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Note

# On reducibility of $n$-ary quasigroups

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**Abstract**

An $n$-ary operation $Q : \Sigma^n \to \Sigma$ is called an $n$-ary quasigroup of order $|\Sigma|$ if in the relation $x_0 = Q(x_1, \ldots, x_n)$ knowledge of any $n$ elements of $x_0, \ldots, x_n$ uniquely specifies the remaining one. $Q$ is permutably reducible if $Q(x_1, \ldots, x_n) = P(R(x_{\sigma(1)}, \ldots, x_{\sigma(k)}), x_{\sigma(k+1)}, \ldots, x_{\sigma(n)})$ where $P$ and $R$ are $(n-k+1)$-ary and $k$-ary quasigroups, $\sigma$ is a permutation, and $1 < k < n$. An $m$-ary quasigroup $S$ is called a retract of $Q$ if it can be obtained from $Q$ or one of its inverses by fixing $n - m > 0$ arguments. We prove that if the maximum arity of a permutably irreducible retract of an $n$-ary quasigroup $Q$ belongs to $\{3, \ldots, n-3\}$, then $Q$ is permutably reducible.


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## 1. Introduction

We continue the investigation of $n$-quasigroups of order 4 that was started in [7,5,8]. The general line of inquiry is the characterization of irreducible $n$-quasigroups (which cannot be represented as a repetition-free superposition of multary quasigroups of smaller orders). For these reasons, we derive a new test for reducibility. In particular, every irreducible $n$-quasigroup does not satisfy the hypothesis of the test; this gives a new necessary condition for an $n$-quasigroup to be irreducible. Although, historically, this work is a part of an investigation of $n$-quasigroups of order 4, the test, which is given in terms of decomposability of retracts, is suitable for any, even infinite, order.

In general, it is very natural to consider possible representations of an $n$-quasigroup as repetition-free superpositions. An extremely useful fact is that there exists a unique (in some sense) canonical decomposition [2] (it is remarkable that this is true for essentially more wide class of functions than the $n$-quasigroups, see [9]). Using the canonical decomposition of an $n$-quasigroup, it is possible to derive decompositions for some of its retracts. The approach of this paper is opposite: using decompositions of some retracts, we reconstruct a decomposition of the original $n$-quasigroup.

Let $\Sigma$ be a nonempty set and $\Sigma^n$ be the set of words of length $n$ over the alphabet $\Sigma$. We assume that $\Sigma$ contains 0; denote $\bar{0} \stackrel{\text{def}}{=} (0, \ldots, 0)$. Let $[n] \stackrel{\text{def}}{=} \{1, \ldots, n\}$.

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**Definition 1** (*n-quasigroup*). An $n$-ary operation $q : \Sigma^n \to \Sigma$ such that in the equality $q(x_1, \ldots, x_n) = x_{n+1}$ knowledge of any $n$ elements of $x_1, \ldots, x_n, x_{n+1}$ uniquely specifies the remaining one is called an *n-ary quasigroup* of order $|\Sigma|$ [1] or simply *n-quasigroup*; we will also use the term *multary quasigroup* when the arity is not specified or inessential.

We see that the definition is symmetric with respect to all variables $x_1, \ldots, x_n, x_{n+1}$, while the form $q(x_1, \ldots, x_n) = x_{n+1}$ is not; this is not handy sometimes. For this reason, we will also use the $(n+1)$-ary predicate $q\langle\cdot\rangle$ instead:

$$q\langle x_1, \ldots, x_n, x_{n+1}\rangle \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad q(x_1, \ldots, x_n) = x_{n+1}. \tag{1}$$

(In fact, the predicate $q\langle\cdot\rangle$ represents the graph of $q$.) We use upper-case letters to name multary quasigroups in predicative form, see the following definition for example. It is also sometimes convenient to talk about $(n-1)$-quasigroups where $n$ is the predicate arity.

By definition, an $n$-quasigroup $q$ in invertible in each place; we will use the notion $\dot{q}$ for the inversion in the first place:

$$\dot{q}(y, x_2 \ldots, x_n) = z \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad q(z, x_2, \ldots, x_n) = y.$$

**Remark 1.** (1) The subset of $\Sigma^{n+1}$ corresponding to an $n$-quasigroup predicate is called a *distance*-2 *MDS code* in the theory of error-correcting codes. Although such codes themselves cannot correct errors, they are useful in constructions of codes with larger distance. (2) The $n$-dimensional value array of an $n$-quasigroup is known as a *Latin hypercube*.

**Definition 2** (*Reducible, irreducible*). An $(n-1)$-quasigroup $M$ is called *reducible* (*irreducible*) iff it can (cannot) be represented as

$$M\langle x_1, \ldots, x_n\rangle \quad \Leftrightarrow \quad K\langle q(x_{\eta(1)}, \ldots, x_{\eta(j)}), x_{\eta(j+1)}, \ldots, x_{\eta(n)}\rangle,$$

where $K$ and $q$ are $(n-j)$- and $j$-quasigroups, $\eta : [n] \to [n]$ is a permutation, and $2 \leqslant j \leqslant n-2$. Note that all binary (as well as 1-ary and 0-ary) quasigroups are irreducible by definition because $2 > n-2$ in this case.

**Remark 2.** Defined as above, the reducibility property does not depend on the order of the arguments of a multary quasigroup. Often (e.g., [1]) by reducibility one means the more strict property, so-called $(i, j)$-reducibility, when $\eta = (i, i+1, \ldots, n, 1, 2, \ldots, i-1)$. We observe this difference to avoid a misunderstanding. In our definition, the reducibility corresponds to the $(i, j, \eta)$-reducibility in [3], where $\eta$ is a permutation.

**Definition 3** (*Isotopic*). Two $n$-quasigroups $Q, Q' : \Sigma^n \to \Sigma$ are called *isotopic* iff

$$Q\langle x_1, \ldots, x_{n+1}\rangle \quad \Leftrightarrow \quad Q'\langle \rho_1(x_1), \ldots, \rho_{n+1}(x_{n+1})\rangle,$$

where $\rho_1, \ldots, \rho_{n+1} : \Sigma \to \Sigma$ are 1-quasigroups (i.e., permutations).

**Definition 4** (*Retract*). If an $l$-ary predicate $K\langle\cdot\rangle$ is obtained by fixing $n-l > 0$ arguments in an $(n-1)$-quasigroup predicate $M\langle\cdot\rangle$, then $K$ is, obviously, a well-defined $(l-1)$-quasigroup; this $(l-1)$-quasigroup is called a *retract* of $M$.

Our goal is to prove the following theorem.

**Theorem 1.** *Let $M : \Sigma^{n-1} \to \Sigma$ be an $(n-1)$-quasigroup. Let $K : \Sigma^{k-1} \to \Sigma$ be a maximal (by arity) irreducible retract of $M$ (note that $3 \leqslant k \leqslant n-1$). Suppose $4 \leqslant k \leqslant n-3$. Then*

$$M\langle \bar{z}\rangle \quad \Leftrightarrow \quad K\langle q^1(\bar{z}^1), \ldots, q^k(\bar{z}^k)\rangle, \tag{2}$$

*where $\bar{z}^1, \ldots, \bar{z}^k$ are nonempty pairwise disjoint collections of variables from $\bar{z}$ and $q^1, \ldots, q^k$ are multary quasigroups.*

**Corollary 1.** *If the maximum arity of an irreducible retract of a given n-quasigroup belongs to* $\{3, \ldots, n-3\}$, *then the n-quasigroup is reducible*.

**Remark 3.** Theorem 1 is not much more stronger than its corollary: indeed, the decomposition (2) exists for every reducible multary quasigroup $M$ and every irreducible retract $K$ that is maximal in the sense that unfixing one or more variables always gives a reducible retract. Such the conclusion can be drawn if we consider a (tree) decomposition of $M$ into superposition of irreducible multary quasigroups; $K$ must be (up to isotopy and changing the order of arguments) an element of the decomposition. More results on the structure of decomposition tree of a reducible multary quasigroup can be found in [2].

**Remark 4.** (1) By numerical reasons [8], almost all $n$-quasigroups of order 4 are irreducible with $k = n-1$.

(2) If $|\Sigma| \equiv 0 \bmod 4$ and $n$ is odd, then there are irreducible $(n-1)$-quasigroups with $k = n-2$ [6]; e.g., the 4-quasigroup with the following value table:

| | | | |
|---|---|---|---|
| 0123 1032 2310 3201 | 1032 0123 3201 2310 | 2301 3210 1023 0132 | 3210 2301 0132 1023 |
| 1032 0123 3201 2310 | 0123 1032 2310 3201 | 3210 2301 0132 1023 | 2301 3210 1023 0132 |
| 2310 3201 0123 1032 | 3201 2310 1032 0123 | 0132 1023 3210 2301 | 1023 0132 2301 3210 |
| 3201 2310 1032 0123 | 2310 3201 0123 1032 | 1023 0132 2301 3210 | 0132 1023 3210 2301 |

(3) If $k=3$, or $k=n-2$ and $n$ is odd, or $k=n-2$ and $|\Sigma| \not\equiv 0 \bmod 4$, then the existence of irreducible $n$-quasigroups is an open question.

In Section 2 we consider several simple statements, which will be used later. Section 3 is the proof of Theorem 1, which consists of several steps, arranged as propositions. In Appendix A we consider the proof of Theorem 1 by the example of a 6-quasigroup. In Appendix B, for convenience, we cite the list of notations.

The results of this paper were announced in [4].

## 2. Auxiliary statements

The following two propositions are straightforward.

**Lemma 1.** *Let K be an l-quasigroup and Q be an* $(n-l)$*-quasigroup. Then*

$$K\langle \bar{x}, Q(\bar{y})\rangle \;\Leftrightarrow\; K(\bar{x}) = Q(\bar{y}) \;\Leftrightarrow\; Q\langle \bar{y}, K(\bar{x})\rangle, \quad \bar{x} \in \Sigma^l, \quad \bar{y} \in \Sigma^{n-l}.$$

**Lemma 2.** *Let* $M' : \Sigma^m \to \Sigma$ *be an m-quasigroup, q be a function from* $\Sigma^k$ *to* $\Sigma$, *and the predicate* $M\langle\cdot\rangle$ *is defined by*

$$M\langle \bar{x}, \bar{y}\rangle \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad M'\langle q(\bar{x}), \bar{y}\rangle, \quad \bar{x} \in \Sigma^k, \quad \bar{y} \in \Sigma^m.$$

*Then M is a well-defined* $(k+m-1)$*-quasigroup if and only if q is a k-quasigroup*.

The next claim means that a reducible $n$-quasigroup can be represented as a superposition of retracts. As a corollary, these retracts uniquely define the multary quasigroup (Lemma 4).

**Lemma 3.** *Let c be a k-quasigroup*, *b be an l-quasigroup. Let*

$$f(\alpha, \bar{\beta}, \bar{\gamma}) \stackrel{\text{def}}{=} c(b(\alpha, \bar{\beta}), \bar{\gamma}), \tag{3}$$

$$c_0(\alpha, \bar{\gamma}) \stackrel{\text{def}}{=} f(\alpha, \bar{0}, \bar{\gamma}), \quad b_0(\alpha, \bar{\beta}) \stackrel{\text{def}}{=} f(\alpha, \bar{\beta}, \bar{0}), \quad a(\alpha) \stackrel{\text{def}}{=} f(\alpha, \bar{0}, \bar{0}), \tag{4}$$

*where* $\alpha \in \Sigma$, $\bar{\beta} \in \Sigma^{l-1}$, $\bar{\gamma} \in \Sigma^{k-1}$. *Then*

$$f(\alpha, \bar{\beta}, \bar{\gamma}) \equiv c_0(a^{-1}(b_0(\alpha, \bar{\beta})), \bar{\gamma}). \tag{5}$$

**Proof.** Substituting (3) into (4) we get $c_0(\cdot,\bar\gamma)\equiv c(b(\cdot,\bar 0),\bar\gamma)$, $b_0(\alpha,\bar\beta)\equiv c(b(\alpha,\bar\beta),\bar 0)$, and $a(\cdot)\equiv c(b(\cdot,\bar 0),\bar 0)$, i.e., $a^{-1}(\cdot)\equiv \dot b(\dot c(\cdot,\bar 0),\bar 0)$. Using these representations, we can verify the validity of (5):

$$c_0(a^{-1}(b_0(\alpha,\bar\beta)),\bar\gamma)\equiv c(b(\dot b(\dot c(c(b(\alpha,\bar\beta),\bar 0),\bar 0),\bar 0),\bar 0),\gamma)\equiv c(b(\alpha,\bar\beta),\bar\gamma)\equiv f(\alpha,\bar\beta,\bar\gamma). \qquad \square$$

**Lemma 4.** *Let $C$, and $\widetilde C$ be $k$-quasigroups, $b$ and $\widetilde b$ be $l$-quasigroups. Suppose*

$$C\langle b(\alpha,\bar 0),\bar\gamma,\delta\rangle \;\Leftrightarrow\; \widetilde C\langle \widetilde b(\alpha,\bar 0),\bar\gamma,\delta\rangle \quad \text{and} \quad C\langle b(\alpha,\bar\beta),\bar 0,\delta\rangle \;\Leftrightarrow\; \widetilde C\langle \widetilde b(\alpha,\bar\beta),\bar 0,\delta\rangle,$$

*where $\alpha,\delta\in\Sigma$, $\bar\beta\in\Sigma^{l-1}$, $\bar\gamma\in\Sigma^{k-1}$. Then $C\langle b(\alpha,\bar\beta),\bar\gamma,\delta\rangle \Leftrightarrow \widetilde C\langle \widetilde b(\alpha,\bar\beta),\bar\gamma,\delta\rangle$.*

## 3. Theorem proof

Given $\bar x=(x_1,x_2,\ldots,x_n)$, we use the following notation: $\bar x^{(k)}\stackrel{\text{def}}{=}(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n)$, $\bar x^{(k)}\#y\stackrel{\text{def}}{=}(x_1,\ldots,x_{k-1},y,x_{k+1},\ldots,x_n)$, and $\bar x^{(l,k)}=\bar x^{(k,l)}\stackrel{\text{def}}{=}\bar x^{(l)(k)}$ provided $k<l$.

Let $M:\Sigma^{n-1}\to\Sigma$ be an $(n-1)$-quasigroup; let $K:\Sigma^{k-1}\to\Sigma$ be an irreducible retract of $M$; and let $k$ be the maximum number for which such retract exists; for the rest of this section we suppose that $4\leqslant k\leqslant n-3$. Without loss of generality we assume that $K\langle x_1,\ldots,x_k\rangle\Leftrightarrow M\langle x_1,\ldots,x_k,0,\ldots,0\rangle$. Put $m\stackrel{\text{def}}{=}n-k$, $\bar x\stackrel{\text{def}}{=}(x_1,\ldots,x_k)$, $\bar y\stackrel{\text{def}}{=}(y_1,\ldots,y_m)$.

In the first four propositions we consider the structure of $k$-ary and $(k-1)$-ary retracts of $M$ with unfixed arguments $x_1,\ldots,x_k$.

**Proposition 1.** *Let $L_{i;\bar y^{(i)}}\langle\bar x,z\rangle\stackrel{\text{def}}{\Longleftrightarrow}M\langle\bar x,\bar y^{(i)}\#z\rangle$ be a retract of $M$. Assume that $K_{\bar y}\langle\bar x\rangle\stackrel{\text{def}}{\Longleftrightarrow}M\langle\bar x,\bar y\rangle\Leftrightarrow L_{i;\bar y^{(i)}}\langle\bar x,y_i\rangle$ is an irreducible retract of $L_{i;\bar y^{(i)}}$ (here we only suppose but do not yet claim that such a retract exists). Then $L_{i;\bar y^{(i)}}$ can be represented as*

$$L_{i;\bar y^{(i)}}\langle x_1,\ldots,x_k,z\rangle \;\Leftrightarrow\; R_{i;\bar y^{(i)}}\langle x_1,\ldots,x_{j-1},q_{i;\bar y^{(i)}}(x_j,z),x_{j+1},\ldots,x_k\rangle, \tag{6}$$

*where $j$ depends (essentially or not) on $i$ and $\bar y^{(i)}$, i.e., $j=j(i,\bar y^{(i)})$, $R_{i;\bar y^{(i)}}$ and $q_{i;\bar y^{(i)}}$ are multary quasigroups.*

**Proof.** The $k$-quasigroup $L_{i;\bar y^{(i)}}$ is reducible because $k<n-1$. But its retract $K_{\bar y}$ obtained by fixing the last variable $z:=y_i$ in $L_{i;\bar y^{(i)}}\langle\cdot\rangle$ is irreducible. So, in any decomposition of $L_{i;\bar y^{(i)}}\langle\bar x,z\rangle$ the variable $z$ must be grouped with exactly one other variable; i.e., $L_{i;\bar y^{(i)}}$ admits one of the two decompositions

$$L_{i;\bar y^{(i)}}\langle x_1,\ldots,x_k,z\rangle \;\Leftrightarrow\; R\langle x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_k,q(x_j,z)\rangle, \tag{7}$$
$$L_{i;\bar y^{(i)}}\langle x_1,\ldots,x_k,z\rangle \;\Leftrightarrow\; Q\langle x_j,z,r(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_k)\rangle, \tag{8}$$

for some 2-quasigroup $q$ ($Q$) and $k$-quasigroup $R$ ($r$). By Lemma 1, (8) implies (7) with $R=r$, $q=Q$. Permuting the arguments in (7), we get the representation (6). $\square$

**Proposition 2.** *All the retracts $K_{\bar y}\langle\bar x\rangle\stackrel{\text{def}}{\Longleftrightarrow}M\langle\bar x,\bar y\rangle$, $\bar y\in\Sigma^m$ are pairwise isotopic and thus irreducible; i.e.,*

$$K_{\bar y}\langle\bar x\rangle \;\Leftrightarrow\; K\langle\rho^1_{\bar y}(x_1),\ldots,\rho^k_{\bar y}(x_k)\rangle, \tag{9}$$

*where $\rho^1_{\bar y},\ldots,\rho^k_{\bar y}$ are permutations $\Sigma\to\Sigma$.*

**Proof.** We prove the proposition by induction on the number of nonzero elements in $\bar y$. The base of induction is $K_{\bar 0}\langle\cdot\rangle\Leftrightarrow K\langle\cdot\rangle$. For the induction step it is sufficient to prove that

$$K_{\bar y''}\langle\bar x\rangle \;\Leftrightarrow\; K_{\bar y'}\langle\bar x^{(j)}\#\rho(x_j)\rangle, \tag{10}$$

where $\bar y'=(y_1,\ldots,y_{i-1},0,0,\ldots,0)$, $\bar y''=(y_1,\ldots,y_{i-1},y_i,0,\ldots,0)$, $j=j(i,\bar y')\in[m]$, $\rho=\rho_{i,\bar y''}$ is a permutation. Then, (10) means that $K_{\bar y'}$ and $K_{\bar y''}$ are isotopic, and from (9) with $\bar y=\bar y'$ we have (9) with $\bar y=\bar y''$, where $\rho^j_{\bar y''}=\rho^j_{\bar y'}\rho$ and $\rho^l_{\bar y''}=\rho^l_{\bar y'}$ for all $l\neq j$.

Let us show (10). Note that $\bar{y}''^{(i)} = \bar{y}'^{(i)} = (y_1, \dots, y_{i-1}, 0, \dots, 0)$. By Proposition 1

$$K_{\bar{y}'}\langle \bar{x}\rangle \;\Leftrightarrow\; M\langle \bar{x}, \bar{y}'\rangle \;\Leftrightarrow\; R_{i;\bar{y}'^{(i)}}\langle x_1, \dots, x_{j-1}, q_{i;\bar{y}'^{(i)}}(x_j, 0), x_{j+1}, \dots, x_k\rangle,$$
$$K_{\bar{y}''}\langle \bar{x}\rangle \;\Leftrightarrow\; M\langle \bar{x}, \bar{y}''\rangle \;\Leftrightarrow\; R_{i;\bar{y}'^{(i)}}\langle x_1, \dots, x_{j-1}, q_{i;\bar{y}'^{(i)}}(x_j, y_i), x_{j+1}, \dots, x_k\rangle,$$

where $j = j(i, \bar{y}'^{(i)})$. We see that (10) holds with $\rho(\cdot) = \dot{q}_{i;\bar{y}'^{(i)}}(q_{i;\bar{y}'^{(i)}}(\cdot, y_i), 0)$. $\square$

Our goal is to show that each of the permutations $\rho^1_{\bar{y}}, \dots, \rho^k_{\bar{y}}$ in (9) essentially depends on its own group of parameters from $\bar{y}$ and these groups are pairwise disjoint. At the first step (which will be used for an induction step later), in Propositions 3 and 4, we will prove that for each $i \in [m]$ there exists a representation like (9) where only one of $\rho^1_{\bar{y}}, \dots, \rho^k_{\bar{y}}$ essentially depends on $y_i$. In the final Proposition 6 we will show (by induction) the existence of such a representation that is common for all $y_i$, $i \in [m]$.

**Proposition 3.** *Each k-quasigroup $L_{i;\bar{y}^{(i)}}\langle \bar{x}, z\rangle \overset{\text{def}}{\Longleftrightarrow} M\langle \bar{x}, \bar{y}^{(i)}\#z\rangle$ can be represented in the form*

$$L_{i;\bar{y}^{(i)}}\langle x_1, \dots, x_k, z\rangle \;\Leftrightarrow\; K\langle p^1_{i;\bar{y}^{(i)}}(x_1), \dots, p^{j-1}_{i;\bar{y}^{(i)}}(x_{j-1}), p_{i;\bar{y}^{(i)}}(x_j, z), p^{j+1}_{i;\bar{y}^{(i)}}(x_{j+1}), \dots, p^k_{i;\bar{y}^{(i)}}(x_k)\rangle, \quad (11)$$

*where $j = j(i, \bar{y}^{(i)})$, $p_{i;\bar{y}^{(i)}}$ is a 2-quasigroup, and $p^t_{i;\bar{y}^{(i)}}$ is a 1-quasigroup (i.e., permutation) for $t \neq j$.*

**Proof.** Fixing $z := 0$ in (6) and applying Proposition 2, we find that for each $i$ and $\bar{y}^{(i)}$ the $(k-1)$-quasigroup $R_{i;\bar{y}^{(i)}}$ in (6) is isotopic to $K$. $\square$

**Proposition 4.** *In Proposition 3 the index j does not depend on $\bar{y}^{(i)}$, i.e., $j = j(i)$.*

**Proof.** Assume the contrary, i.e., there exist $i$, $\bar{y}'^{(i)}$ and $\bar{y}''^{(i)}$ such that $j' \overset{\text{def}}{=} j(i, \bar{y}'^{(i)}) \neq j'' \overset{\text{def}}{=} j(i, \bar{y}''^{(i)})$. Without loss of generality we can assume that $j' = 1$ and $j'' = 2$. So,

$$L_{i;\bar{y}'^{(i)}}\langle x_1, x_2, x_3, \dots, x_k, z\rangle \;\Leftrightarrow\; K\langle p(x_1, z), p^2(x_2), p^3(x_3), \dots, p^k(x_k)\rangle, \quad (12)$$
$$L_{i;\bar{y}''^{(i)}}\langle x_1, x_2, x_3, \dots, x_k, z\rangle \;\Leftrightarrow\; K\langle r^1(x_1), r(x_2, z), r^3(x_3), \dots, r^k(x_k)\rangle. \quad (13)$$

The $k$-quasigroup $K'\langle z, x_2, x_3, \dots, x_k\rangle \overset{\text{def}}{\Longleftrightarrow} L_{i;\bar{y}'^{(i)}}\langle 0, x_2, x_3, \dots, x_k, z\rangle \;\Leftrightarrow\; M\langle \bar{x}^{(1)}\#0, \bar{y}'^{(i)}\#z\rangle$ is isotopic to $K$ (see (12)) and irreducible. By Proposition 2 (taking $x_1 := z$) $K'$ is isotopic to $K''\langle z, x_2, x_3, \dots, x_k\rangle \overset{\text{def}}{\Longleftrightarrow} L_{i;\bar{y}''^{(i)}}\langle 0, x_2, x_3, \dots, x_k, z\rangle \Leftrightarrow M\langle \bar{x}^{(1)}\#0, \bar{y}''^{(i)}\#z\rangle$. But $K''$ is reducible because (13) gives its decomposition when $x_1 = 0$ (here we use the condition $k \geqslant 4$). We get a contradiction. $\square$

Now we see that the function $j(i)$ divides all $y$-variables into $k$ groups, where each group corresponds to an $x$-variable. The next proposition is very important; it consider the structure of a $(k+1)$-ary retract of $M$ with two $y$-variables that belong to different groups. This is the only place where we use the condition $k \neq n-2$; if $k = n-2$, then the proposition does not work, and $M$ can be irreducible, as noted in Remark 4(2).

**Proposition 5.** *Let $j(i') = 1$, $j(i'') = 2$, $v \overset{\text{def}}{=} y_{i'}$, $w \overset{\text{def}}{=} y_{i''}$. Suppose that values of the variables $\bar{y}^{(i',i'')} \in \Sigma^{m-2}$ are fixed, and denote by $N(\bar{x}, v, w)$ the corresponding retract of M. Then*

$$N\langle \bar{x}, v, w\rangle \;\Leftrightarrow\; K\langle o^1(x_1, v), o^2(x_2, w), o^3(x_3), \dots, o^k(x_k)\rangle, \quad (14)$$

*where $o^t$, $t = 1, \dots, k$ are 2- and 1-quasigroups, which depend on the choice of $i', i'', \bar{y}^{(i',i'')}$.*

**Proof.** Recall that for retracts with variables $v, x_1, x_2, \dots, x_k$ or $w, x_1, x_2, \dots, x_k$ we have the decompositions

$$K\langle p(x_1, v), p^2(x_2), \dots, p^k(x_k)\rangle, \quad (15)$$
$$K\langle q^1(x_1), q(x_2, w), q^3(x_3), \dots, q^k(x_k)\rangle, \quad (16)$$

respectively. Consider possible decompositions of $N$. Taking into account that fixing $v$ and $w$ results in an irreducible retract, isotopic to $K$, we can conclude that $N\langle\bar{x}, v, w\rangle$ admits one of the following decompositions:

$$N\langle\bar{x}, v, w\rangle \quad\Leftrightarrow\quad C\langle\bar{x}, b(v, w)\rangle, \tag{17}$$

$$N\langle\bar{x}, v, w\rangle \quad\Leftrightarrow\quad C\langle\bar{x}^{(i)}\#b(x_i, v), w\rangle, \quad i \neq 1, \tag{18}$$

$$N\langle\bar{x}, v, w\rangle \quad\Leftrightarrow\quad C\langle\bar{x}^{(i)}\#b(x_i, w), v\rangle, \quad i \neq 2, \tag{19}$$

$$N\langle\bar{x}, v, w\rangle \quad\Leftrightarrow\quad C\langle\bar{x}^{(i)}\#b(x_i, v, w)\rangle, \tag{20}$$

$$N\langle\bar{x}, v, w\rangle \quad\Leftrightarrow\quad C\langle b(x_1, v), x_2, x_3, \ldots, x_k, w\rangle, \tag{21}$$

$$N\langle\bar{x}, v, w\rangle \quad\Leftrightarrow\quad C\langle x_1, b(x_2, w), x_3, \ldots, x_k, v\rangle. \tag{22}$$

In case (17) $C$ must be reducible, and a decomposition of $C$ provides another decomposition of $N$ (in fact, only (20) is suitable). So, $N$ admits one of (18)–(22). Consider (18). Fixing $x_1$ and $w$ we get a reducible $(k-1)$-ary retract with variables $x_2, \ldots, x_k, v$. But this retract is isotopic to $K$, see (15), which contradicts to the irreducibility of $K$. So, (18) is impossible. Similarly, (19) and (20) lead to contradictions.

Consider (21) (the case (22) is similar). Again, $C$ must be reducible, and a decomposition of $C$ provides another decomposition of $N$. Since (17)–(20) are inadmissible for $N$, the only possibility for $C$ is

$$C\langle u, x_2, x_3, \ldots, x_k, w\rangle \quad\Leftrightarrow\quad C'\langle u, b'(x_2, w), x_3, \ldots, x_k\rangle.$$

In this case

$$N\langle\bar{x}, v, w\rangle \quad\Leftrightarrow\quad C'\langle b(x_1, v), b'(x_2, w), x_3, \ldots, x_k\rangle.$$

Since $C'$ must be isotopic to $K$, the proposition is proved. □

Now we are ready to prove the main theorem. All we need to do is to transform the representation (9) to such a form that for each $i$ only one of $\rho^1_{\bar{y}}, \ldots, \rho^k_{\bar{y}}$ (more exactly, only $\rho^{j(i)}_{\bar{y}}$) essentially depends on $y_i$. For induction needs, we formulate a proposition covering all intermediate cases between Proposition 1 and Theorem 1. So, Theorem 1 is a partial case of the following proposition, which will be proved by induction.

Let the function $j : [m] \to [k]$ be defined as in Proposition 4. Let $\mathbf{i}^t = \{i^t_1, \ldots, i^t_{m_t}\}$ (where $t \in [k]$) be the set of all indexes $i$ such that $j(i)=t$. Obviously, $\bigcup_{t=1}^k \mathbf{i}^t = [m]$ and $\sum_{t=1}^k m_t = m$. For an arbitrary multiindex $\mathbf{i} = \{i_1, \ldots, i_{m'}\} \subseteq [m]$ where $i_1 < i_2 < \cdots < i_{m'}$ we denote $\bar{y}_{\mathbf{i}} \stackrel{\text{def}}{=} (y_{i_1}, \ldots, y_{i_{m'}})$.

**Proposition 6.** *Let $\mathbf{h}^t \subseteq \mathbf{i}^t$, $t = 1, \ldots, k$. Denote $\mathbf{h} = \bigcup_{t=1}^k \mathbf{h}^t$ and $\bar{\mathbf{h}} = [m] \backslash \mathbf{h}$. Then for each $\bar{y}_{\bar{\mathbf{h}}}$ there exist $(1 + |\mathbf{h}^t|)$-quasigroups $q^t_{\bar{y}_{\bar{\mathbf{h}}}}$, $t = 1, \ldots, k$ such that*

$$M\langle\bar{x}, \bar{y}\rangle \quad\Leftrightarrow\quad K\langle q^1_{\bar{y}_{\bar{\mathbf{h}}}}(x_1, \bar{y}_{\mathbf{h}^1}), \ldots, q^k_{\bar{y}_{\bar{\mathbf{h}}}}(x_k, \bar{y}_{\mathbf{h}^k})\rangle. \tag{23}$$

**Proof.** Propositions 3 and 4 imply that the claim holds for $|\mathbf{h}| = 1$. Let this be the induction base.

Assume the claim holds for $|\mathbf{h}| = b$. Let us show that it holds for $\mathbf{h} = \mathbf{g} \subseteq [m]$ where $|\mathbf{g}| = b + 1$. We fix arbitrary different $i', i'' \in \mathbf{g}$ and denote $\mathbf{d} \stackrel{\text{def}}{=} \mathbf{g}\backslash\{i', i''\}$, $\mathbf{d}^t = \mathbf{d} \cap \mathbf{i}^t$. Denote $v \stackrel{\text{def}}{=} y_{i'}$ and $w \stackrel{\text{def}}{=} y_{i''}$. We consider two cases: $j(i') = j(i'')$ and $j(i') \neq j(i'')$.

*Case* 1: Assume $j(i') = j(i'') = 1$, without loss of generality.

By the inductive hypothesis for $\mathbf{h} = \mathbf{d} \cup \{i'\}$, $\bar{\mathbf{h}} = \bar{\mathbf{g}} \cup \{i''\}$, we have

$$M\langle\bar{x}, \bar{y}\rangle \quad\Leftrightarrow\quad K\langle p_w(x_1, \bar{y}_{\mathbf{d}^1}, v), p^2_w(x_2, \bar{y}_{\mathbf{d}^2}), \ldots, p^k_w(x_k, \bar{y}_{\mathbf{d}^k})\rangle, \tag{24}$$

where multary quasigroups $p_w$, $p^t_w$, $t = 2, \ldots, k$ depend also on $\bar{y}_{\bar{\mathbf{g}}}$, i.e., $p^t_w = p^t_{\bar{y}_{\bar{\mathbf{g}}}, w}$.

By the inductive hypothesis for $\mathbf{h} = \mathbf{d} \cup \{i''\}$, $\bar{\mathbf{h}} = \bar{\mathbf{g}} \cup \{i'\}$, we have

$$M\langle \bar{x}, \bar{y}\rangle \quad\Leftrightarrow\quad K\langle r_v(x_1, \bar{y}_{\mathbf{d}^1}, w), r_v^2(x_2, \bar{y}_{\mathbf{d}^2}), \ldots, r_v^k(x_k, \bar{y}_{\mathbf{d}^k})\rangle,$$

where multary quasigroups $r_v$, $r_v^t$, $t = 2, \ldots, k$ depend also on $\bar{y}_{\bar{\mathbf{g}}}$, i.e., $r_v^t = r^t_{\bar{y}_{\bar{\mathbf{g}}}, v}$.

Equating these two representations of $M$ and setting $v := 0$, $\bar{y}_{\mathbf{d}^1} := \bar{0}$, we obtain

$$K\langle p_w(x_1, \bar{0}, 0), p_w^2(x_2, \bar{y}_{\mathbf{d}^2}), \ldots, p_w^k(x_k, \bar{y}_{\mathbf{d}^k})\rangle \quad\Leftrightarrow\quad K\langle r_0(x_1, \bar{0}, w), r_0^2(x_2, \bar{y}_{\mathbf{d}^2}), \ldots, r_0^k(x_k, \bar{y}_{\mathbf{d}^k})\rangle.$$

Changing the variables as $u = p_w(x_1, \bar{0}, 0) \iff x_1 = \dot{p}_w(u, \bar{0}, 0)$, we get

$$K\langle u, p_w^2(x_2, \bar{y}_{\mathbf{d}^2}), \ldots, p_w^k(x_k, \bar{y}_{\mathbf{d}^k})\rangle \quad\Leftrightarrow\quad K\langle r_0(\dot{p}_w(u, \bar{0}, 0), \bar{0}, w), r_0^2(x_2, \bar{y}_{\mathbf{d}^2}), \ldots, r_0^k(x_k, \bar{y}_{\mathbf{d}^k})\rangle.$$

Substituting $p_w(x_1, \bar{y}_{\mathbf{d}^1}, v)$ for $u$, we have

$$\begin{aligned}&K\langle p_w(x_1, \bar{y}_{\mathbf{d}^1}, v), p_w^2(x_2, \bar{y}_{\mathbf{d}^2}), \ldots, p_w^k(x_k, \bar{y}_{\mathbf{d}^k})\rangle\\ &\Leftrightarrow\quad K\langle r_0(\dot{p}_w(p_w(x_1, \bar{y}_{\mathbf{d}^1}, v), \bar{0}, 0), \bar{0}, w), r_0^2(x_2, \bar{y}_{\mathbf{d}^2}), \ldots, r_0^k(x_k, \bar{y}_{\mathbf{d}^k})\rangle.\end{aligned}$$

Since, by (24), the left part is equivalent to $M\langle \bar{x}, \bar{y}\rangle$, we have (23) with $\mathbf{h} = \mathbf{g}$, $\mathbf{h}^1 = \mathbf{d}^1 \cup \{i', i''\}$, $\mathbf{h}^t = \mathbf{d}^t$ for $t \neq 1$, $q^1_{\bar{y}_{\bar{\mathbf{h}}}}(x_1, \bar{y}_{\mathbf{h}^1}) = r_0(\dot{p}_w(p_w(x_1, \bar{y}_{\mathbf{d}^1}, v), \bar{0}, 0), \bar{0}, w)$, and $q^t_{\bar{y}_{\bar{\mathbf{h}}}} = r^t_{\bar{y}_{\bar{\mathbf{d}}}, 0}$ for $t \neq 1$. By Lemma 2, the function $q^1_{\bar{y}_{\bar{\mathbf{h}}}}$ is a multary quasigroup.

*Case* 2: Assume $j(i') = 1$, $j(i'') = 2$, without loss of generality.

By the inductive hypothesis, for every $\bar{y}_{\bar{\mathbf{g}}}$ we have

$$\begin{aligned}M\langle \bar{x}, \bar{y}\rangle \quad&\Leftrightarrow\quad K\langle p_w(x_1, \bar{y}_{\mathbf{d}^1}, v), p_w^2(x_2, \bar{y}_{\mathbf{d}^2}), p_w^3(x_3, \bar{y}_{\mathbf{d}^3}), \ldots, p_w^k(x_k, \bar{y}_{\mathbf{d}^k})\rangle,\\ M\langle \bar{x}, \bar{y}\rangle \quad&\Leftrightarrow\quad K\langle r_v^1(x_1, \bar{y}_{\mathbf{d}^1}), r_v(x_2, \bar{y}_{\mathbf{d}^2}, w), r_v^3(x_3, \bar{y}_{\mathbf{d}^3}), \ldots, r_v^k(x_k, \bar{y}_{\mathbf{d}^k})\rangle.\end{aligned} \tag{25}$$

Repeating steps of Case 1, we derive

$$M\langle \bar{x}, \bar{y}\rangle \quad\Leftrightarrow\quad K\langle s_w(x_1, \bar{y}_{\mathbf{d}^1}, v), r_0(x_2, \bar{y}_{\mathbf{d}^2}, w), r_0^3(x_3, \bar{y}_{\mathbf{d}^3}), \ldots, r_0^k(x_k, \bar{y}_{\mathbf{d}^k})\rangle, \tag{26}$$

where $s_w(x_1, \bar{y}_{\mathbf{d}^1}, v) \stackrel{\text{def}}{=} r_0^1(\dot{p}_w(p_w(x_1, \bar{y}_{\mathbf{d}^1}, v), \bar{y}_{\mathbf{d}^1}, 0), \bar{y}_{\mathbf{d}^1})$. It remains to eliminate the $w$-dependence of the formula in the first position of $K\langle \ldots\rangle$. Put

$$\widetilde{M}\langle \bar{x}, \bar{y}\rangle \quad\stackrel{\text{def}}{\iff}\quad K\langle s_0(x_1, \bar{y}_{\mathbf{d}^1}, v), r_0(x_2, \bar{y}_{\mathbf{d}^2}, w), r_0^3(x_3, \bar{y}_{\mathbf{d}^3}), \ldots, r_0^k(x_k, \bar{y}_{\mathbf{d}^k})\rangle. \tag{27}$$

Setting $w := 0$ in (27) and (26), we find that $\widetilde{M}\langle \bar{x}, \bar{y}^{(i'')}\#0\rangle \Leftrightarrow M\langle \bar{x}, \bar{y}^{(i'')}\#0\rangle$. On the other hand, $s_w(x_1, \bar{y}_{\mathbf{d}^1}, 0) \equiv r_0^1(x_1, \bar{y}_{\mathbf{d}^1})$ by definition of $s_w$; therefore, setting $v := 0$ in (27) and (25), we get $\widetilde{M}\langle \bar{x}, \bar{y}^{(i')}\#0\rangle \Leftrightarrow M\langle \bar{x}, \bar{y}^{(i')}\#0\rangle$. Considering $M$ and $\widetilde{M}$ as 3-quasigroups with the arguments $x_1, x_3, v, w$ and parameters $\bar{x}^{(1,3)}$, $\bar{y}^{(i',i'')}$, and taking into account the decompositions (14) and (27), we see by Lemma 4 (with $\alpha = x_1$, $\bar{\beta} = v$, $\delta = x_3$, $\bar{\gamma} = w$) that $M\langle \bar{x}, \bar{y}\rangle \Leftrightarrow \widetilde{M}\langle \bar{x}, \bar{y}\rangle$. □


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## Appendix A. An example

In this appendix we consider the proof of Theorem 1 (Proposition 6) by the example of a 6-quasigroup $M$. Assume that all 5-ary and 4-ary retracts of $M$ are reducible; and assume that the 3-ary retract $K\langle \bar{x}\rangle \stackrel{\text{def}}{\iff} M\langle \bar{x}, 0, 0, 0\rangle$ is irreducible.

Suppose that some 4-ary retracts of $M$ admit the following decompositions:

$$M\langle \bar{x}, y_1, 0, 0\rangle \;\Leftrightarrow\; R_1\langle q_1(x_1, y_1), x_2, x_3, x_4\rangle,$$

$$M\langle \bar{x}, 0, y_2, 0\rangle \;\Leftrightarrow\; R_2\langle q_2(x_1, y_2), x_2, x_3, x_4\rangle,$$

$$M\langle \bar{x}, 0, 0, y_3\rangle \;\Leftrightarrow\; R_3\langle x_1, q_3(x_2, y_3), x_3, x_4\rangle.$$

By Proposition 2

$$\forall y_1, y_2, y_3 : M\langle \bar{x}, \bar{y}\rangle \;\Leftrightarrow\; K\langle \rho^1_{y_1,y_2,y_3}(x_1), \rho^2_{y_1,y_2,y_3}(x_2), \rho^3_{y_1,y_2,y_3}(x_3), \rho^4_{y_1,y_2,y_3}(x_4)\rangle,$$

where $\rho^1_{y_1,y_2,y_3}, \rho^2_{y_1,y_2,y_3}, \rho^3_{y_1,y_2,y_3}, \rho^4_{y_1,y_2,y_3} : \Sigma \to \Sigma$ are permutations (1-quasigroups). By Propositions 3 and 4 we also have

$$\forall y_2, y_3 : M\langle \bar{x}, \bar{y}\rangle \;\Leftrightarrow\; K\langle p_{1;y_2,y_3}(x_1, y_1), p^2_{1;y_2,y_3}(x_2), p^3_{1;y_2,y_3}(x_3), p^4_{1;y_2,y_3}(x_4)\rangle, \tag{28}$$

$$\forall y_1, y_3 : M\langle \bar{x}, \bar{y}\rangle \;\Leftrightarrow\; K\langle p_{2;y_1,y_3}(x_1, y_2), p^2_{2;y_1,y_3}(x_2), p^3_{2;y_1,y_3}(x_3), p^4_{2;y_1,y_3}(x_4)\rangle, \tag{29}$$

$$\forall y_1, y_2 : M\langle \bar{x}, \bar{y}\rangle \;\Leftrightarrow\; K\langle p^1_{3;y_1,y_2}(x_1), p_{3;y_1,y_2}(x_2, y_3), p^3_{3;y_1,y_2}(x_3), p^4_{3;y_1,y_2}(x_4)\rangle \tag{30}$$

for some 1-quasigroups $p^2_{1;y_2,y_3}, p^3_{1;y_2,y_3}, p^4_{1;y_2,y_3}$, $p^2_{2;y_1,y_3}, p^3_{2;y_1,y_3}, p^4_{2;y_1,y_3}$, $p^1_{3;y_1,y_2}, p^3_{3;y_1,y_2}, p^4_{3;y_1,y_2}$ and 2-quasigroups $p_{1;y_2,y_3}, p_{2;y_1,y_3}, p_{3;y_1,y_2}$. So, $y_1, y_2$ are grouped with $x_1$ and $y_3$ is grouped with $x_2$; i.e., $j(1) = j(2) = 1$, $j(3) = 2$, $\mathbf{i}^1 = \{1, 2\}$, $\mathbf{i}^2 = \{3\}$, $\mathbf{i}^3 = \emptyset$, $\mathbf{i}^4 = \emptyset$.

By Proposition 5 we have

$$\forall y_2 : M\langle \bar{x}, \bar{y}\rangle \;\Leftrightarrow\; K\langle o^1_{y_2}(x_1, y_1), o^2_{y_2}(x_2, y_3), o^3_{y_2}(x_3), o^4_{y_2}(x_4)\rangle \tag{31}$$

for some $o^1_{y_2}, o^2_{y_2}, o^3_{y_2}, o^4_{y_2}$.

From (28)–(30) we see that Proposition 6 holds for $\mathbf{h} = \{1\}$, $\mathbf{h} = \{2\}$, and $\mathbf{h} = \{3\}$.

(1) We will prove that it holds for $\mathbf{h} = \{1, 3\}$. Let $i' = 1$ and $i'' = 3$. Since $j(i') = 1 \neq j(i'') = 2$, we have the situation of Case 2. Equating (28) and (30) and setting $y_1 := 0$ we obtain

$$\begin{aligned}&K\langle p_{1;y_2,y_3}(x_1, 0), p^2_{1;y_2,y_3}(x_2), p^3_{1;y_2,y_3}(x_3), p^4_{1;y_2,y_3}(x_4)\rangle\\ &\quad\Leftrightarrow\; K\langle p^1_{3;0,y_2}(x_1), p_{3;0,y_2}(x_2, y_3), p^3_{3;0,y_2}(x_3), p^4_{3;0,y_2}(x_4)\rangle.\end{aligned}$$

Substituting $x_1 := \dot{p}_{1;y_2,y_3}(u, 0)$ we get

$$\begin{aligned}&K\langle u, p^2_{1;y_2,y_3}(x_2), p^3_{1;y_2,y_3}(x_3), p^4_{1;y_2,y_3}(x_4)\rangle\\ &\quad\Leftrightarrow\; K\langle p^1_{3;0,y_2}(\dot{p}_{1;y_2,y_3}(u, 0)), p_{3;0,y_2}(x_2, y_3), p^3_{3;0,y_2}(x_3), p^4_{3;0,y_2}(x_4)\rangle.\end{aligned}$$

Substituting $u := p_{1;y_2,y_3}(x_1, y_1)$ we get

$$\begin{aligned}&K\langle p_{1;y_2,y_3}(x_1, y_1), p^2_{1;y_2,y_3}(x_2), p^3_{1;y_2,y_3}(x_3), p^4_{1;y_2,y_3}(x_4)\rangle\\ &\quad\Leftrightarrow\; K\langle p^1_{3;0,y_2}(\dot{p}_{1;y_2,y_3}(p_{1;y_2,y_3}(x_1, y_1), 0)), p_{3;0,y_2}(x_2, y_3), p^3_{3;0,y_2}(x_3), p^4_{3;0,y_2}(x_4)\rangle.\end{aligned} \tag{32}$$

Since, by (28), the left part of (32) is equivalent to $M\langle \bar{x}, \bar{y}\rangle$, we have the following:

$$M\langle \bar{x}, \bar{y}\rangle \;\Leftrightarrow\; K\langle s_{y_2,y_3}(x_1, y_1), p_{3;0,y_2}(x_2, y_3), p^3_{3;0,y_2}(x_3), p^4_{3;0,y_2}(x_4)\rangle,$$

where $s_{y_2,y_3}(x_1, y_1) \overset{\text{def}}{=} p^1_{3;0,y_2}(\dot{p}_{1;y_2,y_3}(p_{1;y_2,y_3}(x_1, y_1), 0))$. To eliminate the subindex $y_3$, define

$$\widetilde{M}\langle \bar{x}, \bar{y}\rangle \;\Leftrightarrow\; K\langle s_{y_2,0}(x_1, y_1), p_{3;0,y_2}(x_2, y_3), p^3_{3;0,y_2}(x_3), p^4_{3;0,y_2}(x_4)\rangle. \tag{33}$$

It remains to check that $M$ and $\widetilde{M}$ coincide. Firstly, $M\langle \bar{x}, y_1, y_2, 0\rangle \Leftrightarrow \widetilde{M}\langle \bar{x}, y_1, y_2, 0\rangle$. Secondly, from $s_{y_2,y_3}(x_1, 0) \equiv p^1_{3;0,y_2}(x_1)$ and (30) we derive that $M\langle \bar{x}, 0, y_2, y_3\rangle \Leftrightarrow \widetilde{M}\langle \bar{x}, 0, y_2, y_3\rangle$. For any fixed $x_2, x_4, y_2$ we have decompositions of both $M\langle \bar{x}, \bar{y}\rangle$ and $\widetilde{M}\langle \bar{x}, \bar{y}\rangle$ of type $C(b(x_1, y_1), y_3, x_3)$, see (31) and (33). By Lemma 4 $M\langle \bar{x}, \bar{y}\rangle \Leftrightarrow \widetilde{M}\langle \bar{x}, \bar{y}\rangle$, and, thus, for some $s^1_{y_2}, s^2_{y_2}, s^3_{y_2}, s^4_{y_2}$ we have

$$M\langle \bar{x}, \bar{y}\rangle \quad\Leftrightarrow\quad K\langle s^1_{y_2}(x_1, y_1), s^2_{y_2}(x_2, y_3), s^3_{y_2}(x_3), s^4_{y_2}(x_4)\rangle. \tag{34}$$

(2) Similarly, the statement holds for $\mathbf{h} = \{2, 3\}$, and for some $r^1_{y_1}, r^2_{y_1}, r^3_{y_1}, r^4_{y_1}$ we have

$$M\langle \bar{x}, \bar{y}\rangle \quad\Leftrightarrow\quad K\langle r^1_{y_1}(x_1, y_2), r^2_{y_1}(x_2, y_3), r^3_{y_1}(x_3), r^4_{y_1}(x_4)\rangle. \tag{35}$$

(3) Now, we are ready to prove the statement for $\mathbf{h} = \{1, 2, 3\}$. Let $i' = 1$ and $i'' = 2$. Since $j(i') = j(i'')$, we have the situation of Case 1. The representations (34) and (35) play the role of the induction hypothesis; equating them and setting $y_1 := 0$ we get

$$K\langle s^1_{y_2}(x_1, 0), s^2_{y_2}(x_2, y_3), s^3_{y_2}(x_3), s^4_{y_2}(x_4)\rangle \quad\Leftrightarrow\quad K\langle r^1_0(x_1, y_2), r^2_0(x_2, y_3), r^3_0(x_3), r^4_0(x_4)\rangle.$$

Substitute $x_1 := \dot{s}^1_{y_2}(u, 0)$:

$$K\langle u, s^2_{y_2}(x_2, y_3), s^3_{y_2}(x_3), s^4_{y_2}(x_4)\rangle \quad\Leftrightarrow\quad K\langle r^1_0(\dot{s}^1_{y_2}(u, 0), y_2), r^2_0(x_2, y_3), r^3_0(x_3), r^4_0(x_4)\rangle.$$

Substituting $u := s^1_{y_2}(x_1, y_1)$ and denoting $r(x_1, y_1, y_2) \stackrel{\text{def}}{=} r^1_0(\dot{s}^1_{y_2}(s^1_{y_2}(x_1, y_1), 0), y_2)$, we obtain

$$K\langle s^1_{y_2}(x_1, y_1), s^2_{y_2}(x_2, y_3), s^3_{y_2}(x_3), s^4_{y_2}(x_4)\rangle \quad\Leftrightarrow\quad K\langle r(x_1, y_1, y_2), r^2_0(x_2, y_3), r^3_0(x_3), r^4_0(x_4)\rangle.$$

By (34), the left part is equivalent to $M\langle \bar{x}, \bar{y}\rangle$. Since $r(x_1, y_1, y_2)$ is a 3-quasigroup, by Lemma 2, Theorem 1 for our example is proved.

## Appendix B. Notation list

- $\Sigma$ is a nonempty set; $\Sigma^n$ is the set of $n$-words over $\Sigma$.
- 0 is some fixed element of $\Sigma$; $\bar{0}$ is the all-zero word.
- $[n] \stackrel{\text{def}}{=} \{1, \ldots, n\}$.
- $q\langle x_1, \ldots, x_n, x_{n+1}\rangle \stackrel{\text{def}}{\Longleftrightarrow} q(x_1, \ldots, x_n) = x_{n+1}$.
- $\dot{q}(y, x_2 \ldots, x_n) = z \stackrel{\text{def}}{\Longleftrightarrow} q(z, x_2, \ldots, x_n) = y$.
- If $\bar{x} = (x_1, x_2, \ldots, x_n)$, then
  $\bar{x}^{(k)} \stackrel{\text{def}}{=} (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$,
  $\bar{x}^{(k)}\#y \stackrel{\text{def}}{=} (x_1, \ldots, x_{k-1}, y, x_{k+1}, \ldots, x_n)$,
  $\bar{x}^{(l,k)} = \bar{x}^{(k,l)} \stackrel{\text{def}}{=} \bar{x}^{(l)(k)}$ where $k < l$.

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