula for the envelope shift caused by the interaction of general wave packets.

In the same spirit there are approximation results describing the interaction of KdV-described long waves by higher-order approximations in FPU-lattices [7], in Boussinesq models [12], and in the water wave problem [13].

Notation. Many possibly different constants which can be chosen independently of $0 < \varepsilon \ll 1$ are denoted by C. The space $H^s(m)$ consists of s-times weakly differentiable functions for which $\|u\|_{H^s(m)} = \|u\rho^m\|_{H^s} = (\sum_{j=0}^s \int |\partial_x^j(u\rho^m)|^2\,\mathrm{d}x)^{1/2}$ with $\rho(x) = \sqrt{1 + x^2}$ is finite, where we do not distinguish between scalar and vector-valued functions or real- and complex-valued functions. The space C_b^s consists of s-times continuously differentiable functions for which $\|u\|_{C_b^s} = \sum_{j=0}^s \sup_{x\in\mathbb{R}} |\partial_x^j u|$ is finite. We sometimes write, e.g., $\|u(x)\|_{C_b^s}$ for the C_b^s-norm of the function $x \mapsto u(x)$.

2. Approximate description of internal and interaction dynamics

In this and in the next section we derive approximation equations in order to describe the internal and interaction dynamics of the wave packets. In order to make the concept of internal and interaction dynamics more precise let S_t be the nonlinear evolution operator of the nonlinear wave equation (1). The evolution $S_t(u_A)$ of one single initial wave packet u_A is

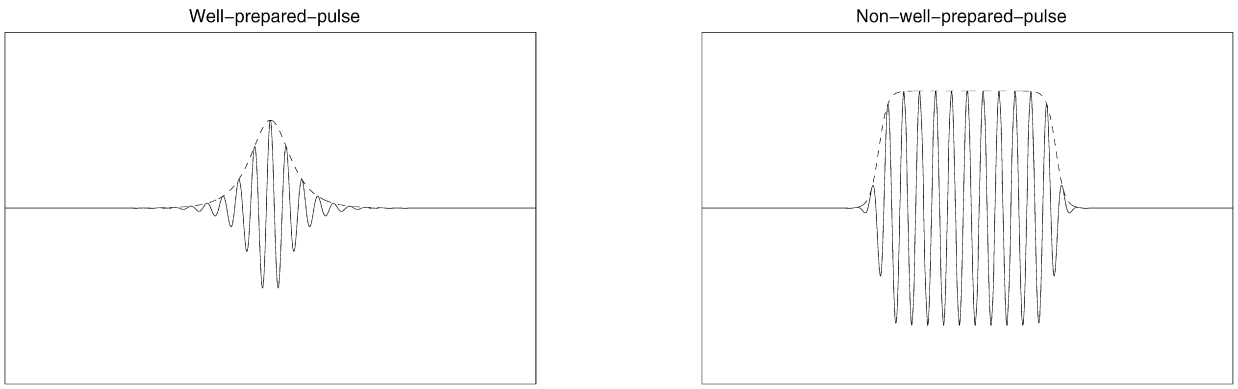

Fig. 1. Left: A well-prepared pulse. The envelope (dashed line) is an NLS 1-soliton. Right: A non-well-prepared pulse. The envelope (dashed line) can be anything "pulse like." Here we chose an almost rectangular envelope.

called internal dynamics. The solution to the sum of two single initial wave packets u_A and u_B evolves as $S_t(u_A + u_B)$. The interaction dynamics is then the difference $S_t(u_A + u_B) - S_t(u_A) - S_t(u_B)$. It is the purpose of this paper to give a precise description of this difference. We are especially interested in improved estimates for carrier and envelope shifts caused by the interaction.

In the following we consider two NLS-described wave packets u_A and u_B which are spatially localized, i.e. for the envelope A there exist $C > 0$ and $m \in \mathbb{N}$ such that $|A(x)| \leqslant C/(1 + |x|)^m$, $x \in \mathbb{R}$, or more precisely, $A \in H^s(m)$, $s > 1/2$.

Approximate description of internal dynamics. In the case of one single wave packet with a wavenumber k_A the dynamics can be described approximately by the ansatz (2). By adding higher-order terms to the ansatz the formal error, or more precisely the later introduced residual, can be made arbitrarily small. The NLS equation is then accompanied by a system of linear PDEs and algebraic equations.

Approximate description of interaction dynamics. In the case of *two-wave propagation* the nonlinearity leads to an interaction between the wave packets which in turn result in a modification of the pure internal dynamics. We improve the ansatz from [3] and seek solutions of the form

$$\varepsilon\Psi = \big(\varepsilon A_1 + \varepsilon^2 A_2 + \varepsilon^3 A_3\big)E + \big(\varepsilon B_1 + \varepsilon^2 B_2 + \varepsilon^3 B_3\big)F + \text{c.c.} + \varepsilon^3 M_{\text{mixed}} \tag{3}$$

where the term M_{mixed} serves to cancel mixed and higher-order harmonic terms in the formal error and where

$$E = \exp\big(\mathrm{i}\big(k_A x - \omega_A t + \varepsilon\Omega_{A,1}(Z_B, T) + \varepsilon^2\Omega_{A,2}(Z_B, T)\big)\big),$$
$$F = \exp\big(\mathrm{i}\big(k_B x - \omega_B t + \varepsilon\Omega_{B,1}(Z_A, T) + \varepsilon^2\Omega_{B,2}(Z_A, T)\big)\big),$$
$$Z_A = \varepsilon\big(x - c_A t + \varepsilon\psi_A(X_B, T)\big), \tag{4}$$
$$Z_B = \varepsilon\big(x - c_B t + \varepsilon\psi_B(X_A, T)\big), \tag{5}$$
$$A_j = A_j(Z_A, T), \qquad B_j = B_j(Z_B, T), \qquad X_A = \varepsilon(x - c_A t), \qquad X_B = \varepsilon(x - c_B t). \tag{6}$$

The internal dynamics of the wave packets will be described by the variables A_j, B_j, $j = 1, 2$, whereas the interaction dynamics is described by the phase shifts $\Omega_{A,j}, \Omega_{B,j}$, $j = 1, 2$ and the envelope shifts ψ_A, ψ_B.

Remark 2.1. a) The ansatz (3) is more general than the one in [3] where we essentially chose A_1 and B_1 in the form of NLS 1-solitons. Here we allow A_1 and B_1 to be more general solutions of the respective NLS equation, see Fig. 1. This then requires the introduction of $\varepsilon^2 A_2 E$, $\varepsilon^2 B_2 F$ to describe the internal dynamics.

b) The phase shifts $\Omega_{A,1}$, $\Omega_{B,1}$ turn out to be real functions. In order to describe the interaction dynamics in more detail than in [3] we additionally introduce phase shift corrections $\Omega_{A,2}$, $\Omega_{B,2}$, which turn out to have an imaginary component, and the envelope shifts ψ_A, ψ_B. These last ones have already been introduced in earlier works like [8] or [11] where they are called pulse shifts. Our aim is to validate formulas for the envelope shifts by rigorous estimates.

c) Furthermore, we change the notation: The variables Y_A, Y_B from [3] are now called A_3, B_3, whereas A_3, B_3 from [3] are here contained in the term M_{mixed}.

d) Finally, in the following we replace the arguments X_B, X_A of ψ_A, ψ_B by Z_B, Z_A. More rigorously we may define Z_A, Z_B implicitly by

$$Z_A = \varepsilon\big(x - c_A t + \varepsilon\tilde{\psi}_A(Z_B)\big), \qquad Z_B = \varepsilon\big(x - c_B t + \varepsilon\tilde{\psi}_B(Z_A)\big). \tag{7}$$

Then ψ_A and $\tilde{\psi}_A$ respectively ψ_B and $\tilde{\psi}_B$ differ by $\mathcal{O}(\varepsilon)$ terms which we may discard for our purposes. Therefore, from now on we write Z_B, Z_A for the arguments of ψ_A, ψ_B, respectively.

Remark 2.2. At this point the notion of an envelope shift is somewhat ambiguous since by Taylor-expansion w.r.t. ψ_A and ψ_B we have with $X_A = \varepsilon(x - c_A t)$, $X_B = \varepsilon(x - c_B t)$,

$$\begin{aligned}\varepsilon\Psi(x,t) = {} & \big(\varepsilon A_1(X_A,T) + \varepsilon^2 A_2(X_A,T) + \varepsilon^3\big(A_3(X_A,T) + \psi_A \partial_1 A_1(X_A,T)\big)\big)E \\ & + \big(\varepsilon B_1(X_B,T) + \varepsilon^2 B_2(X_B,T) + \varepsilon^3\big(B_3(X_B,T) + \psi_B \partial_1 B_1(X_B,T)\big)\big)F + \text{h.o.t.}\end{aligned}$$

The terms $\varepsilon^2 A_2$ and $\varepsilon^2 B_2$ do not contribute to envelope shifts caused by interaction since they are determined by internal dynamics of the individual pulses (see (12)). The term A_3 is of the same order as the envelope shift term $\partial_1 A_1 \psi_A$, i.e., it accounts for both internal and interaction dynamics, but it is neither clear to which amount A_3 describes the interaction, nor in which way—as phase or envelope correction. In other words, it has to be checked, if the derived formulas really quantify the entire envelope shift in the particular order. The validity of the envelope shift formula is investigated numerically in Section 5 and explained analytically in Section 6. This expansion obviously gives an $\mathcal{O}(\varepsilon)$-bound for the envelope shift if we can prove an $\mathcal{O}(\varepsilon^3)$-bound in L^∞ for the terms indicated with h.o.t. and an $\mathcal{O}(1)$-bound for A_3 and B_3. Then the vertical bound $\mathcal{O}(\varepsilon^3)$ only allows a 'horizontal error' of $\mathcal{O}(\varepsilon)$. The required bounds will be proven in Proposition 4.3 and Lemma 4.7.

Remark 2.3. Since $\operatorname{Im}\Omega_{A,2}$, $\operatorname{Im}\Omega_{B,2}$ are supposed to describe interaction dynamics we may assume that $\Omega_{A,2} = \Omega_{B,2} = 0$ initially. Moreover, due to the fact that $\operatorname{Im}\Omega_{A,2}$ and $\operatorname{Im}\Omega_{B,2}$ turn out to be spatially localized, E and F contain only phase shifts for $|x| \to \infty$, i.e. $\operatorname{Im}\Omega_{A,2}$ and $\operatorname{Im}\Omega_{B,2}$ play no role for the envelope shift. In detail, in Lemma 4.6 we prove the $\mathcal{O}(1)$-boundedness of $\Omega_{A,1}$, $\Omega_{B,1}$, $\operatorname{Re}\Omega_{A,2}$ and $\operatorname{Re}\Omega_{B,2}$ in L^∞ and that $\operatorname{Im}\Omega_{A,2}$ and $\operatorname{Im}\Omega_{B,2}$ are $\mathcal{O}(1)$-bounded in $H^s(m)$. Thus, for instance, $|\operatorname{Im}\Omega_{A,2}(Z_B,T)| \leqslant C/(1+\varepsilon|x - c_B t|)^m$ due to Sobolev's embedding theorem for $s > 1/2$. For the same reason we have $|A_j(Z_A,T)| \leqslant C/(1+\varepsilon|x - c_A t|)^m$ and so, for large t, i.e. for $t > 1/\varepsilon$,

$$\big|A_j(Z_A,T)\big| \operatorname{Im}\Omega_{A,2}(Z_B,T) = \mathcal{O}\big((\varepsilon t)^{-m}\big).$$

Moreover, for well-prepared pulses $|A_j(Z_A,T)|\operatorname{Im}\Omega_{A,2}(Z_B,T)$ is exponentially small. According to the last remark, $|A_j(Z_A,T)|\operatorname{Im}\Omega_{A,2}(Z_B,T)$ has to be $o(\varepsilon)$, except during interaction. Thus we require $(\varepsilon t)^{-m} = \mathcal{O}(\varepsilon^{1+\delta m})$ with $\delta > 0$ arbitrary small but fixed. This yields $t \sim \varepsilon^{-(1+1/m+\delta)} \ll \varepsilon^{-2}$ for $m \geqslant 2$. In summary, for $C_1\varepsilon^{-(1+1/m+\delta)} \leqslant t \leqslant C_2\varepsilon^{-2}$ the corrections $\operatorname{Im}\Omega_{A,2}$ and $\operatorname{Im}\Omega_{B,2}$ play no role for the envelope shifts. In case of well-prepared pulses this can be sharpened to $C_1 \ln(\varepsilon)\varepsilon^{-1} \leqslant t \leqslant C_2\varepsilon^{-2}$.

3. Derivation of approximation equations

The so-called residual

$$\operatorname{Res}(\varepsilon\Psi) = -\partial_t^2(\varepsilon\Psi) + \partial_x^2(\varepsilon\Psi) - (\varepsilon\Psi) + (\varepsilon\Psi)^3 \tag{8}$$

describes how much an ansatz $\varepsilon\Psi$ fails to satisfy the nonlinear wave equation (1). Plugging the ansatz (3) into the residual

$$\operatorname{Res}(\varepsilon\Psi) = \sum_{l,m,n} \varepsilon^l \operatorname{Res}_{l,m,n} E^m F^n \tag{9}$$

leads to a number of conditions in order to make the residual as small as possible, in particular to nonlinear Schrödinger equations for A_1 and B_1. (9) is a definition for the terms $\operatorname{Res}_{l,m,n}$ which are the coefficients in an expansion of the residual w.r.t. $\varepsilon^l E^m F^n$.

Remark 3.1. The term $M_{\text{mixed}} = M_{\text{mixed}}(A_1, A_2, A_3, B_1, B_2, B_3, E, F)$ in (3) accounts for terms involving higher-order or mixed harmonics, i.e. for the frequencies which are generated by the nonlinearity according to the formula

$$\big(\varepsilon A_1 E + \varepsilon^2 A_2 E + \varepsilon^3 A_3 E + \varepsilon B_1 F + \varepsilon^2 B_2 F + \varepsilon^3 B_3 F + \text{c.c.}\big)^3 = \sum_{k_1+\ldots+k_{12}=3,\, k_j \geqslant 0} \frac{3!}{k_1!\cdots k_{12}!} (\varepsilon A_1 E)^{k_1} \cdots \big(\varepsilon^3 \overline{B_3 F}\big)^{k_{12}},$$

however without the nonlinear terms generated at E or F. At $\varepsilon^3 E^2 F$ for example the term $A_1^2 B_1$ appears. To cancel this term we extend the ansatz by $\alpha_{21}\varepsilon^3 A_1^2 B_1 E^2 F$ and get an algebraic equation for α_{21} of the form

$$\big(1 + (2\mathrm{i}\omega_A + \mathrm{i}\omega_B)^2 + (2\mathrm{i}k_A + \mathrm{i}k_B)^2\big)\alpha_{21} = 3.$$

The procedure is essentially the same for each such term leading to equations of the form

$$\big(1 + (l\omega_A + j\omega_B)^2 + (lk_A + jk_B)^2\big)\alpha_{lj} = \beta_{lj}.$$

Now M_{mixed} contains all these extensions. Thus, we can concentrate on the remaining terms of the residual.

Remark 3.2. Since A_j, $j=1,2,3$, depend on the same variables and belong to the same harmonic, the subsequent hierarchy of conditions (10)–(15) reappears shifted in order, i.e. the residual actually contains much more terms, for example $\varepsilon^j(-k_A^2+\omega_A^2-1)A_j(Z_A,T)E$, $j=1,2,3$, which we only list for $j=1$. Hence choosing the dispersion relation as solvability condition cancels all these terms. The exact same mechanism holds for the entire system of Eqs. (10)–(15), so we tacitly left out repeated terms to simplify the exposition.

Using the notation $\mathrm{Res}_{l,m,n}$ from (9) for the coefficients of $\varepsilon^l E^m F^n$ we find the subsequent hierarchy of equations.

- At εE we find

$$\mathrm{Res}_{1,1,0}=\big(-k_A^2+\omega_A^2-1\big)A_1(Z_A,T)\stackrel{!}{=}0$$

which yields the linear dispersion relation

$$\omega_A^2=k_A^2+1.$$

- At $\varepsilon^2 E$ we find

$$\mathrm{Res}_{2,1,0}=2\mathrm{i}(k_A-c_A\omega_A)\partial_1 A_1(Z_A,T)\stackrel{!}{=}0$$

which yields the linear group velocity

$$c_A=k_A/\omega_A.$$

- At $\varepsilon^3 E$ we find

$$\mathrm{Res}_{3,1,0}=s_{31}+s_{32}$$

with

$$\begin{aligned}s_{31}&=2\mathrm{i}\omega_A\partial_2 A_1(Z_A,T)+\big(1-c_A^2\big)\partial_1^2 A_1(Z_A,T)+3\big|A_1(Z_A,T)\big|^2A_1(Z_A,T),\\ s_{32}&=\big(2(\omega_A c_B-k_A)(Z_A,T)\partial_1\Omega_{A,1}(Z_B,T)+6\big|B_1(Z_B,T)\big|^2\big)A_1(Z_A,T).\end{aligned}$$

Then $s_{31}\stackrel{!}{=}0$ yields the NLS equation

$$-2\mathrm{i}\omega_A\partial_2 A_1(Z_A,T)=\big(1-c_A^2\big)\partial_1^2 A_1(Z_A,T)+3\big|A_1(Z_A,T)\big|^2A_1(Z_A,T),\tag{10}$$

and $s_{32}\stackrel{!}{=}0$ yields the *phase shift formula*

$$\Omega_{A,1}(Z_B,T)=\frac{3}{k_A-\omega_A c_B}\int^{Z_B}\big|B_1(\zeta,T)\big|^2\,\mathrm{d}\zeta,\tag{11}$$

so $\Omega_{A,1}$ is a real quantity and therefore a *pure phase correction*.

- At $\varepsilon^4 E$ we find

$$\mathrm{Res}_{4,1,0}=2\omega_A\mathrm{i}\partial_2 A_2(Z_A,T)+\big(1-c_A^2\big)\partial_1^2 A_2(Z_A,T)+s_{41}+s_{42}+s_{43}+s_{44}+s_{45}+s_{46}$$

where

$$\begin{aligned}s_{41}&=\big(6A_2(Z_A,T)\overline{A}_1(Z_A,T)+3\overline{A}_2(Z_A,T)A_1(Z_A,T)\big)A_1(Z_A,T),\\ s_{42}&=2c_A\partial_1\partial_2 A_1(Z_A,T),\\ s_{43}&=6\big(B_2(Z_B,T)\overline{B}_1(Z_B,T)+\overline{B}_2(Z_B,T)B_1(Z_B,T)\big)A_1(Z_A,T),\\ s_{44}&=-2\omega_A A_1(Z_A,T)\partial_2\Omega_{A,1}(Z_B,T),\\ s_{45}&=2\mathrm{i}\partial_1 A_1(Z_A,T)\big((k_A-c_B\omega_A)\partial_1\psi_A(Z_B,T)+(1-c_Ac_B)\partial_1\Omega_{A,1}(Z_B,T)\big),\\ s_{46}&=\big(\mathrm{i}\big(1-c_B^2\big)\partial_1^2\Omega_{A,1}(Z_B,T)+2\partial_1\Omega_{A,2}(Z_B,T)(\omega_Ac_B-k_A)\big)A_1(Z_A,T).\end{aligned}$$

We are now left with a linear inhomogeneous evolution equation for A_2,

$$-2\omega_A\mathrm{i}\partial_2 A_2(Z_A,T)=\big(1-{c_A}^2\big)\partial_1^2 A_2(Z_A,T)+s_{41}+s_{42}.\tag{12}$$

Here, no coupling with terms involving B-variables occurs such that A_2 describes internal dynamics of a single pulse.

The terms in s_{45} together with (11) give the *envelope shift formula*

$$\psi_A(Z_B,T)=\frac{3(1-c_Ac_B)}{(c_B\omega_A-k_A)^2}\int^{Z_B}\big|B_1(\zeta,T)\big|^2\,\mathrm{d}\zeta. \tag{13}$$

The terms in $s_{43}+s_{44}+s_{46}$ yield the *second-order correction to the phase shift* in the form

$$\begin{aligned}\partial_1\Omega_{A,2}(Z_B,T)=\frac{1}{2(k_A-\omega_Ac_B)}\big(&\mathrm{i}\big(1-c_B^2\big)\partial_1^2\Omega_{A,1}(Z_B,T)-6\big(B_2(Z_B,T)\overline{B}_1(Z_B,T)+\overline{B}_2(Z_B,T)B_1(Z_B,T)\big)\\&+2\omega_A\partial_2\Omega_{A,1}(Z_B,T)\big).\end{aligned} \tag{14}$$

This equation is separated in real and imaginary part, i.e.,

$$\begin{aligned}\operatorname{Re}\Omega_{A,2}(Z_B,T)&=\int^{Z_B}\frac{1}{2(k_A-\omega_Ac_B)}\big(-6\big(B_2(\zeta,T)\overline{B}_1(\zeta,T)+\overline{B}_2(\zeta,T)B_1(\zeta,T)\big)+2\omega_A\partial_2\Omega_{A,1}(\zeta,T)\big)\,\mathrm{d}\zeta,\\ \operatorname{Im}\Omega_{A,2}(Z_B,T)&=\frac{1}{2(k_A-\omega_Ac_B)}\mathrm{i}\big(1-c_B^2\big)\partial_1\Omega_{A,1}(Z_B,T)=\frac{3\mathrm{i}(1-c_B^2)}{2(k_A-\omega_Ac_B)^2}\big|B_1(Z_B,T)\big|^2.\end{aligned}$$

Only $\operatorname{Re}\Omega_{A,2}$ gives a phase correction, while $\operatorname{Im}\Omega_{A,2}$ is an *amplitude correction*, which however is algebraically small w.r.t. ε except during the collision of wave packets. In order to have an $\mathcal{O}(1)$-bound of $\operatorname{Re}\Omega_{A,2}(Z_B,T)$ subsequently in L^∞ we need that all terms under the integral $\int^{Z_B}\ldots\,\mathrm{d}\zeta$ are spatially localized. This is clear for all terms except $\partial_2\Omega_{A,1}(\zeta,T)$. However, using the NLS equation (10) for B_1 and the representation formula (11) for $\Omega_{A,1}$ we find that

$$\partial_2\Omega_{A,1}(Z_B,T)=\mathrm{i}\frac{3(1-c_A^2)}{2\omega_A(k_A-\omega_Ac_B)}\big(\partial_1B_1(Z_B,T)\overline{B}_1(Z_B,T)-\partial_1B_1(Z_B,T)\overline{B}_1(Z_B,T)\big)$$

is also spatially localized.

- At ε^5E we find

$$\mathrm{Res}_{5,1,0}=2\omega_A\mathrm{i}\partial_2A_3(Z_A,T)+\big(1-c_A^2\big)\partial_1^2A_3(Z_A,T)+s_{51}+s_{52}+s_{53}+s_{54}+s_{55}+s_{56}$$

where

$$\begin{aligned}s_{51}&=\big(6A_3(Z_A,T)\overline{A}_1(Z_A,T)+3\overline{A}_3(Z_A,T)A_1(Z_A,T)\big)A_1(Z_A,T),\\ s_{52}&=-\partial_2^2A_1(Z_A,T)+2c_A\partial_1\partial_2A_2(Z_A,T),\\ s_{53}&=6A_1(Z_A,T)\big|A_2(Z_A,T)\big|^2+3A_2^2(Z_A,T)\overline{A}_1(Z_A,T),\\ s_{54}&=6\big(B_3(Z_B,T)\overline{B}_1(Z_B,T)+\overline{B}_3(Z_B,T)B_1(Z_B,T)\big)A_1(Z_A,T),\\ s_{55}&=\big(1-c_B^2\big)\partial_1A_1(Z_A,T)\partial_1^2\psi_A(Z_B,T)+2(1-c_Ac_B)\partial_1^2A_1(Z_A,T)\partial_1\psi_A(Z_B,T)\\&\quad+\mathrm{i}\big[\big(1-c_B^2\big)A_1(Z_A,T)\partial_1^2\Omega_{A,2}(Z_B,T)+2(1-c_Ac_B)\partial_1A_1(Z_A,T)\partial_1\Omega_{A,2}(Z_B,T)\big]\\&\quad+\big(c_B^2-1\big)A_1(Z_A,T)\big(\partial_1\Omega_{A,1}(Z_B,T)\big)^2,\\ s_{56}&=-2\omega_AA_1(Z_A,T)\partial_2\Omega_{A,2}(Z_B,T)+6\big|B_2(Z_B,T)\big|^2A_1(Z_A,T)\\&\quad+2\mathrm{i}\big[c_B\partial_2A_1(Z_A,T)\partial_1\Omega_{A,1}(Z_B,T)+c_A\partial_1A_1(Z_A,T)\partial_2\Omega_{A,1}(Z_B,T)+c_BA_1(Z_A,T)\partial_1\partial_2\Omega_{A,1}(Z_B,T)\\&\quad+i\omega_A\partial_1A_1(Z_A,T)\partial_2\psi_A(Z_B,T)\big].\end{aligned}$$

The terms s_{51}, s_{52} and s_{53} describe internal dynamics, whereas s_{54}, s_{55} and s_{56} are interaction terms in the same sense as s_{43} and s_{44}. We choose A_3 to satisfy the linear PDE

$$-2\mathrm{i}\omega_A\partial_2A_3(Z_A,T)=\big(1-c_A^2\big)\partial_1^2A_3(Z_A,T)+M_0[A_3,B_3]+I(A_1,A_2,B_1,B_2,\Omega_A,\psi_A) \tag{15}$$

where $M_0[A_3,B_3]=s_{51}+s_{54}$ is linear in its arguments and $I(A_1,A_2,B_1,B_2,\Omega_A,\psi_A)=s_{52}+s_{53}+s_{55}+s_{56}$ contains inhomogeneous terms which are $\mathcal{O}(1)$ bounded on the $\mathcal{O}(1/\varepsilon^2)$-time scale if $A_1,\ldots,\psi_B$ are $\mathcal{O}(1)$ bounded (up to second derivatives for A_1, B_1).

Finally we choose B_1, B_2, B_3, $\Omega_{B,1}$, $\Omega_{B,2}$, and ψ_B to satisfy the counterparts to (10)–(15).

Summary. The hierarchy of approximation equations consist of the linear dispersion relation and the relation for the linear group velocity,

$$\omega_A^2=k_A^2+1,\qquad c_A=k_A/\omega_A,$$

the NLS-equation, two inhomogeneous linear Schrödinger-equations

$$
\begin{aligned}
-2\mathrm{i}\omega_A\partial_2 A_1(Z_A,T) &= \big(1-c_A^2\big)\partial_1^2 A_1(Z_A,T) + 3\big|A_1(Z_A,T)\big|^2 A_1(Z_A,T),\\
-2\omega_A\mathrm{i}\partial_2 A_2(Z_A,T) &= \big(1-{c_A}^2\big)\partial_1^2 A_2(Z_A,T) + s_{41} + s_{42},\\
-2\mathrm{i}\omega_A\partial_2 A_3(Z_A,T) &= \big(1-c_A^2\big)\partial_1^2 A_3(Z_A,T) + M_0[A_3,B_3] + I(A_1,A_2,B_1,B_2,\Omega_A,\psi_A),
\end{aligned}
$$

the phase shift formula, a second-order correction to phase shift and amplitude, and finally the envelope shift formula

$$
\begin{aligned}
\Omega_{A,1}(Z_B,T) &= \frac{3}{k_A-\omega_A c_B}\int^{Z_B}\big|B_1(\zeta,T)\big|^2\,\mathrm{d}\zeta,\\
\partial_1\Omega_{A,2}(Z_B,T) &= \frac{1}{2(k_A-\omega_A c_B)}\big(\mathrm{i}\big(1-c_B^2\big)\partial_1^2\Omega_{A,1}(Z_B,T) - 6\big(B_2(Z_B,T)\overline{B}_1(Z_B,T)+\overline{B}_2(Z_B,T)B_1(Z_B,T)\big)\\
&\qquad + 2\omega_A\partial_2\Omega_{A,1}(Z_B,T)\big),\\
\psi_A(Z_B,T) &= \frac{3(1-c_Ac_B)}{(c_B\omega_A-k_A)^2}\int^{Z_B}\big|B_1(\zeta,T)\big|^2\,\mathrm{d}\zeta.
\end{aligned}
$$

Remark 3.3. Introducing higher-order corrections $\varepsilon^3\Omega_{A,3}$, $\varepsilon^3\Omega_{B,3}$ to the phase and higher-order corrections $\varepsilon^2\psi_{A,2}$, $\varepsilon^2\psi_{B,2}$ to the envelope shift would give similar equations as the ones for $\Omega_{A,2}$ or ψ_A. However, there are three terms in the equation of A_3 which cannot be classified in this way, namely $2(1-c_Ac_B)\partial_1^2A_1(Z_A,T)\partial_1\psi_A(Z_B,T)$, $-2\omega_A A_2(Z_A,T)\partial_2\Omega_{A,1}(Z_B,T)$, and $2\mathrm{i}c_B\partial_2A_1(Z_A,T)\partial_1\Omega_{A,1}(Z_B,T)$. Moreover, $\psi_{A,2}$ is proportional to $\Omega_{A,2}$, i.e. can become imaginary. Hence the introduction of these correction terms would be of no use.

4. Validity of the approximation

As a consequence of the perturbation analysis of the last section the first nonvanishing terms in the residual are formally of order $\mathcal{O}(\varepsilon^6)$. Below we will prove the following:

Lemma 4.1. *Let $s\geqslant 2$, $m\geqslant 2$, $s_A\geqslant s+10$, $k_A\neq k_B$, $k_A,k_B>0$, and let $A_1|_{T=0}$, $B_1|_{T=0}\in H^{s_A}(m)\cap H^{s_A+m}(0)$. Then for all $T_0>0$ there exist $\varepsilon_0>0$, $C>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$ we have*

$$\sup_{t\in[0,T_0/\varepsilon^2]}\big\|\mathrm{Res}(\varepsilon\Psi)\big\|_{H^s}\leqslant C\varepsilon^{11/2}.$$

The difference between the exponents of the formal error $\mathcal{O}(\varepsilon^6)$ and $\mathcal{O}(\varepsilon^{11/2})$ in the lemma follows from the scaling properties of the L^2-norm. The weighted spaces $H^s(m)$ are used to describe analytically the condition that the wave packets are spatially localized. This is needed to estimate the interaction terms like for instance s_{43} and s_{44}. The loss of regularity from s_A to s is explained below.

As a direct consequence of Lemma 4.1 and of the fact that our original system (1) does not contain quadratic terms, with a simple application of Gronwall's inequality [6] it follows that the original system really behaves as predicted by the approximation.

Theorem 4.2. *(Similar to [3, Theorem 3.6].) Under the assumptions of Theorem 4.2, for all $T_0>0$ there exist $\varepsilon_0>0$, $C>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$ we have*

$$\sup_{t\in[0,T_0/\varepsilon^2]}\big\|u(x,t)-\varepsilon\Psi(x,t)\big\|_{H^s}\leqslant C\varepsilon^{7/2}.$$

From Theorem 4.2 we obtain Sobolev's embedding theorem.

Proposition 4.3. *Under the assumptions of Theorem 4.2 we have*

$$\sup_{t\in[0,T_{0/\varepsilon^2}]}\big\|u(x,t)-\varepsilon\Psi(x,t)\big\|_{C_b^{s-1}}\leqslant\varepsilon^{7/2}. \tag{16}$$

As explained in Remark 2.2 this last estimate together with the subsequent Lemma 4.7 allows us to bound the magnitude of the envelope shift by $C\varepsilon$.

Hence it remains to give the proof of Lemma 4.1. The assertion obviously follows if we prove that the approximation equations (10)–(15) possess order $\mathcal{O}(1)$-bounded solutions on the $\mathcal{O}(1/\varepsilon^2)$-time scale. We have to solve three different

kinds of equations. The first set of equations, (10) and (12), describes internal dynamics. Since these two equations are independent of the small parameter $0 < \varepsilon \ll 1$ we have

Lemma 4.4. *Under the assumptions of Theorem* 4.2 *there exists a time* $T_0 > 0$ *such that* (10) *has a unique solution*

$$A_1 \in C\big([0, T_0], H^{s_A}(m) \cap H^{s_A+m}(0)\big).$$

Proof. We apply the variation of constant formula and use the fact that $i\partial_X^2$ is the generator of a strongly continuous semigroup in $H^{s_A}(m) \cap H^{s_A+m}(0)$, cf. [4]. □

Note that T_0 is independent of the weight. This can be proven like in [10, Lemma 6.4] such that the existence time is determined only by the local existence and uniqueness in H^s-spaces. For completeness we remark that the time $T_0 > 0$ can be made arbitrary large by using the global well-posedness [14] in the space L^2 which implies the global well-posedness in H^s-spaces for every $s \geqslant 0$.

Since (12) is a linearized NLS equation for A_2 with $\mathcal{O}(1)$-bounded inhomogeneous terms $s_{41} + s_{42}$ with exactly the same arguments we find

Lemma 4.5. *Under the assumptions of Theorem* 4.2 *the following holds. Let* $A_1 \in C([0, T_0], H^{s_A}(m) \cap H^{s_A+m}(0))$ *be a solution of* (10). *Then for all initial conditions* $A_2|_{T=0} \in H^{s_A}(m) \cap H^{s_A+m}(0)$ *there exists a unique solution of* (12) *with*

$$A_2 \in C\big([0, T_0], H^{s_A-3}(m) \cap H^{s_A-3+m}(0)\big).$$

The loss of regularity for A_2, B_2 comes from the inhomogeneous term s_{42} in (12). The second group of equations, namely (11), (13), and (14), describes the essential interaction dynamics. By pure integration we find

Lemma 4.6. *Under the assumptions of Theorem* 4.2 *the following holds. Let* $A_1, B_1 \in C([0, T_0],\ H^{s_A}(m) \cap H^{s_A+m}(0))$ *be a solution of* (10). *Then*

$$\partial_1 \Omega_{A,1}, \partial_1 \Omega_{B,1}, \partial_{Z_B} \psi_A, \partial_{Z_A} \psi_B, \operatorname{Im} \Omega_{A,2}, \operatorname{Im} \Omega_{B,2} \in C\big([0, T_0], H^{s_A}(m) \cap H^{s_A+m}(0)\big),$$

and $\Omega_{A,1}, \Omega_{B,1}, \operatorname{Re} \Omega_{A,2}, \operatorname{Re} \Omega_{B,2}, \psi_A, \psi_B \in C([0, T_0], C_b^{s_A+m})$.

In terms of local existence and uniqueness and $\mathcal{O}(1)$-boundedness of solutions the only nontrivial equation is (15) which is a linearized NLS equation for A_3 with $\mathcal{O}(1)$-bounded inhomogeneous terms and terms $\varepsilon^{-1}(s_{43} + s_{44})$. Since the last terms are only $\mathcal{O}(\varepsilon^{-1})$ on an $\mathcal{O}(\varepsilon)$-scale w.r.t. T we find

Lemma 4.7. *Assume the conditions of Theorem* 4.2. *Then there exists* $C > 0$ *such that for all* $\varepsilon \in (0, 1]$ *the following holds. System* (15) *with zero initial data has a unique solution* $A_3, B_3 \in C([0, T_0], H^{s_A-6}(m) \cap H^{s_A-6+m}(0))$. *It satisfies*

$$\sup_{0 \leqslant T \leqslant T_0} \big\| (A_3, B_3)(T) \big\|_{H^{s_A-6}(m) \cap H^{s_A-6+m}(0)} \leqslant C.$$

Proof. See [3, Lemma 4.2]. □

Due to the term $\partial_1 \partial_2 A_1$ in the equation for A_2 we have a loss of regularity from A_1 to A_2 of three derivatives. Similarly, the loss of regularity of another three derivatives for A_3, and B_3 comes from the term s_{52}. Finally, the terms with the highest derivative in the residual are $\partial_2^2 A_3$ and $\partial_2^2 B_3$ which gives another loss of four derivatives. Consequently, we have to choose $s_A - s \geqslant 10$.

5. Numerical simulations

Before we discuss the validity of the envelope shift formula (13) we provide some numerical experiments to confirm and illustrate the above analysis. The initial value of the numerical solution is defined on a large grid of equally spaced points x_m, $m \in 1, \ldots, N$, and is defined as

$$u_{\text{num}}(x_m, 0) = \varepsilon \Psi(x_m, 0), \tag{17}$$

where $\varepsilon \Psi(x, t)$ is defined in (3). The numerical solution is generated at equally spaced values of time t_n, $n \in \mathbb{N}$, by integrating (1). The numerical scheme used is accurate enough so that the true differences between the analytical approximate solutions and the actual (numerical in this case) solution can be detected. The scheme also conserves energy which is necessary to have precise estimates, according to the lower bound $|u - u_{\text{approx}}| \geqslant ||u| - |u_{\text{approx}}||$. Due to the multiscale character

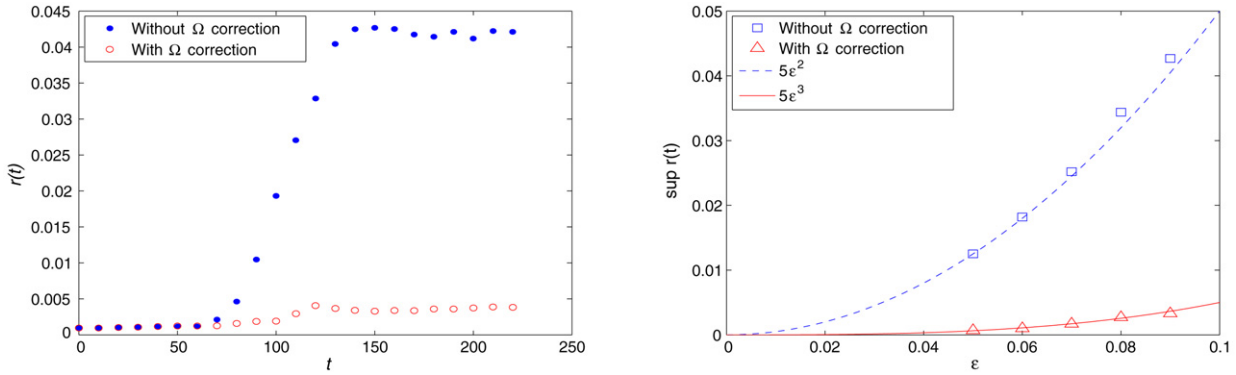

Fig. 2. Left: Plot of error function, $r(t)$, for $\varepsilon = 0.9$. Right: Plot of $\sup_{x\in\mathbb{R}} r(t)$ for various ε.

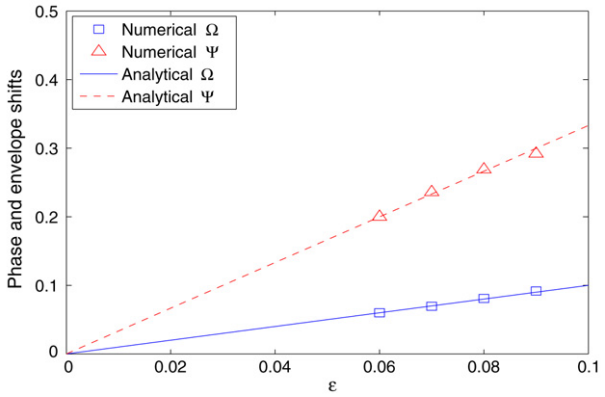

Fig. 3. Plot of numerically computed phase and envelope shifts (markers) and the analytical values (lines); both shifts are $\mathcal{O}(\varepsilon)$ and the analytical shifts are close to the computed ones.

of the problem such numerical computations are CPU and memory intensive and require extended periods of time to run. Therefore, the analytical approximation solution is clearly preferable to the numerical one, which is computed here only to draw comparisons. In the examples below k_A, k_B, x_A, and x_B are chosen such that pulse A will travel through pulse B.

Theorem 4.2 is confirmed numerically by computing the difference

$$r(t_n) = \sup_m \left| u_{\text{num}}(x_m, t_n) - \varepsilon \Psi(x_m, t_n) \right|$$

as a function of time. Moreover, in order to numerically compute the phase shifts and the envelope shifts, the two-pulse solution is compared with the sum of two corresponding single pulse solutions. The phase shift was computed by finding the average difference between adjacent roots of the shifted (two-pulse solutions) and nonshifted solution (two single pulse solutions). For $k_B = 0$ the envelope shift can be estimated by looking at the position of the maximal amplitude. This is due to the fact that the carrier wave with $k_B = 0$ will be identical to its modulating envelope, which makes it easier to detect the actual envelope shift. In the case that $k_B \neq 0$ the envelope was fit with an appropriate function including a parameter for the envelope shift.

The confirmation and quantification of analytical results is as follows.

- It was shown in [3] that if $\varepsilon\Psi$ is the sum of two well-prepared pulses with the corrections $\varepsilon\Omega_{A,1}, \varepsilon\Omega_{B,1}$ for the phase shifts taken into account, then

$$\sup_{t\in[0,\tau_0/\varepsilon^2]} r(t) = \mathcal{O}\left(\varepsilon^3\right).$$

We first numerically confirm this result and compare it to the standard ansatz without phase shift corrections. In the left panel of Fig. 2 a plot of $r(t)$ is shown. Before interaction the difference between the standard and the improved solution is negligible. For times after the interaction the approximate solution with the $\Omega_{A,B}$ corrections is $\mathcal{O}(\varepsilon^3)$ accurate and only $\mathcal{O}(\varepsilon^2)$ without it. This procedure was carried out for various values of ε in order to deduce the asymptotic behavior as $\varepsilon \to 0$. The results are plotted in the right panel of Fig. 2, with the improved approximation clearly superior. Fig. 3 shows a comparison of numerically computed shifts with those predicted by the formulas given in (11) and (13).

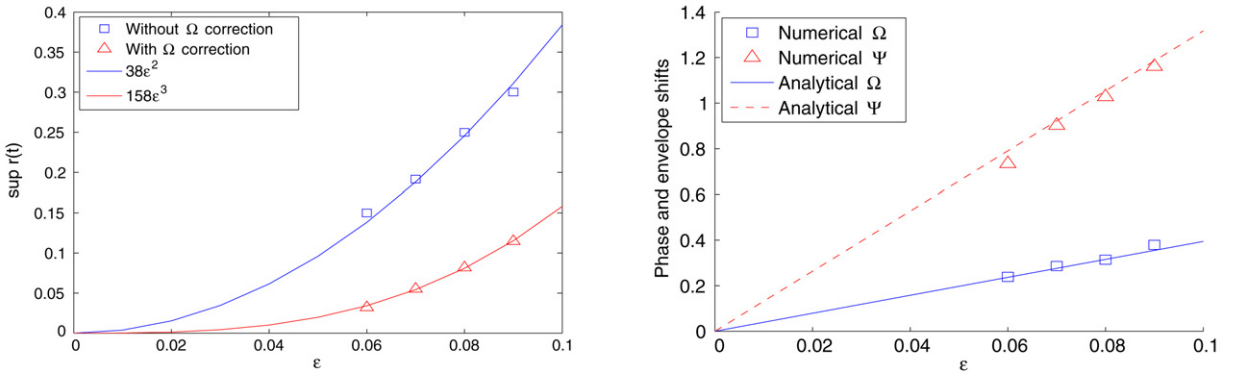

Fig. 4. Left: Plot of error for non-well-prepared pulses. When taking into account the phase shift again an $\mathcal{O}(\varepsilon^3)$ trend is shown. Right: Plot of numerically computed phase and envelope shifts (markers) and analytical values (lines) for non-well-prepared pulses; both shifts are $\mathcal{O}(\varepsilon)$. The formulae and the numerical approximations fit very well.

- We turn our attention to the case where the envelope is not described by a 1-soliton, but rather an arbitrary solution to the NLS equation. These solutions are called non-well-prepared pulses, see Fig. 1 for an example. Using the ansatz (3) we again achieve $\mathcal{O}(\varepsilon^3)$-order accuracy without the assumption that the envelope is well-prepared, see the left panel of Fig. 4. The phase and envelope shifts are computed in the same way as described above. Both exhibit an $\mathcal{O}(\varepsilon)$-trend and both formulae make correct predictions.

6. The validity of the envelope shift formula

As already alluded to in Remark 2.2, it is not clear if the correction term A_3 contributes to the description of the envelope shift or not. Therefore, in order to distinguish between the parts of A_3, which account for internal and interaction dynamics respectively, we introduce the following definition.

Definition 6.1. Let $A_3^{(c)}$, $B_3^{(c)}$ be the solution to the coupled system

$$2\mathrm{i}\omega_A\partial_2 A_3(Z_A,T) = \big(c_A^2-1\big)\partial_1^2 A_3(Z_A,T) - M_0[A_3,B_3] - I(A_1,A_2,B_1,B_2,\Omega_A,\psi_A),$$
$$2\mathrm{i}\omega_B\partial_2 B_3(Z_B,T) = \big(c_B^2-1\big)\partial_1^2 B_3(Z_B,T) - M_0[B_3,A_3] - I(B_1,B_2,A_1,A_2,\Omega_B,\psi_B),$$

and let $A_3^{(u)}$, $B_3^{(u)}$ be the solution to the uncoupled system

$$2\mathrm{i}\omega_A\partial_2 A_3(X_A,T) = \big(c_A^2-1\big)\partial_1^2 A_3(X_A,T) - M_0[A_3,0] - I(A_1,A_2,0,0,0,0),$$
$$2\mathrm{i}\omega_B\partial_2 B_3(X_B,T) = \big(c_B^2-1\big)\partial_1^2 B_3(X_B,T) - M_0[B_3,0] - I(B_1,B_2,0,0,0,0),$$

both with the same initial conditions. We call the envelope shift formula (13) *valid*, if

$$\big\|A_3^{(u)} - A_3^{(c)}\big\|_{C_b^s} \leqslant C\varepsilon^\alpha,$$

for $\alpha > 0$, and respectively for $B_3^{(u)}$, $B_3^{(c)}$.

So the envelope shift formula (13) is only valid, if the correction terms A_3, B_3 only describe internal dynamics (at least in leading order).

Above we found that the coupling terms which distinguish the coupled and uncoupled version of the evolution equations for A_3 are $\mathcal{O}(1)$ on an $\mathcal{O}(\varepsilon)$ time scale w.r.t. T. Thus, using the variation of constant formula we easily get

$$\big\|A_3^{(u)} - A_3^{(c)}\big\|_{C_b^s} = \mathcal{O}(\varepsilon)$$

on every $\mathcal{O}(1)$-time interval. Therefore, for general wave packets the envelope shift formula is valid for all $t \in [1/\varepsilon^{1+2/m+\delta}, T_0/\varepsilon^2]$ for $\delta > 0$ arbitrary small, but fixed, where the lower bound comes from Remark 2.3. As explained in the same remark this result can be improved for well-prepared pulses to $\mathcal{O}(\ln\varepsilon/\varepsilon) \leqslant t \leqslant T_0/\varepsilon^2$.

Conclusions. In leading order the two-wave propagation is given by a linear superposition of the individual waves as long as they are well separated. The nonlinear behavior appears during collision which causes a phase shift that translates into an

envelope shift. Hence, the solitary wave interaction is *elastic* in leading order—a scenario reminiscent of integrable equations. Since

$$\varepsilon g\big(\varepsilon(x+\varepsilon a)\big) - \varepsilon g(\varepsilon x) = \varepsilon g'(\varepsilon x)\varepsilon^2 a + \mathcal{O}\big(\varepsilon\big(\varepsilon^2 a\big)^2\big) = \mathcal{O}\big(\varepsilon^3\big)$$

the estimate (16) immediately shows that an envelope shift larger than $\mathcal{O}(\varepsilon)$ is not possible. This estimate is valid both for well- and non-well-prepared pulses. Moreover the derived envelope shift formula (14) turned out to be valid for well-prepared pulses and general wave packets.

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