Codes from incidence matrices and line graphs of Hamming graphs

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ABSTRACT

We examine the $p$-ary codes, for any prime $p$, that can be obtained from incidence matrices and line graphs of the Hamming graphs, $H(n, m)$, obtaining the main parameters of these codes. We show that the codes from the incidence matrices of $H(n, m)$ can be used for full permutation decoding for all $m, n \geq 3$.

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1. Introduction

Codes obtained from the row span of an incidence matrix $G$ of a regular graph $\Gamma$ of some classes of graphs have been seen to share certain properties that can make such classes useful for practical purposes: if $\Gamma = (V, E)$ is a graph of valency $v$, and $|V| = N$, then $G$ is an $N \times \frac{1}{2} Nv$ matrix of 0s and 1s with rows labelled by the vertices, columns by the edges; the code $C_p(G)$ spanned by the rows of $G$ over $\mathbb{F}_p$, for any prime $p$, might be $\left[ \frac{1}{2} Nv, N, v \right]_p$ or $\left[ \frac{1}{2} Nv, N - 1, v \right]_p$ (if $p = 2$), and every minimum vector a scalar multiple of the incidence vector of the set of edges through a vertex, i.e. a row of $G$. Furthermore, there is often a gap in the weight enumerator between $v$ and $2(v - 1)$, the latter arising from the difference of two rows; see [6,12,13] for two particular classes. If $L(\Gamma')$ denotes the line graph (see Section 2 for the definition) of $\Gamma$, then the code spanned by the rows of an adjacency matrix $A$ of $L(\Gamma')$ over $\mathbb{F}_p$ will be a subcode of $C_2(G)$ if $p = 2$, of minimum weight in the range $[v, 2(v - 1)]$; if $p$ is odd, the code is less interesting, being of minimum weight at most 4 if $\Gamma'$ has a closed path of length 4. We look here at this question for the Hamming graphs.

The Hamming graph $H(n, m)$, for $n, m$ integers, is the graph with vertices the $m^n$ $n$-tuples of $\mathbb{R}^n$, where $R$ is a set of size $m$, and adjacency defined by two $n$-tuples being adjacent if they differ in one coordinate position. For example, the $n$-cube $Q_n$ is $H(n, 2)$ with $R = \mathbb{F}_2$. The number of edges of $H(n, m)$ is $\frac{1}{2} m^n (m - 1) n$. The codes from incidence matrices and line graphs of $H(n, m)$ thus have length $\frac{1}{2} m^n (m - 1) n$.

Our main results are summarized in the following theorem, where, in all cases, if $A$ is a matrix and $p$ a prime, $C_p(A)$ denotes the row span of $A$ over $\mathbb{F}_p$.

**Theorem 1.** Let $G_n(m)$ be an $m^n \times \frac{1}{2} m^n (m - 1) n$ incidence matrix for the Hamming graph $H(n, m)$. Let $L(H(n, m))$ denote the line graph of $H(n, m)$, $A_n(m) a \frac{1}{2} m^n (m - 1) n \times \frac{1}{2} m^n (m - 1) n$ adjacency matrix for $L(H(n, m))$, $L_n(m)$ a $\frac{1}{2} m^n (m - 1) n \times \frac{1}{2} m^n (m - 1) n - 1$ incidence matrix for $L(H(n, m))$, and $f_n(m)$ a $\frac{1}{2} m^n (m - 1) n (m - 1) n - 1 \times \frac{1}{2} m^n (m - 1) n (m - 1) n - 3$ incidence matrix for $L(L(H(n, m)))$. Then
1. For $n \geq 1$, $m \geq 3$, $C_2(G_n(m)) = \left[ \frac{1}{2} m^p(m-1)n, m^n-1, (m-1)n \right]_p$ and $C_p(G_n(m)) = \left[ \frac{1}{2} m^p(m-1)n, m^n, (m-1)n \right]_p$ for $p$ odd. For $m = 2$, $p$ any prime, $C_p(G_n(2)) = \left[ 2^{n-1}n, 2^n-1, n \right]_p$.

For $n \geq 2$, all $p$ and $m \geq 3$, and for $n \geq 3$ and $m = 2$, the minimum words are the non-zero scalar multiples of the rows of $G_n(m)$.

For $n \geq 2$, $C_2(G_n(m))$ has minimum weight 3 for $m \geq 3$; $C_p(G_n(m))$ has minimum weight 4 for $p$ odd, any $m$, and for $p = 2 = m$.

2. For $m$ odd, $C_2(A_n(m)) = C_2(G_n)$. For $m$ even, $C_2(A_n(m))$ is the subcode of $C_2(G_n)$ spanned by the differences of the rows of $G_n$.

For $p$ odd, $C_p(A_n(m))$ has minimum weight at most 4.

3. For $n \geq 3$, $C_2(L_n(2)) = \left[ 2^{n-1}(n-1), 2^{n-1}n-1, 2(n-1) \right]_2$ and $C_p(L_n(2)) = \left[ 2^{n-1}n, 2^{n-1}n, 2(n-1) \right]_p$ for $p$ odd. The minimum words are the scalar multiples of the rows of $L_n(2)$.

For all $p$, $C_p(L_n(2))$ has words of weight 4, and for $p = 2$ it has minimum weight 3.

4. For $n \geq 3$, $C_2(j_n(2)) = \left[ 2^{n-1}(n-1)(2n-3), 2^{n-1}(n-1)-1, 2(2n-3) \right]_2$ and $C_p(j_n(2)) = \left[ 2^{n-1}(n-1)(2n-3)-3, 2^{n-1}(n-1)-2(2n-3) \right]_p$ for $p$ odd. For all $p$ the minimum words are the scalar multiples of the rows of $j_n(2)$.

The automorphism group of these graphs and codes is $S_m \ltimes S_n$. In its action of degree $\frac{1}{2}m^p(m-1)n$ on the edge set of $H(n, m)$, any transitive subgroup of it can be used for full permutation decoding using the code $C_p(G_n(m))$ and any information set.

A transitive subgroup of $S_m \ltimes S_n$ of order $m^p(m-1)n$ is described in Section 9.

We have used Magma [2, 4] for computations for small values of the parameters in order to get an idea of what general results may hold. The proofs of the various parts of the theorem will follow from propositions in the following sections. We give some background definitions and notation in Sections 2 and 3. The result for permutation decoding is in Section 9. Finally, in Section 10 we discuss briefly these ideas applied to the graphs $H^p(n, m)$ from the Hamming association scheme.

2. Background and terminology

The notation for designs and codes is as in [1]. An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{E})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{E}$ is a $t$-$(v, k, \lambda)$ design, if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely $k$ points, and every $t$ distinct points are together incident with precisely $\lambda$ blocks. The design is symmetric if it has the same number of points and blocks. The code $C_2(\mathcal{D})$ of the design $\mathcal{D}$ over the finite field $F$ is the space spanned by the incidence vectors of the blocks over $F$. If $Q \subseteq \mathcal{P}$, then we will denote the incidence vector of $Q$ by $v^Q$, and if $Q = \{P \} \in \mathcal{P}$ where $P \in \mathcal{P}$, then we will write $v^P$ instead of $v^\{P\}$. Thus $C_2(\mathcal{D}) = (v^P \mid B \in \mathcal{B})$, and is a subspace of $F^\mathcal{P}$, the full vector space of functions from $\mathcal{P}$ to $F$. For any $w \in F^\mathcal{P}$ and $P \in \mathcal{P}$, $w(P)$ denotes the value of $w$ at $P$. If $F = \mathbb{F}_p$, then the $p$-rank of the design, written $\text{rank}_p(\mathcal{D})$, is the dimension of its code $C_2(\mathcal{D})$, which we usually write as $C_p(\mathcal{D})$.

All the codes here are linear codes, and the notation $[n, k, d]_q$ will be used for a $q$-ary code $C$ of length $n$, dimension $k$, and minimum weight $d$, where the weight $wt(v)$ of a vector $v$ is the number of non-zero coordinate entries. The support, $\text{Supp}(v)$, of a vector $v$ is the set of coordinate positions where the entry in $v$ is non-zero. So $|\text{Supp}(v)| = wt(v)$. The distance $d(u, v)$ between two vectors $u, v$ is the number of coordinate positions in which they differ, i.e., $wt(u \oplus v)$. A generator matrix for $C$ is a $k \times n$ matrix made up of a basis for $C$, and the dual code $C^\perp$ is the orthogonal under the standard inner product ( ), i.e., $C^\perp = \{ v \in F^n \mid (v, c) = 0 \text{ for all } c \in C \}$. A check matrix for $C$ is a generator matrix for $C^\perp$. The all-one vector will be denoted by $1_n$, and is the vector with all entries equal to 1. If we need to specify the length $m$ of the all-one vector, we write $1_m$. A constant vector is one whose only non-zero entries are 1. Two linear codes of the same length and over the same field are isomorphic if they can be obtained from one another by permuting the coordinate positions. An automorphism of a code $C$ is an isomorphism from $C$ to $C$. The automorphism group will be denoted by Aut($C$). Any code is isomorphic to a code with generator matrix in so-called standard form, i.e., the form $[I_k \mid A]$; a check matrix then is given by $[-A^T \mid I_{n-k}]$. The set of the first $k$ coordinates in the standard form is called an information set for the code, and the set of the last $n-k$ coordinates is the corresponding check set.

The graphs, $\Gamma = (V, E)$ with vertex set $V$ and edge set $E$, discussed here are undirected with no loops. If $x, y \in V$ and $x$ and $y$ are adjacent, we write $[x, y]$ for the edge in $E$ that they define. A graph is regular if all the vertices have the same valency. An adjacency matrix $A$ of a graph with $N$ vertices is an $N \times N$ matrix with entries $a_{ij}$ such that $a_{ij} = 1$ if vertices $v_i$ and $v_j$ are adjacent, and $a_{ij} = 0$ otherwise. An incidence matrix of $\Gamma = (V, E)$ is a $|V| \times |E|$ matrix $B$ with rows labelled by the vertices and columns by the edges and entries $b_{ij} = 1$ if the vertex labelled by row $i$ is on the edge labelled by column $j$, and $b_{ij} = 0$ otherwise. If $\Gamma$ is regular with vertex $v$, then the $1-((|E|, k, 2)$ design with incidence matrix $B$ is called the incidence design of $\Gamma$. The neighbourhood design of a regular graph is the 1-design formed by taking the points to be the vertices of the graph and the blocks to be the sets of neighbours of a vertex, for each vertex, i.e., an adjacency matrix as an incidence matrix for the design. The line graph of a graph $\Gamma = (V, E)$ is the graph $L(\Gamma)$ with $E$ as vertex set and where adjacency is defined so that $e$ and $f$ in $E$, as vertices, are adjacent in $L(\Gamma)$ if $e$ and $f$ as edges of $\Gamma$ share a vertex in $\Gamma$. The code of a graph $\Gamma$ over a finite field $F$ is the row span of an adjacency matrix $A$ over the field $F$, denoted by $C_F(\Gamma)$ or $C_F(A)$. The dimension of the code is the rank of the matrix over $F$, also written $\text{rank}_p(A)$ if $F = \mathbb{F}_p$, in which case we will speak of the $p$-rank of $A$ or $\Gamma$, and write $C_p(\Gamma)$ or $C_p(A)$ for the code. It is also the code over $\mathbb{F}_p$ of the neighbourhood design. Similarly, if $B$ is an incidence
matrix for \( \Gamma \), \( C_\rho(B) \) denotes the row span of \( B \) over \( \mathbb{F}_\rho \) and is the code of the design with blocks the rows of \( B \), in the case that \( \Gamma \) is regular. If \( M \) is an adjacency matrix for \( L(\Gamma) \) where \( \Gamma \) is regular of valency \( k \), with \( N \) vertices, \( e \) edges, then

\[
BB^T = A + kl_N \quad \text{and} \quad B^T B = M + 2I_e. \tag{1}
\]

**Permutation decoding**, first developed by MacWilliams [16], involves finding a set of automorphisms of a code called a PD-set. The method is described fully in MacWilliams and Sloane [17, Chapter 16, p. 513] and Huffman [5, Section 8]. In [10,15] the definition of PD-sets was extended to that of s-PD-sets for s-error-correction:

**Definition 1.** If \( C \) is a \( t \)-error-correcting code with information set \( I \) and check set \( C \), then a PD-set for \( C \) is a set \( \delta \) of automorphisms of \( C \) which is such that every \( t \)-set of coordinate positions is moved by at least one member of \( \delta \) into the check positions \( C \).

For \( s \leq t \) an \( s \)-PD-set is a set \( \delta \) of automorphisms of \( C \) which is such that every \( s \)-set of coordinate positions is moved by at least one member of \( \delta \) into \( C \).

The algorithm for permutation decoding is given in [5] and requires that the generator matrix is in standard form. Furthermore, there is a combinatorial lower bound for \(|\delta|\); see [9,18], or [5].

### 3. Hamming graphs

The Hamming graph \( H(n, m) \), for \( n, m \) integers, is the graph with vertices the \( m^n \)-n-tuples of \( R^n \), where \( R \) is a set of size \( m \) which will take to be a ring with identity (in particular \( \mathbb{Z}_m \) or \( \mathbb{F}_m \) when \( m \) is a prime power) and adjacency defined by two \( n \)-tuples being adjacent if they differ in one coordinate position. It is a regular graph of valency \((m - 1)n\) and \( \frac{1}{2}m^n(m - 1)n \) edges. Edges will be denoted \([x, x + e] \) where \( x, e \in R^n \) and \( wt(e) = 1 \). As usual we will denote the standard basis for \( R^n \) by \( e_1, \ldots, e_n \), so \( e_1 = (1, 0, \ldots, 0) \), for example. It is well known that \( Aut(H(n, m)) = S_m \wr S_n \) (see [3]), where \( S_n \) is the symmetric group on the \( n \) coordinate positions of \( R^n \), now acting naturally on the \( n \)-tuples, and \( S_m \) acts on the elements of \( R \). Thus by Whitney [19], \( Aut(L(H(n, m))) = S_m \wr S_n \) where \( L(H(n, m)) \) is the line graph of \( H(n, m) \).

An incidence matrix for \( H(n, m) \) will be denoted by \( G_\alpha(m) \) and the 1-design defined by taking for points the set \( \mathcal{P}_1 \) of edges of \( H(n, m) \) and for blocks the rows of \( G_\alpha(m) \), will be denoted by \( g_\mu(m) \). Thus if \( x \in R^n \), it defines the block \( \bar{x} \), where

\[
\bar{x} = \{[x, x + e] : e \in R^n, wt(e) = 1\}. \tag{2}
\]

Thus \( g_\mu(m) \) is a 1-(\( \frac{1}{2}m^n \times (m - 1)n \), \( (m - 1)n \)) design.

The following general result about the automorphism groups holds:

**Lemma 1.** Let \( \Gamma = (V, E) \) be a regular graph with \(|V| = N, |E| = e \) and valency \( v \). Let \( \overline{g} \) be the 1- \((e, v, 2) \) incidence design from an incidence matrix \( G \) for \( \Gamma \). Then \( Aut(\Gamma) = Aut(\overline{g}) \).

**Proof.** Denote the set of points of \( \overline{g} \) by \( \mathcal{P} = \{E\} \) and the blocks of \( \overline{g} \) by \( \mathcal{B} \). Thus for \( P \in \mathcal{V}, \mathcal{P} \) is the set of edges through \( P \).

First suppose \( \alpha \in Aut(\Gamma) \). Then \( \alpha \) maps edges to edges so it acts naturally on \( \mathcal{P} \). Naturally its action on \( \mathcal{B} \) is defined so that \( (\mathcal{P} \alpha) = \mathcal{P} \alpha = \mathcal{P} \). So if \( X \in \mathcal{P} \), then \( X = [P, Q], X^{\alpha} = [P^{\alpha}, Q^{\alpha}] \), so \( X^{\alpha} = (\mathcal{P} \alpha) = \mathcal{P} \alpha = \mathcal{P} \). So \( \alpha \in Aut(\overline{g}) \).

Now suppose \( \alpha \in Aut(\overline{g}) \). Then \( \alpha \) acts on \( \mathcal{P} \), the edges of \( \Gamma \), and on the blocks \( \mathcal{B} \), so that for \( X = E, X^{\alpha} \in \mathcal{B} \). We need to define \( \alpha \) to act on vertices \( P \in \mathcal{V} \), and we do this by defining \( P^{\alpha} = Q \) if \( (\mathcal{P} \alpha = \mathcal{P} \). If \( [P, Q] \in \mathcal{B} \) then we need to show that \( [P^{\alpha}, Q^{\alpha}] \in \mathcal{B} \). Now \( [P, Q] \in \mathcal{P}, \mathcal{B} \), so \( [P, Q] = [P^{\alpha}, Q^{\alpha}] \). So \( [P, Q] = [P^{\alpha}, R] = [Q, S] \). Since \( P \neq Q \), so that \( P^{\alpha} \neq Q^{\alpha} \), we must have \( [P, Q] = [P^{\alpha}, Q^{\alpha}] \), so that \( P^{\alpha} \) and \( Q^{\alpha} \) are together on an edge, and hence \( \alpha \in Aut(\Gamma) \). ■

This shows that \( Aut(g_\mu(m)) = S_m \wr S_n \). Note that this is in contrast to the automorphism group of the neighbourhood design, which may well be larger; see [7, Proposition 3], where \( m = 2 \) and the group is larger, and [8, Proposition 3], where \( m 
eq 2 \) and the group is the same as that of the graph.

For \( H(n, m) \) we assume a natural ordering on the elements of \( R \). An \( m^n \times \frac{1}{2}m^n(m - 1)n \) incidence matrix \( G_\alpha(m) \) can be written in the following way: take the natural ordering of the rows corresponding to the \( m \)-ary representation of the natural numbers from 0 to \( m^n - 1 \), and divide the rows into \( m \) sections \( R_i, i = 0, \ldots, m - 1 \), where the rows in \( R_i \) are labelled by the vectors \( x = (x_1, \ldots, x_n) \) with \( x_n = i \). The columns are ordered so that we first take all the edges between vertices in the rows from \( R_0 \), followed by those edges between vertices from rows \( R_1 \), up to all those in \( R_{m - 1} \). Then take all edges between vertices in rows \( R_0 \) and \( R_1 \), then \( R_0 \) and \( R_2 \), up to \( R_0 \) and \( R_{m - 1} \), then \( R_2 \) and \( R_3 \), and so on inductively until finally \( R_{m - 2} \) and \( R_{m - 1} \). Thus an incidence matrix \( G_\alpha(m) \) will have the following form:

\[
G_\alpha(m) =
\begin{bmatrix}
G_{n-1}(m) & 0 & 0 & 0 & \cdots & I & I & I & \cdots & 0 & 0 & 0 \\
0 & G_{n-1}(m) & 0 & 0 & \cdots & I & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & G_{n-1}(m) & 0 & \cdots & I & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & G_{n-1}(m) & 0 & 0 & \cdots & 0 & f & I \\
\end{bmatrix}.
\]
where \( I = I_{m-1} \) and there are \( m - 1 \) of them in each of the \( m \) sets \( R_i \) of rows. The columns are also in blocks, and there are \( \binom{m+1}{2} \) of them. There are \( m \) column blocks \( G_i \) for which the only entry is \( G_{n-1} \), and these are the first \( m \) column blocks, \( C_1, \ldots, C_m \).

For example, for \( m = 2 \) and 3,

\[
G_2(2) = \begin{bmatrix}
G_{n-1}(2) & 0 & I \\
0 & G_{n-1}(2) & I
\end{bmatrix},
\quad G_n(3) = \begin{bmatrix}
G_{n-1}(3) & 0 & 0 & I & I & 0 \\
0 & G_{n-1}(3) & 0 & I & 0 & I \\
0 & 0 & G_{n-1}(3) & 0 & I & I
\end{bmatrix},
\]

where \( I = I_{2n-1} \) in \( G_2(2) \) and \( I = I_{2n-1} \) in \( G_3(3) \).

A general observation about the codes from incidence matrices is in the following lemma:

**Lemma 2.** Let \( \Gamma = (V, E) \) be a graph such that \( V = V_1 \cup V_2 \) (disjoint union) and every \( x \in V_1 \) is on an edge with some \( y \in V_2 \). Let \( \Gamma_1 = (V_1, E_1) \) where \( E_1 \) is the set of edges between vertices in \( V_1 \). If \( G \) is an incidence matrix for \( \Gamma, \Gamma_1 \) an incidence matrix for \( \Gamma_1, p \) any prime, then \( \text{rank}_p(G) \leq \text{rank}_p(G_1) + |V_2| \).

**Proof.** We order the rows and columns of \( G \) as follows: for the rows we take \( V_1 \) followed by \( V_2 \). For the columns we first take all the edges between members of \( V_1 \), and then follow with edges between \( V_1 \) and \( V_2 \), and finally edges between vertices in \( V_2 \). Then

\[
G = \begin{bmatrix} G_1 & X & 0 \\ 0 & Y & Z \end{bmatrix},
\]

where \( Y \) has only one entry 1 in each column and there are at least \( |V_2| \) such columns, so the rank of the second set of rows is at least \( |V_2| \), and clearly also at most \( |V_2| \). The top set has rank at least that of \( G_1 \). \( \square \)

The following general result for connected graphs is from [14]:

**Result 1.** Let \( \Gamma = (V, E) \) be a graph, \( G \) an incidence matrix for \( \Gamma, C_p(G) \) the row-span of \( G \) over \( \mathbb{F}_p \). If \( \Gamma \) is connected then \( \dim(C_p(G)) = |V| - 1 \), and if \( \Gamma \) is connected and has a closed path of odd length \( \geq 3 \), then \( \dim(C_p(G)) = |V| \) for odd \( p \).

### 4. Codes from an incidence matrix for \( H(n, m) \)

In this section we consider the codes \( C_p(G_n(m)) \). All the notation will be as defined in Section 3. We take \( R \) to be a ring, in fact usually \( \mathbb{Z}_m \), or a field if \( m \) is a prime power.

We first need a lemma.

**Lemma 3.** Let \( \Gamma \) be a graph, \( G \) an incidence matrix for \( \Gamma \), and \( (P, Q, R, S) \) a closed path in \( \Gamma \). For any prime \( p \) if \( C = C_p(G) \), then

\[
u = v^{[P,Q]} + v^{[R,S]} - v^{[P,S]} - v^{[Q,R]} \in C^⊥.
\]

In particular, for \( p \) any prime, \( m \geq 2 \), then for \( n \geq 2 \), \( C_p(G_n(m)) \) contains the weight-4 word

\[
u(x, x + e, x + f) = v^{[x,x+e]} - v^{[x,x+f]} - v^{[x+e,x+e+f]} + v^{[x+e,x+e+f]} \quad \text{for} \quad x \in R^n, wt(e) = wt(f) = 1, e \neq f. \]

where \( x \in R^n \), \( wt(e) = wt(f) = 1, e \neq f. \) Further, \( C_p(G_n(m)) \) has minimum weight 4 for \( p \) odd, any \( m \), and for \( p = 2 = m \); \( C_2(G_n(m)) \) has minimum weight 3 for \( m \geq 3 \).

**Proof.** For the first statement, note that it is clear that \( (u, r) = 0 \) for any row \( r \) of \( G \). Then note that \( (x, x + e, x + e + f, x + f) \) is a closed path in \( H(n, m) \) for \( n \geq 2 \). It is easy to verify that \( C^⊥ \) cannot contain vectors of weight 2. Vectors of weight 3 can only occur in the case \( p = 2, m \geq 2 \) and will have the form \( v^{[x,x+\alpha \epsilon]} + v^{[x,x+\beta \epsilon]} + v^{[x+\alpha \epsilon,x+\beta \epsilon]} \), where \( wt(e) = 1, \alpha, \beta \in R \), and \( \epsilon \neq 0 \).

We prove our results for \( m = 2 \) separately as these have already been studied for \( p = 2 \) (see [6]), and we can use a different proof in this case.

**Proposition 1.** Let \( G_2(2) \) be a \( 2^m \times 2^{n-1} \) incidence matrix for \( H(n, 2) \).

For \( p \) any prime, \( C_p(G_n(2)) = \mathbb{Z}_p \). For \( n \leq 3 \) the minimum words are the scalar multiples of the rows of \( G_n \) and \( C_p(G_n(2)) \) is spanned by the weight-4 vectors of Eq. (5).

For \( n \geq 3 \), \( \text{Aut}(C_p(G_n(2))) = S_2 \times S_n \), where \( T_n \) is the translation group on \( \mathbb{F}_2^n \).

**Proof.** For the dimension, it is clearly true for \( n = 1 \). Assume it is true for \( n - 1 \), where \( n \geq 2 \). Then it easily follows from Eq. (4) that the rank of \( G_n \) is \( 2^n - 1 \), by induction.

Write \( C = C_p(G_n(2)) \). To prove the statement concerning the minimum weight, we show that the weight of a word in the dual to \( C^⊥ \) must be at least \( n \) by examining combinatorial properties of the supports of the weight-4 vectors. Let \( \mathcal{B}_n \) be the set of supports of the vectors \( u(a, b, c) \) as defined in Eq. (5). Then \( \mathcal{B}_n \) is a \( 1 \times (2^{n-1} - 1, 4, r) \) design, where \( r = (n - 1) \), since the blocks contain \( \{x, x + e_i\} \) are \( u(x, x + e_i, x + e_j) \) where \( i \neq j \). Furthermore, any two points are together on one or no blocks, since two points determine the block.
Let $w \in C$ and $\text{Supp}(w) = \delta$, where $|\delta| = s$. Let $P = [0, e_1] \in \delta$. Suppose that in $\delta$ there are $k$ points of the type that are on a block with $P$, and $\ell$ that are not. Then $s = k + \ell + 1$. Counting blocks of $\mathcal{B}_n$ through the point $P$, suppose that there are $z_i$ that meet $\delta$ in $i$ points. Then $z_0 = z_1 = z_0 = 0$ for $i \geq 5$, since $w$ cannot meet a block of $\mathcal{B}_n$ only once. Thus $r = z_2 + z_3 + z_4$ and $z_2 + 2z_3 + 3z_4 = k = s - 1 - \ell$. Thus $r = n - 1 \leq z_2 + 2z_3 + 3z_4 \leq (s - 1)$ so $s \geq n$. Since there are vectors of weight $n$, this is the minimum weight. Suppose $s = n$. Thus the inequalities above are equalities and so $z_2 = z_4 = 0$, and $\ell = 0$. Thus $\delta$ consists of $[0, e_1]$ and points of the form $[0, e_1], [e_1, e_1 + e_2], [e_1, e_1 + e_3]$, and each block of $\mathcal{B}_n$ meets $\delta$ exactly twice, since this argument applies to any point of $\delta$. Furthermore, since $\ell = 0$ for each point of $\delta$, any two points of $\delta$ are on a block. This implies that $s = 0$ or $e_1$, and since $n$ is the minimum weight, $w = uu_0$ or $aa_1$.

That the vectors of weight 4 span $C_1$ follows in the same way for the binary codes, as proved in [6, Proposition 15], by finding $\dim(C_1)$ linearly independent weight-4 vectors of this form.

For the statement regarding the automorphism group of $C_p(G_n(2))$, we know that for $n \geq 3$ the words of weight $n$ are the scalar multiples of the rows of $G_n(2)$, i.e. of the incidence vectors of the blocks of $g_p(n)(2)$, and since any automorphism of the code must preserve weight classes, we see that the blocks of the design are preserved, and thus we have an automorphism of the design. Now use Lemma 1. ■

**Note.** The group $S_2 \wr S_n = T \times S_n$ is known as the generalized symmetric group.

Now we take $m \geq 3$. Notice that $H(1, m) = K_m$, where $K_m$ is the complete graph on $m$ vertices, and thus $G_1(m)$ is an $m \times \binom{m}{2}$ incidence matrix for $K_m$. This matrix is written as $M_m$ in [12] where it is proved that, writing

$$G_1(m) = M_m = \begin{bmatrix} M_{m-1} & I_{m-1} \\ 0 \cdots 0 \\ 1 \cdots I \end{bmatrix},$$

for $m \geq 3$, where $M_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $\dim(C_p(M_m)) = m$ for $m \geq 3$ and $p$ an odd prime, and that for $m \geq 4$ the minimum weight of $C_p(M_m)$ is $m - 1$ for $m \geq 4$. It is well known that $\dim(C_2(M_m)) = m - 1$, the minimum weight of $C_2(M_m)$ is $m - 1$, and the words of weight $m - 1$ are the rows of $M_m$. We need these facts in the proposition below, where we exclude the case $m = 2$ since this has been examined in Proposition 1 and elsewhere (see [6]).

**Proposition 2.** Let $G_n(m)$ be an $m^n \times \frac{1}{2}m^n(m - 1)n$ incidence matrix for $H(n, m)$.

For $n \geq 2$, $p$ odd, $m \geq 3$,

- $C_p(G_n(m)) = \left[ \frac{1}{2}m^n(m - 1)n, m^n, (m - 1)n \right]_{p}$;
- $C_2(G_n(m)) = \left[ \frac{1}{2}m^n(m - 1)n, m^n - 1, (m - 1)n \right]_{2}$.

For all $p$, the minimum vectors are the non-zero scalar multiples of the rows of $G_n(m)$. For $n \geq 2$, $\text{Aut}(C_p(G_n(m))) = S_m \wr S_n$.

**Proof.** Recall that $\Gamma = H(n, m)$ has $m^n$ vertices, valency $(m - 1)n$ and $\frac{1}{2}m^n(m - 1)n$ edges. We take $G_n(m)$ as written in Eq. (3).

The statement about the dimension of the codes is clear since $H(n, m)$ is clearly connected so we can use Result 1 immediately for $p = 2$, and also for $p$ odd since $(0, e_1, ae_1)$ for $a \neq 0, 1$ is a closed path of length 3.

Each $w \in C_p(G_n(m))$ is written as a concatenation of vectors $w_i$, for $1 \leq i \leq \binom{m+1}{2}$, where $w_i$ has length $\frac{1}{2}m^n(m - 1)(n - 1) i(n - 1)$ if $1 \leq i \leq m$ and length $m^{n - 1}$ for $i > m$.

**Case (i); $p$ odd**

For $p$ odd, we need to look first at $m = 3$, since $C_p(G_1(3))$ is the full space $F_2^3$, of minimum weight 1. So we consider $G_2(3)$ in order to establish an induction base.

$$G_2(3) = \begin{bmatrix} G_1(3) & 0 & 0 & 1 & 1 \\ 0 & G_1(3) & 0 & 1 & 0 \\ 0 & 0 & G_1(3) & 1 & 1 \end{bmatrix}, \quad \text{where} \quad G_1(3) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad I = I_3.$$

First observe that the sum of any two multiples of rows of $G_1(3)$ has weight at least 2, i.e. three rows are needed to get weight 1 from $G_1(3)$. We will show that the minimum weight of $C_p(G_2(3))$ is $(m - 1)n = 4$ and that the minimum words are the scalar multiples of the rows of $G_2(3)$.

As mentioned before, we label the column blocks as $C_i$ for $i = 1, \ldots, 6$. We write $w \in C_p(G_2(3))$ as $w = (w_1, w_2, w_3, w_4, w_5, w_6)$ where $w_i$ is that part of $w$ in the column block $C_i$. If two or three rows from $R_0$ are taken then we get a vector of weight at least 6, and similarly for $R_1, R_2$. Similarly if at least one row from each of $R_0$ and $R_1$ are taken, and similarly for the other pairs. Now take $w$ a sum of rows from each of $R_0, R_1, R_2$. Let $r_i = 1, 2, 3$ denote the rows of $G_1(3)$, and $r_i$ for $i = 1, \ldots, 9$ the rows of $G_2(3)$. If $w = \sum_{i=1}^3 a_i r_i + \sum_{i=1}^3 \beta_i r_{i+3} + \sum_{i=1}^3 \gamma_i r_{i+6}$ then $w_1 = \sum_{i=1}^3 a_i r_i, w_2 = \sum_{i=1}^3 \beta_i r_i, w_3 = \sum_{i=1}^3 \gamma_i r_i, w_4 = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \alpha_3 + \beta_3), w_5 = (\alpha_1 + \gamma_1, \alpha_2 + \gamma_2, \alpha_3 + \gamma_3), w_6 = (\beta_1 + \gamma_1, \beta_2 + \gamma_2, \beta_3 + \gamma_3)$. Clearly $w(tw_i) \geq 1$ for $i = 1, 2, 3$ (since the rows of $G_1(3)$ are linearly independent); if $w(tw_4) < 3$ then $a_1 = -\beta_1$, say, and $\beta_1 + \gamma_1$ and $\alpha_1 + \gamma_1 (= -\beta_1 + \gamma_1)$ are not both zero unless $\alpha_1 = \beta_1 = \gamma_1 = 0$ in which case $w(tw_i) \geq 2$ for $i = 1, 2, 3$.


and so $\text{wt}(w) \geq 6$. So we have $\text{wt}(w) \geq 4$. If we have equality then, since either $w_2$ or $w_5$ is not zero, we have $w_4 = 0$ and the same argument will yield that $\alpha_2 = \beta_2 = \gamma_2 = 0$, giving a contradiction. Thus we have the result for $m = 3$ and $n = 2$. We can now turn to the general value of $m \geq 3$, $p$ odd. To establish the induction base, we know that for $m \geq 4$ the minimum weight of $G_1(m)$ is $m - 1$. So we can start our induction at $n = 2$ for $m = 3$ and at $n = 1$ for $m \geq 4$. Suppose the assertion is true for $n - 1$. For the minimum weight, we have $w = (w_1, \ldots, w_m) \in C_p(G_n(m))$ where $N = (m + 1)$. We consider combinations of rows from $R_i$ for $i = 0, \ldots, m - 1$. If $w$ is a sum of $k \geq 2$ rows from $R_0$, then $\text{wt}(w) = \text{wt}(w_1) + k(m - 1) \geq (m - 1)(n - 1) + k(m - 1) = (m - 1)n + (m - 1)(k - 1) > (m - 1)n$. Similarly for all the $R_i$, $i = 1, \ldots, m - 1$. If $w$ is a combination of rows from at least two blocks $R_i$, then the components from the $G_{m-1}(m)$ will provide weight at least $2(m - 1)(n - 1) > (m - 1)n$ unless $n = 2$ and so $m \geq 4$ and we are taking just two blocks of rows, so that $\text{wt}(w) \geq 2(m - 1)(n - 1) + 2(m - 2)$, since the blocks containing $f_{m-1}$ overlap in only one column block. Thus $\text{wt}(w) > (m - 1)n$ in this case too. This completes the proof for $p$ odd.

Case (ii): $p = 2$

For $p = 2$, note that any $m^n - 1$ rows of $G_n(m)$ are linearly independent. We can proceed directly by induction, supposing the assertion is true for $n = 1$, from known results concerning $M_m$. Suppose it is true for $n - 1$, where $n - 1 \geq 1$. A sum of rows from $R_0$ will yield zero from the component in $G_{n-1}(m)$ only if all the rows are taken in the sum, in which case the weight is at least $(m - 1)m^{n-1} - (m - 1)n$ since $m^{n-1} \geq 3^{n-1} > n$ for $n \geq 2$. If $w$ is a combination of rows from at least two blocks $R_i$ and if at least two of the components from $G_{n-1}(m)$ are non-zero then we argue as we did in the case of $p$ odd. So the only case that needs consideration is that case when all the components, or all but one, from the $G_{n-1}(m)$ are zero. If $w$ is a sum of all the rows in $k$ blocks of rows $R_j$, $j = 1, \ldots, k$, where $m > k \geq 2$, then $\text{wt}(w) = k(m - 1)m^{n-1} > (m - 1)n$. Similarly if we take fewer than all the rows from one block $R_j$, but all the rows in some other blocks $R_j$. In all events, $\text{wt}(w) > (m - 1)n$ if more than one row is taken. This completes the proof for $p = 2$.

The statement regarding the automorphism group of $C_p(G_n(m))$ follows as in Proposition 1.  

5. Line graphs

We make a few general observations about line graphs and their associated codes.

Let $\mathcal{G} = (V, E)$ be a regular graph with vertex set $V$, edge set $E$, and $N = |V|, |E| = e$, valency $v$. $L(G)$ denotes the line graph of $G$. Then $e = \frac{1}{2}Nv$. We write $L_1 = L(G)$, and recursively $L_i = L(L_{i-1})$ for $i \geq 2$, and we can write $L_0 = G$. Let $L_1 = (V_1, E_1)$ and write $v_1$ for the valency of $L_1$. Then by definition $|V_1| = |E_{i-1}|$ and $v_1 = 2(v_{i-1} - 1)$ for $i \geq 1$, where $V_0 = V$, $E_0 = E$ and $v_0 = v$. It follows that, for $m \geq 1$,

$$
|V_m| = e \prod_{i=0}^{m-2} (2^i v - 2^{i+1} + 1), \quad |E_m| = e \prod_{i=0}^{m-1} (2^i v - 2^{i+1} + 1), \quad v_m = 2^{(m-1)v - 2^m} + 1.
$$

Example 1. Let $G = H(n, 2) = Q_n$, then $N = 2^n, v = n, e = 2^n - n$. Thus for $m \geq 1$,

$$
|V_m| = 2^{n-1}n \prod_{i=0}^{m-2} (2^i n - 2^{i+1} + 1), \quad |E_m| = 2^{n-1}n \prod_{i=0}^{m-1} (2^i n - 2^{i+1} + 1), \quad v_m = 2^{2^{n-1}n - 2^m + 1}.
$$

For the Hamming graphs in general, $L(H(n, m))$ has $\frac{1}{2}m^n(m - 1)n$ vertices, valency $2((m - 1)n - 1)$ and $\frac{1}{2}m^n(m - 1)n((m - 1)n - 1)$ edges.

We will denote the neighbourhood design of $L(H(n, m))$ by $L_m$, where the block defined by the point $[x, x + e]$ is denoted by $[x, x + e]$ and given by

$$
[x, x + e] = \{[x, x + f] | \text{wt}(f) = 1, f \neq e\} \cup \{[x + e, x + e + f] | \text{wt}(f) = 1, f \neq e\}.
$$

The design $L_m$ is a symmetric 1-$(\frac{1}{2}m^n(m - 1)n, 2, (m - 1)n - 1, 2((m - 1)n - 1))$ design.

Below we note some general observations about the codes from line graphs. From the observation that if $(P_1, P_2, \ldots, P_r)$ is a closed path in $G$, where $r \geq 3$, then $([P_1, P_2], [P_2, P_3], \ldots, [P_r, P_1])$ is a closed path in $L(G)$, so we have immediately from Lemma 3, if $(P, Q, R, S)$ is a closed path in $G$, $G_1$ an incidence matrix for $L(G)$, $p$ any prime, then

$$
$$

From this follows:

Lemma 4. If $G$ is a graph with closed paths of length 4, and $C$ is the $p$-ary code from an incidence matrix for $G$, $p$ any prime, then $C^T$ has minimum weight at most 4. Furthermore, if $G_1$ is an incidence matrix for $L(G)$, then $C_p(G_1)^T$ has minimum weight at most 4, for any $i \geq 1$.

Also from Lemma 3, if $G = (V, E), P \in V$, and $[Q, R, S, T]$ neighbours of $P$, then if $p$ is any prime and $G_1$ an incidence matrix for $L(G)$, it follows that

$$
$$
Recall that if \([P, Q]\) is an edge in \(I^*\) then the block of the symmetric design from the neighbours of each point of the line graph \(L(I^*)\) is denoted by \([P, Q] = ([P, R] \mid R \neq Q) \cup ([R, Q] \mid R \neq P)\).

**Proposition 3.** Let \(I^*\) be a graph and \((P, Q, R, S)\) a closed path in \(I^*\), \(p\) an odd prime. Then

\[
\nu^{[P, Q]} + \nu^{[R, S]} - \nu^{[P, S]} - \nu^{[Q, R]} \in C_p(L(I^*)�)
\]

**Proof.** It can be verified directly that

\[
\nu^{[P, Q]} + \nu^{[R, S]} - \nu^{[P, S]} - \nu^{[Q, R]} = -2(\nu^{[P, Q]} + \nu^{[R, S]} - \nu^{[P, S]} - \nu^{[Q, R]}�)
\]

and this is not 0 since \(p\) is odd. \(\blacksquare\)

6. Codes from an adjacency matrix for \(L(H(n, m))\)

In this section we look at \(C_p(L(H(n, m))�), i.e. codes from an adjacency matrix for the line graph of \(H(n, m)\). We can use our results from Section 4 to study these codes. Here \(G_n = G_n(m)\) will be a \(m^n \times \frac{1}{2}m^n(m-1)n\) incidence matrix for \(H(n, m)\) given in the form of Eq. (3), and \(A_n = A_n(m)\) will be the corresponding \(\frac{1}{2}m^n(m-1)n \times \frac{1}{2}m^n(m-1)n\) incidence matrix for \(L(H(n, m))\). As usual

\[
G_n^T G_n = A_n + 2I_{\frac{1}{2}m^n(m-1)n}.
\]

Further, we write \(G_n = [\varnothing_{n-1}, \mathcal{J}_{n-1}]\), as in Eq. (3), where \(\varnothing_{n-1} = m^n \times \frac{1}{2}m^n(m-1)n\) (1 - 1) and \(\mathcal{J}_{n-1} = m^n \times \frac{1}{2}m^n(m-1)n\) and

\[
\varnothing_{n-1} = \begin{bmatrix}
G_{n-1} & 0 & 0 & \cdots & \cdots & 0
0 & G_{n-1} & 0 & \cdots & \cdots & 0
0 & 0 & G_{n-1} & \cdots & \cdots & 0
\vdots & \vdots & \vdots & \ddots & \cdots & \vdots
0 & 0 & 0 & \cdots & \cdots & G_{n-1}
\end{bmatrix},
\]

\[
\mathcal{J}_{n-1} = \begin{bmatrix}
I & I & \cdots & 0 & 0 & 0
I & 0 & \cdots & 0 & 0 & 0
0 & I & \cdots & 0 & 0 & 0
0 & 0 & I & \cdots & 0 & 0
\vdots & \vdots & \vdots & \ddots & \cdots & \vdots
0 & 0 & 0 & \cdots & I & I
\end{bmatrix},
\]

where \(I = I_{m^n(m-1)}\). There are \(m\) blocks of rows and \(m\) blocks of columns in \(\varnothing_{n-1}, m - 1\) copies of \(I\) in each block of rows of \(\mathcal{J}_{n-1},\) which has \(m\) blocks of rows and \((\binom{m}{2})\) blocks of columns.

We will also write

\[
\mathcal{A}_{n-1} = \begin{bmatrix}
A_{n-1} & 0 & 0 & \cdots & \cdots & 0
0 & A_{n-1} & 0 & \cdots & \cdots & 0
0 & 0 & A_{n-1} & \cdots & \cdots & 0
\vdots & \vdots & \vdots & \ddots & \cdots & \vdots
0 & 0 & 0 & \cdots & \cdots & A_{n-1}
\end{bmatrix},
\]

a symmetric \(\frac{1}{2}m^n(m-1)n - 1 \times \frac{1}{2}m^n(m-1)n - 1\) matrix, partitioned into \(m\) blocks of rows and columns. Then it is easy to verify that, for \(n \geq 2\) and all \(m\), with notation as above,

\[
A_n = \begin{bmatrix}
\mathcal{A}_{n-1} & \mathcal{J}_{n-1}^T \varnothing_{n-1}^{-1} & \mathcal{J}_{n-1}^T \mathcal{J}_{n-1}^{-1}
\end{bmatrix},
\]

where \(\mathcal{J}_{n-1}^T \mathcal{J}_{n-1}^{-1}\) is a symmetric \(m^{-1}(\binom{m}{2}) \times m^{-1}(\binom{m}{2})\) matrix with \(2(m-2)\) entries 1 in every row and column.

**Proposition 4.** For \(n \geq 2\), any \(m \geq 2\), let \(G_n(m)\) be an incidence matrix for \((H(n, m), A_n(m)\) an adjacency matrix for the line graph \(L(H(n, m))\).

For \(m\) odd, \(C_2(A_n(m)) = C_2(G_n(m))\). For \(m\) even, \(C_2(A_n(m))\) is the subcode of \(C_2(G_n(m))\) spanned by the differences of the rows of \(G_n(m)\).

For \(p\) an odd prime, \(C_p(A_n(m))\) has minimum weight at most 4, and for \(n \geq 3\) and \(m = 2\), \(C_p(A_n(2)) \supseteq C_p(G_n(2))\).

**Proof.** We write \(G = G_n(m), A = A_n(m)\). Over \(\mathbb{F}_2\) we have \(G^T G = A\). Thus \(C_2(A) \subseteq C_2(G)\). By **Proposition 2** \(C_2(G)\) has rank \(m^n - 1\). If \(J = J_{m^n}\) is the all-one vector of length \(m^n\) then \(J G = 0\).

Let \(V\) be the row span of \(G^T\) over \(\mathbb{F}_2\) and \(C = C_2(A)\). Then \(\dim V = m^n - 1\). The map \(\tau : V \rightarrow C\) is defined by \(\tau : v \mapsto v \leftrightarrow (v_1, \ldots, v_{m^n}) G\), so that \(V \tau = C\) and \(\dim C + \dim \ker(\tau) = \dim V = m^n - 1\). A vector \(v\) is in the kernel if and only if \(v \in V\) and \(v G = 0\), and since \(J G = 0\), we need determine when \(J \in V\).

Since \(G^T\) is spanned by vectors of weight 2, it is an even-weight binary code. If \(m\) is odd then \(m^n\) is odd and hence \(J_{m^n} \notin V\) and thus \(\dim C = m^n - 1\) and \(C = C_2(G)\). If \(m\) is even then \(m - 1\) is odd and if all the rows of \(\mathcal{J}_{n-1}^T\) are added, we get \(J_{m^n} \in V\). Hence \(\dim(C) = m^n - 2\). Since this is the same as the dimension of the code spanned by the differences of the rows of \(G\), we have the result.
7. Codes from an incidence matrix for $L(H(n, 2))$

We now consider the row span over $\mathbb{F}_p$, for any prime $p$, of an incidence matrix $L_n(2)$ for the line graph $L(H(n, 2))$ of $H(n, 2)$ for $n \geq 1$.

We consider a $2^{n-1}n \times 2^{n-1}n(n-1)$ incidence matrix $L_n(2)$ of $L(H(n, 2))$ with a particular ordering on the rows and columns. Each row of $L_n(2)$ has $2(n-1)$ entries equal to 1, including the case $n = 1$ in which case the line graph has no edges.

Let $G_n = G_n(2)$ as given in Eq. (4), with rows and columns ordered as described in Section 3. Now for the rows of $L_n(2)$ we use the same ordering we had for the columns of $G_n(2)$. We call these sets $R_1, R_2, R_3$. For the columns, we assume we have an ordering for $L_{n-1}(2)$, and for the first four columns for $L_n(2)$ we insert $L_{n-1}(2)$ in the rows $R_i$. For the next set of columns, we insert $L_{n-1}(2)$ in $R_2$ to show the edges $[x + e_n, y + e_n], [x + e_n, z + e_n]$, where $[x, y], [x, z]$ is an edge in $L(H(n, 2))$.

For the next columns we take all edges between those points in $R_1$ and $R_3$, followed by those in $R_2$ and $R_3$. We need to start with $L_2(2)$, and from our ordering for $G_2(2)$ we can order the columns so that $L_2(2)$ is as follows:

$$G_2(2) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad L_2(2) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$  \hspace{1cm} (12)

Thus, representing 0 by the empty space for clarity,

$$L_3(2) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \hspace{1cm} (13)$$

In general, for $n \geq 2$, recalling that for $n = 2$ the matrix $L_{n-1}(2)$ has no columns,

$$L_n(2) = \begin{bmatrix} L_{n-1}(2) & 0 & 0 & X & 0 \\ 0 & L_{n-1}(2) & 0 & W & 0 \\ 0 & 0 & Y & Z & 0 \end{bmatrix},$$  \hspace{1cm} (14)

where $X, W$ are $2^{n-2}(n-1) \times 2^{n-2}(n-1)$ matrices, and $Y, Z$ are $2^{n-1} \times 2^{n-1}(n-1)$ matrices. Further, every column of $X, Y, Z, W$ has exactly one non-zero entry 1 in it, and every row of $X$ and $W$ has precisely two non-zero entries 1 in it, while every row of $Y$ and $Z$ has precisely $n-1$ non-zero entries 1 in it. Clearly every row of $L_{n-1}(2)$ has 2$(n-2)$ entries equal to 1. We label the column blocks $C_i$ for $i = 1, 2, 3, 4$.

We first need a lemma that will be used as our induction base. The notation is as given above.

**Lemma 5.** $C_2(L_3(2)) = [24, 11, 4]_2$ and for $p$ odd $C_p(L_3(2)) = [24, 12, 4]_p$. The vectors of weight 4 are the scalar multiples of the rows of $L_3(2)$ for all $p$.

**Proof.** For $p = 2$ we can simply use Magma to verify this for the binary code. Thus take $p$ odd. It is easy to see that $C_p(L_2(2))$ has dimension 3 and minimum weight 2. However, not all the weight-2 vectors are scalar multiples of the rows. Further, writing the rows of $L_2(2)$ as $r_i$ for $1 \leq i \leq 4$, any three of the rows are linearly independent, and

$$u = \sum_{i=1}^{4} \alpha_i r_i = (\alpha_1 + \alpha_3, \alpha_1 + \alpha_4, \alpha_2 + \alpha_3, \alpha_2 + \alpha_4) = 0$$

only if $\alpha_1 = \alpha_2 = -\alpha_3 = -\alpha_4$.

Using Eq. (13), we label the submatrix of the first four rows of $L_3(2)$ as $R_1$, the next four as $R_2$ and the last four as $R_3$. Let $C = C_p(L_3(2))$. Then any vector $w \in C$ can be viewed as a concatenation of vectors in the four partitions of the columns. So we write $w = (w_1, w_2, w_3, w_4)$ where $w_1$ and $w_2$ are in $\mathbb{F}_p^4$ and $w_3$ and $w_4$ are in $\mathbb{F}_p^8$. Thus $wt(w) = wt(w_1) + wt(w_2) + wt(w_3) + wt(w_4)$.

To show that $C$ has dimension 12, notice that $[0, e_1], [e_1, e_1 + e_2], [e_1 + e_2, e_1 + e_2 + e_3], [e_1 + e_2 + e_3, e_1 + e_3], [e_1 + e_3, e_1]$ is a closed path of odd length 5, so we can use Result 1 since the graph is clearly connected.
To prove the statement about the minimum weight and words we take \( w \) to be a sum of \( k \geq 1 \) non-zero scalar multiples of rows from \( R_1 \). Then \( \text{wt}(w) = \text{wt}(w_1) + 2k \). If \( w_1 = 0 \) then from the above discussion we must have \( k = 4 \) and thus \( \text{wt}(w) = 8 \). If \( w_1 \neq 0 \) then \( \text{wt}(w) \geq 2 + 2k \geq 4 \), with equality only when \( k = 1 \) and we have a multiple of a row. The same argument applies to a sum of rows from \( R_2 \), since \( W \) is equivalent to \( X \). If \( w \) is a sum of \( k \geq 1 \) non-zero scalar multiples of rows from \( R_3 \), then \( \text{wt}(w) = 4k \geq 4 \) with equality only if \( k = 1 \).

If \( w \) is a sum of \( k \geq 1 \) non-zero scalar multiples of rows from \( R_1 \) and \( m \geq 1 \) non-zero scalar multiples of rows from \( R_2 \), then \( \text{wt}(w) = \text{wt}(w_1) + \text{wt}(w_2) + 2k + 2m \geq 2(k + m) \geq 4 \) with equality only if \( k = m = 1 \) and \( w_1 = w_2 = 0 \), and this cannot happen. If \( w \) is a sum of \( k \geq 1 \) non-zero scalar multiples of rows of \( R_1 \) and \( m \geq 1 \) non-zero scalar multiples of rows of \( R_3 \), then \( \text{wt}(w) = \text{wt}(w_1) + \text{wt}(w_3) + 2m \). If \( w_1 = 0 \) then \( k = 4 \) and \( \text{wt}(w) = \text{wt}(w_3) + 2m \geq 4m \) if \( m \geq 3 \). If \( m = 1 \) then \( \text{wt}(w_3) \geq 6 \) since \( w_3 = x(x_1, 1, 1, 1, -1, -1, -1) + \beta y \), where \( y \) is a row of \( Y \) and has weight 2. If \( m = 2 \) then \( w_3 \neq 0 \) so all words we get are of weight greater than 4. So \( \text{wt}(w_1) \geq 2 \), and so \( \text{wt}(w) \geq 4 \) with equality only if \( m = 1 \), \( \text{wt}(w_1) = 2 \) and \( w_3 = 0 \). The latter is impossible since the rows of \( X \) with one row of \( Y \) are linearly independent, where \( X \) and \( Y \) are as shown in Eq. (13). A similar argument applies to \( w \) a sum of rows from \( R_2 \) and \( R_3 \). Thus we cannot get vectors of weight 4 this way.

Finally we take \( w \) to be a sum of \( k \geq 1 \) non-zero scalar multiples of rows from \( R_1 \) and \( j \geq 1 \) non-zero scalar multiples of rows from \( R_2 \) and \( m \geq 1 \) non-zero scalar multiples of rows of \( R_3 \). Then \( \text{wt}(w) = \text{wt}(w_1) + \text{wt}(w_2) + \text{wt}(w_3) + \text{wt}(w_4) \). Denoting the 12 rows of \( L_3 \) by \( r_i \) for \( 1 \leq i \leq 12 \), write \( w = \sum_{i=1}^{4} a_i r_i + \sum_{i=1}^{4} \beta_i r_{4+i} + \sum_{i=1}^{4} \gamma_i r_{8+i} \) where \( k \) of the \( a_i \), \( m \) of the \( \beta_i \) and \( j \) of the \( \gamma_i \) are not zero. From Eq. (13) we see that

\[
\text{wt}(w) = \text{wt}(w_1) + \text{wt}(w_2) + \text{wt}(w_3) + \text{wt}(w_4).
\]

If \( w_3 = 0 \) then \( w_1 = \alpha \gamma_1, \alpha_1 + \gamma_2, \alpha_2 + \gamma_3, \alpha_3 + \gamma_4, \alpha_4 + \gamma_5, \alpha_5 + \gamma_6, \alpha_6 + \gamma_7, \alpha_7 + \gamma_8, \gamma_9 \). From this we see that \( \text{wt}(w) \geq 4 \). Similarly,

\[
\text{wt}(w_3) + \text{wt}(w_4) = \text{wt}(w_1) + \text{wt}(w_2) + \text{wt}(w_3) + \text{wt}(w_4).
\]

If \( w_3 + w_4 = 0 \) then \( \text{wt}(w) \geq 6 \). If \( w_3 + w_4 \neq 0 \) then \( \text{wt}(w) \geq 8 \). As a sum of rows \( w \), we see that \( w_3 + w_4 \neq 0 \). From this we see that \( \text{wt}(w) 
\text{Proposition 5. For } n \geq 1, \text{ let } L_n(2) \text{ be a } 2^{n-1} n \times 2^{n-1} n(n - 1) \text{ incidence matrix for } L(H(n, 2)).

For } n \geq 3,
\begin{itemize}
\item \( C_2(L_n(2)) = [2^{n-1} n(n - 1), 2^{n-1} n - 1, 2(n - 1)_{2}]; \)
\item \( C_p(L_n(2)) = [2^{n-1} n(n - 1), 2^{n-1} n, 2(n - 1)_{p}] \) for } p \text{ odd.
\end{itemize}

For all } p, \text{ the minimum words are the scalar multiples of the rows of } L_n(2).

For } p = 2, n \geq 3, \text{ it has minimum weight } 3. \text{ For } n \geq 3, \text{ Aut}(C_p(L_n(2))) = S_2 \times S_2.
\textbf{Proof. Case (i): } p \text{ odd
We prove the result stated by induction, having established it in Lemma 5 for } n = 3.

Suppose the statement is true for } n = 3, \text{ where } n \geq 4, \text{ and consider } L_n(2) \text{ as shown in Eq. (14). We use the same constructions and labelling as for the proof of the case } n = 3. \text{ Thus } w = (w_1, w_2, w_3, w_4). \text{ Let } C = C_p(L_n(2)). \text{ It is clear that}
\dim(C) = 2^{n-1} n \text{ since the rows from } R_1 \text{ and } R_2 \text{ give dimension } 2^{n-1} (n - 1), \text{ by induction, and those from } R_3 \text{ give dimension } 2^{n-1} \text{ since each column has exactly one entry of } 1 \text{ in it, and each row has at least one entry } 1.

If } w \text{ is a sum of } k \geq 1 \text{ non-zero scalar multiples of rows } r_i \text{ from } R_1, \text{ then } \text{wt}(w) = \text{wt}(w_1) + \text{wt}(w_2) \geq 2(n - 2) + 2k \text{ since } X \text{ has every row containing exactly two entries } 1, \text{ and every column exactly one entry } 1. \text{ If } k \geq 2 \text{ then } \text{wt}(w) \geq 2n, \text{ so we only get weight } 2(n - 1) \text{ when } k = 1. \text{ The same argument holds for } w \text{ a sum of rows in } R_3. \text{ If } w \text{ is a sum of } k \neq 0, \text{ non-zero scalar multiples of rows from } R_3, \text{ then } \text{wt}(w) = 2(n - 1)k \geq 2(n - 1), \text{ (since } Y, Z \text{ each have their rows consisting of } n - 1 \text{ entries equal to } 1, \text{ with equality only if } k = 1 \text{ and } w \text{ is a scalar multiple of a row.}

If } w \text{ is a sum of } k \geq 1 \text{ non-zero scalar multiples of rows from } R_1 \text{ and } m \geq 1 \text{ non-zero scalar multiples of rows from } R_2, \text{ then } \text{wt}(w) = \text{wt}(w_1) + \text{wt}(w_2) + 2k + 2m \geq 2(n - 1). \text{ If } w \text{ is a sum of } k \geq 1 \text{ non-zero scalar multiples of rows from } R_1 \text{ and } m \geq 1 \text{ non-zero scalar multiples of rows from } R_3, \text{ then } \text{wt}(w) = \text{wt}(w_1) + \text{wt}(w_3) + m(n - 1) \geq 2(n - 2) + \text{wt}(w_2) + m(n - 1) \geq 3n - 5 > 2n - 2 \text{ for } n \geq 4. \text{ So no weight } 2(n - 1) \text{ can arise from this combination of rows, and similarly from } R_2 \text{ and } R_3.

Finally we take } w \text{ to be a sum of } k \neq 0 \text{ non-zero scalar multiples of rows from } R_1 \text{ and } j \neq 0 \text{ non-zero scalar multiples of rows from } R_3, \text{ then } \text{wt}(w) = \text{wt}(w_1) + \text{wt}(w_3) + \text{wt}(w_4) \geq 4(n - 2) > 2n - 4 \text{ for } n \geq 4. \text{ This completes the proof for } p \text{ odd.
Case (ii): \( p = 2 \)

Again by Lemma 5 we have the result for \( n = 3 \), so we use induction. So suppose \( n \geq 4 \) and that it is true for \( n - 1 \), and consider \( L_0(2) \) as shown in Eq. (14). We use the same constructions and labelling as before. Thus \( w = (w_1, w_2, w_3, w_4) \). Let \( C = C_0(L_0(2)) \). It is clear that \( \dim(C) \leq 2n-1 - n - 1 \) since the sum of all the rows is 0. Further notice that any \( 2n-1 - n - 1 \) rows of \( L_n(2) \) are linearly independent. By induction the dimension is at least this as the rows from \( R_1 \) give dimension at least \( 2n-2(n - 1) - 1 \), and the those from \( R_2 \) and \( R_3 \) give dimension \( 2n-2(n - 1) - 2n-1 \), respectively, since each column has exactly one entry of 1 in it, and each row has at least one entry 1.

If \( w \) is a sum of \( k \geq 1 \) rows \( r_i \) from \( R_1 \), then \( \text{wt}(w) = \text{wt}(w_1) + \text{wt}(w_3) \geq \text{wt}(w_1) + 2k \) since \( X \) has every row containing exactly two entries 1, and every column exactly one entry 1. If \( w_1 = 0 \) then \( k = 2n-2(n - 1) \) and \( \text{wt}(w) = 2n-1(n - 1) - 2(n - 1) \). Otherwise \( \text{wt}(w) \geq 2(n - 2) + 2k > 2(n - 2) + 2 = 2(n - 1) \) with equality only if \( k = 1 \). The same argument holds for a sum of rows in \( R_2 \). If \( w \) is a sum of \( k \neq 0 \) rows from \( R_2 \), then \( \text{wt}(w) = 2(n - 1)k \geq 2(n - 1) \), (since \( Y, Z \) each have their rows consisting of \( n - 1 \) entries equal to 1), with equality only if \( k = 1 \) and \( w \) is a scalar multiple of a row.

If \( w \) is a sum of \( k \neq 0 \) rows from \( R_1 \) and \( m \neq 0 \) rows from \( R_2 \) then \( \text{wt}(w) = \text{wt}(w_1) + \text{wt}(w_2) + 2k + 2m \). If \( w_1 = 0 \) then \( k = 2n-2(n - 1) \) so \( \text{wt}(w) \geq 2n-1(n - 1) - 2 > 2(n - 1) \). If \( w_2 = 0 \) then \( \text{wt}(w) = 2(n - 2) + 2k \geq 2(n - 2) - 2 = 2(n - 1) \) with equality only if \( k = 1 \). So we can assume neither is 0, in which case \( \text{wt}(w) \geq 4(n - 2) + 2 > 2n - 2 \) for \( n \geq 4 \). Now take \( w \) to be a sum of \( k \geq 1 \) rows from \( R_1 \) and \( m \geq 1 \) rows from \( R_2 \). Then \( \text{wt}(w) = \text{wt}(w_1) + \text{wt}(w_3) + m(n - 1) \). If \( w_1 = 0 \) then \( k = 2n-2(n - 1) \) and \( \text{wt}(w_2) = 2n-1(n - 1) - 2(n - 1) \) otherwise \( \text{wt}(w_1) \geq 2(n - 2) \) and \( \text{wt}(w_2) \geq 2(n - 2) \), since each column has \( n - 1 \) entries equal to 1, with equality only if \( k = 1 \) and \( w \) is a scalar multiple of a row.

Finally we take \( w \) to be a sum of \( k \geq 1 \) rows from \( R_1 \) and \( j \geq 1 \) rows from \( R_2 \) and \( m \geq 1 \) rows from \( R_3 \). Then if \( w_1, w_2 \neq 0 \), \( \text{wt}(w) = \text{wt}(w_1) + \text{wt}(w_2) + \text{wt}(w_3) + \text{wt}(w_4) \) \( \geq 4(n - 2) + 2 + 2n - 2 = 2n - 4 \) for \( n \geq 4 \). If \( w_2 = 0 \) then \( k = 2n-2(n - 1) \) and \( \text{wt}(w_3) = 2n-1(n - 1) - m(n - 1) \) and \( \text{wt}(w_4) \geq 2(n - 2) + 2n-1(n - 1) - m(n - 1) + \text{wt}(w_4) > 2(n - 1) \) unless \( m = 2n-1 \) and \( \text{wt}(w_4) = 0 \). If \( m = 2n-1 \) then \( j = 2n-2(n - 1) + w \) and \( w = 0 \). If \( \text{wt}(w_4) = 2 \) then \( j = 2n-2(n - 1) - 1 \) and \( w \) is the row of \( R_2 \) that was left out of the sum. Finally, if \( w_1, w_2 = 0 \) then \( \text{wt}(w) = 2(n - 2) - (m - 1) > 2(n - 1) \). If \( m < 2n-2(n - 1) \) then \( w \) is the row of \( R_3 \) that was omitted from the sum, and if \( m = 2n-1 \), then \( w = 0 \). This completes all cases and the induction.

For the statement about the dual codes, that the minimum weight is at most 4 follows from Lemma 4. For \( p = 2 \), for any \( x \in \mathbb{R}^n \), the set \([\{x, x + e_1\}, [x, x + e_2]\], \([x, x + e_1], [x, x + e_3]\], \([x, x + e_2], [x, x + e_3]\]) forms the support of a word of weight 3 in the dual code.

The statement about the automorphism group of the code follows from Whitney’s Theorem (see Section 3), Lemma 1, and the proof of the similar statement in Proposition 1. □

**Note.** We will not consider here the general case of an incidence matrix \( L_n(m) \) for \( L(H(n, m)) \) for \( m \geq 3 \), although all indications (for example, with Magma) are that similar results hold.

8. Codes from an incidence matrix for \( L_2(H(n, 2)) \)

Recall that we write \( J_n(m) \) for a \( \frac{1}{2}m^n(m - 1)(m(n - 1) - 1)(m(n - 1) - 1)(m(n - 1) - 1) \) incidence matrix for \( L(L(H(n, m))) = L_2(H(n, m)) \). We restrict our attention to \( m = 2 \), i.e. we consider the graph \( L(H(n, 2)) \). We let \( J_n(2) \) be an incidence matrix for this graph, using the same ordering for the rows as for the columns of \( L_n(2) \), and ordering the columns using the same algorithm as we used for \( L_n(2) \). Then, from Example 1, \( L_2(H(n, 2)) \) has valency \( 4n - 6 \) and \( J_n(2) \) is \( 2n-1(n - 1) \times 2n-1(n - 1) \times (2n - 3) \), of the form

\[
J_n(2) = \begin{bmatrix}
J_{n-1}(2) & 0 & X & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & J_{n-2}(2) & 0 & W & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & Y_1 & 0 & Y_2 & 0 & Y_3 & 0 & 0 \\
0 & 0 & 0 & Z_1 & 0 & Z_2 & Z_3 & 0 & 0
\end{bmatrix},
\]

where \( X, W \) are \( 2n-2(n - 1)(n - 2) \times 2n-2(n - 1)(n - 2) \) matrices; \( Y_1, Z_1 \) are \( 2n-1(n - 1) \times 2n-1(n - 1) \) matrices; \( Y_2, Z_2 \) are \( 2n-1(n - 1) \times 2n-1(n - 1) \) matrices; \( Y_3, Z_3 \) are \( 2n-1(n - 1) \times 2n-1(n - 1) \) matrices. Further, every column of \( X, Y_1, Y_3, Z_1, Z_3 \), \( W \) has exactly one non-zero entry 1 in it, and every column of \( Y_2, Z_2 \) has two non-zero entries 1 in it; every row of \( X \) and \( W \) has precisely four non-zero entries 1 in it; every row of \( Y_1 \) and \( Z_1 \) has precisely \( 2(n - 2) \) non-zero entries 1 in it; every row of \( Y_2 \) and \( Z_2 \) has precisely \( (n - 1) \) non-zero entries 1 in it; every row of \( Y_3 \) and \( Z_3 \) has precisely \( (n - 1) \) non-zero entries 1 in it.

For example, for \( n = 3 \),

\[
J_2(2) = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}, \quad X = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix},
\]
Proposition 6. For \( n \geq 2 \), let \( J_n(2) \) be a \( 2^{n-1}n(n-1) \times 2^{n-1}n(n-1) \) incidence matrix for \( L(L(H(n, 2))) = L_2(H(n, 2)) \).

For \( n \geq 3 \),

- \( C_p(J_n(2)) = [2^{n-1}n(n-1)(2n-3), 2^{n-1}n(n-1) - 1, 2(2n-3)]_2 \);
- \( C_p(J_n(2)) = [2^{n-1}n(n-1)(2n-3), 2^{n-1}n(n-1) - 1, 2(2n-3)]_p \) for \( p \) odd.

For \( n \geq 3 \) and all \( p \) the minimum words are the scalar multiples of the rows of \( J_n(2) \), and \( \text{Aut}(C_p(J_n(2))) = S_2 : S_n \).
Proof. We write $J_n = J_n(2)$. Much of the proof of this mirrors that of Proposition 5.

We proceed by induction, since we have an induction base from Lemma 6. The statement about the dimension of the codes follows immediately. For the minimum weight, we need to separate our proofs for $p$ odd or $p = 2$, since the submatrix $J_{n-1}$ can give $wt(w_1) = 0$ when taking a collection of rows from $R_1$ in the case $p = 2$ but not in the case $p$ odd.

Case (i): $p$ odd

Take $p$ odd and assume we have the result for $n - 1 \geq 3$. Let $w \in C_p(J_n)$. We assume that the minimum weight of $C_p(J_{n-1})$ is $4n - 10$ and the words of this weight are multiples of the rows of $J_{n-1}$.

First we take $w$ to be a sum of $k \geq 1$ non-zero scalar multiples of rows from $R_1$. Then $wt(w) = wt(w_1) + 4k \geq 4n - 10 + 4k \geq 4n - 6$ with equality only if $k = 1$. The same argument applies to a sum of rows from $R_2$, since $W$ is equivalent to $X$. If $w$ is a sum of $k \geq 1$ non-zero scalar multiples of rows from $R_3$ or $R_4$, then $wt(w) = (2(n - 2) + 2(n - 1))k \geq 4n - 6$ with equality only if $k = 1$.

Now take $w$ to be a sum of $k \geq 1$ non-zero scalar multiples of rows from $R_1$ and $m \geq 1$ non-zero scalar multiples of rows from $R_3$. Then $wt(w) = wt(w_1) + wt(w_3) + 2m(n - 1) \geq 4n - 10 + 2(n - 1) > 4n - 6$ for $n \geq 3$. Rows from $R_2$ and $R_4$ behave similarly. If $w$ is a sum of $k \geq 1$ non-zero scalar multiples of rows from $R_1$ and $m \geq 1$ non-zero scalar multiples of rows from $R_4$, then $wt(w) = wt(w_1) + 4k + m(4n - 6) > 4n - 6$, and similarly for $R_2$ and $R_3$. For $k \geq 1$ rows from $R_1$ and $m \geq 1$ rows from $R_2$ we have $wt(w) = wt(w_2) + wt(w_2) + 4k + 4m \geq 2(4n - 10) + 4(k + m) > 4n - 6$; for $k$ from $R_3$ and $m$ from $R_4$, $wt(w) = (2(n - 2) + (n - 1))k + (2(n - 2) + (n - 1)m) + wt(w_7) \geq 6n - 10 > 4n - 6$ for $n > 2$.

Next take $w$ to be a sum of $k \geq 1$ non-zero scalar multiples of rows from $R_1$, $m \geq 1$ from $R_3$, and $\ell \geq 1$ from $R_4$. Then $wt(w) = wt(w_1) + wt(w_3) + (n - 1)m + (3n - 5)\ell$ and $wt(w) \geq 4n - 10 + (n - 1) + (3n - 5) > 4n - 6$ for $n \geq 3$. If $w$ is a sum of a non-zero scalar multiples of $k \geq 1$ rows from $R_1$, $\ell \geq 1$ from $R_4$, and $m \geq 1$ from $R_3$, then $wt(w) = wt(w_1) + wt(w_2) + wt(w) + 2(n - 2) + m\geq 4k \geq 2(4n - 10) + 2n - 2 > 4n - 6$ for $n > 3$.

Finally, if $w$ is a sum of $k \geq 1$ non-zero scalar multiples of rows from $R_1$, $m \geq 1$ from $R_3$, $\ell \geq 1$ from $R_4$, and $r \geq 1$ from $R_4$, then $wt(w) = wt(w_1) + wt(w_2) + wt(w_2) + wt(w_4) + (n - 1)w + (n - 1)r + wt(w_7) \geq 2(4n - 10) + 2(n - 1) > 4n - 6$ for $n > 3$. This completes the proof for $p$ odd.

Case (ii): $p = 2$

For $p = 2$ the argument needs to include the possibility that $w_1$ or $w_3$ is $0$, in which case $w_1$ or $w_3$ is the all-one vector of length $2n(n - 1)(n - 2) > 4n - 6$ for $n \geq 3$. Again this only happens if all the rows from $R_3$ or $R_2$ are taken. Assume we have the result for $n - 1 \geq 3$. Let $w \in C_2(J_n)$. We assume that the minimum weight of $C_2(J_{n-1})$ is $4n - 10$ and the words of this weight are multiples of the rows of $J_{n-1}$. Note that $J_{n-1}$ has $2^n - 2(n - 1) = 2(n - 2)$ rows and that the number of columns of $X$ is $2^n(n - 1) > 4n - 6$ for $n \geq 3$.

First we take $w$ to be a sum of $k \geq 1$ non-zero scalar multiples of rows from $R_1$. Then $wt(w) = wt(w_1) + 4k \geq 4n - 10 + 4k \geq 4n - 6$ with equality only if $k = 1$, unless $w_1 = 0$, in which case $k = 2^n - 2(n - 1) = 2(n - 2)$. Then $wt(w) = 2^n(n - 1) > 4n - 6$ for $n \geq 3$. The same argument applies to a sum of rows from $R_2$, since $W$ is equivalent to $X$. If $w$ is a sum of $k \geq 1$ non-zero scalar multiples of rows from $R_3$ or $R_4$, then $wt(w) = (2(n - 2) + 2(n - 1))k \geq 4n - 6$ with equality only if $k = 1$.

If $w$ is a sum of $k \geq 1$ non-zero scalar multiples of rows from $R_1$ and $m \geq 1$ non-zero scalar multiples of rows from $R_3$ then $wt(w) = wt(w_1) + wt(w_3) + 2m(n - 1) \geq 4n - 10 + 2(n - 1) > 4n - 6$ for $n \geq 3$ if $w_1 \neq 0$.

If $w_1 = 0$ then $k = 2^n - 2(n - 1)$, and if $m = 2$ then $wt(w) \geq 4(n - 1) > 4n - 6$. Thus $m = 1$ and $wt(w_3) = 2^n(n - 1)(n - 2) - 2(n - 2)$, so $wt(w) = 2^n(n - 1)(n - 2) - 2(n - 2) + 2(n - 1) > 4n - 6$ for $n \geq 3$.

If $w$ is a sum of $k \geq 1$ non-zero scalar multiples of rows from $R_1$ and $m \geq 1$ non-zero scalar multiples of rows from $R_4$, then $wt(w) = wt(w_1) + 4k + m(4n - 6) > 4n - 6$, and similarly for $R_2$ and $R_3$. For $k \geq 1$ rows from $R_1$ and $m \geq 1$ from $R_2$ we have $wt(w) = wt(w_1) + wt(w_3) + 4k + 4m \geq 2(4n - 10) + 4(k + m) \geq 4n - 6$ if $w_1, w_2 \neq 0$.

If $w_1 = 0$ then $wt(w_3) = 2^n(n - 1)(n - 2) - 4n - 6$ for $n \geq 3$. This completes the proof.

The statement about the automorphism group of the code follows from Whitney's Theorem (see Section 3), Lemma 1, and the proof of the similar statement in Proposition 1.
9. Permutation decoding

In [11, Lemma 7] the following was proved:

**Result 2.** Let $C$ be a code with minimum distance $d$, $I$ an information set, $E$ the corresponding check set and $\mathcal{P} = I \cup E$. Let $A$ be an automorphism group of $C$, and $n$ the maximum of $|\mathcal{O} \cap I|/|\mathcal{O}|$, where $\mathcal{O}$ is a $A$-orbit. If $s = \min \left( \left\lfloor \frac{n}{2} \right\rfloor - 1, \left\lfloor \frac{n+1}{2} \right\rfloor \right)$, then $A$ is an $s$-PD-set for $C$.

Note that this result is true for any information set. If the group $A$ is transitive then $|\mathcal{O}|$ is the degree of the group and $|\mathcal{O} \cap I|$ is the dimension of the code.

If we take $C = C_p(G_n(m))$, the degree is $\frac{1}{2}m^n(m-1)n$ and the dimension is $m^n - 1$. Certainly the group $S_m \wr S_n$ is transitive, and has transitive subgroups. One of the smallest order should be taken to minimize the size of the PD-set. If this is done then $C_p(G_n(m))$ can be used to its full error-correction capability with this group as PD-set since in all cases $s$ in Result 2 is $\left\lfloor \frac{(m-1)n}{2} + 1 \right\rfloor$.

For example, if $R = \mathbb{Z}_m$, then

$$T_{a_1, \ldots, a_n} : (x_1, \ldots, x_n) \mapsto (x_1 + a_1, \ldots, x_n + a_n)$$

mapping $\mathbb{Z}_m^n$ to itself, and if, for $a \in \mathbb{Z}_m^*$, $\sigma = (1, \ldots, n) \in S_n$, acting as a permutation matrix on $\mathbb{Z}_m^n$, then

$$A = \{ T_x \mid x \in \mathbb{Z}_m^n \{ bo^i \mid b \in \mathbb{Z}_m^* \}, 1 \leq i \leq n \}$$

is a transitive subgroup group of order $m^n(m-1)n$. This follows since if $P = [0, a e_i]$, $Q = [x, x + be_i]$, where $a, b \neq 0$, $a, b \in \mathbb{Z}_m$, then $\tau = a^{-1}be_i T_{e_1, \ldots, e_n}$ will have $Pr = Q$.

We take $m > 2$ here since $m = 2$ has been covered in [6].

**Proposition 7.** For $n \geq 3$, $m \geq 3$ any transitive subgroup of $S_m \wr S_n$ of degree $\frac{1}{2}m^n(m-1)n$ is a PD-set for the code $C_p(G_n(m))$, where $C_2(G_n(m)) = \left[ \frac{1}{2}m^n(m-1)n, m^n - 1, (m-1)n \right]_2$ and $C_p(G_n(m)) = \left[ \frac{1}{2}m^n(m-1)n, m^n, (m-1)n \right]_p$ for $p$ odd, for any information set.

**Proof.** We use Result 2 and the propositions and lemmas we have obtained for the dimensions of the codes and their minimum weights.\[\blacksquare\]

The proof of **Theorem 1** now follows from the propositions in the preceding sections.

10. Further classes of graphs

The codes from the incidence matrices of line graphs $L_t(H(n, m))$ for $t \geq 1$ and $m \geq 3$, or for $t \geq 3$ and $m \geq 2$, behave similarly in all cases where we have tested them computationally. However they become too large and complicated to handle in a manner similar to the one we have employed in this paper, so a more general approach might be needed for these.

The definition of the Hamming graphs $H(n, m)$ can be extended to the class of graphs $H^k(n, m)$, for $k, n, m \geq 1$, i.e. the graphs with vertices the $m^n$-tuples in $R^n$, and adjacency defined by two vertices in $R^n$ being adjacent if they differ in $k$ coordinate positions. (These are the graphs in the Hamming association scheme.) We can examine codes from an $m^n \times \frac{1}{2}m^n(m-1)^k \binom{n}{k}$ incidence matrix for $H^k(n, m)$ and also the incidence matrices of their line graphs. These codes appear to share properties similar to those we mentioned in Section 1 and that we have established here for the $H(n, m)$. Computations with Magma confirm this. We examine these codes in a forthcoming paper.

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**References**