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Asymptotic Bounds for the Eigenvalues of Vibrating Systems

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1. INTRODUCTION

The characteristic frequencies of a vibrating string with fixed ends, of length l, and with nonnegative integrable density $\rho(x)$, $x \in [0, l]$, are determined by the ordered eigenvalues

$$0 < \lambda_1[\rho] < \lambda_2[\rho] < \cdots < \lambda_n[\rho] < \cdots$$

of the differential system

$$\begin{aligned} u'' + \lambda\rho(x)u &= 0, \qquad x \in [0, l], \\ u(0) = u(l) &= 0. \end{aligned} \tag{1}$$

We consider these eigenvalues as functionals of the density ρ. In previous work, the extreme values of $\lambda_n[\rho]$ were investigated under the condition that the density function ρ be restricted to lie in a given class of functions. For example, the class might be all the bounded, concave, or convex functions with the condition that $\int_0^l \rho(x)\, dx = M$, a constant. It has been shown that there exists functions ρ_n^+ and ρ_n^- in certain such classes of functions E such that $\lambda_n[\rho_n^-] \leqslant \lambda_n[\rho] \leqslant \lambda_n[\rho_n^+]$ for all $\rho \in E$. But for many such classes, the extremizing functions ρ_n^+, ρ_n^-, $(n > 1)$ have been completely determined only up to a given subclass of functions. (See [1], [2].) From these, the extreme values can hopefully be computed for a given n, if n is not too large.

In this paper, a general method is presented for finding the limit of convergent subsequences of $\{\rho_n^+\}$ and $\{\rho_n^-\}$ when they exist. In these cases, it is then possible to find information concerning the extreme values $\lambda_n[\rho_n^+]$ and $\lambda_n[\rho_n^-]$ and the corresponding functions ρ_n^+, ρ_n^- for n sufficiently large. There is a close connection between bounds for eigenvalues and bounds for the number of zeros of solutions of second order equations. Our results give information concerning this problem as a by-product.

In particular, we will consider the following classes of functions: A given measurable function ρ defined on $[0, l]$ is defined to be in the

(i) class K if $0 \leqslant \rho(x) \leqslant H$ and $\int_0^l \rho(x)\,dx = M$, where H and M are constants,

(ii) monotone class E_1 if $\rho \in K$ and is monotone increasing

(iii) convex class E_2 if $\rho \in K$ and is convex, i.e.,

$$\rho[\theta x + (1-\theta)\,y] \leqslant \theta\rho(x) + (1-\theta)\,\rho(y), \qquad (0 \leqslant \theta \leqslant 1).$$

(iv) concave class E_3 if $\rho \in K$ and is concave (a concave function is automatically bounded).

(v) Lipschitz class E_4 if $\rho \in K$ and satisfies a Lipschitz condition with constant L.

In Section 2, a theorem is proved which relates limits of subsequences of the maximizing functions $\{\rho_n{}^+\}$ to the function $\rho_0{}^+$ which minimizes

$$J[\rho] = \int_0^l \sqrt{\rho(x)}\,dx$$

over a given subclass of functions in K. A corresponding result is proved for $\rho_n{}^-$. This result is then applied to the classes E_i $(i = 1, 2, 3, 4)$ to show that the minimizing functions $\rho_n{}^-$ have the property that $\lim_{n\to\infty} \rho_n{}^-(x) = M/l$ almost everywhere on $[0, l]$. Lower bounds for $\lambda_n[\rho]$ for n sufficiently large and $\rho \in E_i$ $(i = 1, 2, 3, 4)$ are obtained as a corollary to this result.

In Section 3, some results are given concerning upper bounds for $\lambda_n[\rho]$, $(n = 1, 2, \ldots)$ when ρ is in the Lipschitz class E_4. The main theorem of Section 2 is then applied to this case to give information about the maximizing functions $\rho_n{}^+$ for large n. In Section 4, some properties of the maximizing functions $\rho_n{}^+$ for the classes of functions E_i $(i = 1, 2, 3)$ are given for n sufficiently large, and in Section 5 some applications are discussed.

2. Asymptotic Bounds

The main results in this paper are based on the following:

THEOREM 1. *Let C be any subclass of functions from the class*

$$K = \left\{\rho : 0 \leqslant \rho \leqslant H, \int_0^l \rho(x)\,dx = M\right\}$$

which is sequentially compact in the sense of pointwise convergence almost everywhere. Then there exist functions $\rho_n{}^-$, $\rho_n{}^+ \in C$ such that

$$\lambda_n[\rho_n{}^-] \leqslant \lambda_n[\rho] \leqslant \lambda_n[\rho_n{}^+] \tag{2}$$

for all $\rho \in C$. *Furthermore, if* $\{\rho_{n_k}^+\}$ *is a subsequence of* $\{\rho_n^+\}$ *such that* $\lim \rho_{n_k}^+ = \rho_0^+$, *then* ρ_0^+ *is a minimizing function for* $J[\rho] = \int_0^l \sqrt{\rho(x)}\, dx$ *for all* $\rho \in C$. *The corresponding statement holds for the maximum of* $J[\rho]$ *over* C *and the minimizing functions* $\{\rho_n^-\}$.

Proof. The existence of the functions ρ_n^+ and ρ_n^- is a consequence of the sequential compactness of the class C (see [3], p. 166). To show that $J[\rho] \leqslant J[\rho_0^-]$ for all $\rho \in C$, let $\{\rho_{n_k}^-\}$ be a convergent subsequence of $\{\rho_n^-\}$. Then by Egoroff's theorem, it is known that for any $\epsilon > 0$ there is a set $A_\epsilon \subset [0, l]$ of measure less than ϵ and an integer N_ϵ such that $|\rho_0^-(x) - \rho_{n_k}^-(x)| < \epsilon$ if $x \notin A_\epsilon$ and $k \geqslant N_\epsilon$. Defining functions U_ϵ and L_ϵ by

$$L_\epsilon(x) = \begin{cases} 0 & x \in A_\epsilon, \\ \max\{0, \rho_0^-(x) - \epsilon\} & x \notin A_\epsilon, \end{cases}$$

and

$$U_\epsilon(x) = \begin{cases} H & x \in A_\epsilon, \\ \rho_0^-(x) + \epsilon & x \notin A_\epsilon, \end{cases}$$

we have $L_\epsilon(x) \leqslant \rho_{n_k}^-(x) \leqslant U_\epsilon(x)$ for $x \in [0, 1]$ and $k \geqslant N_\epsilon$. By the comparison theorem for eigenvalues ([4], p. 411), it follows that

$$\lambda_{n_k}[L_\epsilon] \geqslant \lambda_{n_k}[\rho_{n_k}^-] \geqslant \lambda_{n_k}[U_\epsilon].$$

Recalling that

$$\lim \lambda_n[\rho]/n^2 = \left(\pi \Big/ \int_0^l \sqrt{\rho(x)}\, dx\right)^2, \tag{3}$$

we see that these inequalities imply that

$$\frac{\pi^2}{(\int_0^l \sqrt{L_\epsilon(x)}\, dx)^2} \geqslant \limsup_{k\to\infty} \frac{\lambda_{n_k}(\rho_{n_k}^-)}{n_k^2} \geqslant \liminf_{k\to\infty} \frac{\lambda_{n_k}(\rho_{n_k}^-)}{n_k^2} \geqslant \frac{\pi^2}{(\int_0^l \sqrt{U_\epsilon(x)}\, dx)^2}. \tag{4}$$

By the dominated convergence theorem $\int_0^l \sqrt{L_\epsilon(x)}\, dx$ and $\int_0^l \sqrt{U_\epsilon(x)}\, dx$ converge to $\int_0^l \sqrt{\rho_0^-(x)}\, dx$ as $\epsilon \to 0$. Inequality (4) then implies

$$\lim_{k\to\infty} \frac{\lambda_{n_k}[\rho_{n_k}^-]}{n_k^2} = \left(\frac{\pi}{\int_0^l \sqrt{\rho_0^-(x)}\, dx}\right)^2. \tag{5}$$

Now, by (2) we have $\lambda_{n_k}(\rho_{n_k}^-) \leqslant \lambda_{n_k}[\rho]$ for all $\rho \in C$. Dividing this inequality by n_k^2 and taking the limit as $k \to \infty$, we obtain from (3) and (5) the inequality

$$\left(\frac{\pi}{\int_0^l \sqrt{\rho_0^-(x)}\, dx}\right)^2 \leqslant \left(\frac{\pi}{\int_0^l \sqrt{\rho(x)}\, dx}\right)^2.$$

Hence, $J[\rho] \leqslant J[\rho_0^-]$ for all $\rho \in C$. The proof of the remainder of the theorem is analogous and will be omitted.

To apply Theorem 1 to a given class of functions, it must be shown that the class is sequentially compact. For the cases to be considered here, this is assured by the following

LEMMA. *Let the sequence of functions $\{\rho_n\}$ defined on $[0, l]$ be uniformly bounded and of uniformly bounded variation. Then there is a subsequence $\{\rho_{n_k}\}$ and a function ρ_0 of bounded variation on $[0, l]$ such that $\rho_{n_k}(x) \to \rho_0(x)$ almost everywhere as $k \to \infty$.*

For a proof, see [5], p. 425. We now show that Theorem 1 and the above lemma yield the following

THEOREM 2. *Let C be any one of the classes of functions E_i $(i = 1, 2, 3, 4)$. Let ρ_n^- denote the minimizing function for $\lambda_n[\rho]$ over the class C. Then*

$$\lim_{n\to\infty} \rho_n^-(x) = M/l$$

almost everywhere.

Proof. The lemma yields the fact that each of the E_i and, hence, also C are sequentially compact. Suppose then that $\{\rho_{n_k}^-\}$ is a convergent subsequence of $\{\rho_n^-\}$ and ρ_0^- is the limit of this subsequence. Then by Theorem 1, ρ_0^- maximizes $J[\rho]$ over the class C. But Schwarz' inequality implies

$$J^2[\rho] = \left(\int_0^l \sqrt{\rho(x)}\, dx\right)^2 \leqslant \int_0^l \rho(x)\, dx \cdot \int_0^l dx = M \cdot l$$

with equality if, and only if, $\rho \equiv M/l$. It follows that $\rho_0^-(x) \equiv M/l$ and that every convergent subsequence of $\{\rho_n^-\}$ converges to this value. Consequently, we must have $\lim_{n\to\infty} \rho_n^-(x) = M/l$ almost everywhere.

It is not possible, in general, to replace the convergence almost everywhere by ordinary convergence. It can be shown that for the class of concave functions, for example, $\rho_n^-(0) = \rho_n^-(l) = 0$ for all n.

This theorem has the following immediate

COROLLARY. *For any* $\epsilon > 0$, *the inequality*

$$\lambda_n[\rho] \geqslant \lambda_n[\rho_n^-] \geqslant \lambda_n[\epsilon + M/l] = \frac{n^2\pi^2}{l^2}\left(\frac{1}{\epsilon + M/l}\right)$$

holds for n sufficiently large for $\rho \in E_i$ $(i = 1, 2, 3, 4)$.

For by looking at the form of the minimizing ρ_n^- for each of the class E_i (see [1], [6]) it can be seen that for arbitrary $\epsilon > 0$, $\rho_n^-(x) \leqslant M/l + \epsilon$ for all $x \in [0, l]$ when n is sufficiently large. The comparison theorem then yields the result.

3. THE LIPSCHITZ CLASS

In this section, we consider the problem of finding upper bounds for $\lambda_n[\rho]$ when ρ belongs to the Lipschitz class E_4. We first give an upper bound for $\lambda_1[\rho]$. As a by-product, we obtain an upper bound for the lowest eigenvalue of a string with one end fixed and the other end free. These results will be given in terms of the fundamental pair of solutions U_1 and U_2 of Airy's equation $u'' + xu = 0$, where $U_1(0) = 1$, $U_1'(0) = 0$ and $U_2(0) = 0$, $U_2'(0) = 1$ (see [7]).

THEOREM 3. *Let* $\lambda_1[\rho]$ *be the lowest eigenvalue of the system* (1) *where* $\rho \in E_4$, i.e., $|\rho(x) - \rho(y)| \leqslant L\,|x - y|$ *and* $\int_0^l \rho(x)\,dx = M$. *Then*

$$\lambda_1[\rho]\, l^3 L \leqslant S_1^3(Ll^2/M), \tag{6}$$

where $S_1(K)$ *is the least positive root of*

$$U_2\left[S\left(\frac{1}{4} + \frac{1}{K}\right)\right] U_1'\left[S\left(\frac{1}{K} - \frac{1}{4}\right)\right] - U_1\left[S\left(\frac{1}{K} + \frac{1}{4}\right)\right] U_2'\left[S\left(\frac{1}{K} - \frac{1}{4}\right)\right] = 0 \tag{7}$$

when $Ll^2 \leqslant 4M$ *and of*

$$U_1(S/\sqrt{K}) = 0 \tag{8}$$

when $Ll^2 > 4M$. *Moreover, equality holds if, and only if,* $\rho \equiv \rho_1^+$ *where*

$$\rho_1^+(x) = \begin{cases} L(l/4 - x) + M/l & 0 \leqslant x \leqslant l/2, \\ \rho_1^+(l - x) & l/2 \leqslant x \leqslant l, \end{cases} \tag{9}$$

when $Ll^2 \leqslant 4M$ *and*

$$\rho_1^+(x) = \begin{cases} -Lx + \sqrt{ML}, & 0 \leqslant x \leqslant \sqrt{M/L}, \\ 0, & \sqrt{M/L} \leqslant x \leqslant l/2, \\ \rho_1^+(l-x), & l/2 \leqslant x \leqslant l, \end{cases} \tag{10}$$

when $Ll^2 > 4M$.

Proof. Let $\rho \in E_4$ and consider the function ρ_1 defined by

$$\rho_1(x) = \tfrac{1}{2}[\rho(x) + \rho(l-x)].$$

Then $\rho_1 \in E_4$ and is symmetric about the point $x = l/2$. Let $\lambda_1[\rho_1]$ be the lowest eigenvalue of the system (1) with ρ replaced by ρ_1. It is shown in [3], p. 174, that $\lambda_1[\rho_1] \geqslant \lambda_1[\rho]$. To prove the theorem, we show that $\lambda_1[\rho_1^+] \geqslant \lambda_1[\rho_1]$ where ρ_1^+ is defined by (9) or (10) depending on the magnitude of l^2L/M. Because of the symmetry of ρ_1, $\lambda_1[\rho_1]$ is also the lowest eigenvalue of the system

$$\begin{gathered} u'' + \mu\rho(x)u = 0, \qquad x \in [0, l/2], \\ u(0) = u'(l/2) = 0, \end{gathered} \tag{11}$$

when ρ is taken to be equal to ρ_1 on the interval $[0, l/2]$. A similar statement holds for $\lambda_1[\rho_1^+]$. Hence, we may use the following comparison theorem due to Nehari [8].

THEOREM. *Let* ρ_1 *and* q *be nonnegative continuous functions defined on* $[0, l/2]$ *such that*

$$\int_x^{l/2} \rho_1(x)\,dx \geqslant \int_x^{l/2} q(x)\,dx. \tag{12}$$

If $\mu_1[\rho_1]$ *and* $\mu_1[q]$ *are the lowest eigenvalues of the system* (11) *with* ρ *replaced by* ρ_1 *and* q, *respectively, then*

$$\mu_1[\rho_1] \leqslant \mu_1[q]. \tag{13}$$

To show that (12) holds, we note that $\int_0^{l/2} \rho_1(x)\,dx = \int_0^{l/2} q(x)\,dx = M/2$. Since ρ_1 and q are nonnegative and continuous, they must have at least one positive common value for some point x. If a is such a point, the Lipschitz condition implies that

$$-L(x-a) \leqslant \rho_1(x) - \rho_1(a) \leqslant L(x-a)$$

when $x > a$. Consequently, $q(a) = \rho_1(a)$ implies

$$\rho_1(x) \geqslant L(a-x) + q(a) = q(x)$$

for $x > a$ and wherever $q(x) > 0$. Similarly, for all $x < a$ it follows that $\rho(x) \leqslant L(x - a) + q(a) = q(x)$. Thus, the inequality (12) holds and $\mu_1[\rho_1] \leqslant \mu_1[q]$. Solving the system (11) for $\mu_1[q]$ in terms of Airy functions, we get Eq. (7) or (8), depending on the magnitude of l^2L/M, for $S_1(l^2L/M)$, and find that $\mu_1[q] = \lambda_1[q] = S_1^3(l^2L/M)/l^3L$. The inequality (6) then follows and the theorem is proved.

It is clear that our proof also yields an upper bound for $\mu_1[\rho]$ when ρ is restricted to satisfy the Lipschitz condition and $\int_0^{l/2} \rho(x)\, dx = M/2$. In [6] the problem of finding lower bounds for $\lambda_n[\rho]$ when $\rho \in E_4$ was solved completely. It was shown that the density ρ_n^- which minimizes $\lambda_n[\rho]$ over the class E_4 is a regular "saw-tooth" function of n teeth with the point of each tooth coinciding with an antinode of the corresponding eigenfunction. One might conjecture, in the light of previous results of Krein [3] and Schwarz [9], that the density which maximizes $\lambda_n[\rho]$ over the class E_4 would again be a regular saw-tooth function with the point of each tooth coinciding with a node of the corresponding eigenfunction. We now show that this is not the case.

We first prove the following

THEOREM 4. *Let $\lambda_n[\rho]$ be the nth eigenvalue of the system (1) where $\rho \in E_4$, the Lipschitz class. Then there is a function ρ_n^+ in the Lipschitz class such that $\lambda_n[\rho_n^+] \geqslant \lambda_n[\rho]$ where ρ_n^+ is a nonnegative continuous saw-tooth function of the form*

$$\rho_n^+(x) = \begin{cases} r_0 - Lx, & 0 \leqslant x \leqslant \beta_1, \\ r_1 + L(x - \alpha_1), & \beta_1 \leqslant x \leqslant \alpha_1, \\ r_1 - L(x - \alpha_1), & \alpha_1 \leqslant x \leqslant \beta_2, \\ \cdots & \cdots \\ r_n + L(x - l), & \beta_n \leqslant x \leqslant l = \alpha_n \end{cases}$$

with $0 \leqslant \alpha_1 \leqslant \beta_1 \leqslant \alpha_2 \cdots < \beta_n \leqslant \alpha_n = l$.

It should be noted that α_i, β_i are not necessarily equally spaced and that they and the r_i are such that $\rho_n^+(x)$ is continuous. Also note that some of the intervals (α_i, β_i), and (β_i, α_i) may be degenerate.

The existence of ρ_n^+ such that $\lambda_n[\rho_n^+] = \max_{E_4} \lambda_n[\rho]$ is proved in Theorem 1. To show that ρ_n^+ has the specified form, we use the first variation $\delta\lambda_n[\rho]$ of $\lambda_n[\rho]$ within the class of functions E_4. It is shown in [2], p. 1186 that this is given by

$$\delta\lambda_n[\rho] = -\lambda_n[\rho] \int_0^l (\delta\rho)\, u_n^2\, dx,$$

where u_n is the normalized eigenfunction corresponding to $\lambda_n[\rho]$ and $\rho + \delta\rho \in E_4$, i.e., $\delta\rho$ is such that $\int_0^l \delta\rho\, dx = 0$ and $\rho + \delta\rho$ satisfies the

Lipschitz condition of E_4. Applying this variational formula to $\lambda_n[\rho_n^+]$, we must have

$$\delta\lambda_n[\rho_n^+] = -\lambda_n[\rho_n^+] \int_0^l (\delta\rho_n^+)\, u_n^2\, dx \leqslant 0.$$

The theorem will be proved provided we show that if ρ_n^+ does not have the form specified in the theorem, then there is a $\delta\rho_n^+$ within E_4 such that $\delta\lambda_n[\rho_n^+] > 0$. Suppose there is a nonnegative function $r(x)$, $x \in [0, l]$ of the form defined by (14) such that $r \in E_4$ and such that

$$\int_0^l r u_n^2\, dx < \int_0^l \rho_n^+ u_n^2\, dx.$$

Then $\delta\rho_n^+(x) = \epsilon[r(x) - \rho_n^+(x)]$ has the property that

$$\int_0^l \delta\rho_n^+ u_n^2\, dx < 0,$$

where $0 < \epsilon < 1$. Also, $\rho_n^+ + \delta\rho_n^+$ satisfies the Lipschitz condition of E_4 and it follows that $\delta\lambda_n[\rho_n^+] > 0$.

It remains to show there is such a function $r(x)$ in E_4. Rather than carry out the details of construction of such a function, which are more tedious than enlightening, we refer the reader to a similar construction for the class of convex functions which is described in [2], p. 1193.

The following theorem yields some information about the nature of the maximizing functions ρ_n^+ for n sufficiently large.

THEOREM 5. *Let ρ_n^+ be the maximizing function of $\lambda_n[\rho]$ over the Lipschitz class E_4 and let $\{\rho_{n_k}^+\}$ be a convergent subsequence of the sequence of functions $\{\rho_n^+\}$. Then $\lim_{k\to\infty} \rho_{n_k}^+(x)$ equals $r(x)$ or $r(l - x)$, where r is defined by*

$$r(x) = L(x - l/2) + M/l$$

if $L < 2M/l^2$, and by

$$r(x) = \begin{cases} 0 & 0 \leqslant x \leqslant l - \sqrt{2M/L}, \\ L(x - l) + \sqrt{2ML} & l - \sqrt{2M/L} \leqslant x \leqslant l \end{cases}$$

if $L \geqslant 2M/l^2$.

Proof. In view of Theorem 1, we need only show that the minimum of $J[\rho]$ is given by $r(x)$ and $r(l - x)$, and only by these two functions.

Let ρ be any function in E_4. The rearrangement of ρ into increasing order is defined to be the function $\bar{\rho}$ on $[0, l]$ with values given by $\bar{\rho}(x) = m^{-1}(x)$, where $m(y)$ denotes the measure of the set $\{x : \rho(x) \geqslant y\}$ and m^{-1} is its inverse (see [10], p. 276). It follows from this definition that $J[\rho] = J[\bar{\rho}]$ and that $\int_0^l \rho(x)\,dx = \int_0^l \bar{\rho}(x)\,dx = M$. It is also true that $\bar{\rho}$ satisfies the same Lipschitz condition as ρ so that $\bar{\rho} \in E_4$. Hence, to find the minimum of $J[\rho]$ over the class E_4, we may assume that ρ is increasing. Using the same argument presented in the proof of Theorem 3, we also have that $\int_a^x \rho(t)\,dt \geqslant \int_a^x r(t)\,dt$ for all $x \in [0, l]$.

We now apply the following result due to Hardy, Littlewood, and Polya (see [10], p. 170, [11], p. 152).

THEOREM (H-L-P). *If f, g are increasing functions and $\int_a^x g(t)\,dt \geqslant \int_a^x f(t)\,dt$ for all $x \in [a, b]$, then*

$$\int_a^b \Phi[g(x)]\,dx \geqslant \int_a^b \Phi[f(x)]\,dx,$$

where Φ is a strictly concave continuous function. Equality occurs if, and only if, $\int_a^x g(t)\,dt \equiv \int_a^x f(t)\,dt$ for $x \in [a, b]$.

Hence, we have

$$\int_0^l \sqrt{\rho(x)}\,dx \geqslant \int_0^l \sqrt{r(x)}\,dx$$

with equality only when $\int_a^x \rho(x)\,dx = \int_a^x r(x)\,dx$, i.e., when $\rho(x) \equiv r(x)$ for all $x \in [0, l]$ if ρ is an increasing function. The only other function for which equality can occur is $r(l - x)$. For if ρ_0 is such that

$$\int_0^l \sqrt{\rho_0(x)}\,dx = \int_0^l \sqrt{r(x)},$$

then the rearrangement of $\rho_0(x)$ into increasing order must be identical with $r(x)$. Hence, $max_x\, \rho_0(x) = \max_x r(x)$. But the Lipschitz condition then implies that $\int_0^l \rho_0(x)\,dx > \int_0^l r(x)\,dx$, except when $\rho_0(x) = r(1 - x)$. This is a contradiction and the theorem is proved.

It is an immediate consequence of Theorem 5 that the functions $\rho_n^+ \in E_4$ $(n = 1, 2, \ldots)$ cannot all be regular saw-tooth functions described in the remark prefacing Theorem 4. And, in fact, only a finite number can be of this form. For if there were such an infinite subsequence of $\{\rho_n^+\}$, it is easily seen that it would converge to the constant value M/l which contradicts Theorem 5.

4. Asymptotic Upper Bounds

In this section, we apply Theorem 1 to give results on upper bounds of $\lambda_n[\rho]$ when ρ belongs to each of the classes E_1, E_2, and E_3. We first consider the monotone increasing class E_1. We have

THEOREM 6. *Let $\rho_n{}^+$ be the maximizing function of $\lambda_n[\rho]$ over the increasing class E_1. Let $\{\rho_{n_k}^+\}$ be a convergent subsequence of $\{\rho_n{}^+\}$. Then* $\lim_{k\to\infty} \rho_{n_k}(x) = r(x)$ *where*

$$r(x) = \begin{cases} 0 & 0 \leqslant x \leqslant l - \dfrac{M}{H}, \\ H & l - \dfrac{M}{H} \leqslant x \leqslant l. \end{cases} \tag{15}$$

Proof. By Theorem 1, we need only show that if $\rho(x) \in E_1$, then $J[\rho] > J[r]$ unless $\rho(x) = r(x)$ almost everywhere. But we have immediately that $\int_0^x \rho(x)\, dx \geqslant \int_0^x r(x)\, dx$ so that again by the theorem of H-L-P, we have the desired result.

For the concave case, E_3, we have

THEOREM 7. *Let $\rho_n{}^+$ be the maximizing function of $\lambda_n[\rho]$ over the concave class E_3. Let $\{\rho_{n_k}^+\}$ be a convergent subsequence of $\{\rho_n{}^+\}$. Then there is a number t in $[0, l]$ such that* $\lim_{k\to\infty} \rho_{n_k}^+(x) = r_t(x)$, *where*

$$r_t(x) = \begin{cases} \dfrac{2M}{lt}\, x & 0 \leqslant x \leqslant t, \\ \dfrac{2M}{l(l-t)}\,(l - x) & t < x \leqslant l. \end{cases} \tag{16}$$

Proof. The rearrangement of a concave function into increasing order preserves the concavity. As noted in the proof of Theorem 5, to minimize $J[\rho]$ we note that $J[\rho] = J[\rho^*]$, where ρ^* is the rearrangement of ρ into increasing order. Hence, we may assume that ρ is increasing. We also note that r_l is an increasing function and that $\int_0^x \rho(x)\, dx \geqslant \int_0^x r_l(x)\, dx$. The theorem of H-L-P then yields the inequality $J[\rho] \geqslant J[r_l]$ with equality only if $\rho \equiv r_l$. But we also have that $J[r_t] = J[r_l]$ for all $t \in [0, l]$. Hence, the minimizing function of $J[\rho]$ for $\rho \in E_3$ is not unique but belongs to the family $\{r_t\}$ since the only concave rearrangements of r_l are in $\{r_t\}$.

Finally, we have the theorem for the convex case E_2. Since the proof of this theorem involves no new ideas, we give no proof.

THEOREM 8. *Let $\rho_n{}^+$ be the maximizing functions of $\lambda_n[\rho]$ ($n = 1, 2, \ldots$) over the bounded convex class E_2. Let $\{\rho_{n_k}^+\}$ be a convergent subsequence of $\{\rho_n{}^+\}$. Then there is a number t in $[0, l]$ such that* $\lim \rho_{n_k}^+(x) = r_t(x)$, *where*

$$r_t(x) = \begin{cases} H - \left(\dfrac{H-h}{t}\right) x & 0 \leqslant x \leqslant t, \\ \left(\dfrac{H-h}{l-t}\right)(x - l) + H & t \leqslant x \leqslant l, \end{cases} \tag{17}$$

with $h = (2M/l) - H$ *when* $Hl \leqslant 2M$ *and*

$$r_t(x) = \begin{cases} H - Hx/t & 0 \leqslant x \leqslant t, \\ 0 & t \leqslant x \leqslant s, \\ H + H(x-l)/(s-l) & s \leqslant x \leqslant l, \end{cases} \tag{18}$$

with s, t *determined by* $Ht + H(l-s) = 2M$ *when* $Hl > 2M$.

5. Remarks

We conclude with some remarks based on the theorems of the previous section and on the ideas considered there.

It is shown in Theorem 1 of [2] that maximizing function ρ_n^+ of $\lambda_n[\rho]$ over the increasing class E_1 is an increasing step function with at least one and at most n jumps. Theorem 6 shows that, however many jumps ρ_n^+ has, as $n \to \infty$, these must either tend to reduce to a single jump or there is eventually only one jump or possibly both of these events occur for different subsequences of $\{\rho_n^+\}$.

Corresponding remarks can be made about the maximizing functions ρ_n^+ in the convex class E_2 and the concave class E_3, except that in these cases the limiting function r_t is apparently not unique. By further restricting these classes, however, we have the following results.

(1) Let $\lambda_n[\rho_n^+] = \max \lambda_n[\rho]$, where the maximum is taken over those functions in the convex class E_2 which are increasing. The functions ρ_n^+ are known to be piecewise linear with at most $n + 1$ pieces (see [2], p. 1193). It can be seen from the proof of Theorem 8 that every convergent subsequence $\{\rho_{n_k}^+\}$ from $\{\rho_n^+\}$ is such that $\lim_{k\to\infty} \rho_{n_k}^+(x) = r_0(x)$, where r_0 is defined by (17) or (18) with $t = 0$. There is a corresponding result when the functions E_2 are required to be symmetric. In this case, the limit function is given by (17) with $t = l/2$ or by (18) with $(s + t)/2 = l/2$.

(2) Let $\lambda_n[\rho_n^+] = \max \lambda_n[\rho]$, where the maximum is taken over those functions in the concave class E_3 which are symmetric. The functions ρ_n^+ are known to be piecewise linear functions whose graphs are made up of at most n linear segments (see [2], p. 1196). It can be seen from the proof of Theorem 7, every convergent subsequence $\{\rho_{n_k}^+\}$ of $\{\rho_n^+\}$ converges to the function $r_{l/2}$ defined (16) in Theorem 7. A corresponding statement holds if the functions of E_3 are increasing or decreasing.

(3) Finally, one may consider the more general self-adjoint eigenvalue problem

$$y'' + \Lambda P(x) y = 0$$
$$\alpha_{11} y(0) + \alpha_{12} y'(0) + \alpha_{13} y(l) + \alpha_{14} y'(l) = 0$$
$$\alpha_{21} y(0) + \alpha_{22} y'(0) + \alpha_{23} y(l) + \alpha_{24} y'(l) = 0$$

with

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} = \begin{vmatrix} \alpha_{13} & \alpha_{14} \\ \alpha_{23} & \alpha_{24} \end{vmatrix}.$$

Denoting the n-th eigenvalue of this system by $\Lambda_n(P)$, it follows ([4], p. 415) that

$$\lim_{n \to \infty} \frac{\Lambda_n(P)}{n^2} = \left(\frac{\pi}{\int_0^l \sqrt{P(x)}\, dx} \right)^2.$$

Thus, Theorem 1 holds if we replace $\lambda_n(P)$ by $\Lambda_n(P)$. It is now obvious that Theorems 2, 5, 6, 7, and 8 will hold with $\lambda_n(P)$ replaced by the more general $\Lambda_n(P)$, and the proofs will be identical to those given for $\lambda_n(P)$.

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