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Linearization near an Integral Manifold*

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1. INTRODUCTION

Suppose the linear equation $y' = A(t)\,y$ has an exponential dichotomy and suppose the functions $f(t, x, y)$ and $g(t, x, y)$ are continuous and small Lipschitzian in x and y. Then the system of differential equations,

$$\begin{aligned} x' &= f(t, x, y), \\ y' &= A(t)\,y + g(t, x, y), \end{aligned}$$

has an integral manifold given by $y = v(t, x)$. In this paper, which continues the work in [1], we show that there is a continuous function of (t, x, y) that is a homeomorphism of the (x, y) space for each fixed t, sending the solutions of this system onto the solutions of the linearized system

$$\begin{aligned} x' &= f(t, x, v(t, x)), \\ y' &= A(t)\,y. \end{aligned}$$

The stationary point and periodic orbit are included as special cases, and thus the linearization theorems of Hartman and of Irwin [2] are generalized. Our treatment is also more general in that the vector fields we consider are just Lipschitzian and our equations are nonautonomous. This work also overlaps the work of Pugh and Shub [3] and, although more general in the above-mentioned aspects, it is less general in that the integral manifolds considered are not as general as theirs.

2. STATEMENT OF THE THEOREM

If x is in R^m and y is in R^n, then we denote their norms by $|x|$ and $|y|$ and, if A is an $n \times n$ matrix, we denote its operator norm by $|A|$. If $h(t)$

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is a continuous vector function of t in R, we make extensive use of the norms

$$\| h \| = \sup_{-\infty < t < \infty} \{| h(t)| e^{\alpha |t|}\},$$

$$\| h \|^+ = \sup_{t \geqslant 0}\{| h(t)| e^{\alpha t}\}, \qquad \| h \|^- = \sup_{t \leqslant 0}\{| h(t)| e^{-\alpha t}\},$$

where $\alpha > 0$.

Suppose that $A(t)$ is a matrix function defined and continuous for all t on the real line R. Then we say that the linear differential equation

$$y' = A(t)\, y \tag{1}$$

has an exponential dichotomy if it has a fundamental matrix $Y(t)$ such that

$$\begin{aligned} | Y(t)\, PY^{-1}(s)| &\leqslant K \exp(-2\alpha(t-s)) \qquad \text{for } s \leqslant t, \\ | Y(t)\, (I-P)\, Y^{-1}(s)| &\leqslant K \exp(-2\alpha(s-t)) \qquad \text{for } s \geqslant t, \end{aligned} \tag{2}$$

where P is a projection ($P^2 = P$) and K, α are positive constants.

Now we state our theorem.

THEOREM. *Suppose $A(t)$ is a continuous matrix function such that the linear equation* (1) *has a fundamental matrix $Y(t)$ satisfying* (2). *Suppose $f(t, x, y)$ is a continuous function of $R \times R^m \times R^n$ into R^m such that*

$$| f(t, x_1, y_1) - f(t, x_2, y_2)| \leqslant q_1 \,| x_1 - x_2 | + N \,| y_1 - y_2 | \tag{3}$$

for all t, x_1, x_2, y_1, y_2, and suppose $g(t, x, y)$ is a continuous function of $R \times R^m \times R^n$ into R^n such that

$$\begin{aligned} | g(t, x, y)| &\leqslant \mu, \\ | g(t, x_1, y_1) - g(t, x_2, y_2)| &\leqslant q_2[| x_1 - x_2 | + | y_1 - y_2 |] \end{aligned} \tag{4}$$

for all t, x, y, x_1, x_2, y_1, y_2. Then, if

$$q_1 \leqslant \alpha/4, \qquad q_2 \leqslant \min\{\alpha^2/32NK, \alpha/8K\},$$

(i) *there exists a continuous function $v(t, x)$ of $R \times R^m$ into R^n satisfying*

$$| v(t, x)| \leqslant K\mu\alpha^{-1}, \qquad | v(t, x_1) - v(t, x_2)| \leqslant 8K\alpha^{-1}q_2 \,| x_1 - x_2 |$$

for all t, x, x_1, x_2 such that $y = v(t, x)$ determines an integral manifold for the system,

$$\begin{aligned} x' &= f(t, x, y), \\ y' &= A(t)\, y + g(t, x, y); \end{aligned} \tag{5}$$

i.e., if $x(t)$ is a solution of the equation

$$x' = f(t, x, v(t, x)),$$

then $x(t)$, $v(t, x(t))$ is a solution of (5). *Moreover, if $x(t)$, $y(t)$ is a solution of* (5) *such that* $\sup_{-\infty<t<\infty} |y(t)| < \infty$, *then $y(t) = v(t, x(t))$ for all t.*

(ii) *There exists a continuous function*

$$H(t, x, y) = (H_1(t, x, y), H_2(t, x, y))$$

of $R \times R^m \times R^n$ onto $R^m \times R^n$ such that, if $x(t)$, $y(t)$ is a solution of (5), *then $H_1(t, x(t), y(t))$, $H_2(t, x(t), y(t))$ is a solution of the system*

$$\begin{aligned} x' &= f(t, x, v(t, x)), \\ y' &= A(t)\, y. \end{aligned} \tag{6}$$

For each fixed t, $H_t(x, y) = H(t, x, y)$ is a homeomorphism of $R^m \times R^n$.

$$L(t, x, y) = (L_1(t, x, y), L_2(t, x, y)) = H_t^{-1}(x, y)$$

is continuous in $R \times R^m \times R^n$ and, if $z(t)$, $w(t)$ is any solution of (6), *then $L_1(t, z(t), w(t))$, $L_2(t, z(t), w(t))$ is a solution of* (5).

We omit the proof of the first part of the theorem since it is a well-known result (see, for example, [4, 5]).

The basic idea of the proof of the second part of the theorem is as follows. Given a solution $x(t)$, $y(t)$ of (5), we find another solution $\hat{x}(t)$, $\hat{y}(t)$ of (5) on the "stable manifold" of (5), i.e., $\hat{y}(t)$ is bounded as $t \to +\infty$, such that $x(t)$, $y(t)$ approach $\hat{x}(t)$, $\hat{y}(t)$ exponentially as $t \to -\infty$. Then we find a solution $z(t)$ on the integral manifold of (5) such that $\hat{x}(t)$ approaches $z(t)$ exponentially as $t \to +\infty$, and we take $w(t)$ to be the unique solution of (1) such that $w(t) - y(t)$ is bounded on the real line. Then

$$H_1(t, x(t), y(t)) = z(t) \qquad \text{and} \qquad H_2(t, x(t), y(t)) = w(t).$$

Our basic tool in the proof is Lemma 2, which is proved in the next section.

3. Lemmas

To prove Lemma 2, we need the following.

LEMMA 1. *Let $f(t, x)$, $k(t, x)$ be continuous functions of $R \times R^m$ into R^m such that*

$$| k(t, x_1) - k(t, x_2)| \leqslant \nu \, | x_1 - x_2 | \tag{7}$$

for all t, x_1, x_2. Suppose the equation $x' = f(t, x)$ has a solution $x(t)$ such that

$$\sup_{t\geqslant 0}\{| f(t, x(t)) - k(t, x(t))| e^{\alpha t}\} < \infty.$$

Then, if $\nu < \alpha$, there exists a unique solution $z(t)$ of

$$x' = k(t, x) \tag{8}$$

such that

$$\sup_{t\geqslant 0}\{| z(t) - x(t)| e^{\alpha t}\} < \infty.$$

Moreover, if $k(t, x) = k(t, x, \zeta)$, $x(t) = x(t, \zeta)$, $f(t, x) = f(t, x, \zeta)$ depend continuously on a parameter ζ in a subset of some Euclidean space such that ν in (7) is independent of ζ and

$$\sup_{t\geqslant 0}\{| f(t, x(t, \zeta), \zeta) - k(t, x(t, \zeta), \zeta)| e^{\alpha t}\} < \infty$$

uniformly with respect to ζ, then the solution $z(t) = z(t, \zeta)$ found above is continuous in (t, ζ).

Proof. Put

$$M = \sup_{t\geqslant 0}\{| f(t, x(t)) - k(t, x(t))| e^{\alpha t}\}.$$

Let $\mathscr{S}$ be the set of continuous functions $z(t)$, defined on $t \geqslant 0$, such that $\| z - x \|^+ < \infty$. If z is in $\mathscr{S}$, we put

$$Z(t) = x(t) - \int_t^\infty [k(s, z(s)) - f(s, x(s))]\, ds,$$

so that

$$Z'(t) = k(t, z(t)).$$

Then

$$\begin{aligned} | Z(t) - x(t)| &\leqslant \int_t^\infty \nu\, | z(s) - x(s)| + | k(s, x(s)) - f(s, x(s))|\, ds \\ &\leqslant (\nu\alpha^{-1} \| z - x \|^+ + \alpha^{-1} M)\, e^{\alpha t} \end{aligned}$$

so that $\| Z - x \|^+ < \infty$. Hence Z is in $\mathscr{S}$. Also, if z_1, z_2, are in $\mathscr{S}$ with Z_1, Z_2 corresponding to them as above, then

$$Z_1(t) - Z_2(t) = - \int_t^\infty [k(s, z_1(s)) - k(s, z_2(s))]\, ds,$$

so that

$$\| Z_1 - Z_2 \|^+ \leqslant \alpha^{-1}\nu \| z_1 - z_2 \|^+.$$

Hence, the mapping $z \to Z$ is a contraction on $\mathscr{S}$, made into a complete metric space by $d(z_1, z_2) = \| z_1 - z_2 \|^+$, and so it has a unique fixed point $z(t)$ which is clearly a solution of (8) and satisfies $\| z - x \|^+ < \infty$.

Now let $z_1(t)$ be another solution of (8) such that $\| z_1 - x \|^+ < \infty$. Then, if n is a positive integer,

$$z_1(t) = z(t) + z_1(n) - z(n) - \int_t^n [k(s, z_1(s)) - k(s, z(s))] \, ds.$$

$\| z_1 - z \|^+ < \infty$ so that $| z_1(n) - z(n)| \leqslant \| z_1 - z \|^+ e^{-\alpha n} \to 0$ as $n \to \infty$. Hence, letting $n \to \infty$, we obtain

$$z_1(t) = z(t) - \int_t^\infty [k(s, z_1(s)) - k(s, z(s))] \, ds.$$

(The infinite integral exists since $\| z_1 - z \|^+ < \infty$.) Then

$$\| z_1 - z \|^+ \leqslant \alpha^{-1} \nu \| z_1 - z \|^+,$$

which implies $\| z_1 - z \|^+ = 0$ and so $z_1(t) = z(t)$ for all t.

Suppose now $k(t, x, \zeta)$, $f(t, x, \zeta)$, $x(t, \zeta)$ depend continuously on ζ. We replace $\mathscr{S}$ by the set of continuous functions $z(t, \zeta)$ defined for all ζ and $t \geqslant 0$ such that

$$\sup_{\zeta, t \geqslant 0} \{| z(t, \zeta) - x(t, \zeta)| \, e^{\alpha t}\} < \infty.$$

Then we prove, almost exactly as before, that the integral equation

$$z(t, \zeta) = x(t, \zeta) - \int_t^\infty [k(s, z(s, \zeta), \zeta) - f(s, x(s, \zeta), \zeta)] \, ds$$

has a unique solution $z(t, \zeta)$ in $\mathscr{S}$. Then, for each ζ, $z(t) = z(t, \zeta)$ is the unique solution of $x' = k(t, x, \zeta)$ such that $\sup_{t \geqslant 0}\{| z(t) - x(t, \zeta)| \, e^{\alpha t}\} < \infty$ and, moreover, $z(t, \zeta)$ is continuous in (t, ζ) for all ζ and $t \geqslant 0$. From standard theorems on continuous dependence on a parameter and initial values, it then follows that $z(t, \zeta)$ is continuous for all ζ and all t.

LEMMA 2. *Suppose $A(t)$ is a continuous matrix function such that the linear equation* (1) *has a fundamental matrix $Y(t)$ satisfying* (2) *and suppose f and g are continuous functions satisfying* (3) *and* (4).

Let $x(t) = x(t, \xi, \eta, \tau), y(t) = y(t, \xi, \eta, \tau)$ be the solution of (5) *such that $x(\tau) = \xi$, $y(\tau) = \eta$, and let $q(t) = q(t, \zeta)$ be, for each ζ in some Euclidean space, a solution of the linear equation* (1) *such that $q(t, \zeta)$ is continuous in (t, ζ). Then, if*

$$q_1 \leqslant \alpha/2, \qquad q_2 \leqslant \min\{\alpha/8K, \alpha^2/16NK\}, \tag{9}$$

there exists, for each fixed (ξ, η, τ, ζ) *such that* $y(t, \xi, \eta, \tau) - q(t, \zeta)$ *is bounded in* $t \geqslant 0$, *a unique solution* $\hat{x}(t) = \hat{x}(t, \xi, \eta, \tau, \zeta)$, $\hat{y}(t) = \hat{y}(t, \xi, \eta, \tau, \zeta)$ *of* (5) *such that*

$$\sup_{t\geqslant 0}\{|\hat{x}(t) - x(t)|\, e^{\alpha t}\} < \infty, \qquad \sup_{-\infty<t<\infty} |\hat{y}(t) - q(t)| < \infty.$$

Moreover, $\hat{x}(t, \xi, \eta, \tau, \zeta)$, $\hat{y}(t, \xi, \eta, \tau, \zeta)$, *are continuous in* $(t, \xi, \eta, \tau, \zeta)$, *and the inequalities*

$$\begin{aligned} |\hat{x}(t) - x(t)| &\leqslant 8NK\alpha^{-1}\, |\hat{y}(s) - y(s)| \exp(-\alpha(t-s)) \qquad \text{if} \quad s \leqslant t, \\ |\hat{y}(t) - y(t)| &\leqslant 4K\, |\hat{y}(s) - y(s)| \exp(-\alpha(t-s)) \qquad \text{if} \quad s \leqslant t \end{aligned} \tag{10}$$

are satisfied.

Proof. We first of all suppose that ξ, η, τ, ζ are fixed. Then let $\mathscr{S}$ be the set of all continuous vector functions $w(t)$ such that

$$\sup_{-\infty<t<\infty} |w(t) - q(t)| \leqslant K\mu\alpha^{-1} \qquad \text{and} \qquad \|w - y\|^{+} < \infty.$$

If w is in $\mathscr{S}$, then according to Lemma 1, since

$$|f(t, x(t), y(t)) - f(t, x(t), w(t))| \leqslant N\, \|w - y\|^{+}\, e^{-\alpha t} \qquad \text{for } t \geqslant 0,$$

there exists a unique solution $z(t)$ of $x' = f(t, x, w(t))$ such that $\|z - x\|^{+} < \infty$. $z(t)$ satisfies the integral equation

$$z(t) = x(t) - \int_t^\infty [f(s, z(s), w(s)) - f(s, x(s), y(s))]\, ds. \tag{11}$$

Now let $v(t)$ be the unique bounded solution of

$$v' = A(t)\, v + g(t, z(t), w(t)),$$

i.e.,

$$\begin{aligned} v(t) = &\int_{-\infty}^{t} Y(t)\, PY^{-1}(s)\, g(s, z(s), w(s))\, ds \\ &- \int_t^\infty Y(t)\, (I - P)\, Y^{-1}(s)\, g(s, z(s), w(s))\, ds, \end{aligned}$$

so that $|v(t)| \leqslant K\mu\alpha^{-1}$. Finally, put

$$W(t) = q(t) + v(t).$$

Then

$$| W(t) - q(t)| \leqslant K\mu\alpha^{-1},$$
$$W'(t) = A(t) W(t) + g(t, z(t), w(t)),$$

and

$$u(t) = W(t) - y(t) = W(t) - q(t) + q(t) - y(t)$$

is bounded in $t \geqslant 0$ and is a solution of

$$u' = A(t) u + h(t), \tag{12}$$

where

$$h(t) = g(t, z(t), w(t)) - g(t, x(t), y(t))$$

so that

$$\| h \|^+ \leqslant q_2[\| z - x \|^+ + \| w - y \|^+].$$

Then

$$u(t) = Y(t) PY^{-1}(0) u(0) + \int_0^t Y(t) PY^{-1}(s) h(s) ds$$
$$- \int_t^\infty Y(t) (I - P) Y^{-1}(s) h(s) ds,$$

so that

$$\| u \|^+ \leqslant K | u(0)| + 2K\alpha^{-1} \| h \|^+; \tag{13}$$

i.e., $\| W - y \|^+ < \infty$. Hence, W is in $\mathscr{S}$.

We make $\mathscr{S}$ into a complete metric space by giving it the metric $d(w_1, w_2) = \| w_1 - w_2 \|$, well defined since $\| w_1 - w_2 \|^+ < \infty$ and $| w_1(t) - w_2(t)| \leqslant 2K\mu\alpha^{-1}$ if $t \leqslant 0$. In the above, let z_1, z_2, W_1, W_2 correspond to w_1, w_2. Then, from (11),

$$z_1(t) - z_2(t) = - \int_t^\infty [f(s, z_1(s), w_1(s)) - f(s, z_2(s), w_2(s))] ds$$

for all t. So

$$| z_1(t) - z_2(t)| \leqslant q_1 \int_t^\infty | z_1(s) - z_2(s)| ds + N\alpha^{-1} \| w_1 - w_2 \| e^{-\alpha t}.$$

By an elementary Gronwall lemma-type argument, this implies

$$| z_1(t) - z_2(t)| \leqslant N(\alpha - q_1)^{-1} \| w_1 - w_2 \| e^{-\alpha t} \qquad \text{for all } t.$$

Thus

$$\| z_1 - z_2 \| \leqslant 2N\alpha^{-1} \| w_1 - w_2 \|.$$

Now $u(t) = W_1(t) - W_2(t)$ is the unique solution of (12) such that $\| u \| < \infty$, where

$$h(t) = g(t, z_1(t), w_1(t)) - g(t, z_2(t), w_2(t))$$

so that

$$\| h \| \leqslant q_2[\| z_1 - z_2 \| + \| w_1 - w_2 \|] \leqslant q_2(1 + 2N\alpha^{-1}) \| w_1 - w_2 \| .$$

We have

$$u(t) = \int_{-\infty}^{t} Y(t) PY^{-1}(s) h(s) \, ds - \int_{t}^{\infty} Y(t) (I - P) Y^{-1}(s) h(s) \, ds$$

and, hence,

$$\| u \| \leqslant 2K\alpha^{-1} q_2(1 + 2N\alpha^{-1}) \| w_1 - w_2 \| .$$

Thus,

$$\| W_1 - W_2 \| \leqslant \tfrac{1}{2} \| w_1 - w_2 \| .$$

That is, the mapping $w \to W$ of $\mathscr{S}$ into itself is a contraction and so has a unique fixed point $\hat{y}(t)$. Then

$$| \hat{y}(t) - q(t) | \leqslant K\mu\alpha^{-1},$$

and

$$\hat{y}'(t) = A(t) \hat{y}(t) + g(t, \hat{x}(t), \hat{y}(t)),$$

where $\hat{x}(t)$ is the unique solution of

$$\hat{x}'(t) = f(t, \hat{x}(t), \hat{y}(t))$$

such that $\| \hat{x} - x \|^+ < \infty$. So $\hat{x}(t), \hat{y}(t)$ is the required solution of (5).

Now we prove uniqueness. Let $x_1(t)$, $y_1(t)$ be another solution of (5) such that $\| x_1 - x \|^+ < \infty$ and $\sup_{-\infty < t < \infty} | y_1(t) - q(t) | < \infty$. Then $x_1(t)$, $y_1(t)$ and $\hat{x}(t)$, $\hat{y}(t)$ are two solutions of (5) such that

$$\| x_1 - \hat{x} \|^+ < \infty \qquad \text{and} \qquad \sup_{-\infty < t < \infty} | y_1(t) - \hat{y}(t) | < \infty.$$

Now $u(t) = y_1(t) - \hat{y}(t)$ is a bounded solution of

$$u' = A(t) u + h(t) + F(t, u),$$

where

$$h(t) = g(t, x_1(t), y_1(t)) - g(t, \hat{x}(t), y_1(t))$$

so that

$$| h(t) | \leqslant q_2 \| x_1 - \hat{x} \|^+ e^{-\alpha t} \qquad \text{if} \qquad t \geqslant 0$$

and

$$F(t, u) = g(t, \hat{x}(t), \hat{y}(t) + u) - g(t, \hat{x}(t), \hat{y}(t))$$

so that

$$| F(t, u) | \leqslant q_2 | u | .$$

Hence, by a slight generalization of Theorem 13 in [6, p. 80],

$$|y_1(t) - \hat{y}(t)| = |u(t)| \leqslant 2K\{|u(0)| + 2\alpha^{-1}\|h\|^+\} e^{-\alpha t} \quad \text{if} \quad t \geqslant 0.$$

Then, as in the uniqueness in Lemma 1, we can show that

$$x_1(t) - \hat{x}(t) = -\int_t^\infty [f(s, x_1(s), y_1(s)) - f(s, \hat{x}(s), \hat{y}(s))]\, ds.$$

Also, $u(t) = y_1(t) - \hat{y}(t)$ is the unique bounded solution of

$$u' = A(t)\, u + g(t, x_1(t), y_1(t)) - g(t, \hat{x}(t), \hat{y}(t)).$$

Thus, almost exactly as in our proof that the mapping was a contraction, it follows that $\|x_1 - \hat{x}\| < \infty$, $\|u\| < \infty$ and $\|x_1 - \hat{x}\| \leqslant 2N\alpha^{-1}\|u\|$, $\|u\| \leqslant 2Kq_2\alpha^{-1}[\|x_1 - \hat{x}\| + \|u\|]$. These imply $\|x_1 - \hat{x}\| = \|u\| = 0$ and hence the uniqueness.

We now prove the inequalities (10). Writing

$$\|h\|_s = \sup_{t \geqslant s}\{|h(t)| \exp(\alpha(t - s))\}$$

for fixed s, it follows again from the slight generalization of Theorem 13 in [6, p. 80] that, if $s \leqslant t$,

$$|\hat{y}(t) - y(t)| \leqslant 2K[|\hat{y}(s) - y(s)| + 2q_2\alpha^{-1}\|\hat{x} - x\|_s] \exp(-\alpha(t - s)).$$

So

$$\|\hat{y} - y\|_s \leqslant 2K\, |\hat{y}(s) - y(s)| + 4Kq_2\alpha^{-1}\|\hat{x} - x\|_s. \tag{14}$$

Now

$$\hat{x}(t) = x(t) - \int_t^\infty [f(u, \hat{x}(u), \hat{y}(u)) - f(u, x(u), y(u))]\, du.$$

Thus,

$$\|\hat{x} - x\|_s \leqslant \alpha^{-1}q_1\|\hat{x} - x\|_s + N\alpha^{-1}\|\hat{y} - y\|_s. \tag{15}$$

Combining (14) and (15), we obtain

$$\|\hat{x} - x\|_s \leqslant 8NK\alpha^{-1}\, |\hat{y}(s) - y(s)|, \qquad \|\hat{y} - y\|_s \leqslant 4K\, |\hat{y}(s) - y(s)|,$$

and hence the inequalities.

Finally, we prove the continuity of $\hat{x}(t, \xi, \eta, \tau, \zeta)$, $\hat{y}(t, \xi, \eta, \tau, \zeta)$. We choose any bounded subset $\mathscr{B}$ in the (ξ, η, τ, ζ) space. Then $q(0, \zeta) - y(0, \xi, \eta, \tau)$ is bounded in $\mathscr{B}$. We let $\mathscr{S}$ be the set of vector functions $w(t, \xi, \eta, \tau, \zeta)$, continuous in $R \times \mathscr{B}$, such that

$$|w(t, \xi, \eta, \tau, \zeta) - q(t, \zeta)| \leqslant K\mu\alpha^{-1}$$

and

$$\sup_{\mathscr{B}} \sup_{t \geqslant 0}\{|w(t, \xi, \eta, \tau, \zeta) - y(t, \xi, \eta, \tau)|\, e^{\alpha t}\} < \infty.$$

If w is in $\mathscr{S}$, we define $z(t) = z(t, \xi, \eta, \tau, \zeta)$, for each fixed (ξ, η, τ, ζ), as the unique solution of

$$x' = f(t, x, w(t, \xi, \eta, \tau, \zeta))$$

such that

$$\sup_{t \geq 0}\{| z(t) - x(t, \xi, \eta, \tau)| e^{\alpha t}\} < \infty,$$

where $z(t, \xi, \eta, \tau, \zeta)$ is continuous by Lemma 1, and then we put

$$W(t, \xi, \eta, \tau, \zeta) = q(t, \zeta) + v(t, \xi, \eta, \tau, \zeta),$$

where $v(t) = v(t, \xi, \eta, \tau, \zeta)$ is, for each fixed (ξ, η, τ, ζ), the unique bounded solution of

$$v' = A(t)\, v + g(t, z(t, \xi, \eta, \tau, \zeta), w(t, \xi, \eta, \tau, \zeta)),$$

where $v(t, \xi, \eta, \tau, \zeta)$ is continuous by Lemma 1 in [1]. Then we can prove W is in $\mathscr{S}$ in the same way as before, where we make essential use of the boundedness of $q(0, \zeta) - y(0, \xi, \eta, \tau)$ in (13). Finally, we prove that the mapping $w \to W$ is a contraction on $\mathscr{S}$ with the metric

$$d(w_1, w_2) = \sup_{R \times \mathscr{B}}\{| w_1(t, \xi, \eta, \tau, \zeta) - w_2(t, \xi, \eta, \tau, \zeta)| e^{\alpha t}\}.$$

The fixed point of $w \to W$ is, by the uniqueness in the first part, $\hat{y}(t, \xi, \eta, \tau, \zeta)$ which is therefore continuous in $R \times \mathscr{B}$, and it also follows that $\hat{x}(t, \xi, \eta, \tau, \zeta)$ is continuous in $R \times \mathscr{B}$. But this holds for any bounded set $\mathscr{B}$ in the (ξ, η, τ, ζ) space. So $\hat{x}$, $\hat{y}$ are continuous everywhere. Thus the proof of Lemma is complete.

Remark. Let $x(t) = x(t, \xi, \eta, \tau)$, $y(t) = y(t, \xi, \eta, \tau)$ be the solution of (5) such that $x(\tau) = \xi$, $y(\tau) = \eta$. Then

$$q(t) = Y(t)\, PY^{-1}(t) \left\{ y(t) - \int_{-\infty}^{t} Y(t)\, PY^{-1}(s)\, g(s, x(s), y(s))\, ds \right\}$$

is a solution of (1), $q(t) = q(t, \xi, \eta, \tau)$ is a continuous function of (t, ξ, η, τ), and

$$y(t) - q(t) = Y(t)\,(I - P)\, Y^{-1}(t)\, y(t) + \int_{-\infty}^{t} Y(t)\, PY^{-1}(s)\, g(s, x(s), y(s))\, ds$$

is bounded in $t \leq 0$. So, if the conditions (9) hold, it follows from the obvious version of Lemma 2 corresponding to $t \to -\infty$ that there exists a unique solution $\hat{x}(t) = \hat{x}(t, \xi, \eta, \tau)$, $\hat{y}(t) = \hat{y}(t, \xi, \eta, \tau)$ of (5) such that $\| \hat{x} - x \|^{-} < \infty$, $\sup_{-\infty < t < \infty} | \hat{y}(t) - q(t)| < \infty$, and $\hat{x}$, $\hat{y}$ are both continuous in (t, ξ, η, τ).

Now $|\hat{y}(t)| \leqslant |\hat{y}(t) - q(t)| + |q(t)|$ is bounded in $t \geqslant 0$ so that $\hat{x}(t), \hat{y}(t)$ is on the "stable manifold" of (5). Furthermore, from (10), if $s \geqslant t$,

$$\begin{aligned} |\hat{x}(t) - x(t)| &\leqslant 8NK\alpha^{-1} |\hat{y}(s) - y(s)| \exp(-\alpha(s-t)), \\ |\hat{y}(t) - y(t)| &\leqslant 4K |\hat{y}(s) - y(s)| \exp(-\alpha(s-t)). \end{aligned} \tag{16}$$

Conversely, let $x_1(t), y_1(t)$ be any other solution on the "stable manifold" of (5), i.e., $y_1(t)$ is bounded in $t \geqslant 0$, such that

$$\| x_1 - x \|^- < \infty, \sup_{t \leqslant 0} | y_1(t) - y(t)| < \infty.$$

Then, if $t \geqslant 0$, $|y_1(t) - q(t)| \leqslant |y_1(t)| + |q(t)|$, which is bounded, and, if $t \leqslant 0$, $|y_1(t) - q(t)| \leqslant |y_1(t) - y(t)| + |y(t) - q(t)|$, which is also bounded, so that $\sup_{-\infty<t<\infty} |y_1(t) - q(t)| < \infty$. Thus it follows from the uniqueness in Lemma 2 that $x_1(t) = \hat{x}(t)$, $y_1(t) = \hat{y}(t)$.

Summing up, we have proved that *if* $x(t) = x(t, \xi, \eta, \tau)$, $y(t) = y(t, \xi, \eta, \tau)$ *is the solution of* (5) *such that* $x(\tau) = \xi$, $y(\tau) = \eta$, *then under the conditions* (9) *on* q_1, q_2 *there is a unique solution* $\hat{x}(t) = \hat{x}(t, \xi, \eta, \tau)$, $\hat{y}(t) = \hat{y}(t, \xi, \eta, \tau)$ *of* (5) *for which* $\sup_{t\geqslant 0} |\hat{y}(t)| < \infty$ *such that*

$$\sup_{t\leqslant 0} \{| \hat{x}(t) - x(t)| e^{-\alpha t}\} < \infty, \qquad \sup_{t \leqslant 0} |\hat{y}(t) - y(t)| < \infty.$$

Moreover, the inequalities (16) *are satisfied and* $\hat{x}$, $\hat{y}$ *are continuous functions of* (t, ξ, η, τ).

4. Proof of the Theorem

Let $x(t), y(t)$ be the solution of (5) such that $x(\tau) = \xi$, $y(\tau) = \eta$. Then there exists a unique solution $\hat{x}(t), \hat{y}(t)$ on the "stable manifold" of (5) such that $\| \hat{x} - x \|^- < \infty$, $\sup_{t\leqslant 0} |\hat{y}(t) - y(t)| < \infty$. Now, using Lemma 2 with $q(t) = 0$, let $z(t), w_1(t)$ be the unique solution of (5) such that $\| z - \hat{x} \|^+ < \infty$, $\sup_{-\infty<t<\infty} | w_1(t)| < \infty$. Finally, take $w(t)$ as the unique solution of the linear equation (1) such that $\sup_{-\infty<t<\infty} | w(t) - y(t)| < \infty$. Then

$$\begin{aligned} w(t) = y(t) - \Big[&\int_{-\infty}^{t} Y(t) PY^{-1}(s) g(s, x(s), y(s))\, ds \\ &- \int_t^{\infty} Y(t)(I-P) Y^{-1}(s) g(s, x(s), y(s))\, ds \Big], \end{aligned}$$

so that

$$| w(t) - y(t)| \leqslant K\mu\alpha^{-1}. \tag{17}$$

We put $H_1(\tau, \xi, \eta) = z(\tau)$, $H_2(\tau, \xi, \eta) = w(\tau)$. From Lemma 2, $z(\tau)$ is a continuous function of $(\tau, \hat{x}(\tau), \hat{y}(\tau))$ and $\hat{x}(\tau), \hat{y}(\tau)$ are, in turn, continuous functions of (τ, ξ, η). Also, $w(\tau)$ is a continuous function of (τ, ξ, η). Thus H_1, H_2 are both continuous, and from (17) we have

$$| H_2(\tau, \xi, \eta) - \eta | \leqslant K\mu\alpha^{-1}.$$

Furthermore, it is clear from our definitions of H_1, H_2 that

$$H_1(t, x(t), y(t)) = z(t), \qquad H_2(t, x(t), y(t)) = w(t) \quad \text{for all } t.$$

Conversely, let $z(t), w(t)$ be the solution of (6) such that $z(\tau) = \xi$, $w(\tau) = \eta$. Then, applying Lemma 2 to $z(t)$, $v(t, z(t))$ as the solution of (5) and to $Y(t)\, PY^{-1}(t)\, w(t)$ as the solution of (1), there exists a unique solution $\tilde{z}(t)$, $\tilde{w}(t)$ of (5) such that

$$\| \tilde{z} - z \|^+ < \infty, \qquad \sup_{-\infty<t<\infty} | \tilde{w}(t) - Y(t)\, PY^{-1}(t)\, w(t)| < \infty.$$

Finally, applying the version of Lemma 2 corresponding to $t \to -\infty$ to $\tilde{z}(t)$, $\tilde{w}(t)$ as the solution of (5) and $w(t)$ as the solution of (1), there exists a unique solution $x(t)$, $y(t)$ of (5) such that

$$\| x - \tilde{z} \|^- < \infty, \qquad \sup_{-\infty<t<\infty} | y(t) - w(t)| < \infty.$$

Now we put $L_1(\tau, \xi, \eta) = x(\tau)$, $L_2(\tau, \xi, \eta) = y(\tau)$. L_1, L_2 are continuous as H_1, H_2 are and, moreover,

$$L_1(t, z(t), w(t)) = x(t), \qquad L_2(t, z(t), w(t)) = y(t) \quad \text{for all } t.$$

All that remains to prove is that $H = (H_1, H_2)$ and $L = (L_1, L_2)$ are inverses of each other for fixed t; i.e., $L_t \circ H_t = H_t \circ L_t =$ the identity, where

$$H_t(x, y) = (H_1(t, x, y), H_2(t, x, y))$$

and

$$L_t(x, y) = (L_1(t, x, y), L_2(t, x, y)).$$

So let $x(t)$, $y(t)$ be a solution of (5) with

$$z(t) = H_1(t, x(t), y(t)), \qquad w(t) = H_2(t, x(t), y(t))$$

the corresponding solution of (6). We show, first, that $\hat{x}(t) = \tilde{z}(t)$, $\hat{y}(t) = \tilde{w}(t)$, where $\hat{x}(t)$, $\hat{y}(t)$ comes from $x(t)$, $y(t)$ and $\tilde{z}(t)$, $\tilde{w}(t)$ comes from $z(t)$, $w(t)$ as in the above. Since

$$\| \tilde{z} - \hat{x} \|^+ \leqslant \| \tilde{z} - z \|^+ + \| z - \hat{x} \|^+$$

and

$$\begin{aligned} | \tilde{w}(t) - \hat{y}(t)| \leqslant{} & | \tilde{w}(t) - Y(t)\, PY^{-1}(t)\, w(t)| + | Y(t)\, PY^{-1}(t)\, (w(t) - y(t))| \\ & + \left| \int_{-\infty}^{t} Y(t)\, PY^{-1}(s)\, g(s, x(s), y(s))\, ds \right| \\ & + \Big| \hat{y}(t) - Y(t)\, PY^{-1}(t) \\ & \times \left\{ y(t) - \int_{-\infty}^{t} Y(t)\, PY^{-1}(s)\, g(s, x(s), y(s))\, ds \right\} \Big| \end{aligned}$$

so that $\| \tilde{z} - \hat{x} \|^+ < \infty$ and $\sup_{-\infty<t<\infty} | \tilde{w}(t) - \hat{y}(t)| < \infty$, it follows as in the proof of the uniqueness in Lemma 2 that $\hat{x}(t) = \tilde{z}(t)$, $\hat{y}(t) = \tilde{w}(t)$. Now, denoting the solution of (5) corresponding to the solution $z(t)$, $w(t)$ of (6) by

$$\bar{x}(t) = L_1(t, z(t), w(t)), \qquad \bar{y}(t) = L_2(t, z(t), w(t)),$$

we have

$$\| \bar{x} - x \|^- \leqslant \| \bar{x} - \tilde{z} \|^- + \| \hat{x} - x \|^- < \infty$$

and

$$\sup_{-\infty<t<\infty} | \bar{y}(t) - y(t)| \leqslant \sup_{-\infty<t<\infty} | \bar{y}(t) - w(t)| + \sup_{-\infty<t<\infty} | w(t) - y(t)| < \infty$$

so that it follows again that $\bar{x}(t) = x(t)$, $\bar{y}(t) = y(t)$. This implies that $L_t \circ H_t$ in the identity for all t. Similarly, we can prove that $H_t \circ L_t$ is the identity for all t. Thus the proof of the theorem is complete.

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