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Null Controllability of the Heat Equation as Singular Limit of the Exact Controllability of Dissipative Wave Equation Under the Bardos-Lebeau-Rauch Geometric Control Condition

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Abstract—We extend the result of the null controllability property of the heat equation, obtained as limit, when ε tends to zero, of the exact controllability of a singularly perturbed damped wave equation depending on a parameter $\varepsilon > 0$, described in [1], to bounded domains which satisfy the Bardos-Lebeau-Rauch geometric control condition [2]. We add to the method of Lopez, Zhang and Zuazua in [1] an explicit in $\varepsilon > 0$ observability estimate for the singularly perturbed damped wave equation under the Bardos-Lebeau-Rauch geometric control condition. Here the geometric conditions are more optimal than in [1] and the proof is simpler than in [1]. Instead of using global Carleman inequalities as in [1], we apply an integral representation formula. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION AND RESULTS

This paper is devoted to complete the results in [1] on the controllability of the following damped, singularly perturbed wave equation depending on a parameter $\varepsilon > 0$:

$$\begin{aligned} \varepsilon\partial_t^2 u - \Delta u + \partial_t u &= f_\varepsilon 1_{|\omega}, && \text{in } \Omega \times \,]0,T[\,, \\ u &= 0, && \text{on } \partial\Omega \times]0,T[, \\ u(\cdot,0) = u_0, \quad \partial_t u(\cdot,0) &= u_1, && \text{in } \Omega. \end{aligned} \tag{1.1}$$

In (1.1), Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$, with a smooth boundary $\partial\Omega$, $u = u(x,t)$ is the state to be controlled, $f_\varepsilon = f_\varepsilon(x,t)$ is the control, and $1_{|\omega}$ denotes the characteristic function of the open subset ω of Ω, where the control is supported. The measure of the cost of controllability is given by the following assertion: given any $T > 0$, there exist two positive constants $\varepsilon(T) > 0$

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and $C(\varepsilon, T) > 0$, such that

$$\|f_\varepsilon\|_{L^2(\omega\times]0,T[)} \leq C(\varepsilon, T)\left\|u_0, \sqrt{\varepsilon} u_1\right\|_{H_0^1(\Omega)\times L^2(\Omega)}, \qquad \forall\,(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega), \tag{1.2}$$

for all $0 < \varepsilon < \varepsilon(T)$, where the control $f_\varepsilon \in L^2(\omega\times]0,T[)$ is constructed such that the solution of (1.1) satisfies $u(x,T) = \partial_t u(x,T) = 0$ in Ω. The formal limit, as ε tends to zero, of (1.1) is the controlled heat equation with initial datum $u_0 \in H_0^1(\Omega)$.

The method described in [1] allows us to prove that, given any $T > 0$, system (1.1) is exactly controllable for $\varepsilon > 0$ sufficiently small with a uniform bound of the cost of controllability if there exists $T_o = T_o(\Omega, \omega) > 0$ such that for all $T > T_o$, there exist positive constants $C_1, C_2 > 0$ such that for any L^2-solution of the following damped wave equation depending of the parameter $k \in \mathbb{R}$:

$$\begin{aligned} \partial_t^2\psi - \Delta\psi + k\partial_t\psi = 0, &\qquad \text{in } \Omega\times]0,T[\,, \\ \psi = 0, &\qquad \text{on } \partial\Omega\times]0,T[\,, \\ \psi(\cdot,0) = \psi_0, \quad \partial_t\psi(\cdot,0) = \psi_1, &\qquad \text{in } \Omega, \end{aligned} \tag{1.3}$$

we have

$$\|\psi_0\|_{L^2(\Omega)}^2 + \|\psi_1\|_{H^{-1}(\Omega)}^2 \leq C(k)\int_0^T\int_\omega |\psi(x,t)|^2\,dx\,dt, \qquad \forall\, k \in \mathbb{R}, \tag{1.4}$$

with $C(k) = C_1 e^{C_2|k|}$ where the two positive constants C_1, C_2 do not depend on k.

Under the following geometric control hypothesis of the work of Bardos, Lebeau and Rauch in [2],

(i) there is no infinite order of contact between the boundary $\partial\Omega$ and the bicharacteristic of $\partial_t^2 - \Delta$; (1.5)
(ii) any generalized bicharacterisic of $\partial_t^2 - \Delta$ parametrized by $t \in]0, T_c[$ meets ω.

It is well known that (1.4) holds true, but with a nonexplicit, in $k \in \mathbb{R}$, constant $C(k)$ and with $T = T_c$. Besides, it has been proved in [1] that $C(k) = C_1 e^{C_2|k|}$ by using global Carleman inequalities with the hypothesis given by the multiplier techniques and with $T_o > T_c$. Our goal is to prove the results in [1] under the geometric control hypothesis (1.5) without using global Carleman inequalities.

The main result of this paper is the following theorem which asserts that under the geometric control hypothesis of the work of Bardos, Lebeau and Rauch, there exists a control, for system (1.1), which has a uniformly bounded cost.

THEOREM 1. *Let hypothesis (1.5) be satisfied. Let $T > 0$. Then there exists $\varepsilon(T) > 0$ such that for any $0 < \varepsilon < \varepsilon(T)$, system (1.1) is exactly controllable in time T and the constant $C(\varepsilon, T)$ in (1.2) remains bounded as ε tends to zero. Furthermore, for any $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ fixed, the controls f_ε of (1.1) may be chosen such that f_ε tends to f in $L^2(\omega\times]0,T[)$ as ε tends to zero, f being a null control for the limit heat equation with initial datum u_0.*

In this way, we answer an open question in [1]. To prove Theorem 1, we propose the following observability estimate for any L^2-solution of the damped wave equation (1.3) depending of the parameter $k \in \mathbb{R}$.

THEOREM 2. *Let hypothesis (1.5) be satisfied. Then for all $T > 4T_c$, there exists a positive constant $C > 0$ such that*

$$\|\psi_0\|_{L^2(\Omega)}^2 + \|\psi_1\|_{H^{-1}(\Omega)}^2 \leq Ce^{C|k|}\int_0^T\int_\omega |\psi(x,t)|^2\,dx\,dt, \tag{1.6}$$

for every solution of (1.3) and all $k \in \mathbb{R}$.

The paper is organized as follows. In Section 2, we establish two lemmas on observability estimates which are easily obtained from the work of Zuazua in [3] for the one-dimensional case and from the work of Bardos, Lebeau and Rauch in [2] for the multidimensional case. In Section 3, we prove Theorem 2. The key point to prove (1.6) is an integral representation formula which was already exploited in [4] for the Schrödinger control problems. Such analogous formulas are also used in [5] to study inverse problems for equations of parabolic or elliptical types. The proof of Theorem 1, which is omitted here, is then deduced from the work of Lopez, Zhang and Zuazua in [1] by using the three-step controllability method: let the time interval $[0,T]$ be divided into three subintervals $[0,T/3]$, $[T/3,2T/3]$, and $[2T/3,T]$. In the first step, the parabolic projection of the solution of (1.1) is controlled uniformly in ε to zero on $[0,T/3]$. In the second interval, $[T/3,2T/3]$, system (1.1) evolves freely without control, so that the size of the solution at time $t=2T/3$ becomes exponentially small with the dissipation. In the final step, a control is constructed by the HUM method of Lions [6] and from Theorem 2 to steer to zero the whole solution.

2. PRELIMINARY LEMMAS

In this section, we describe two lemmas which can be easily obtained by well-known results. The first one concerns an explicit in k observability estimate for system (1.3) in one space dimension and is deduced from [3, Theorem 4]. The second one is an application of the observability estimate in [2] for the wave equation when the geometrical control condition (1.5) is assumed.

LEMMA 1. *Let $\delta>0$. Let us consider an interval $D\subset\mathbb{R}$ given by $D=]-L;3L[$, where $L>0$. Then for all $T>4(L+\delta)$, there exist a constant $C>0$ and a function χ defined by $\chi=\chi(\ell)\in C_0^\infty(]L;3L[)$, $\chi(\ell)=1$ for $\ell\in]L+\delta;2L[$, such that*

$$\|z_0\|^2_{H_0^1(D)}+\|z_1\|^2_{L^2(D)}\le Ce^{C|k|}\int_0^T\|\chi z(\cdot,t)\|^2_{H_0^1(D)}\,dt,\tag{2.1}$$

respectively,

$$\|z_0\|^2_{H^2(D)\cap H_0^1(D)}+\|z_1\|^2_{H_0^1(D)}\le Ce^{C|k|}\int_0^T\|\chi z(\cdot,t)\|^2_{H_0^2(D)}\,dt,\tag{2.2}$$

for every $z=z(\ell,t)$ H^1-solution, (respectively, H^2-solution) of

$$\begin{aligned}\partial_t^2 z-\partial_\ell^2 z+k\partial_t z&=0, && \text{in } (\ell,t)\in D\times]0,T[,\\ z(\ell=-L,t)=z(\ell=3L,t)&=0, && \text{for } t\in]0,T[,\\ z(\cdot,t=0)=z_0,\partial_t z(\cdot,t=0)&=z_1, && \text{in } D,\end{aligned}\tag{2.3}$$

where $(z_0,z_1)\in H_0^1(D)\times L^2(D)$ (respectively, $(z_0,z_1)\in H^2(D)\cap H_0^1(D)\times H_0^1(D)$) and for all $k\in\mathbb{R}$.

LEMMA 2. *Let hypothesis (1.5) be satisfied. Then there exists a constant $C>0$ such that*

$$\begin{aligned}\|\psi_0\|^2_{L^2(\Omega)}+\|\psi_1\|^2_{H^{-1}(\Omega)}&\le Ce^{C|k|}\int_0^{T_c}\int_\omega|\psi(x,t)|^2\,dx\,dt\\ &+Ce^{C|k|}\left(\|\psi_0\|^2_{H^{-1}(\Omega)}+\left\|(-\Delta)^{-1}(\psi_1+k\psi_0)\right\|^2_{L^2(\Omega)}\right),\end{aligned}\tag{2.4}$$

for every solution of (1.3) and all $k\in\mathbb{R}$.

PROOF OF LEMMA 1. Let $\varphi(\ell,t)=e^{(1/2)kt}z(\ell,t)$, then the solution φ, with initial data $(\varphi(\cdot,0),\partial_t\varphi(\cdot,0))=(\varphi_0,\varphi_1)$, solves

$$\begin{aligned}\partial_t^2\varphi-\partial_\ell^2\varphi-\frac14k^2\varphi&=0, && \text{in } (\ell,t)\in D\times]0,T[,\\ \varphi(\ell=-L,t)=\varphi(\ell=3L,t)&=0, && \text{for } t\in]0,T[,\\ \varphi_0=z_0,\quad \varphi_1&=\frac12kz_0+z_1, && \text{in } D.\end{aligned}\tag{2.5}$$

By applying the results of Zuazua in [3, Theorem 4 and Lemma 2, p. 121], we deduce that there exists $C > 0$ such that

$$\|\varphi_0\|^2_{L^2(D)} + \|\varphi_1\|^2_{H^{-1}(D)} \le Ce^{C|k|} \int_\delta^{T-\delta} \int_{L+\delta}^{2L} |\varphi(\ell,t)|^2 \, d\ell \, dt, \tag{2.6}$$

for every $k \in \mathbb{R}$.

Consequently, we have

$$\|\varphi_0\|^2_{L^2(D)} + \|\varphi_1\|^2_{H^{-1}(D)} \le Ce^{C|k|} \int_0^{T} \int_{L}^{3L} |\beta(t)\,\alpha(\ell)\,\varphi(\ell,t)|^2 \, d\ell \, dt, \tag{2.7}$$

where $\alpha = \alpha(\ell) \in C_0^\infty(]L + (\delta/2); 2L + \delta[)$, $\alpha \ge 0$, $\alpha(\ell) = 1$ for $\ell \in]L + \delta; 2L[$, and $\beta = \beta(t) \in C_0^\infty(]0; T[)$, $\beta \ge 0$, $\beta(t) = 1$ for $t \in]\delta; T - \delta[$.

By applying the operator ∂_t on the equations of system (2.5), we deduce from (2.7) that

$$\begin{aligned} &\|\varphi_0\|^2_{H_0^1(D)} + \|\varphi_1\|^2_{L^2(D)} \\ &\le Ce^{C|k|} \int_0^T \int_L^{3L} \left(|\beta(t)\,\alpha(\ell)\,\partial_t\varphi(\ell,t)|^2 + |\beta(t)\,\alpha(\ell)\,\varphi(\ell,t)|^2 \right) d\ell \, dt. \end{aligned} \tag{2.8}$$

Now, from (2.5) and by using integrations by parts, we easily obtain the following estimate:

$$\int_0^T \int_L^{3L} |\beta(t)\,\alpha(\ell)\,\partial_t\varphi(\ell,t)|^2 \, d\ell \, dt \le C \left|1 + k^2\right| \int_0^T \|\varphi(\cdot,t)\|^2_{H^1(]L+(\delta/2);2L+\delta[)} \, dt. \tag{2.9}$$

Finally,

$$\|\varphi_0\|^2_{H_0^1(D)} + \|\varphi_1\|^2_{L^2(D)} \le Ce^{C|k|} \int_0^T \|\chi\varphi(\cdot,t)\|^2_{H_0^1(D)} \, dt, \tag{2.10}$$

for all $k \in \mathbb{R}$, where the constant $C > 0$ does not depend on k and where the function χ is defined by $\chi = \chi(\ell) \in C_0^\infty(]L; 3L[)$, $\chi \ge 0$, $\chi(\ell) = 1$ for $\ell \in]L + \delta/2; 2L + \delta[$. Then, from (2.10), we prove (2.1). Also, (2.2) easily follows by applying the operator ∂_t^2 on system (2.5) and from (2.6).

PROOF OF LEMMA 2. We will begin to prove Lemma 2 with more regular initial data. More precisely, we have the following result.

LEMMA 3. *Let hypothesis (1.5) be satisfied. Then there exists a constant $C > 0$ such that for all $k \in \mathbb{R}$, for all initial data $(w_0, w_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the solution of the problem*

$$\begin{aligned} \partial_t^2 w - \Delta w + k\partial_t w = 0, &\qquad \text{in } \Omega \times]0,T[, \\ w = 0, &\qquad \text{on } \partial\Omega \times]0,T[, \\ w(\cdot,0) = w_0, \quad \partial_t w(\cdot,0) = w_1, &\qquad \text{in } \Omega, \end{aligned} \tag{2.11}$$

satisfies

$$\begin{aligned} \|w_0\|^2_{H^1(\Omega)} + \|w_1\|^2_{L^2(\Omega)} &\le Ce^{C|k|} \int_0^{T_c} \int_\omega |\partial_t w(x,t)|^2 \, dx \, dt \\ &\quad + Ce^{C|k|} \left(\|w_0\|^2_{L^2(\Omega)} + \|w_1\|^2_{H^{-1}(\Omega)} \right). \end{aligned} \tag{2.12}$$

The proof of Lemma 2 follows from Lemma 3 with

$$w(x,t) = \int_0^t \psi(x,s)\, ds - (-\Delta)^{-1} \left(\psi_1(x) + k\psi_0(x)\right). \tag{2.13}$$

PROOF OF LEMMA 3. First, we observe that it is sufficient to prove (2.12) for all $k > 0$, because the case $k < 0$ is easily deduced by a symmetry argument on the variable t. Let $v(x,t) = e^{(1/2)kt}w(x,t)$, then the solution $v = v(x,t)$ solves

$$\begin{aligned} \partial_t^2 v - \Delta v &= \frac{1}{4}k^2 v, && \text{in } \Omega \times]0,T[, \\ v &= 0, && \text{on } \partial\Omega \times]0,T[, \\ v(\cdot,0) = w_0, \quad \partial_t v(\cdot,0) &= w_1 + \frac{1}{2}kw_0, && \text{in } \Omega. \end{aligned} \tag{2.14}$$

By applying the observability estimate from the work of Bardos, Lebeau and Rauch [2], to the solution $v(x,t)$ of the wave equation with a second member given by $(1/4)k^2v(x,t)$, we have $\exists\, c(T_c) > 0$,

$$\begin{aligned} &\|w_0\|^2_{H^1(\Omega)} + \left\| w_1 + \frac{1}{2}kw_0 \right\|^2_{L^2(\Omega)} \\ &\leq c(T_c)\int_0^{T_c}\int_\omega |\partial_t v(x,t)|^2\, dx\, dt + c(T_c)\int_0^{T_c}\int_\Omega \left|k^2 v(x,t)\right|^2 dx\, dt. \end{aligned} \tag{2.15}$$

Coming back to the solution $w(x,t)$ and using the relation

$$\frac{1}{2}\frac{d}{dt}\left(\|\partial_t w\|^2_{H^{-1}(\Omega)} + \|w\|^2_{L^2(\Omega)}\right) = -k\,\|\partial_t w\|^2_{H^{-1}(\Omega)} \leq 0, \tag{2.16}$$

we finally deduce that

$$\begin{aligned} \|w_0\|^2_{H^1(\Omega)} + \|w_1\|^2_{L^2(\Omega)} &\leq Ce^{C|k|}\int_0^{T_c}\int_\omega |\partial_t w(x,t)|^2\, dx\, dt \\ &\quad + Ce^{C|k|}\left(\|w_0\|^2_{L^2(\Omega)} + \|w_1\|^2_{H^{-1}(\Omega)}\right), \end{aligned} \tag{2.17}$$

where the constant $C > 0$ does not depend on k. This concludes the proof of Lemma 3.

3. PROOF OF THEOREM 2

From Lemma 2, we deduce that in order to prove Theorem 2, it is sufficient to demonstrate that if hypothesis (1.5) is satisfied, then for all $\delta > 0$, for all $T > 4(T_c + \delta)$, there exists a constant $C > 0$ such that

$$\|\psi_0\|^2_{H^{-1}(\Omega)} + \left\|(-\Delta)^{-1}(\psi_1 + k\psi_0)\right\|^2_{L^2(\Omega)} \leq Ce^{C|k|}\int_0^T\int_\omega |\psi(x,t)|^2\, dx\, dt, \tag{3.1}$$

for every solution of (1.3) and all $k \in \mathbb{R}$.

By a duality argument, we observe that, in order to prove the observability estimate (3.1), it is sufficient to solve an exact controllability problem. Indeed, if for all initial data $(W_0, W_1) \in H^2 \cap H_0^1(\Omega) \times H_0^1(\Omega)$, there exists a control $g \in L^2(\omega\times]0,T[)$ such that the solution $W = W(x,t)$ of the following system:

$$\begin{aligned} \partial_t^2 W - \Delta W - k\partial_t W &= g1_{|\omega}, && \text{in } \Omega \times]0,T[, \\ W &= 0, && \text{on } \partial\Omega \times]0,T[, \\ W(\cdot,0) = W_0, \quad \partial_t W(\cdot,0) &= W_1, && \text{in } \Omega, \end{aligned} \tag{3.2}$$

satisfies $W(\cdot,T) = \partial_t W(\cdot,Tg) = 0$, and

$$\|g\|^2_{L^2(\omega\times]0,T[)} \leq Ce^{C|k|}\left(\|\Delta W_0\|^2_{L^2(\Omega)} + \|W_1\|^2_{H_0^1(\Omega)}\right), \tag{3.3}$$

then it is sufficient to choose

$$W_0 = (-\Delta)^{-1}(-\Delta)^{-1}(\psi_1 + k\psi_0),$$
$$W_1 = (-\Delta)^{-1}\psi_0, \tag{3.4}$$

in the following duality relation:

$$\int_\Omega \{-W_1(x)\psi_0(x) + W_0(x)[\psi_1(x) + k\psi_0(x)]\}\,dx = \int_0^T \int_\omega g(x,t)\,\psi(x,t)\,dx\,dt, \tag{3.5}$$

to conclude the proof of the observability estimate (3.1).

Our goal is now to prove the following exact controllability property.

PROPOSITION 1. *Let hypothesis (1.5) be satisfied. Let $\delta > 0$. Then, for all $T > 4(T_c + \delta)$, for all initial data $(W_0, W_1) \in H^2 \cap H_0^1(\Omega) \times H_0^1(\Omega)$, there exists a control $g \in L^2(\omega \times]0,T[)$ such that the solution $W = W(x,t)$ of problem (3.2) satisfies $W(\cdot,T) = \partial_t W(\cdot,T) = 0$ and estimate (3.3) holds for all $k \in \mathbb{R}$, where the constant $C > 0$ does not depend on k.*

PROOF OF PROPOSITION 1. We decompose the solution $W = W(x,t)$ of (3.2) as $W = \Phi + \Psi$, where the solutions $\Phi = \Phi(x,t)$ and $\Psi = \Psi(x,t)$ satisfy the two following lemmas.

LEMMA 4. *Let hypothesis (1.5) be satisfied. Let $\delta > 0$. Then, for all $T > 4(T_c+\delta)$, for all initial data $W_1 \in H_0^1(\Omega)$, there exists a control $g_1 \in L^2(\omega \times]0,T[)$ such that the solution $\Phi = \Phi(x,t)$ of the problem*

$$\begin{aligned} \partial_t^2\Phi - \Delta\Phi - k\partial_t\Phi = g_1 1_{|\omega}, &\qquad \text{in } \Omega \times]0,T[, \\ \Phi = 0, &\qquad \text{on } \partial\Omega \times]0,T[, \\ \Phi(\cdot,0) = 0, \quad \partial_t\Phi(\cdot,0) = W_1, &\qquad \text{in } \Omega, \end{aligned} \tag{3.6}$$

satisfies $\Phi(\cdot,T) = \partial_t\Phi(\cdot,T) = 0$ and the estimate

$$\|g_1\|^2_{L^2(\omega\times]0,M[)} \le Ce^{C|k|}\,\|W_1\|^2_{H_0^1(\Omega)} \tag{3.7}$$

holds for all $k \in \mathbb{R}$ where the constant $C > 0$ does not depend on k.

LEMMA 5. *Let hypothesis (1.5) be satisfied. Let $\delta > 0$. Then, for all $T > 4(T_c + \delta)$, for all initial data $W_0 \in H^2 \cap H_0^1(\Omega)$, there exists a control $g_2 \in L^2(\omega \times]0,T[)$ such that the solution $\Psi = \Psi(x,t)$ of the problem*

$$\begin{aligned} \partial_t^2\Psi - \Delta\Psi - k\partial_t\Psi = g_2 1_{|\omega}, &\qquad \text{in } \Omega \times]0,T[, \\ \Psi = 0, &\qquad \text{on } \partial\Omega \times]0,T[, \\ \Psi(\cdot,0) = W_0, \quad \partial_t\Psi(\cdot,0) = 0, &\qquad \text{in } \Omega, \end{aligned} \tag{3.8}$$

satisfies $\Psi(\cdot,T) = \partial_t\Psi(\cdot,T) = 0$ and the estimate

$$\|g_2\|^2_{L^2(\omega\times]0,M[)} \le Ce^{C|k|}\,\|\Delta W_0\|^2_{L^2(\Omega)} \tag{3.9}$$

holds for all $k \in \mathbb{R}$ where the constant $C > 0$ does not depend on k.

PROOF OF LEMMA 4. We construct $\Phi = \Phi(x,t)$ the state to be controlled, solution of (3.6), with the following integral representation formula:

$$\Phi(x,t) = \int_{\mathbb{R}} F(\ell,t)\,y(x,\ell)\,d\ell, \tag{3.10}$$

where the solutions $y:(x,\ell)\in\Omega\times\mathbb{R}_\ell\mapsto y(x,\ell)$ and $F:(\ell,t)\in\mathbb{R}_\ell\times]0,T[\mapsto F(t,\ell)$ satisfy the two following control problems:

$$\begin{aligned}
\partial_\ell^2 y-\Delta y &= h1_{|\omega}, \qquad \text{in } \Omega\times\{-L<\ell<L\},\\
y &= 0, \quad \text{on } \partial\Omega\times]-L,L[,\\
y(\cdot,\ell=0) &= W_1\in H_0^1(\Omega),\quad \partial_\ell y(\cdot,\ell=0)=0, \qquad \text{in } \Omega,\\
y(x,\ell) &= \partial_\ell y(x,\ell)\equiv 0, \quad \text{for } (x,\ell)\in\Omega\times(]-\infty,-L]\cup[L,+\infty[),
\end{aligned}\tag{3.11}$$

$$\begin{aligned}
&\partial_t^2F-\partial_\ell^2F-k\partial_tF=\chi\varrho 1_{|]L,3L[}, && \text{in } (\ell,t)\in]-L,3L[\times]0,T[,\\
&F(\ell,t=0)=0, && \text{for } \ell\in]-L,3L[,\\
&\partial_tF(\cdot,t=0),=\delta(\cdot), && \text{in } \mathbb{R}_\ell,\\
&F(\ell,t=T)=\partial_tF(\ell,t=T)=0, && \text{for } \ell\in]-L,3L[,
\end{aligned}\tag{3.12}$$

with the following estimates:

$$\|h\|^2_{L^2(\omega\times]-L,L[)}\le C\,\|W_1\|^2_{H_0^1(\Omega)},\tag{3.13}$$

$$\|F\|^2_{L^2(]-L,L[\times]0,T[)}\le Ce^{c|k|},\tag{3.14}$$

for all $k\in\mathbb{R}$ where the constant $C>0$ does not depend on k.

The existence of the solution $y=y(x,\ell)$ is obtained by a simple reflection argument as a consequence of the theorem of Bardos, Lebeau and Rauch [2] on the exact controllability for hyperbolic equations with the geometrical control condition (1.5) and with $L=T_c$. Estimate (3.13) is a direct consequence of the HUM method of Lions [6]. The existence of the solution $F=F(\ell,t)$ and estimate (3.14) comes from Lemma 1 with estimate (2.1) and from the HUM method. F is solution of the damped wave controlled equation with a second member which is the localized control function $\chi\varrho$. In the integrations by parts, the term $\chi\varrho$ disappears because y is null on the support of the control function $\chi\varrho$.

Proof of Lemma 5. We construct $\Psi=\Psi(x,t)$ the state to be controlled, solution of (3.8), with the following integral representation formula:

$$\Psi(x,t)=\int_{\mathbb{R}}F(\ell,t)\,y(x,\ell)\,d\ell,\tag{3.15}$$

where the solutions $y:(x,\ell)\in\Omega\times\mathbb{R}_\ell\mapsto y(x,\ell)$ and $F:(\ell,t)\in\mathbb{R}_\ell\times]0,T[\mapsto F(t,\ell)$ satisfy the two following control problems:

$$\begin{aligned}
\partial_\ell^2 y-\Delta y &= h1_{|\omega}, \qquad \text{in } \Omega\times\{-L<\ell<L\},\\
y &= 0, \qquad \text{on } \partial\Omega\times]-L,L[,\\
y(\cdot,\ell=0) &= W_0\in H^2(\Omega)\cap H_0^1(\Omega),\quad \partial_\ell y(\cdot,\ell=0)=0, \qquad \text{in } \Omega,\\
y(x,\ell) &= \partial_\ell y(x,\ell)\equiv 0, \qquad \text{for } (x,\ell)\in\Omega\times(]-\infty,-L]\cup[L,+\infty[),
\end{aligned}\tag{3.16}$$

$$\begin{aligned}
&\partial_t^2F-\partial_\ell^2F-k\partial_tF=\chi\varrho 1_{|]L,3L[}, && \text{in } (\ell,t)\in]-L,3L[\times]0,T[,\\
&F(\cdot,t=0)=\delta(\cdot), && \text{in } \mathbb{R}_\ell,\\
&\partial_tF(\ell,t=0)=0, && \text{for } \ell\in]-L,3L[,\\
&F(\ell,t=T)=\partial_tF(\ell,t=T)=0, && \text{for } \ell\in]-L,3L[,
\end{aligned}\tag{3.17}$$

with the following estimates:

$$\|h\|^2_{H_0^1(]-L,L[;L^2(\omega))}\le C\,\|\Delta W_0\|^2_{L^2(\Omega)},\tag{3.18}$$

$$\|F\|^2_{C(]0,T[;H^{-1}(\mathbb{R}_\ell))}\le Ce^{c|k|},\tag{3.19}$$

for all $k\in\mathbb{R}$ where the constant $C>0$ does not depend on k.

The existence of the solution $y = y(x, \ell)$ is obtained by a simple reflection argument as a consequence of the theorem of Bardos, Lebeau and Rauch [2] on the exact controllability for hyperbolic equations with the geometrical control condition (1.5) and with $L = T_c$. Estimate (3.18) is a direct consequence of the HUM method of Lions [6] with a weaker norm. The existence of the solution $F = F(\ell, t)$ and estimate (3.17) comes from Lemma 1 with estimate (2.2) and from the HUM method. F is the solution of the damped wave controlled equation with a second member which is the localized control function. We observe that y is null on the support of the control function $\chi\varrho$.

This concludes the proof of Proposition 1.

Finally, we have proved that under the geometric control condition (1.5), let $\delta > 0$, then for all $T > 4T_c + \delta$, there exists a positive constant $C > 0$ such that

$$\|\psi_0\|^2_{L^2(\Omega)} + \|\psi_1\|^2_{H^{-1}(\Omega)} \leq Ce^{C|k|} \int_0^T \int_\omega |\psi(x,t)|^2 \, dx \, dt, \tag{3.20}$$

for every solution of (1.3) and all $k \in \mathbb{R}$.

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