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# Combinatorics of diagonally convex directed polyominoes

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## Abstract

A new bijection between the diagonally convex directed (dcd-) polyominoes and ternary trees makes it possible to enumerate the dcd-polyominoes according to several parameters (sources, diagonals, horizontal and vertical edges, target cells). For a part of these results we also give another proof, which is based on Raney's generalized lemma. Thanks to the fact that the diagonals of a dcd-polyomino can grow at most by one, the problem of  $q$ -enumeration of this object can be solved by an application of Gessel's  $q$ -analog of the Lagrange inversion formula.

## Résumé

Une nouvelle bijection entre les polyominos dirigés diagonalement convexes (polyominos d.d.c.) et les arbres ternaires permet l'énumération des polyominos d.d.c. suivant plusieurs paramètres (sources, diagonales, arêtes horizontales et verticales, cellules cibles). Pour une partie de ces résultats nous donnons une preuve supplémentaire, qui est basée sur le lemme généralisée de Raney. Grâce au fait que les diagonales d'un polyomino d.d.c. croissent au plus d'une unité, leur  $q$ -énumération peut être résolue en utilisant le  $q$ -analogue de la formule d'inversion de Lagrange dû à Gessel.

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## 1. Definitions, conventions and notations

### 1.1. Binomial coefficients

Generally, we adopt the convention: if a binomial coefficient has a negative numerator or denominator, then the value of the coefficient is zero. Exceptionally, for those binomial coefficients which are indicated by an arrow  $\nearrow$  we stipulate:  $\binom{-1}{-1}_{\nearrow} = 1$ .

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The *Gaussian polynomials* are defined by

$$\begin{bmatrix} k \\ r \end{bmatrix} = \frac{(1 - q^k)(1 - q^{k-1}) \cdots (1 - q^{k-r+1})}{(1 - q)(1 - q^2) \cdots (1 - q^r)}.$$

If  $k < 0$  or  $r < 0$ , we agree that  $\begin{bmatrix} k \\ r \end{bmatrix} = 0$ . Again, the only exception is  $\begin{bmatrix} -1 \\ -1 \end{bmatrix} = 1$ .

### 1.2. Formal sums

Let  $h(z) = h(z; q)$  be a formal power series in  $z$ , whose coefficients are formal Laurent series in  $q$ . For  $n \geq 0$  we set

$$\langle z^n \rangle h(z) := \text{the coefficient of } z^n \text{ in } h(z),$$

$$h^{[n]}(z) := h(z)h(qz) \cdots h(q^{n-1}z),$$

$$\tilde{h}^{[n]}(z) := h(z; q^{-1})h(q^{-1}z; q^{-1}) \cdots h(q^{-(n-1)}z; q^{-1}).$$

By convention, the empty products defining  $h^{[0]}(z)$  and  $\tilde{h}^{[0]}(z)$  are equal to 1.

### 1.3. Sets of integers

For  $n \in \mathbb{N}$ ,  $\underline{n}$  denotes the set  $\{i \in \mathbb{N}: 1 \leq i \leq n\}$ .

By the *plane lattice* we mean the set  $\mathbb{Z} \times \mathbb{Z}$ .

### 1.4. Lattice paths

Unless the contrary is explicitly stated, the lattice paths occurring in this paper are on the step-set  $\{(1, 0), (0, 1)\}$ . A path with vertices  $v_0, v_1, \dots, v_n$  is '1/2-good' if all the vertices  $v_1, \dots, v_n$  lie in the half-plane  $y < \frac{1}{2}x$ .

### 1.5. Ternary trees

Given a ternary tree  $\bar{T}$ , we first visit the root and then traverse its subtrees from left to right. Let  $u$  and  $v$  be two vertices of  $\bar{T}$ . We put  $u < v$  iff the first visit to  $u$  precedes the first visit to  $v$ . Thus we obtain the *prefix order* on  $\bar{T}$  (Fig. 1, left). Further, we say that  $l$  is an *odd* (resp. *even*) leaf of  $\bar{T}$  if  $|\{k \text{ leaf of } \bar{T}: k \leq l\}|$  is an odd (resp. even) number. We call a leaf  $l$  a *final leaf* if there are no internal nodes  $v$  such that  $l < v$  (Fig. 1, right).

### 1.6. Directed animals

Let  $\hat{A}$  be a finite subset of the plane lattice. A nonempty intersection between  $\hat{A}$  and a diagonal line  $y = -x + b$  ( $b \in \mathbb{Z}$ ) is called a *diagonal* of  $\hat{A}$ . We shall say that the points of the southwesternmost (resp. northeasternmost) diagonal of  $\hat{A}$  are the

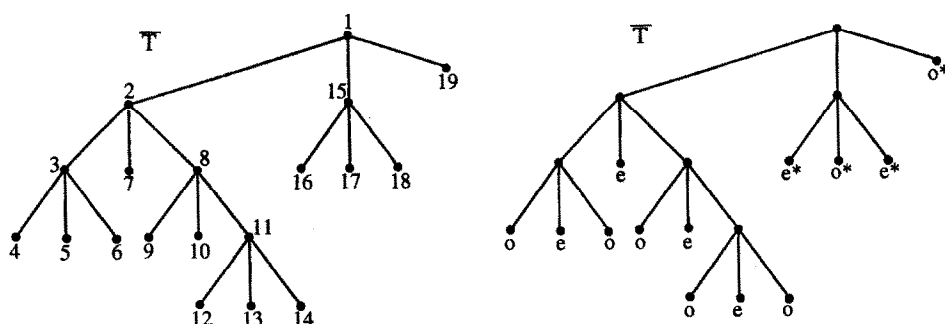


Fig. 1. A ternary tree  $\bar{T}$ . Left: The vertices are labeled after the prefix order. Right: The odd and even leaves are labeled by  $o$  and  $e$ . The final leaves are starred as well.

SW-points (resp. NE-points) of  $\hat{A}$ . The set  $\hat{A}$  is a *directed animal* if every point of  $\hat{A}$  can be reached from some of the SW-points of  $\hat{A}$  by a lattice path whose vertices all lie in  $\hat{A}$ . If  $\hat{A}$  is a directed animal, the SW-points of  $\hat{A}$  are called the *sources* and the NE-points of  $\hat{A}$  are called the *target points*. A directed animal whose diagonals consist of consecutive points is called a *diagonally convex* directed animal (a *dcd-animal* for short).

A *cell* is a unit square  $[i, i + 1] \times [j, j + 1]$ , where  $i, j \in \mathbb{Z}$ .

### 1.7. Directed animalinoes

Suppose we are given a directed animal  $\hat{A}$  and we replace each point  $p \in \hat{A}$  by the elementary cell of which  $p$  is the bottom left corner. We say that the resulting plane figure  $\bar{A}$  is a *directed animalino*. By definition, the diagonals (resp. sources, target cells) of a directed animalino are what the diagonals (resp. sources, target points) of the underlying directed animal become in consequence of the thickening. The directed animalinoes obtained from the dcd-animals will be called *dcd-animalinoes*. See Fig. 2.

A *polyomino* is a finite union of cells which has a connected interior.

By a *directed* (resp. *dcd-*) *polyomino* we mean a directed (resp. dcd-) animalino which is a polyomino. From Fig. 5 we see that not every directed animalino is a directed polyomino. This is related to the fact that every point of a directed animal is required to be connected to some of the sources, but the sources are not required to be connected among themselves. However, all the 1-source directed animalinoes are directed polyominoes. Further, when dealing with the directed objects having several sources it is natural to begin with deriving the results for the animalinoes, and then obtain the corresponding results for the polyominoes.

The main topic of this paper is the enumeration of the dcd-polyominoes. The dcd-animals are admittedly a more classical combinatorial object, but they have a flaw: being a dcd-animal a finite set of points, it is immaterial to count its horizontal

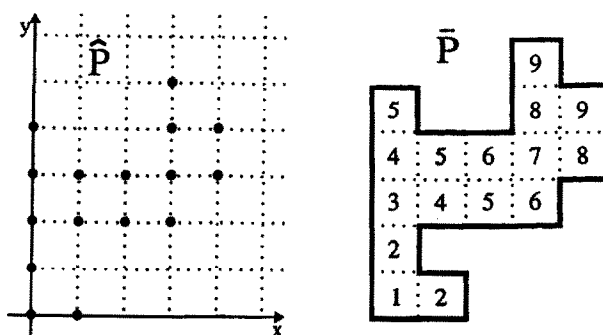


Fig. 2. Left: a dcd-animal with one source, nine diagonals and two target points. Right: the corresponding dcd-animalino.

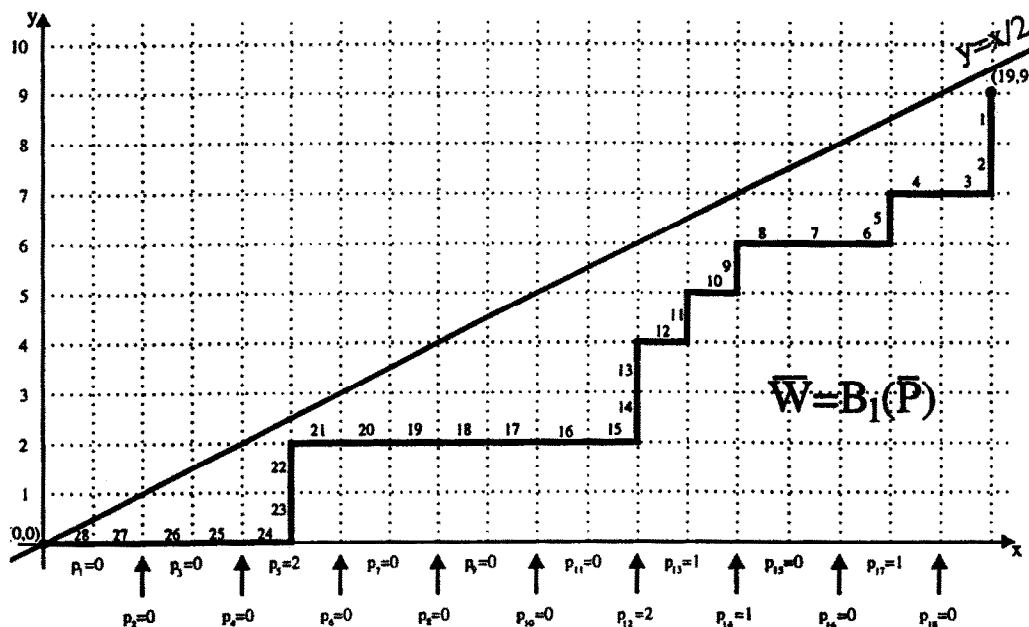


Fig. 3. The path  $\bar{W} = B_1(\bar{P})$ , where  $\bar{P}$  is the polyomino of Fig. 2. The steps of  $\bar{W}$  are numbered in the reverse order because this will help us to draw the tree  $B_2(\bar{W})$  in the next figure.

and vertical edges. But in the setting of dcd-polyominoes the enumeration by the perimeter can be legitimately carried out, and it gives interesting results as well.

As usual in the literature, when enumerating the figures like animals, animalinoes and polyominoes we make no difference between two of them which can be transformed into one other by a translation.

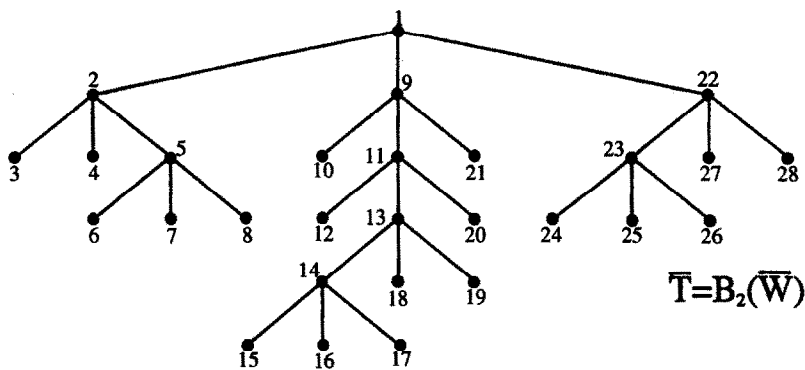


Fig. 4. The ternary tree  $T = B_2(\bar{W})$ , where  $\bar{W}$  is the path shown in the preceding figure.

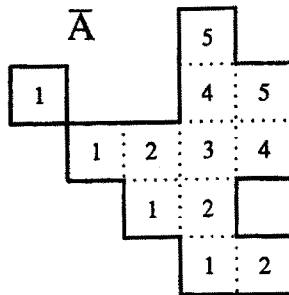


Fig. 5.

## 2. Introduction

Polyominoes are used in physics and chemistry to model crystal growth, polymers, etc. Despite strenuous efforts over the past 40 years, counting the general polyominoes remains an unsolved problem. However, considerable progress has been made in solving various simpler, but nontrivial models. For instance, nice results are known for the classes of parallelogram, column-convex, convex, directed and diagonally convex directed polyominoes. See [2] or [18] for a survey.

The dcd-polyominoes model was used for the first time by the physicists Privman and Švrakić [14] and [15, p. 99], who obtained the area generating function for the 1-source case. The enumeration by the perimeter was carried out later by Delest and Fédou [13] and Penaud [13].

In Section 3 we introduce the *sequences of losses*, which are a kind of coding for the dcd-polyominoes. In Section 4 we define a new bijection between the

dcd-polyominoes and ternary trees. This bijection is helpful in the perimeter enumeration of dcd-polyominoes (Section 5). In Section 6 the main theorem of Section 5.1 is proved again by using Raney's generalized lemma. Section 7 is about the  $q$ -enumeration (i.e. the area and perimeter enumeration) of dcd-polyominoes. Gessel's  $q$ -Lagrange inversion formula plays a decisive part in this context.

### 3. The sequence of losses and some facts about it

Let  $\bar{P}$  be a 1-source dcd-polyomino with  $k$  diagonals. Let, for  $j \in \underline{k}$ ,  $X_j$  and  $Y_j$  be the maximal abscissa and the maximal ordinate of the  $j$ th diagonal of  $\bar{P}$ . Observe that, for  $2 \leq i \leq k$ , the largest possible values of  $X_i$  and  $Y_i$  are  $X_{i-1} + 1$  and  $Y_{i-1} + 1$ , respectively. Consider the sequence  $\langle p_1, \dots, p_{2k} \rangle$  defined by

$$p_1 = p_2 = 0,$$

$$p_{2i-1} = X_{i-1} + 1 - X_i \quad (2 \leq i \leq k), \quad p_{2i} = Y_{i-1} + 1 - Y_i \quad (2 \leq i \leq k).$$

Evidently, this is a sequence of nonnegative integers. For  $2 \leq i \leq k$ ,  $p_{2i-1}$  (resp.  $p_{2i}$ ) is the number of cells that could have been included in, but yet remained below (resp. above) the  $i$ th diagonal of  $\bar{P}$ . For this reason we shall call  $\langle p_1, \dots, p_{2k} \rangle$  the sequence of losses of  $\bar{P}$ .

**Example 1.** The sequence of losses of the dcd-polyomino on the right-hand side of Fig. 2 is

$$p_1 = \dots = p_4 = 0, \quad p_5 = 2, \quad p_6 = \dots = p_{11} = 0, \quad p_{12} = 2,$$

$$p_{13} = p_{14} = 1, \quad p_{15} = p_{16} = 0, \quad p_{17} = 1, \quad p_{18} = 0.$$

For instance,  $p_{11}$  is zero and  $p_{12}$  is two because the sixth diagonal occupies all the available sites at the bottom and leaves two sites free at the top.

Now it is useful to state two simple propositions about the sequences of losses, which can easily be proved by induction on  $k$ . The first proposition tells us how the area and perimeter of a dcd-polyomino can be read off from its sequence of losses.

**Proposition 1.** Let  $\bar{P}$  be a 1-source dcd-polyomino with  $k$  diagonals and let  $\langle p_1, \dots, p_{2k} \rangle$  be its sequence of losses. Then

- (a) for  $j \in \underline{k}$ , the  $j$ th diagonal of  $\bar{P}$  contains  $j - \sum_{i=1}^{2j} p_i$  cells;
- (b) the boundary of  $\bar{P}$  consists of  $2|\{j \in \underline{k}: p_{2j-1} = 0\}|$  horizontal edges and  $2|\{j \in \underline{k}: p_{2j} = 0\}|$  vertical edges.

The second proposition gives a characterization of those integer sequences which are sequences of losses.

**Proposition 2.** *A sequence of nonnegative integers  $\langle p_1, \dots, p_{2k} \rangle$  is the sequence of losses of some 1-source dcd-polyomino if and only if  $\sum_{i=1}^{2j} p_i < j$  ( $\forall j \in \underline{k}$ ).*

#### 4. A new bijection between the 1-source dcd-polyominoes and ternary trees

Using Schützenberger's methodology, Delest and Fédou, in [3], obtained the following interesting result: the number of 1-source dcd-polyominoes with  $k$  diagonals is equal to  $(1/(3k+1))\binom{3k+1}{k}$ , which is also the number of ternary trees with  $k$  internal nodes. A result such as this naturally calls for a bijective proof and two such proofs have actually been given in [3, 13]. Nevertheless, we believe that our simple new bijection between the 1-source dcd-polyominoes and ternary trees still deserves to be mentioned.

First we need some notation.

**Notation 1.** We write  $\mathcal{P}(k)$  to denote the family of 1-source dcd-polyominoes with  $k$  diagonals. By  $\mathcal{W}(k)$  we denote the family of 1/2-good paths which start at  $(0, 0)$  and terminate at  $(2k+1, k)$ . By  $\mathcal{T}(k)$  we denote the family of ternary trees with  $k$  internal nodes.

Our bijection consists of two steps.

*Step 1:* Let  $\bar{P}$  be an element of  $\mathcal{P}(k)$  and let  $\langle p_1, \dots, p_{2k} \rangle$  be its sequence of losses. We associate with  $\bar{P}$  the lattice path  $B_1(\bar{P})$  which starts at  $(0, 0)$  with a horizontal step, makes  $p_i$  vertical steps with abscissa  $i$  ( $\forall i \in \underline{2k}$ ) and ends at  $(2k+1, k)$ .

The northmost points of  $B_1(\bar{P})$  with abscissas  $2j-1$  and  $2j$  ( $j \in \underline{k}$ ) are  $Q_j = (2j-1, \sum_{i=1}^{2j-1} p_i)$  and  $R_j = (2j, \sum_{i=1}^{2j} p_i)$ , respectively. By Proposition 2,  $\sum_{i=1}^{2j-1} p_i \leq \sum_{i=1}^{2j} p_i < j$ . So the ordinates of  $Q_j$  and  $R_j$  are at most  $j-1$  and those two points lie below the line  $y = \frac{1}{2}x$ . This shows that  $B_1(\bar{P})$  is a 1/2-good path.

Let  $\bar{P} \in \mathcal{P}(k)$  and let  $\bar{W} = B_1(\bar{P})$ . The sequence of losses of  $\bar{P}$  can be uniquely read off from  $\bar{W}$  and  $\bar{P}$  is uniquely determined by its sequence of losses. Therefore,  $B_1$  is an injection. Further,  $B_1$  is also a surjection, as it can be readily shown with the aid of Proposition 2. Thus we have

**Proposition 3.**  *$B_1$  is a bijection between  $\mathcal{P}(k)$  and  $\mathcal{W}(k)$ .*

*Step 2:* What remains to do now is to define a bijection between the families  $\mathcal{W}(k)$  and  $\mathcal{T}(k)$ . Since the lattice paths are certainly a more classical combinatorial object than the dcd-polyominoes, at this stage it is useful to refer to the literature. So in [4] one can find a list of bijections between the ordered trees with  $n$  edges and various

other objects. For instance, there is a bijection between those trees and the set of lattice paths from  $(0, 0)$  to  $(n, n)$  which entirely lie in the half-plane  $y \geq x$ . Adapting (with no claim to originality) this bijection to our case, where the ordered trees are ternary, we arrived at the following conclusion.

**Proposition 4.** *For every path  $\bar{W} \in \mathcal{W}(k)$  there is a unique ternary tree  $B_2(\bar{W}) \in \mathcal{T}(k)$  such that the  $i$ th ( $i \in \underline{3k+1}$ ) vertex of  $B_2(\bar{W})$  in prefix order is an internal node iff the  $i$ th step from the endpoint of  $\bar{W}$  is a vertical step.*

*Moreover,  $B_2$  is a bijection between  $\mathcal{W}(k)$  and  $\mathcal{T}(k)$ .*

Quite obviously, Propositions 3 and 4 imply the following theorem.

**Theorem 1.** *The composition  $B_2 \circ B_1$  is a bijection between  $\mathcal{P}(k)$  and  $\mathcal{T}(k)$ .*

Thus, we have got a new bijection between the 1-source dcd-polyominoes and ternary trees. An example is shown in Figs. 2–4.

## 5. The new bijection put to use

### 5.1. Enumeration by the perimeter in the case of one source

**Notation 2.** By  $\mathcal{P}(k, l, m, e)$  we denote the set of 1-source dcd-polyominoes with  $k$  diagonals,  $2l$  horizontal edges,  $2m$  vertical edges and  $e$  target cells. By  $\mathcal{T}(k, l, m, e)$  we denote the set of ternary trees  $\bar{T}$  which have the following properties:

- (i)  $\bar{T}$  has  $k$  internal nodes,
- (ii) the event ‘the prefix order successor of an even (resp. odd) leaf of  $\bar{T}$  is again a leaf’ takes place  $l$  (resp.  $m$ ) times,
- (iii) the left branch of  $\bar{T}$  is of length  $e$ .

**Proposition 5.**  *$B_2 \circ B_1$  is a bijective correspondence between  $\mathcal{P}(k, l, m, e)$  and  $\mathcal{T}(k, l, m, e)$ .*

**Proof.** The statement can be established by using Proposition 1 and closely examining the correspondence  $B_2 \circ B_1$ . Indeed, let  $\bar{P} \in \mathcal{P}(k, l, m, e)$  and let  $\langle p_1, \dots, p_{2k} \rangle$  be the sequence of losses of  $\bar{P}$ . Let  $\bar{T} = (B_2 \circ B_1)(\bar{P})$ . Since  $\bar{P}$  has  $e$  cells in the last ( $k$ th) diagonal, Proposition 1(a) tells us that  $k - \sum_{i=1}^{2k} p_i = e$ . The path  $\bar{W} = B_1(\bar{P})$  goes along the line  $x = 2k$  up to the point  $(2k, \sum_{i=1}^{2k} p_i) = (2k, k - e)$  and then it moves one step to the right. Hence, to reach its endpoint  $(2k + 1, k)$ , the path  $\bar{W}$  must make another  $e$  upward steps with abscissa  $2k + 1$ . This implies that the prefix order list of vertices of  $\bar{T} = B_2(\bar{W})$  begins with  $e$  internal nodes. In other words, the left branch of  $\bar{T}$  is of length  $e$ . Thus, we have seen that  $\bar{T}$  possesses the property (iii). The proof that it also possesses the property (ii) is left to the reader.  $\square$



So we have  $|\mathcal{P}(k, l, m, e)| = |\mathcal{T}(k, l, m, e)|$ . Let

$$F(d, x, y, t) = \sum_{k, l, m, e \geq 1} |\mathcal{P}(k, l, m, e)| d^k x^l y^m t^e, \quad (1)$$

and

$$f(d, x, y) = F(d, x, y, 1). \quad (2)$$

In order to get an algebraic equation for the function  $F$ , we now partition the nontrivial ternary trees into eight classes  $\mathcal{T}_{000}, \mathcal{T}_{001}, \dots, \mathcal{T}_{111}$ : the trees belonging to the class  $\mathcal{T}_{\alpha\beta\gamma}$  have a nontrivial left (resp. middle, right) subtree if and only if  $\alpha$  (resp.  $\beta, \gamma$ ) is 1. Recall that a ternary tree with  $k$  internal nodes has  $2k + 1$  leaves, so that the number of leaves is always odd. Hence, the last leaf of the left (resp. middle, right) subtree of a ternary tree  $\bar{T}$  is an odd (resp. even, odd) leaf of  $\bar{T}$ . With this in mind, it is not hard to read off from Fig. 6 that the contributions to  $F$  are:

$$\begin{aligned} & \text{from } \mathcal{T}_{000}: dtxy & \text{from } \mathcal{T}_{001}: dt y f \\ & \text{from } \mathcal{T}_{010}: dt x f(d, y, x) & \text{from } \mathcal{T}_{011}: dt f(d, y, x) f \\ & \text{from } \mathcal{T}_{100}: dt x y F & \text{from } \mathcal{T}_{101}: dt y F f \\ & \text{from } \mathcal{T}_{110}: dt x F f(d, y, x) & \text{from } \mathcal{T}_{111}: dt F f(d, y, x) f. \end{aligned}$$

But clearly, there is a bijection between  $\mathcal{P}(k, l, m, e)$  and  $\mathcal{P}(k, m, l, e)$  (reflection in the line  $y = x$ ). Consequently, we have  $F(d, x, y, t) = F(d, y, x, t)$  and  $f(d, x, y) = f(d, y, x)$ . Using this remark, we find that the eight contributions to  $F$  sum up to

$$F = dt(F + 1)(f + x)(f + y), \quad (3)$$

whence

$$f = d(f + 1)(f + x)(f + y). \quad (4)$$

Let  $f_1 = f/(1 + f)$ . Solving (3) for  $F$  and using (4) we obtain

$$F = \frac{td(f + x)(f + y)}{1 - td(f + x)(f + y)} = \frac{tf/(1 + f)}{1 - tf/(1 + f)} = \frac{tf_1}{1 - tf_1}. \quad (5)$$

Hence the coefficients of  $F$  and  $f_1$  are related by

$$\langle d^k x^l y^m t^e \rangle F = \langle d^k x^l y^m t^e \rangle \sum_{i \geq 1} t^i f_1^i = \langle d^k x^l y^m t^e \rangle t^e f_1^e = \langle d^k x^l y^m \rangle f_1^e. \quad (6)$$

Dividing (4) by  $f + 1$  and using  $f = f_1/(1 - f_1)$ , we get

$$f_1 = d[f_1/(1 - f_1) + x][f_1/(1 - f_1) + y]. \quad (7)$$

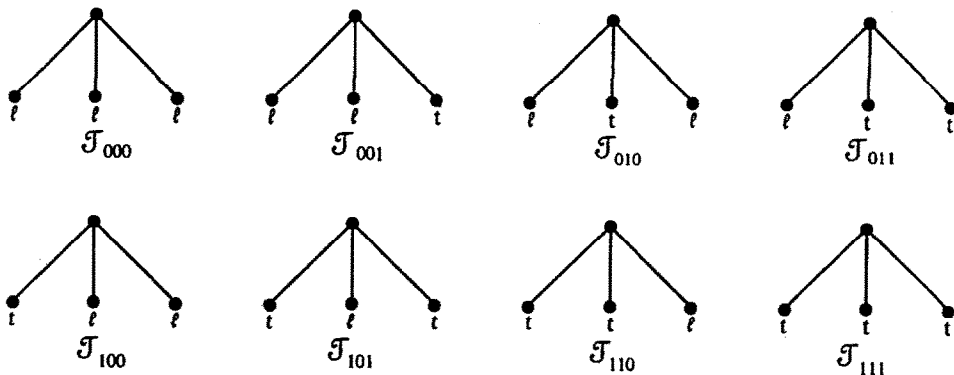


Fig. 6. The eight classes of nontrivial ternary trees. 't' and 'T' stand for a leaf and a nontrivial ternary tree, respectively.

**Theorem 2.** *The number of 1-source dcd-polyominoes having  $k$  diagonals,  $2l$  horizontal edges,  $2m$  vertical edges and  $e$  target cells is equal to*

$$\frac{e}{k} \binom{k-e-1}{2k-l-m-1} \binom{k}{l} \binom{k}{m}. \quad (8)$$

**Proof.** Applying the Lagrange inversion formula (see, e.g., [9]) to (7), we obtained (8) as the coefficient of  $d^k x^l y^m$  in  $f_1^e$ . By (6), (1) and Notation 2, this coefficient of  $f_1^e$  is equal to the number of dcd-polyominoes which have the required properties.  $\square$

**Proposition 6.** *The number of 1-source dcd-polyominoes having  $k$  diagonals,  $2l$  horizontal edges and  $2m$  vertical edges is*

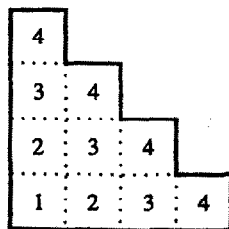
$$\frac{1}{k} \binom{k}{2k-l-m+1} \binom{k}{l} \binom{k}{m}. \quad (9)$$

**Proof.** Formula (9) can be proved in two ways: either by using Vandermonde's convolution to sum over  $e \geq 1$  in (8), or by applying the Lagrange inversion formula to (4).  $\square$

Thus, we have generalized the results of Delest and Fédou [3] and Penaud [13], who obtained the coefficients of  $F(d, 1, 1, 1)$ ,  $F(1, x, x, 1)$  and  $F(d, t, 1, 1)$ .

## 5.2. Enumeration by the perimeter in the case of several sources

**Notation 3.** We write  $\mathcal{A}(r, k, l, m, e)$  (resp.  $\mathcal{P}(r, k, l, m, e)$ ) for the set of dcd-animalinoes (resp. dcd-polyominoes) which have  $r$  sources,  $k$  diagonals,  $2l$  horizontal edges,  $2m$  vertical edges and  $e$  target cells. The symbol  $\Delta$ , denotes the dcd-polyomino

Fig. 7. The triangle  $\Delta_4$ .

with  $r$  diagonals and with  $i$  cells in the  $i$ th diagonal, for every  $i \in \underline{r}$  (Fig. 7). By  $\mathcal{T}(k, l, m, e, u)$  we denote the subfamily of  $\mathcal{T}(k, l, m, e)$  consisting of those trees which have precisely  $u$  final leaves.

Let

$$A(s, d, x, y, t) = \sum_{r, k, l, m, e \geq 1} |\mathcal{A}(r, k, l, m, e)| s^r d^k x^l y^m t^e \quad (10)$$

and

$$P(s, d, x, y, t) = \sum_{r, k, l, m, e \geq 1} |\mathcal{P}(r, k, l, m, e)| s^r d^k x^l y^m t^e. \quad (11)$$

Let  $r \in \mathbb{N}$  be fixed. To every  $\bar{A} \in \mathcal{A}(r, k, l, m, e)$  we associate the 1-source dcd-polyomino  $\text{Cl}_r(\bar{A})$  obtained by replacing the first diagonal of  $\bar{A}$  by the 'triangle'  $\Delta_r$  (Fig. 8). The polyomino  $\text{Cl}_r(\bar{A})$  has  $k + r - 1$  diagonals. It again has  $2l$  horizontal edges,  $2m$  vertical edges and  $e$  target cells. Next, for  $i \in \underline{r}$ ,  $\text{Cl}_r(\bar{A})$  has  $i$  cells in the  $i$ th diagonal. This means that the sequence of losses of  $\text{Cl}_r(\bar{A})$  begins with at least  $2r$  zeros.

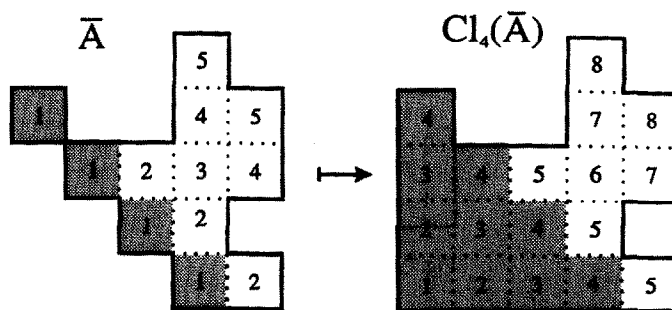
**Proposition 7.**  $\text{Cl}_r$  is a bijection between  $\mathcal{A}(r, k, l, m, e)$  and the set of 1-source polyominoes which belong to  $\mathcal{P}(k + r - 1, l, m, e)$  and have a sequence of losses with at least  $2r$  initial zeros.

Further examination of the bijection  $B_2 \circ B_1$  (defined in Section 4) yields the following result.

**Proposition 8.** The set of 1-source polyominoes which belong to  $\mathcal{P}(k + r - 1, l, m, e)$  and have a sequence of losses with at least  $2r$  initial zeros is in bijection with  $\bigcup_{u=2r+1}^{\infty} \mathcal{T}(k + r - 1, l, m, e, u)$ .

Propositions 7 and 8 imply

$$|\mathcal{A}(r, k, l, m, e)| = \sum_{u \geq 2r+1} |\mathcal{T}(k + r - 1, l, m, e, u)|. \quad (12)$$

Fig. 8. The animalinoes  $\bar{A}$  and  $Cl_4(\bar{A})$ .

Now we shall first derive the function

$$H(d, x, y, t, z) = \sum_{k, l, m, e, u \geq 1} |\mathcal{T}(k, l, m, e, u)| d^k x^l y^m t^e z^u. \quad (13)$$

Knowing  $H$ , we shall be able to obtain  $A$ ; knowing  $A$ , we shall be able to obtain  $P$ .

Let  $h(d, x, y, z) = H(d, x, y, 1, z)$  and  $\bar{h}(d, x, y, z) = h(d, y, x, z)$ . Note that  $h(d, x, y, 1)$  is the function  $f(d, x, y)$  of Section 5.1. Also note that  $f(d, x, y)$  is equal to  $f(d, y, x)$ , but  $h(d, x, y, z)$  is not equal to  $h(d, y, x, z)$ . Now we rescan the eight classes of ternary trees (Fig. 6) reading off their contributions to the function  $H$ . The contributions are:

$$\begin{array}{ll} \text{from } \mathcal{T}_{000}: dtxyz^3 & \text{from } \mathcal{T}_{001}: dtyh \\ \text{from } \mathcal{T}_{010}: dtxz\bar{h} & \text{from } \mathcal{T}_{011}: dtfh \\ \text{from } \mathcal{T}_{100}: dtxyz^2H & \text{from } \mathcal{T}_{101}: dtyFh \\ \text{from } \mathcal{T}_{110}: dtxzF\bar{h} & \text{from } \mathcal{T}_{111}: dtFfh. \end{array}$$

Summing these contributions we find

$$(1 - dtxyz^2)H = dtxyz^3 + dt(F + 1)[(f + y)h + xz\bar{h}]. \quad (14)$$

Putting  $t = 1$  in (14) produces

$$[1 - dxyz^2 - d(f + 1)(f + y)] \cdot h - dxz(f + 1) \cdot \bar{h} = dxyz^3. \quad (15)$$

Since the function  $f$  is (implicitly) given by (4), we may regard (15) as a linear equation in the two unknowns  $h$  and  $\bar{h}$ . Swapping  $x$  and  $y$  in (15), we obtain another linear equation

$$-dyz(f + 1) \cdot h + [1 - dxyz^2 - d(f + 1)(f + x)] \cdot \bar{h} = dxyz^3. \quad (16)$$

Now we solve the linear system (15), (16) and then work out the expression  $(f + y)h + xz\bar{h}$ , which appears on the right-hand side of (14). Using (4) to simplify the

numerator, we find

$$(f+y)h + xz\bar{h} = \frac{dxyz^3[(xz+y)(1-dxyz^2) + dxyz^2]}{[1-dxyz^2-d(f+1)(f+x)][1-dxyz^2-d(f+1)(f+y)]-d^2xyz^2(f+1)^2}. \quad (17)$$

Next we multiply the numerator and denominator of (17) by  $(f+x)(f+y) - xyz^2$ , use (4) to simplify the denominator and patiently calculate to reach the formula

$$(f+y)h + xz\bar{h} = \frac{dz^3[(xz+y)(1-dxyz^2) + dxyz^2][(f+x)(f+y) - xyz^2]}{(1-dxyz^2+dxz^2)(1-dxyz^2+dyz^2) - z^2(1-dxyz^2)^2}. \quad (18)$$

Formulas (14) and (18), along with an application of (3) to simplify the numerator, produce

$$H = \frac{dxyz^3t}{1-dxyz^2t} + \frac{dz^3[(xz+y)(1-dxyz^2) + dxyz^2][(1-dxyz^2t)F - dxyz^2t]}{[(1-dxyz^2+dxz^2)(1-dxyz^2+dyz^2) - z^2(1-dxyz^2)^2](1-dxyz^2t)}. \quad (19)$$

Now that we know  $H$ , our next goal is to find the function  $A$ . It turns out that we get the desired result if we compute the following sequence of functions:

$$I(d, x, y, t, z) = [H(d, x, y, t, z) - H(d, x, y, t, -z)]/2, \quad (20)$$

$$J(d, x, y, t, z) = [H(d, x, y, t, z) + H(d, x, y, t, -z)]/2, \quad (21)$$

$$K(d, x, y, t, z) = zI + J, \quad (22)$$

$$L(s, d, x, y, t) = K(d, x, y, t, s^{1/2}), \quad (23)$$

$$M(s, d, x, y, t) = (sF - L)/(1 - s), \quad (24)$$

$$N(s, d, x, y, t) = d \cdot M(sd^{-1}, d, x, y, t). \quad (25)$$

Namely, the above definitions imply

$$\langle s^r d^k x^l y^m t^e \rangle N = \sum_{u \geq 2r+1} \langle d^{k+r-1} x^l y^m t^e z^u \rangle H. \quad (26)$$

Now it follows from (26), (13), (12) and (10) that

$$\begin{aligned} \langle s^r d^k x^l y^m t^e \rangle N &= \sum_{u \geq 2r+1} |\mathcal{F}(k+r-1, l, m, e, u)| \\ &= |\mathcal{A}(r, k, l, m, e)| = \langle s^r d^k x^l y^m t^e \rangle A. \end{aligned}$$

Hence  $N = A$ . Noting that, we proceed to the effective computation of  $I, J, \dots, N = A$  from (19)–(25). In conclusion, the following lemma is obtained.

**Lemma 1.** *We have*

$$A = \frac{s[F - xyst/(1 - xyst)]}{[1 + xs/(1 - xys)][1 + ys/(1 - xys)] - sd^{-1}}, \quad (27)$$

where  $F$  is given by (4) and (5).

Now there is yet one goal ahead of us: we have to find the function  $P$ . But we are already near to this goal. Namely, the function  $P$  is closely related to the function  $A$ , as it will shortly be seen.

For  $\bar{A}$  a dcd-animalino with at least  $n$  diagonals, we shall (again) write  $X_n(\bar{A})$  and  $Y_n(\bar{A})$  to denote the maximal abscissa and the maximal ordinate of the  $n$ th diagonal of  $\bar{A}$ . Let  $\mathcal{A}_+$  be the set of dcd-animalinoes which have at least two diagonals. For  $i, j \in \mathbb{N}_0$ , let

$$\mathcal{A}_{ij} = \{\bar{A} \in \mathcal{A}_+ : X_1(\bar{A}) + 1 - X_2(\bar{A}) = i, Y_1(\bar{A}) + 1 - Y_2(\bar{A}) = j\}. \quad (28)$$

To put it into words,  $\mathcal{A}_{ij}$  is the set of those dcd-animalinoes whose second diagonal misses to occupy  $i$  of the sites available on the bottom side and  $j$  of the sites available on the top side. The collection of sets  $\{\mathcal{A}_{ij} : i, j \in \mathbb{N}_0\}$  is evidently a partition of  $\mathcal{A}_+$ . Next, as soon as  $i \geq 2$  or  $j \geq 2$  (or both), every  $\bar{A} \in \mathcal{A}_{ij}$  has some cut points in its first diagonal. So in this case the elements of  $\mathcal{A}_{ij}$  are not polyominoes. On the other hand, if  $\bar{B} \in \mathcal{A}_{00} \cup \mathcal{A}_{01} \cup \mathcal{A}_{10} \cup \mathcal{A}_{11}$ , then  $\bar{B}$  has no cut points in the first diagonal. Moreover, it is not hard to see that  $\bar{B}$  is in fact a polyomino.

Let  $\mathcal{P}$  be the set of all dcd-polyominoes. Obviously,  $\mathcal{P}$  contains only one 1-diagonal polyomino: it is the one-cell polyomino  $\Delta_1$ . The dcd-polyominoes with two or more diagonals lie in  $\mathcal{A}_+$ . Hence, by the above discussion, they lie in  $\mathcal{A}_{00} \cup \mathcal{A}_{01} \cup \mathcal{A}_{10} \cup \mathcal{A}_{11}$ . Thus, we realize that  $\{\{\Delta_1\}, \mathcal{A}_{00}, \mathcal{A}_{01}, \mathcal{A}_{10}, \mathcal{A}_{11}\}$  is a partition of  $\mathcal{P}$ . Each set in this partition accounts for a part of the function  $P$ . Concretely,

$\{\Delta_1\}$  gives  $dxyst$ ,

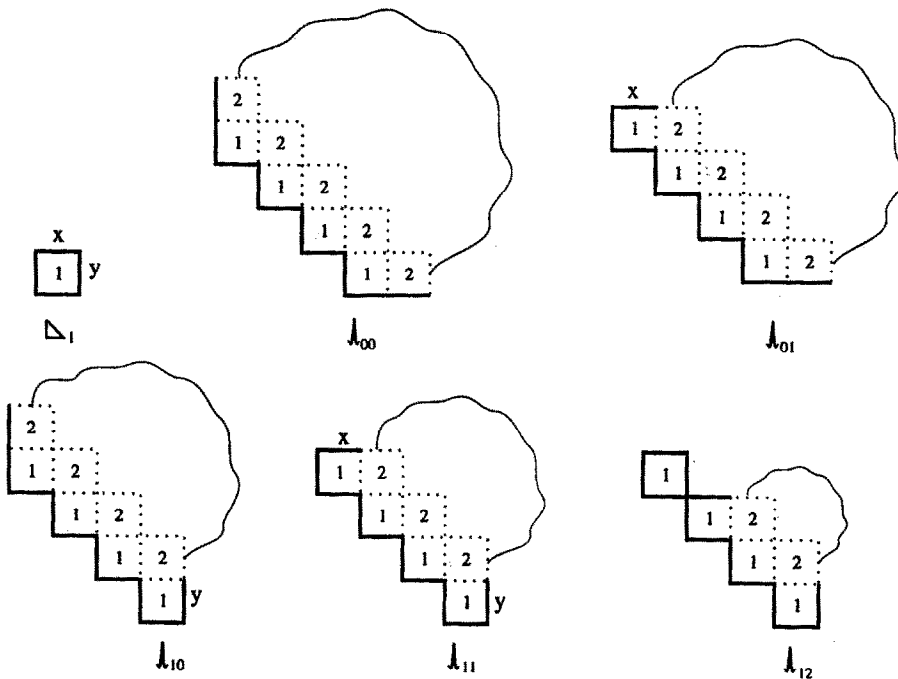
$\mathcal{A}_{00}$  gives  $ds^{-1}(A - sF)$ ,  $\mathcal{A}_{01}$  gives  $dxA$ ,

$\mathcal{A}_{10}$  gives  $dyA$ ,  $\mathcal{A}_{11}$  gives  $dxysA$ ,

as may be established with the aid of Fig. 9. Putting the five parts of  $P$  together, we find

$$P = ds^{-1}(1 + xs)(1 + ys)A - d(F - xyst). \quad (29)$$

Combining (29) with Lemma 1, the following theorem is obtained.

Fig. 9. Shape of the elements of  $\mathcal{A}_{ij}$ .

**Theorem 3.** *We have*

$$P = \frac{d(1 + xs)(1 + ys)[F - xyst/(1 - xyst)]}{[1 + xs/(1 - xys)][1 + ys/(1 - xys)] - sd^{-1}} - d(F - xyst), \quad (30)$$

where  $F$  is given by (4) and (5).

## 6. A proof which rests on Raney's generalized lemma

In this section we will give an easy new proof of Theorem 2 of Section 5.1. The proof makes use of a simple, but powerful counting tool, which Graham et al. call Raney's generalized lemma [11, p. 348; 16]. In fact, the first to discover this lemma were Dvoretzky and Motzkin in 1947 ([7]; also see [5]). But since we shall carry the statement of the lemma over from 'Concrete Mathematics', we have chosen to call the lemma by the name given to it in that book. For the sake of completeness, we also state Raney's ungeneralized lemma [11, p. 345].

**Lemma 2 (Raney's lemma).** *If  $\langle x_1, x_2, \dots, x_n \rangle$  is any sequence of integers whose sum is  $+1$ , exactly one of the cyclic shifts*

$$\langle x_1, x_2, \dots, x_n \rangle, \langle x_2, \dots, x_n, x_1 \rangle, \dots, \langle x_n, x_1, \dots, x_{n-1} \rangle$$

*has all of its partial sums positive.*

**Lemma 3** (Raney's generalized lemma). *If  $\langle x_1, x_2, \dots, x_n \rangle$  is any sequence of integers with  $x_j \leq 1$  for all  $j$ , and with  $x_1 + x_2 + \dots + x_n = e > 0$ , then exactly  $e$  of the cyclic shifts*

$$\langle x_1, x_2, \dots, x_n \rangle, \langle x_2, \dots, x_n, x_1 \rangle, \dots, \langle x_n, x_1, \dots, x_{n-1} \rangle$$

*have all positive partial sums.*

**Example 2.** Consider the sequence  $\langle 1, -1, 1, 1, 0, 1, -2, 1, -1, 1 \rangle$ . Its cyclic shifts are

$$\begin{aligned} &\langle 1, -1, 1, 1, 0, 1, -2, 1, -1, 1 \rangle, & \langle -1, 1, 1, 0, 1, -2, 1, -1, 1, 1 \rangle, \\ \rightarrow &\langle 1, 1, 0, 1, -2, 1, -1, 1, 1, -1 \rangle, & \langle 1, 0, 1, -2, 1, -1, 1, 1, -1, 1 \rangle, \\ &\langle 0, 1, -2, 1, -1, 1, 1, -1, 1, 1 \rangle, & \langle 1, -2, 1, -1, 1, 1, -1, 1, 1, 0 \rangle, \\ &\langle -2, 1, -1, 1, 1, -1, 1, 1, 0, 1 \rangle, & \langle 1, -1, 1, 1, -1, 1, 1, 0, 1, -2 \rangle, \\ &\langle -1, 1, 1, -1, 1, 1, 0, 1, -2, 1 \rangle, & \rightarrow \langle 1, 1, -1, 1, 1, 0, 1, -2, 1, -1 \rangle, \end{aligned}$$

and only the two marked  $\rightarrow$  have all partial sums positive.

The reader will note that Lemma 2 holds for arbitrary integers, while in Lemma 3 it is assumed that the integers are smaller or equal than 1. So it might seem that the case  $e = 1$  of Lemma 3 is not so general as Lemma 2. But in spite of the appearances, these two statements are equivalent (why?).

An application of Lemma 2 to the enumeration of column-convex directed polyominoes was given in [8]. Here we could use Lemma 2 to enumerate the family  $\mathcal{P}(k, l, m, 1)$ . But we shall rather use Lemma 3, which allows us to enumerate the family  $\mathcal{P}(k, l, m, e)$  for arbitrary  $e \in \mathbb{N}$ .

Let us write  $\mathcal{P}(\dots)$  as abbreviation for  $\mathcal{P}(k, l, m, e)$ .

Now we define a kind of 'Raney bijection' between  $\mathcal{P}(\dots)$  and certain family of integer sequences. Afterwards we enumerate this family of sequences with the aid of Lemma 3.

To a dcd-polyomino  $\bar{P} \in \mathcal{P}(\dots)$ , whose sequence of losses is  $\langle p_1, \dots, p_{2k} \rangle$ , we associate the sequence  $R(\bar{P})$  defined by

$$R(\bar{P}) = \langle 1, -p_1, -p_2, 1, -p_3, -p_4, \dots, 1, -p_{2k-1}, -p_{2k} \rangle. \quad (31)$$

Clearly,  $R$  is an injection. Next, using Propositions 1 and 2, we can readily characterize the image of  $\mathcal{P}(\dots)$  under  $R$ . It is the set of integer sequences

$$\langle 1, a_1, a_2, 1, a_3, a_4, \dots, 1, a_{2k-1}, a_{2k} \rangle \quad (32)$$

such that

- (i)  $a_i \leq 0$  ( $\forall i \in \underline{2k}$ );
  - (ii)  $|\{i \in \underline{k} : a_{2i-1} = 0\}| = l$ ;    (iii)  $|\{i \in \underline{k} : a_{2i} = 0\}| = m$ ;
  - (iv)  $\sum_{i=1}^j (1 + a_{2i-1} + a_{2i}) > 0$  ( $\forall j \in \underline{k}$ );    (v)  $\sum_{i=1}^k (1 + a_{2i-1} + a_{2i}) = e$ .
- Observe that (v) can be rewritten as  $\sum_{j=1}^{2k} a_j = e - k$ .



**Definition 1.** Let  $\bar{s}$  be a sequence of the form (32). A *nest* of  $\bar{s}$  is any of the subsequences  $(a_1, a_2), (a_3, a_4), \dots, (a_{2k-1}, a_{2k})$ . The *principal* cyclic shifts of  $\bar{s}$  are the cyclic shifts

$$\langle 1, a_1, a_2, 1, a_3, a_4, \dots, 1, a_{2k-1}, a_{2k} \rangle, \langle 1, a_3, a_4, \dots, 1, a_{2k-1}, a_{2k}, 1, a_1, a_2 \rangle, \dots \\ \dots, \langle 1, a_{2k-1}, a_{2k}, 1, a_1, a_2, \dots, 1, a_{2k-3}, a_{2k-2} \rangle.$$

Let  $\mathcal{S}(\dots)$  be the set containing the integer sequences of the form (32), which have the properties (i), (ii), (iii) and (v) (property (iv) is not required). Observe that, for  $\bar{s} \in \mathcal{S}(\dots)$ , the principal cyclic shifts of  $\bar{s}$  also belong to  $\mathcal{S}(\dots)$ . The enumeration of the set  $\mathcal{S}(\dots)$  is easy. To define an element of  $\mathcal{S}(\dots)$  we have to

- (a) decide which  $l$  of the nests will have first component zero,
- (b) decide which  $m$  of the nests will have second component zero,
- (c) define the  $2k - l - m$  yet undefined terms of the sequence so that they be negative integers which add up to  $-(k - e)$ .

(a) can be done in  $\binom{k}{l}$  ways, (b) in  $\binom{k}{m}$  ways, and (c) in  $\binom{k-e-1}{2k-l-m-1}_\infty$  ways. Hence,

$$|\mathcal{S}(\dots)| = \binom{k}{l} \binom{k}{m} \binom{k-e-1}{2k-l-m-1}_\infty. \quad (33)$$

By Lemma 3, a sequence  $\bar{s} \in \mathcal{S}(\dots)$  has exactly  $e$  cyclic shifts with all positive partial sums. But these  $e$  cyclic shifts cannot be but principal, because every nonprincipal cs begins with a nonpositive term. Being principal, these  $e$  cs's belong to  $\mathcal{S}(\dots)$ . Having all positive partial sums, they possess property (iv) as well. So these  $e$  principal cs's of  $\bar{s}$  belong to  $R(\mathcal{P}(\dots))$ . Also, it is immediate that the remaining  $k - e$  principal cs's of  $\bar{s}$  do not belong to  $R(\mathcal{P}(\dots))$ .

Imagine all the elements of  $\mathcal{S}(\dots)$  together with all of their principal cyclic shifts being listed in an  $|\mathcal{S}(\dots)|$  by  $k$  array. Since the elements of  $R(\mathcal{P}(\dots))$  occur  $e$  times in each row, they occur  $e|\mathcal{S}(\dots)|$  times in the whole array. Since the columns of the array are permutations of  $\mathcal{S}(\dots) \supseteq R(\mathcal{P}(\dots))$ , the elements of  $R(\mathcal{P}(\dots))$  occur  $|R(\mathcal{P}(\dots))|$  times in each column and  $k|R(\mathcal{P}(\dots))|$  times in the whole array. Therefore,  $k|R(\mathcal{P}(\dots))| = e|\mathcal{S}(\dots)|$  and

$$|\mathcal{P}(\dots)| = |R(\mathcal{P}(\dots))| = \frac{e}{k} |\mathcal{S}(\dots)| = \frac{e}{k} \binom{k-e-1}{2k-l-m-1}_\infty \binom{k}{l} \binom{k}{m}.$$

This completes the new proof of Theorem 2.

## 7. The $q$ -enumeration

To begin with, we introduce the notation for some sets of dcd-animalinoes and for the generating functions of these sets. It is to be understood that these generating functions are in the following four variables:  $d$  = diagonals,  $x$  = 1/2 horizontal perimeter,  $y$  = 1/2 vertical perimeter,  $q$  = area. Instead of  $\varphi(d, x, y, q)$  we usually write  $\varphi$  or  $\varphi(d)$ .

**Notation 4.** As before,  $\Delta_1$  denotes the one-cell polyomino. By  $\mathcal{X}_\beta$  we denote the set of one-source dcd-polyominoes with  $\beta$  target cells.  $\mathcal{X}_{\alpha\beta}$  stands for the subset of  $\mathcal{X}_\beta$  containing those polyominoes which have  $\alpha$  cells in the next to last diagonal.

For the generating functions of the sets  $\mathcal{X}_\beta$  and  $\mathcal{X}_{\alpha\beta}$  we write  $f_\beta$  and  $f_{\alpha\beta}$ , respectively.

**Notation 5.** The number of diagonals, horizontal perimeter, vertical perimeter and area of a given dcd-animalino  $\bar{A}$  will be denoted by  $D(\bar{A})$ ,  $H(\bar{A})$ ,  $V(\bar{A})$  and  $\text{Area}(\bar{A})$ , respectively.

Let  $\bar{P}$  be an element of  $\mathcal{X}_e$ . Put  $z(0) = 0$ . For  $j \in \underline{e}$ , we define  $z(j) = \max\{z \in D(\bar{P}) : \text{the } z\text{th diagonal of } \bar{P} \text{ contains at most } j \text{ cells}\}$ .

Let  $i \in \underline{e}$  be fixed. Let  $\Pi_i(\bar{P})$  be the union of the diagonals  $z(i-1)+1, z(i-1)+2, \dots, z(i)$  of  $\bar{P}$ . Clearly,  $\Pi_i(\bar{P})$  is a dcd-animalino with at least  $i$  cells in every diagonal. Next, since the diagonals of  $\bar{P}$  grow at most by one, the first and the last diagonal of  $\Pi_i(\bar{P})$  contain exactly  $i$  cells each. (Incidentally, the animalino  $\Pi_i(\bar{P})$  in most cases happens to be a polyomino. To be specific,  $\Pi_i(\bar{P})$  is not a polyomino iff  $i > 1$  and  $z(i-1)+1 = z(i)$ .) Let  $\pi_i(\bar{P})$  be that what remains of  $\Pi_i(\bar{P})$  after we cut off the  $i-1$  top cells from each of its diagonals. It is easy to see that  $\pi_i(\bar{P}) \in \mathcal{X}_1$ .

Thus, we have associated with  $\bar{P} \in \mathcal{X}_e$  the  $e$ -tuple  $\pi(\bar{P}) = (\pi_1(\bar{P}), \dots, \pi_e(\bar{P})) \in \mathcal{X}_1^e$ . See Fig. 10.

Clearly,  $D(\bar{P}) = \sum_{i=1}^e D(\pi_i(\bar{P}))$ . The sequence of losses of  $\bar{P}$  can be obtained from those of  $\pi_i(\bar{P})$ 's by concatenation. Hence by Proposition 1(b),  $H(\bar{P}) = \sum_{i=1}^e H(\pi_i(\bar{P}))$  and  $V(\bar{P}) = \sum_{i=1}^e V(\pi_i(\bar{P}))$ . But with the area the things are different:

$$\text{Area}(\bar{P}) = \sum_{i=1}^e [\text{Area}(\pi_i(\bar{P})) + (i-1)D(\pi_i(\bar{P}))]. \quad (34)$$

The above properties of the decomposition  $\pi: \mathcal{X}_e \rightarrow \mathcal{X}_1^e$  lead us to the conclusion

$$f_e(d) = f_1(d)f_1(qd) \cdots f_1(q^{e-1}d) = f_1^{[e]}(d) \quad (\forall e \in \mathbb{N}). \quad (35)$$

We see that the function  $f_1$  is standing out among the  $f_e$ 's. So let us take a closer look at  $f_1$ .

Since the sets  $\{\Delta_1\}$  and  $\mathcal{X}_{e1}(e \in \mathbb{N})$  form a partition of  $\mathcal{X}_1$ , we have  $f_1(d) = dqxy + \sum_{e \geq 1} f_{e1}(d)$ . Then, Fig. 11 should suffice to convince the reader that  $f_{e1}(d) = dq(x+y+e-1)f_e(d)$ . These remarks together with (35) imply

$$f_1(d) = dq \left\{ xyf_1^{[0]}(d) + \sum_{e \geq 1} (x+y+e-1)f_1^{[e]}(d) \right\}. \quad (36)$$

At first glance, it seems that we shall have to be very ingenious to solve this challenging functional equation. But fortunately we need not bother too much, because Gessel's  $q$ -analog of the Lagrange inversion formula applies to our case. Indeed, Gessel [10] has proved the following theorem.

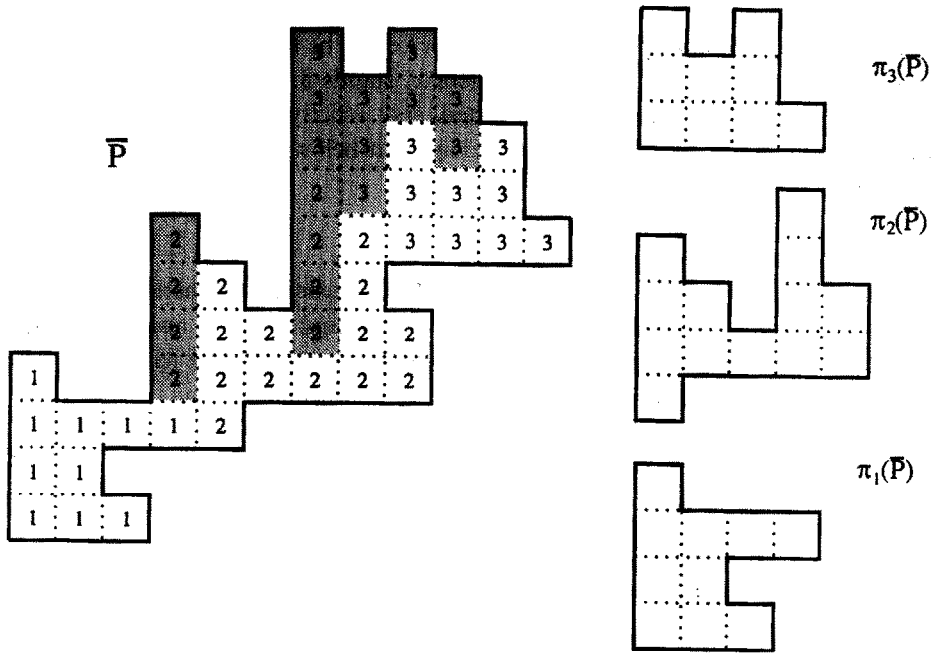


Fig. 10. The decomposition  $\pi$ . Left: the cells of  $\Pi_i(\bar{P})$  ( $i = 1, 2, 3$ ) are labeled  $i$ . The shaded cells are those being cancelled from the  $\Pi_i$ 's to obtain  $\pi_i$ 's. Right: the triple  $\pi_1(\bar{P})$ ,  $\pi_2(\bar{P})$ ,  $\pi_3(\bar{P})$ .

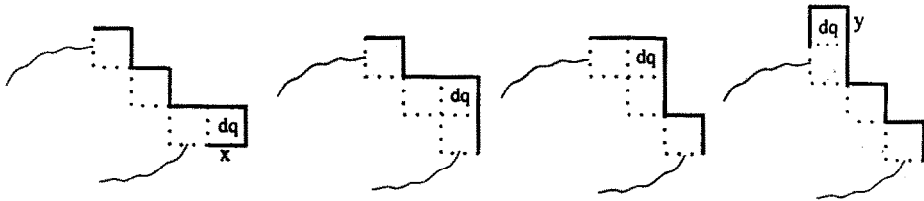


Fig. 11. The four types of elements of  $\mathcal{Z}_{3,1}$ . Their contributions to  $f_{3,1}$  are, from left to right,  $dqxf_3$ ,  $dqf_3$ ,  $dqyf_3$ . Thus,  $f_{3,1} = dq(x + y + 2)f_3$ .

**Theorem 4.** ( $q$ -analogue of the Lagrange inversion formula). Let  $f_1(d) = f_1(d, q)$  satisfy

$$f_1(d) = dq \sum_{e \geq 0} g_e f_1^{[e]}(d), \quad (37)$$

where the  $g_e$  are indeterminates. Let  $g(t) = \sum_{e \geq 0} g_e t^e$ . Then for  $n, e \geq 0$ ,

$$\langle d^n \rangle \frac{f_1^{[e]}(d)}{1 - \delta(d)} = q^{(n+1)n/2} \langle t^{n-e} \rangle \tilde{g}^{[n]}(q^{-1}t),$$

where

$$\delta(d) = d \sum_{i,j \geq 0} g_{i+j+1} f_1^{[i]}(d) \tilde{f}_1^{[j]}(d).$$

**Remark 1.** Let us say a few words about the lattice path interpretation that the function  $f_1$  is given in the proof of Theorem 4. Let  $\mathcal{W}$  be the family of all lattice paths over the step-set  $\{x_e: e \in \mathbb{Z}, e \geq -1\}$ , where  $x_e$  stands for  $(1, e)$ . Let  $\mathcal{F} \subseteq \mathcal{W}$  be the family of paths which, after having started, stay strictly below the horizontal line through the starting point and terminate one unit below that line. It is readily seen that every  $w \in \mathcal{F}$  has a unique factorization

$$w = w_1 w_2 \cdots w_e x_{e-1}, \quad (38)$$

where  $e \in \mathbb{N}_0$  and  $w_1, \dots, w_e \in \mathcal{F}$ . Next, let  $\varphi_1$  be the generating function for  $\mathcal{F}$  in the following (commuting) variables:  $d$  = number of steps;  $q$  = the number of lattice points above a path, and below or on the horizontal line through its starting point;  $g_e$  = the number of steps  $x_{e-1}$  ( $\forall e \in \mathbb{N}_0$ ). From (38) it follows that

$$\varphi_1(d) = dq \sum_{e \geq 0} g_e \varphi_1^{[e]}(d). \quad (39)$$

Finally, (37) and (39) imply  $\varphi_1 = f_1$ .

**Corollary 1.** If  $f_1(d) = f_1(d, q)$  satisfies (37), then for  $e \geq 0$ ,

$$f_1^{[e]}(d) = \frac{\sum_{n \geq 0} d^n q^{(n+1)n/2} \langle t^{n-e} \rangle \tilde{g}^{[n]}(q^{-1}t)}{\sum_{n \geq 0} d^n q^{(n+1)n/2} \langle t^n \rangle \tilde{g}^{[n]}(q^{-1}t)},$$

where  $g(t)$  is as in Theorem 4.

**Proof.**

$$f_1^{[e]}(d) = \frac{f_1^{[e]}(d)/[1 - \delta(d)]}{f_1^{[0]}(d)/[1 - \delta(d)]} = \frac{\sum_{n \geq 0} d^n \langle d^n \rangle f_1^{[e]}(d)/[1 - \delta(d)]}{\sum_{n \geq 0} d^n \langle d^n \rangle f_1^{[0]}(d)/[1 - \delta(d)]},$$

and the assertion follows from applying Theorem 4 to the numerator and denominator of the latter fraction.  $\square$

We see that Eq. (36) is of the type (37), with  $g_0 = xy$  and  $g_e = x + y + e - 1$  for  $e \in \mathbb{N}$ . Thus, we have

$$\begin{aligned} g(t) &= xy t^0 + \sum_{e \geq 1} (x + y + e - 1) t^e \\ &= xy \frac{[1 + (1-x)x^{-1}t][1 + (1-y)y^{-1}t]}{(1-t)^2}. \end{aligned} \quad (40)$$

Applying two identities for Gaussian polynomials given in Macdonald's book [12, p. 18, Example 3], we find that ( $\forall m \in \mathbb{Z}, n \in \mathbb{N}_0$ )

$$\begin{aligned} \langle t^m \rangle \tilde{g}^{[n]}(q^{-1}t) &= \sum_{i, j, k \geq 0} \begin{bmatrix} i+n-1 \\ n-1 \end{bmatrix} \begin{bmatrix} m+n-i-j-k-1 \\ n-1 \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} \\ &\quad \times q^{[j(j-1)+k(k-1)]/2 - mn} (1-x)^j x^{n-j} (1-y)^k y^{n-k}. \end{aligned} \quad (41)$$

Eq. (36), Corollary 1, (41) and the relation  $f_e(d) = f_1^{[e]}(d)$  put together give the following theorem.

**Theorem 5.** *We have*

$$f_e(d) = \frac{\sum_{i,j,k,n \geq 0} \begin{bmatrix} i+n-1 \\ n-1 \end{bmatrix} \begin{bmatrix} 2n-e-i-j-k-1 \\ n-1 \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} \cdot B}{\sum_{i,j,k,n \geq 0} \begin{bmatrix} i+n-1 \\ n-1 \end{bmatrix} \begin{bmatrix} 2n-i-j-k-1 \\ n-1 \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} \cdot C},$$

where

$$B = q^{[j(j-1)+k(k-1)-n(n-2e-1)]/2} d^n (1-x)^j x^{n-j} (1-y)^k y^{n-k},$$

and

$$C = q^{[j(j-1)+k(k-1)-n(n-1)]/2} d^n (1-x)^j x^{n-j} (1-y)^k y^{n-k}.$$

If we set  $x = y = 1$ , the formula for  $f_e(d)$  considerably simplifies, because only the  $j = k = 0$  terms survive.

**Corollary 2.** *In the case  $x = y = 1$  we have*

$$f_e(d) = \frac{\sum_{i,n \geq 0} \begin{bmatrix} i+n-1 \\ n-1 \end{bmatrix} \begin{bmatrix} 2n-e-i-1 \\ n-1 \end{bmatrix} q^{-n(n-2e-1)/2} d^n}{\sum_{i,n \geq 0} \begin{bmatrix} i+n-1 \\ n-1 \end{bmatrix} \begin{bmatrix} 2n-i-1 \\ n-1 \end{bmatrix} q^{-n(n-1)/2} d^n}.$$

Our Theorem 5 and Corollary 2 are an improvement of the related results due to Privman and Švrakić [14] and [15, p. 99]. Also note that the above formulas for  $f_e(d)$  are what Bousquet-Mélou and Fédou [1] would call formulas perfectly developed in  $d$ .

The extension of the  $q$ -enumeration to the case of  $r > 1$  sources presents some difficulties, because Theorem 4 can no longer be applied. In fact, these difficulties can be overcome by further use of the ideas of Gessel's proof, but this requires strenuous effort. So we shall stop here for the moment, and the case of  $r > 1$  sources will be treated in a future paper of ours.

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