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On the sign-imbalance of partition shapes

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Abstract

Let the *sign* of a standard Young tableau be the sign of the permutation you get by reading it row by row from left to right, like a book. A conjecture by Richard Stanley says that the sum of the signs of all SYTs with n squares is $2^{\lfloor n/2 \rfloor}$. We present a stronger theorem with a purely combinatorial proof using the Robinson–Schensted correspondence and a new concept called chess tableaux.

We also prove a sharpening of another conjecture by Stanley concerning weighted sums of squares of sign-imbalances. The proof is built on a remarkably simple relation between the sign of a permutation and the signs of its RS-corresponding tableaux.

MSC: primary: 06A07; secondary: 05E10

Keywords: Inversion; Tableau; Shape; Domino; Fourling; Sign-balanced; Sign-imbalance; Robinson–Schensted correspondence; Row insertion; Chess tableau

1. Introduction

Young tableaux are simple combinatorial objects with complex properties. They play a central role in the theory of symmetric functions (see [1]) so they have been studied a lot, but the subject is still very much alive. Recently, Richard Stanley came up with a very nice conjecture on Young tableaux:

Let the sign of a standard Young tableau be the sign of the permutation you get by reading it row by row from left to right, like a book. The sum of the signs of all SYTs with n squares is $2^{\lfloor n/2 \rfloor}$.

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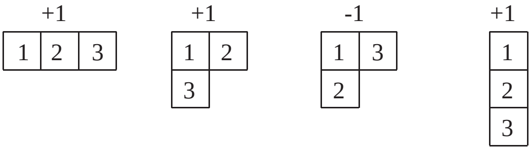
If we take $n = 3$ for example, there are four SYTs:



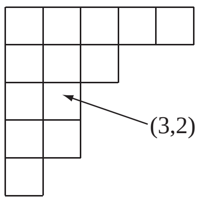
Their signs sum up to $2 = 2^{\lfloor 3/2 \rfloor}$.

The above conjecture is just a special case of another one which Stanley gave in [9] (our Conjecture 3.1(a)). That conjecture was proved by Lam [2] but we will prove an even stronger theorem (our Theorem 3.3). Part (b) of the same conjecture is also proved in a stronger version (our Theorems 3.4 and 3.5).

To settle the conjectures we use two tools: the Robinson–Schensted correspondence, and a new concept called chess tableaux. Some of our results in developing these tools have the flavour of an ad hoc lemma, but Proposition 5.3, which is a link between signs of tableaux and signs of permutations, may be of interest in its own right.

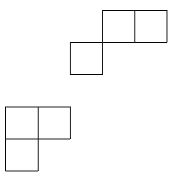
2. Preliminaries

An *n-shape* $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a graphical representation (a Ferrers diagram) of an integer partition of $n = \sum_i \lambda_i$. We write $\lambda \vdash n$ and we will not distinguish the partition itself from its shape. The *coordinates* of a square is the pair (r, c) where r and c are the row and column indices. Example:

(5, 3, 2, 2, 1) = 

The *conjugate* λ' of a shape λ is the reflection of λ in the main diagonal, i.e. exchanging rows and columns.

A shape λ is a *subshape* of a shape μ if $\lambda_i \leqslant \mu_i$ for all i. For any subshape $\lambda \subseteq \mu$ the *skew* shape μ/λ is μ with λ deleted. Example:

(5, 3, 2, 2, 1)/(3, 2, 2) = 

A *domino* is a rectangle consisting of two squares. By $v(\lambda)$ we will denote the maximal number of disjoint vertical dominoes that fit in the shape λ. We let $h(\lambda) = v(\lambda')$.

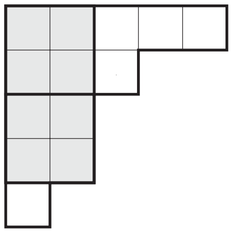


Fig. 1. The shaded squares form the fourling body and the white squares are the strip. Here $d(\lambda) = 2$ and $\mathrm{vs}(\lambda) = \mathrm{hs}(\lambda) = 1$.

A *fourling* is a 2×2-square. The maximal number of disjoint fourlings that fit in a shape λ is denoted by $d(\lambda)$. A *fourling shape* is a (possibly empty) shape consisting of fourlings. The *fourling body* $\mathrm{fb}(\lambda)$ of a shape λ is its largest fourling subshape. The remaining squares form the *strip* of the shape. By $\mathrm{vs}(\lambda)$ we will denote the maximal number of disjoint vertical dominoes that fit in the strip of λ. We let $\mathrm{hs}(\lambda) = \mathrm{vs}(\lambda')$. See Fig. 1.

A *tableau* on an *n*-shape λ is a labelling of the squares of λ with *n* different integers such that every integer is greater than its neighbours above and to the left. A *standard Young tableau* (SYT) on an *n*-shape is a tableau with the numbers $[n] = \{1, 2, \ldots, n\}$. We let $\mathrm{SYT}(\lambda)$ denote the set of SYTs on the shape λ. Here is an example:

1	4	6	7	10
2	5	9		
3	11			
8	13			
12				

The shape of a tableau T is denoted by $\mathrm{sh}(T)$.

By a *k-word* we will mean a sequence of k integers, all different. A *sorted word* is a strictly increasing sequence of integers. The *sign* of a word $w = w_1 w_2 \cdots w_k$ is $(-1)^{|\{(i,j)\,:\,i<j,\ w_i>w_j\}|}$, so it is $+1$ for an even number of inversions, -1 otherwise.

The *sign* $\mathrm{sgn}(T)$ of a tableau T is the sign of the word you get by reading the integers row by row, from left to right and from top to bottom, like a book. Our exale tableau has 18 inversions, so $\mathrm{sgn}(T) = +1$. The *sign-imbalance* I_λ of a shape λ is the sum of the signs of all SYTs on that shape.

Definition 2.1.

$$I_\lambda = \sum_{T \in \mathrm{SYT}(\lambda)} \mathrm{sgn}(T).$$

3. Stanley's conjecture and our results

Richard Stanley gave the following conjecture in [9].

Conjecture 3.1.

(a) *For every* $n \geqslant 0$

$$\sum_{\lambda \vdash n} q^{v(\lambda)} t^{d(\lambda)} x^{h(\lambda)} I_\lambda = (q + x)^{\lfloor n/2 \rfloor}.$$

(b) *If* $n \not\equiv 1 \pmod 4$

$$\sum_{\lambda \vdash n} (-1)^{v(\lambda)} t^{d(\lambda)} I_\lambda^2 = 0.$$

The special case $t = 0$ of (a) goes like this:

Proposition 3.2. *For all* $n \geqslant 0$ *we have*

$$\sum_{\lambda=(n-i,1^i)} q^{v(\lambda)} x^{h(\lambda)} I_\lambda = (q + x)^{\lfloor n/2 \rfloor},$$

where λ *ranges over all hooks* $(n - i, 1^i)$, $0 \leqslant i \leqslant n - 1$.

It tells us that the right-hand side $(q + x)^{\lfloor n/2 \rfloor}$ comes from the *hooks*, i.e. the fourling-free shapes, and was proved twice by Stanley in [9, Proposition 3.4]. We give a third proof in Section 6.

The rest of (a) says that, for fixed $d \geqslant 1$, h and v, the sum of the sign-imbalances of all n-shapes λ with $v(\lambda) = v$, $h(\lambda) = h$ and $d(\lambda) = d$ vanishes.

Part (a) of the conjecture has been proved by Lam [2]. We will prove a stronger version of part (a) which lets us fix not only the number of fourlings but the whole fourling shape:

Theorem 3.3. *Given a non-empty fourling shape* D *and non-negative integers* h, v *and* s,

$$\sum I_\lambda = 0$$

where the sum is taken over all shapes λ *with fourling body* D, s *squares in the strip,* $\mathrm{hs}(\lambda) = h$, *and* $\mathrm{vs}(\lambda) = v$.

The proof will be found in Section 6 and is purely combinatorial. Fig. 2 shows an example.

In the same spirit, we have the following theorem which is a sharpening of (b) when n is even.

Theorem 3.4. *Given a fourling shape* D *and an even integer* $n \geqslant 0$,

$$\sum (-1)^{v(\lambda)} I_\lambda^2 = 0,$$

where the sum is taken over all n*-shapes* λ *with* $\mathrm{fb}(\lambda) = D$.

We will prove it in Section 5.

The next theorem, which we prove in Section 4, covers the rest of (b).

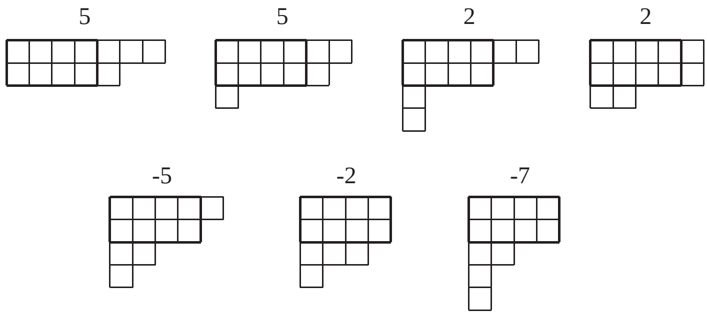


Fig. 2. The imbalances of the 12-shapes λ with fourling body and $\mathrm{vs}(\lambda) = \mathrm{hs}(\lambda) = 1$. You can check that their sum vanishes.

Theorem 3.5. *If $n \equiv 2$ or $n \equiv 3 \pmod 4$*

$$\sum_{\lambda \vdash n} (-1)^{v(\lambda)} F(\lambda) = 0$$

for any function $F : \{n\text{-}shapes\} \to \mathbb{C}$ such that $F(\lambda) = F(\lambda')$ and $I_\lambda = 0 \Rightarrow F(\lambda) = 0$ for all n-shapes λ.

Choosing $F(\lambda) = t^{d(\lambda)} I_\lambda^2$ proves (b) for $n \equiv 2$ and $n \equiv 3 \pmod 4$ since $|I_\lambda| = |I'_\lambda|$ (see e.g. Stanley [9] or our Proposition 6.6). Thus we have proved all parts of Stanley's conjecture.

Finally, the special case $t = 1$ of (b) will be proved also without the assumption $n \not\equiv 1 \pmod 4$:

Theorem 3.6. *For all $n \geqslant 2$*

$$\sum_{\lambda \vdash n} (-1)^{v(\lambda)} I_\lambda^2 = 0.$$

This was proved independent of us by Reifegerste [3, Theorem 5.1]. Stanley proved it for even n [9, Theorem 3.2(b)].

The rest of this paper is composed as follows. In Section 4 we introduce the concept of a chess tableau and prove Theorem 3.5. In Section 5 we show how the signs of tableaux and permutations are related by the Robinson–Schensted correspondence. The most important result is Proposition 5.3 which we use to prove Theorems 3.6 and 3.4. Finally, in Section 6 we prove Theorem 3.3 using chess tableaux and the RS-correspondence.

4. Chess tableaux and Theorem 3.5

When working on sums of tableau signs one is naturally led to use domino tableaux (see [9,6]). In this paper we choose a similar approach which turns out to be more successful in settling the conjectures.

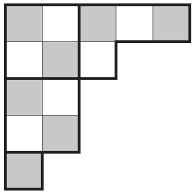


Fig. 3. The white strip squares count the strip dominoes, $\mathrm{vs}(\lambda) + \mathrm{hs}(\lambda) = 2$.

A *chess colouring* of a shape is a colouring of the squares such that a square (r, c) is black if $r + c$ is even and white if $r + c$ is odd. From now on we will frequently refer to white and black squares of a shape, implicitly meaning the chess colouring. A *chess tableau* is an SYT with odd integers in black squares and even in white.

Lemma 4.1. *Given a shape λ, $\sum_{T \in \mathrm{SCT}(\lambda)} \mathrm{sgn}(T) = I_\lambda$, where $\mathrm{SCT}(\lambda)$ is the set of chess tableaux on λ.*

Proof. There is a sign-alternating involution on the non-chess SYTs: Given a non-chess SYT there are at least two consecutive integers of the same colour. Choose the least such pair and switch the integers. This is allowed unless they are horizontal or vertical neighbours, which they are not since neighbours have different colours. $\square$

Proposition 4.2. *If λ is a shape with s strip squares, $I_\lambda \neq 0$ only if it has equally many white and black squares or one more black square. This implies that* $\mathrm{hs}(\lambda) + \mathrm{vs}(\lambda) = \lfloor s/2 \rfloor$.

Proof. Let B and W be the number of black, respectively, white squares in the strip of λ. By Lemma 4.1 we must have $B = W$ or $W + 1$ if $I_\lambda \neq 0$ (otherwise there are no chess tableaux). Every white strip square belongs to a certain strip domino, namely the one with the black square above or to the left, so $W = \mathrm{hs}(\lambda) + \mathrm{vs}(\lambda)$, see Fig. 3. Thus, for a λ with $I_\lambda \neq 0$ we have $\mathrm{hs}(\lambda) + \mathrm{vs}(\lambda) = \lfloor s/2 \rfloor$. $\square$

Proof of Theorem 3.5. We show that if λ is an n-shape with $n \equiv 2$ or $n \equiv 3 \pmod 4$, either $I_\lambda = 0$ or $v(\lambda) \not\equiv h(\lambda) \pmod 2$. This implies that the non-vanishing terms $(-1)^{v(\lambda)} F(\lambda)$ come in cancelling pairs $(-1)^{v(\lambda)} F(\lambda) + (-1)^{v(\lambda')} F(\lambda')$.

Suppose $I_\lambda \neq 0$ and let s be the number of strip squares in λ. Since the fourling body consists of fourlings we have $s \equiv 2$ or $s \equiv 3 \pmod 4$. By Proposition 4.2 we can assume that $\mathrm{hs}(\lambda) + \mathrm{vs}(\lambda) = \lfloor s/2 \rfloor$ which is odd. The fourling body has equally many horizontal and vertical dominoes so $v(\lambda) \not\equiv h(\lambda) \pmod 2$. $\square$

5. Robinson–Schensted correspondence and Theorems 3.6 and 3.4

Given a tableau T and a number a different from all numbers in T, by (*row*) *insertion* of a into T we mean the usual Robinson–Schensted insertion (see for example [8, p. 316]) resulting in a tableau $(T \leftarrow a)$ with one more square x than T. By (*row*) *extraction* of x

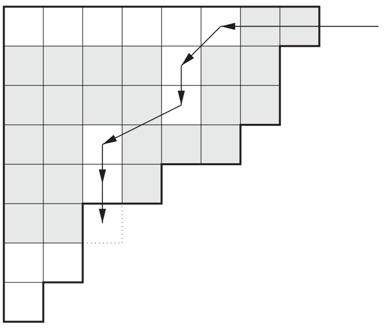


Fig. 4. Insertion of a number. The shaded squares are counted by $\sum_{i=2}^{k}(\lambda_{i-1} - c_{i-1} + c_i - 1)$ in the proof.

we mean the reverse process resulting in T and a. Insertion of a word into a tableau means insertion of the integers in the word one by one from left to right.

We will use the following lemma later on.

Lemma 5.1. *Given a tableau T and integers $a \neq b$ different from all entries in T, the square* $\mathrm{sh}(T \leftarrow ab)/\mathrm{sh}(T \leftarrow a)$ *appears in a column somewhere to the right of* $\mathrm{sh}(T \leftarrow a)/\mathrm{sh}(T)$ *if and only if $a < b$.*

Proof. Suppose that $a < b$. We can insert the two numbers in parallel row by row. If a is greater than every number in the first row, the squares $x = \mathrm{sh}(T \leftarrow a)/\mathrm{sh}(T)$ and $y = \mathrm{sh}(T \leftarrow ab)/\mathrm{sh}(T \leftarrow a)$ will be placed rightmost in that row with y to the right of x. If a pops a number a_2 in the first row, b will either terminate leaving y rightmost in the first row or pop a number $b_2 > a_2$. The if part of the lemma follows by induction. The converse is proved similarly. $\square$

The next lemma tells us what insertion does to the sign of the tableau.

Lemma 5.2. *If T is a tableau and a is a number different from all entries in T,*

$$\mathrm{sgn}(T \leftarrow a) = (-1)^{l+w+u}\,\mathrm{sgn}(T),$$

where l is the number of entries in T less than a, w is 0 *if* $\mathrm{sh}(T \leftarrow a)/\mathrm{sh}(T)$ *is black and* 1 *if it is white, and u is the number of squares in rows above* $\mathrm{sh}(T \leftarrow a)/\mathrm{sh}(T)$.

Proof. Let $\lambda = \mathrm{sh}(T)$ and look at Fig. 4. During the insertion $a_1 = a$ pops a number a_2 at $(1, c_1)$ which pops a number a_3 at $(2, c_2)$ and so on. Finally the number a_k fills a new square $(k, c_k) = \mathrm{sh}(T \leftarrow a)/\mathrm{sh}(T)$. For $2 \leqslant i \leqslant k$, the move of a_i multiplies the sign of the tableau by $(-1)^{\lambda_{i-1}-c_{i-1}+c_i-1}$. Summation yields

$$\sum_{i=2}^{k}(\lambda_{i-1} - c_{i-1} + c_i - 1) = c_k - c_1 + \sum_{i=1}^{k-1}(\lambda_i - 1) = u - k + 1 + c_k - c_1.$$

The placing of $a = a_1$ in the first row multiplies the sign of the tableau by $(-1)^{l-c_1+1}$, so the total factor is $(-1)^{u-k+1+c_k-c_1+l-c_1+1} = (-1)^{u+l+c_k+k} = (-1)^{u+l+w}$. $\square$

Now the following natural question arises: How is the sign property transferred by the RS-correspondence? The answer is quite beautiful:

Proposition 5.3. *In the RS-correspondence $\pi \leftrightarrow (P, Q)$ we have*

$$\operatorname{sgn}(\pi) = (-1)^{v(\lambda)} \operatorname{sgn}(P) \operatorname{sgn}(Q),$$

where λ is the shape of P and Q.

Proof. Suppose we have inserted the first k numbers in π yielding tableaux P^k and Q^k on the shape λ^k, and $\operatorname{sgn}(\pi_1 \cdots \pi_k) = (-1)^{v(\lambda^k)} \operatorname{sgn}(P^k) \operatorname{sgn}(Q^k)$. This is certainly true for $k = 0$. Now we argue by induction over k. We insert the next number π_{k+1} and look at what happens according to Lemma 5.2. We get $\operatorname{sgn}(P^{k+1}) = (-1)^{l+w+u} \operatorname{sgn}(P^k)$, and if λ^{k+1}/λ^k has coordinates (r, c) we get $\operatorname{sgn}(Q^{k+1}) = (-1)^{k-u-c+1} \operatorname{sgn}(Q^k) = (-1)^{k-u-w+r+1} \operatorname{sgn}(Q^k)$ since w is congruent to $r + c$ modulo 2. Whether a new vertical domino will fit in λ^{k+1} is only dependent on r, so $(-1)^{v(\lambda^{k+1})} = (-1)^{r+1}(-1)^{v(\lambda^k)}$. Finally, $\operatorname{sgn}(\pi_1 \cdots \pi_{k+1}) = (-1)^{k-l} \operatorname{sgn}(\pi_1 \cdots \pi_k)$.

Putting it all together yields at last

$$\begin{aligned}
\operatorname{sgn}(\pi_1 \cdots \pi_{k+1}) &= (-1)^{k-l} \operatorname{sgn}(\pi_1 \cdots \pi_k) = (-1)^{k-l}(-1)^{v(\lambda^k)} \operatorname{sgn}(P^k) \operatorname{sgn}(Q^k) \\
&= (-1)^{r+1}(-1)^{v(\lambda^k)}(-1)^{l+w+u} \operatorname{sgn}(P^k)(-1)^{k-u-w+r+1} \operatorname{sgn}(Q^k) \\
&= (-1)^{v(\lambda^{k+1})} \operatorname{sgn}(P^{k+1}) \operatorname{sgn}(Q^{k+1}). \qquad \square
\end{aligned}$$

The above result was also found by Reifegerste [3, Theorem 4.3] independent of us.

Remark. If we specialise to the RS-bijection $\pi \leftrightarrow (P, P)$ between involutions $\pi \in S_n$ and n-SYTs P, Proposition 5.3 gives that $\operatorname{sgn}(\pi) = (-1)^{v(\operatorname{sh}(P))}$. This is also a simple consequence of a theorem by Schützenberger [5, p. 127] (see also [8, Exercise 7.28 a]) stating that the number of fix points in π equals the number of columns of P of odd length.

As a simple consequence of Proposition 5.3 we get Theorem 3.6.

Proof of Theorem 3.6. By Proposition 5.3 we have

$$\begin{aligned}
\sum_{\lambda \vdash n} (-1)^{v(\lambda)} I_\lambda^2 &= \sum_{\lambda \vdash n} (-1)^{v(\lambda)} \left(\sum_{P \in \mathrm{SYT}(\lambda)} \operatorname{sgn}(P) \right)^2 \\
&= \sum_{\lambda \vdash n} \sum_{P, Q \in \mathrm{SYT}(\lambda)} (-1)^{v(\lambda)} \operatorname{sgn}(P) \operatorname{sgn}(Q) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) = 0. \qquad \square
\end{aligned}$$

To prove Theorem 3.4 we will need the following much stronger theorem which is proved in a manner similar to what we did above.

Theorem 5.4. *Given a set B of black squares and an even integer $n \geqslant 0$,*

$$\sum (-1)^{v(\lambda)} I_\lambda^2 = 0,$$

where the sum is taken over all n-shapes λ whose black squares are exactly the ones in B.

Proof. Let A be the set of shapes whose black squares are exactly the ones in B. For an n-SYT Q, let $Q \setminus n$ denote the $(n-1)$-SYT we get by deleting the number n from Q. If Q is a chess tableau, $\mathrm{sh}(Q) \in A \Leftrightarrow \mathrm{sh}(Q \setminus n) \in A$ since $\mathrm{sh}(Q)$ and $\mathrm{sh}(Q \setminus n)$ contain exactly the same set of black squares (remember that n is even). Then, by Lemma 4.1,

$$\sum_{\substack{\lambda \vdash n \\ \lambda \in A}} (-1)^{v(\lambda)} I_\lambda^2 = \sum_{\lambda \vdash n} (-1)^{v(\lambda)} I_\lambda \sum_{\substack{Q \in \mathrm{SCT}(\lambda) \\ \mathrm{sh}(Q \setminus n) \in A}} \mathrm{sgn}(Q).$$

Now we take any n-shape λ and compute its contribution to the sum. If λ does not have equally many white and black squares, $I_\lambda = 0$ by Proposition 4.2 and the contribution is zero. If λ has equally many white and black squares, then, for $Q \in \mathrm{SYT}(\lambda)$, Q is a chess tableau if and only if $Q \setminus n$ is a chess tableau. Thus, we can write our expression in a slightly different way:

$$\sum_{\lambda \vdash n} (-1)^{v(\lambda)} I_\lambda \sum_{\substack{Q \in \mathrm{SYT}(\lambda) \\ Q \setminus n \text{ is a chess tableau} \\ \mathrm{sh}(Q \setminus n) \in A}} \mathrm{sgn}(Q).$$

By Proposition 5.3 this equals

$$\sum_{\lambda \vdash n} \sum_{\substack{P, Q \in \mathrm{SYT}(\lambda) \\ Q \setminus n \text{ is a chess tableau} \\ \mathrm{sh}(Q \setminus n) \in A}} (-1)^{v(\lambda)} \mathrm{sgn}(P) \, \mathrm{sgn}(Q) = \sum_{\pi \in S} \mathrm{sgn}(\pi),$$

where $S \subseteq S_n$ is the set of permutations corresponding to n-tableaux P and Q such that $Q \setminus n$ is a chess tableau whose shape is in A. (Note that we do not require that Q is a chess tableau.)

For an n-permutation π, let π' be the $(n-1)$-permutation defined by

$$\pi'_i = \begin{cases} \pi_i & \text{if } \pi_i < \pi_n, \\ \pi_i - 1 & \text{if } \pi_i > \pi_n. \end{cases}$$

We can consider the set S_n of n-permutations as a disjoint union $S_n = \bigcup_{\rho \in S_{n-1}} S_n^\rho$, where $S_n^\rho = \{\pi \in S_n : \pi' = \rho\}$. In the RS-correspondence $\pi \to (P, Q)$ the locations of the first $n-1$ numbers in Q are only dependent on π'. Thus we can write S as a disjoint union $S = \bigcup_{\rho \in S'} S_n^\rho$, where S' is the set of $(n-1)$-permutations corresponding to a chess Q-tableau whose shape is in A. But $\sum_{\pi \in S_n^\rho} \mathrm{sgn}(\pi) = 0$ since we can choose the last element π_n in an even number of ways. $\square$

Finally we show that Theorem 3.4 is a simple consequence of the above theorem.

Proof of Theorem 3.4. Note that it is impossible to change the fourling body of a shape by adding or removing only white squares.

Let B_λ denote the set of black squares in a shape λ and let $\mathcal{B} = \{B_\lambda : \lambda \vdash n,\ \mathrm{fb}(\lambda) = D\}$. Then

$$\sum_{\substack{\lambda\vdash n\\ \mathrm{fb}(\lambda)=D}} (-1)^{v(\lambda)} I_\lambda^2 = \sum_{B\in\mathcal{B}} \sum_{\substack{\lambda\vdash n\\ B_\lambda=B}} (-1)^{v(\lambda)} I_\lambda^2 = 0$$

by Theorem 5.4. □

6. The proofs of Proposition 3.2 and Theorem 3.3

First some definitions:

Definition 6.1. Given an n-shape λ and an integer $k \geqslant 0$, let $\mathcal{T}_{\lambda,k}$ be the set of tableaux on λ with numbers in $[n+k]$.

Given $T \in \mathcal{T}_{\lambda,k}$, let *the complementary k-word $w_{T,k}$ of T* be the sorted k-word of the elements of $[n+k]$ not in T.

Let $\mathrm{SW}_{i,j}$ denote the set of sorted j-words with letters in $[i]$.

Given a k-word w, let $\sigma(w) = (-1)^L$, where $L = \sum_{i=1}^{k}(w_i - 1)$.

Given a skew shape μ/λ, let $\tau(\mu/\lambda) = (-1)^{W+U}$, where W is the number of white squares in μ/λ and U is the number of square pairs $(x, y) \in \lambda \times \mu/\lambda$ with x in a row somewhere above y.

Lemma 6.2. *Let λ be an n-shape. Insertion of $w_{T,k}$ into T gives a bijection between $\mathcal{T}_{\lambda,k}$ and the set of SYTs on $(n+k)$-shapes $\mu \supseteq \lambda$ with $v(\mu/\lambda) = 0$. We have*

$$\mathrm{sgn}(T \leftarrow w_{T,k}) = \sigma(w_{T,k})\tau(\mathrm{sh}(T \leftarrow w_{T,k})/\lambda)\,\mathrm{sgn}(T). \tag{1}$$

Fig. 5 shows an example.

Proof. Let $T \in \mathcal{T}_{\lambda,k}$ and let $\mu = \mathrm{sh}(T \leftarrow w_{T,k})$. By Lemma 5.1 the extra squares μ/λ will appear from left to right, without any vertical dominoes. The inverse of the insertion is extraction of the squares μ/λ from right to left. Clearly it is a bijection. Eq. (1) follows from iteration of Lemma 5.2, where L stems from l, W from w, and U from u. □

Lemma 6.3.

$$\sum_{w\in\mathrm{SW}_{i,j}} \sigma(w) = \begin{cases} 0 & \textit{if } i \textit{ is even and } j \textit{ is odd},\\ (-1)^{\lfloor j/2\rfloor}\binom{\lfloor i/2\rfloor}{\lfloor j/2\rfloor} & \textit{otherwise}. \end{cases}$$

Proof. By definition, we have $\sigma(w) = (-1)^L$, where $L = (w_1 - 1) + \cdots + (w_j - 1)$. Since $\sigma(w_1 w_2 \cdots w_j) \neq \sigma((w_1 + 1)w_2 \cdots w_j)$ we only have to consider words in which

Let $\lambda = (5, 2, 2, 1)$ and $k = 3$. If we take, for example,

$$T = \begin{array}{ccccc} 2 & 4 & 6 & 9 & 13 \\ 3 & 5 & & & \\ 8 & 11 & & & \\ 12 & & & & \end{array}$$

then $w_{T,3} = 1\ 7\ 10$ and insertion yields

$$(T \leftarrow w_{T,3}) = \begin{array}{ccccc} 1 & 4 & 6 & 7 & 10 \\ 2 & 5 & 9 & 13 & \\ 3 & 11 & & & \\ 8 & & & & \\ 12 & & & & \end{array}$$

We get $L = (1-1)+(7-1)+(10-1) = 15$, so $\sigma(1\ 7\ 10) = (-1)^L = -1$. Among the three extra squares only one is white, so $W = 1$. The number of original squares in rows above the extra squares is 10, 5 and 5, so $U = 20$, and $\tau(\mathrm{sh}(T \leftarrow w_{T,3})/\lambda) = (-1)^{W+U} = (-1)^{1+20} = -1$. The lemma says that $\mathrm{sgn}(T \leftarrow w_{T,3}) = \sigma\tau\,\mathrm{sgn}(T)$. We check that T has 11 inversions and $(T \leftarrow w_{T,3})$ has 21, so it seems alright.

Fig. 5. Example of Lemma 6.2.

$w_1 + 1 = w_2$ and this value is even. By iteration of this argument we see that we only have to consider words in which $w_{2k-1} + 1 = w_{2k}$ for $1 \leqslant k \leqslant \lfloor j/2 \rfloor$ and these values are even. Every such pair gives an odd contribution to L.

If j is odd, the last letter w_j may be anywhere in the interval $(w_{j-1}, i]$. Since we have $\sigma(w_1 \cdots w_n) \neq \sigma(w_1 \cdots (w_n + 1))$ only words with $w_n = i$ odd remain. Then w_n gives an even contribution to L so we can ignore it.

Thus, if i is even and j is odd the sum vanishes, otherwise we can place the $\lfloor j/2 \rfloor$ pairs in $\lfloor i/2 \rfloor$ positions, and we get $(-1)^{\lfloor j/2 \rfloor}\binom{\lfloor i/2 \rfloor}{\lfloor j/2 \rfloor}$. $\square$

Remark. A referee has pointed out that, using q-binomial coefficients, the sum in Lemma 6.3 can be written as

$$(-1)^{\binom{j}{2}} \begin{bmatrix} i \\ j \end{bmatrix}_{q=-1}.$$

This follows from the bijection between sorted words $w_1 w_2 \cdots w_j \in \mathrm{SW}_{i,j}$ and weakly increasing sequences $0 \leqslant w_1 - 1 \leqslant w_2 - 2 \leqslant \cdots \leqslant w_j - j \leqslant i - j$, and from the fact that q-binomial coefficients enumerate lattice paths by area.

Proposition 6.4. *Given an n-shape λ whose strip consists of vertical dominoes, and a non-negative integer k, let H_λ be the set of $(n+k)$-shapes $\mu \supseteq \lambda$ with $\mathrm{fb}(\mu) = \mathrm{fb}(\lambda)$, $\mathrm{vs}(\mu) = \mathrm{vs}(\lambda)$, and $\mathrm{hs}(\mu) = \lfloor k/2 \rfloor$. Then*

$$\sum_{\mu \in H_\lambda} I_\mu = \binom{n/2 + \lfloor k/2 \rfloor}{\lfloor k/2 \rfloor} I_\lambda.$$

Proof. Put $m = n + k$ and let $H_\lambda^* \supseteq H_\lambda$ be the set of m-shapes $\mu \supseteq \lambda$ with $\mathrm{fb}(\mu) = \mathrm{fb}(\lambda)$ and $\mathrm{vs}(\mu) = \mathrm{vs}(\lambda)$, i.e. the set of m-shapes $\mu \supseteq \lambda$ with $v(\mu/\lambda) = 0$. By Proposition 4.2 all $\mu \in H_\lambda^* \setminus H_\lambda$ have $I_\mu = 0$. Now we apply Lemma 6.2 to $\mathcal{T}_{\lambda,k}$ and get

$$\sum_{\mu \in H_\lambda} I_\mu = \sum_{\substack{T \in \mathcal{T}_{\lambda,k} \\ \mathrm{sh}(T \leftarrow w_{T,k}) \in H_\lambda}} \sigma(w_{T,k}) \tau(\mathrm{sh}(T \leftarrow w_{T,k})/\lambda) \,\mathrm{sgn}(T). \tag{2}$$

If $\mathrm{sh}(T \leftarrow w_{T,k}) \in H_\lambda$ we have $W = \lfloor k/2 \rfloor$ (by the proof of Proposition 4.2) and U is even in Definition 6.1, which means that $\tau(\mathrm{sh}(T \leftarrow w_{T,k})/\lambda) = (-1)^{\lfloor k/2 \rfloor}$. By first considering a summation of $\sigma(w_{T,k}) \,\mathrm{sgn}(T)$ over the whole set H_λ^* and then removing the contribution from $H_\lambda^* \setminus H_\lambda$, we can write (2) as

$$(-1)^{\lfloor k/2 \rfloor} \left(\sum_{w \in \mathrm{SW}_{m,k}} \sigma(w) \sum_{\substack{T \in \mathcal{T}_{\lambda,k} \\ w_{T,k} = w}} \mathrm{sgn}(T) - \sum_{\mu \in H_\lambda^* \setminus H_\lambda} \sum_{\substack{T \in \mathcal{T}_{\lambda,k} \\ \mathrm{sh}(T \leftarrow w_{T,k}) = \mu}} \sigma(w_{T,k}) \,\mathrm{sgn}(T) \right)$$

which equals

$$(-1)^{\lfloor k/2 \rfloor} \left(\sum_{w \in \mathrm{SW}_{m,k}} \sigma(w) I_\lambda - \sum_{\mu \in H_\lambda^* \setminus H_\lambda} \frac{I_\mu}{\tau(\mu/\lambda)} \right) = (-1)^{\lfloor k/2 \rfloor} I_\lambda \sum_{w \in \mathrm{SW}_{m,k}} \sigma(w)$$

since $I_\mu = 0$ for $\mu \in H_\lambda^* \setminus H_\lambda$. By Lemma 6.3, $\sum_{w \in \mathrm{SW}_{m,k}} \sigma(w) = (-1)^{\lfloor k/2 \rfloor} \binom{n/2 + \lfloor k/2 \rfloor}{\lfloor k/2 \rfloor}$ which gives the desired result. $\square$

Proposition 3.2 is now proved "for free":

Proof of Proposition 3.2. If $h + v = \lfloor n/2 \rfloor$, applying Proposition 6.4 to (1^{2v}) and $k = n - 2v$ yields the coefficient of $q^v x^h$:

$$\sum_{\mu \in H_{(1^{2v})}} I_\mu = \binom{v+h}{h} I_{(1^{2v})} = \binom{v+h}{h}.$$

By Proposition 4.2, the coefficient of $q^v x^h$ vanishes if $h + v \neq \lfloor n/2 \rfloor$. $\square$

For the proof of Theorem 3.3 we will need the following observation.

Lemma 6.5. *A non-empty fourling shape D has zero sign-imbalance, $I_D = 0$.*

Proof. By Lemma 4.1 we only have to consider chess tableaux. But there are no chess tableaux on a non-empty fourling shape since all outer corners (squares without neighbours below or to the right) are black and the last number is even. $\square$

We will also need the following fundamental proposition.

Proposition 6.6. *For all shapes λ we have*

$$I_{\lambda'} = (-1)^{d(\lambda)} I_\lambda.$$

Proof. Let $x = (r_x, c_x)$ and $y = (r_y, c_y)$ be two squares in λ sorted so that $r_x \leqslant r_y$. After transposition x becomes (c_x, r_x) and y becomes (c_y, r_y) in λ'. The book permutation order between x and y is changed if and only if $r_x < r_y$ and $c_x > c_y$. Thus $I_{\lambda'} = (-1)^p I_\lambda$, where p is the number of pairs (x, y) of squares in λ with x north-east of y.

Let n be the number of squares in λ. By Proposition 4.2 we can assume that λ has $\lfloor n/2 \rfloor$ white squares. Take any n-SYT T on λ. For each number i in T, let p_i be the number of north-east pairs containing i and a smaller number. It is easy to see that if i is in the square (r, c) we have $p_i = i - rc = (i+1) - (r+c+(r-1)(c-1))$, where $r+c$ is odd if the square is white and even if it is black, while $(r-1)(c-1)$ is odd if and only if the square is the south-east corner of a fourling in the fourling body. Thus, $p = \sum_{i=1}^n p_i \equiv \frac{n(n+3)}{2} + \lfloor n/2 \rfloor + d(\lambda) \pmod 2$, since there are $\lfloor n/2 \rfloor$ white squares in λ. But $\frac{n(n+3)}{2} + \lfloor n/2 \rfloor = \lfloor n(n+4)/2 \rfloor$ is always even, so $p \equiv d(\lambda) \pmod 2$. $\square$

Finally we have all the tools we need.

Proof of Theorem 3.3. By Proposition 4.2, we can assume that $h + v = \lfloor s/2 \rfloor$. Let V be the set of shapes with fourling body D, $2v$ squares in the strip, and v vertical strip dominoes. First we will show that $\sum_{\lambda \in V} I_\lambda = 0$. Let $V' = \{\lambda' : \lambda \in V\}$. By Proposition 6.6, $\sum_{\lambda \in V} I_\lambda = (-1)^{d(D)} \sum_{\lambda \in V'} I_\lambda$, so it suffices to show that the latter sum vanishes. Applying Proposition 6.4 to D' and $k = 2v$ yields

$$\sum_{\lambda \in V'} I_\lambda = \sum_{\lambda \in H_{D'}} I_\lambda = \binom{2d(D) + v}{v} I_{D'} = 0$$

by Lemma 6.5. Finally, we apply Proposition 6.4 to every $\lambda \in V$ and $k = s - 2v$, and get

$$\sum_{\lambda \in V} \sum_{\mu \in H_\lambda} I_\mu = \binom{2d(D) + v + h}{h} \sum_{\lambda \in V} I_\lambda = 0. \qquad \square$$

7. Possible generalizations

The concept of sign-imbalance generalizes naturally to general finite posets. Note that an SYT is a linear extension of the partial order on the squares implied by coordinate pairs.

Let P be an n-element poset and let $\omega : P \to [n] = \{1, 2, \ldots, n\}$ be a bijection called the *labelling* of P. A *linear extension* of P is an order preserving bijection $f : P \to [n]$. If we regard f as a permutation π_f of $[n]$ given by $\pi_f(i) = \omega(f^{-1}(i))$ we can talk about the sign of f. The *sign-imbalance* of P is the sum of the signs of all linear extensions of P. If the sign-imbalance of P is zero we say that P is *sign-balanced*.

Note that the sign of a linear extension depends on the labelling ω. However, this dependence is not essential since changing the labelling of P simply multiplies π_f by a fixed permutation. For instance, the sign-imbalance of P is defined up to a sign without specifying ω, and the notion of sign-balance is completely independent of the labelling.

There has been some work (see [9]) considering sign-imbalances of general posets and identifying the sign-balanced ones. Unfortunately, the approach taken in this paper does not seem applicable to this more general question.

If we specialise to partition shapes, however, we hope that our Robinson–Schensted technique will be useful in future research. Some things to do:

- Characterise the sign-balanced partition shapes. There are some theorems on sign-balanced posets (see [9]); a complete characterisation in the special case of partition shapes may shed some light on this more general question.
- Find the "best" version of Theorem 3.3, i.e. find the smallest classes of n-shapes whose imbalance sum vanishes. This is a generalization of the above and, as Fig. 2 shows, there is still work to do.
- Find a nice formula for I_λ, maybe in the same spirit as the hook length formula. This may very well be impossible, as Stanley points out [9, p. 14].
- Study the imbalance of skew partitions. This is an interesting issue since most structural properties of partitions generalize to skew partitions, including the RS-correspondence (see e.g. [4]).

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