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Journal of Functional Analysis 236 (2006) 59–77

JOURNAL OF Functional Analysis

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On a class of free Lévy laws related to a regression problem

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Received 17 November 2004; accepted 27 September 2005

Available online 18 April 2006

Communicated by Dan Voiculescu

Abstract

The free Meixner laws arise as the distributions of orthogonal polynomials with constant-coefficient recursions. We show that these are the laws of the free pairs of random variables which have linear regressions and quadratic conditional variances when conditioned with respect to their sum. We apply this result to describe free Lévy processes with quadratic conditional variances, and to prove a converse implication related to asymptotic freeness of random Wishart matrices.

Keywords: Free Meixner law; Free cumulants

1. Introduction

The family of classical Meixner laws was discovered by Meixner [34] who described the class of orthogonal polynomials $p_n(x)$ with generating function of the form $\sum_{n=0}^{\infty} t^n p_n(x)/n! = f(t)e^{xu(t)}$; it turns out that up to affine transformations of x, they correspond to one of the six classical probability measures: Gaussian, Poisson, gamma, Pascal (negative binomial), binomial, or a two-parameter hyperbolic secant density which Schoutens [41] calls the (classical) Meixner law. (Some authors consider only five probability measures that correspond to non-degenerate

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1 Research partially supported by KBN grant #2PO3A00723, EU Network grant HPRN-CT-2002-00279.
2 Research partially supported by NSF grants #INT-0332062, #DMS-0504198, and by C.P. Taft Memorial Fund.

doi:10.1016/j.jfa.2005.09.010

polynomials, see [19, p. 163].) Laha and Lukacs [27] arrived at the same family through the quadratic regression property. Morris [35] reinterpreted the latter result into the language of natural exponential families, and pointed out the connection to orthogonal polynomials. Wesołowski [48] proved that the five infinitely-divisible classical Meixner laws are the laws of stochastic processes with linear regressions and quadratic conditional variances when the conditional variances depend only on the increments of the process.

The free Meixner systems of polynomials were introduced by Anshelevich [3] and Saitoh, Yoshida [40]. Combining [40] with [3, Theorem 4], up to affine transformations the corresponding probability measures associated with the free Meixner system can be classified into six types: Wigner's semicircle, free Poisson, free Pascal (free negative binomial), free Gamma, a law that we will call pure free Meixner, or a free binomial law. The first five of these laws are infinitely-divisible with respect to the additive free convolution and correspond to the five classical laws, even though Anshelevich [3, p. 241] observed that this correspondence does not follow the Bercovici–Pata bijection [8]. Anshelevich [3, Remark 6] points out the q-Meixner systems of polynomials that interpolate between the classical case $q = 1$ and the free case $q = 0$. The interpolating q-deformed Meixner laws appear also as the laws of classical stochastic processes with linear regressions and quadratic conditional variances in [16]. To facilitate the comparison with the latter paper, in this note we parameterize the free Meixner family so that our parameters a, b correspond to the parameters θ, τ in [16] when $q = 0$ and the univariate distributions of the processes are taken at time $t = 1$. In Remark 5.4 we observe another parallel between the free and classical cases on the level of cumulants.

In this note we relate free Meixner laws directly to free probability via regression properties (12) and (13). We use the regression characterization to point out that the $\boxplus$-infinitely divisible members of the free Meixner family are the distributions of the free Lévy processes with linear regressions and quadratic condition variances, which is a free counterpart of the classical result [47, Theorem 2.1]. We also prove a free analog of the classical characterization [39, Theorem 2] of Wishart matrices by the independence of the sum and the quotient.

We conclude this section with the remark that some of the laws in the free Meixner family appeared under other names in the literature. The semicircle law is the law of the free Brownian motion [10, Section 5.3] and appeared as the limiting distribution of the eigenvalues of random matrices [49]. The free Poisson law [45, Section 2.7] is also know as Marchenko–Pastur law [32]. The free pure Meixner type law appears under the name "Continuous Binomial" in [3, Theorem 4], and for $a = 0$ appears as a Brown measure in [22, Example 5.6]. A sub-family of free Meixner laws appears in random walks on the free group in Kesten [26], and as Gaussian and Poisson laws in the generalized limit theorems [12,14].

2. Free Meixner laws

2.1. Free cumulants

We assume that our probability space is a von Neumann algebra $\mathcal{A}$ with a normal faithful tracial state $\tau : \mathcal{A} \to \mathbb{C}$, i.e., $\tau(\cdot)$ is linear, continuous in weak* topology, $\tau(ab) = \tau(ba)$, $\tau(\mathbb{I}) = 1$, $\tau(aa^*) \geqslant 0$ and $\tau(aa^*) = 0$ implies $a = 0$. A (noncommutative) random variable $\mathbb{X}$ is a self-adjoint ($\mathbb{X} = \mathbb{X}^*$) element of $\mathcal{A}$.

The joint moments of random variables $\mathbb{X}_1, \mathbb{X}_2, \ldots, \mathbb{X}_k \in \mathcal{A}$ are complex numbers $\tau(\mathbb{X}_1\mathbb{X}_2 \ldots \mathbb{X}_k)$. Since the sequence of univariate moments $\{\tau(\mathbb{X}^n)\colon n = 0, 1, \ldots\}$ is real, positive-definite, and bounded by $\|\mathbb{X}\|^n$, there is a unique probability measure μ, called the law of $\mathbb{X}$, such that $\tau(\mathbb{X}^n) = \int x^n\,d\mu$. (In more algebraic versions of non-commutative probability, a law of $\mathbb{X}$ is a unital linear functional μ on the algebra of polynomials $\mathbb{C}(\mathbb{X})$ in one variable.)

The concept of freeness was introduced by Voiculescu, see [45] and the references therein. We are interested in free random variables, and for our purposes the combinatorial approach of Speicher [42] is convenient. Recall that a partition $\mathcal{V} = \{B_1, B_2, \ldots\}$ of $\{1, 2, \ldots, n\}$ is crossing if there are $i_1 < j_1 < i_2 < j_2$ such that i_1, i_2 are in the same block B_r of $\mathcal{V}$, j_1, j_2 are in the same block $B_s \in \mathcal{V}$ and $B_r \neq B_s$. By $\mathcal{NC}(n)$ we denote the set of all non-crossing partitions of $\{1, 2, \ldots, n\}$.

Let $\mathbb{C}\langle\mathbb{X}_1, \ldots, \mathbb{X}_k\rangle$ denote the non-commutative polynomials in variables $\mathbb{X}_1, \ldots, \mathbb{X}_k$. The free (non-crossing) cumulants are the k-linear maps $R_k : \mathbb{C}\langle\mathbb{X}_1, \ldots, \mathbb{X}_k\rangle \to \mathbb{C}$ defined recurrently by the formula that connects them with moments:

$$\tau(\mathbb{X}_1\mathbb{X}_2\ldots\mathbb{X}_n) = \sum_{\mathcal{V}\in\mathcal{NC}(n)} R_{\mathcal{V}}(\mathbb{X}_1, \mathbb{X}_2, \ldots, \mathbb{X}_n), \tag{1}$$

where

$$R_{\mathcal{V}}(\mathbb{X}_1, \mathbb{X}_2, \ldots, \mathbb{X}_n) = \prod_{B\in\mathcal{V}} R_{|B|}(\mathbb{X}_j\colon\ j \in B),$$

see [42]. We will also write $R_n(\mathbb{X})$ for $R_n(\mathbb{X}, \mathbb{X}, \ldots, \mathbb{X})$, and $R_n(\mu)$ for $R_n(\mathbb{X})$ when μ is the law of $\mathbb{X}$.

Random variables $\mathbb{X}_1, \mathbb{X}_2, \ldots, \mathbb{X}_k \in \mathcal{A}$ are free if for every $n \geqslant 1$ and every non-constant choice of $\mathbb{Z}_1, \ldots, \mathbb{Z}_n \in \{\mathbb{X}_1, \mathbb{X}_2, \ldots, \mathbb{X}_k\}$ we have

$$R_n(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_n) = 0,$$

see [42]. The free convolution $\mu \boxplus \nu$ of measures μ, ν is the law of $\mathbb{X} + \mathbb{Y}$ where $\mathbb{X}, \mathbb{Y}$ are free and have laws μ, ν, respectively.

The R-transform of a random variable $\mathbb{X}$ is

$$r_{\mathbb{X}}(z) = \sum_{k=0}^{\infty} R_{k+1}(\mathbb{X}) z^k.$$

It linearizes the additive free convolution, $r_{\mu\boxplus\nu}(z) = r_\mu(z) + r_\nu(z)$, see [24,45,46].

2.2. Free Meixner laws

We are interested in the two-parameter family of probability measures $\{\mu_{a,b}\colon\ a \in \mathbb{R}, b \geqslant -1\}$ with the Cauchy–Stieltjes transform

$$\int_{\mathbb{R}} \frac{1}{z-y}\mu_{a,b}(dy) = \frac{(1+2b)z + a - \sqrt{(z-a)^2 - 4(1+b)}}{2(bz^2 + az + 1)}, \tag{2}$$

which we will call the free (standardized, i.e. with mean zero and variance one) Meixner laws. The absolutely continuous part of $\mu_{a,b}$ is

$$\frac{\sqrt{4(1+b)-(x-a)^2}}{2\pi(bx^2+ax+1)}$$

on $a-2\sqrt{1+b}\leqslant x\leqslant a+2\sqrt{1+b}$; the measure may also have one atom if $a^2>4b\geqslant 0$, and a second atom if $-1\leqslant b<0$. For more details, see [40], who analyze monic orthogonal polynomials which satisfy constant-coefficient recurrences; in our parametrization the monic orthogonal polynomials p_n with respect to $\mu_{a,b}$ satisfy the recurrence

$$xp_n(x)=p_{n+1}(x)+ap_n(x)+(1+b)p_{n-1}(x),\quad n\geqslant 2, \tag{3}$$

with the perturbed initial terms $p_0(x)=1$, $p_1(x)=x$, $p_2(x)=x^2-ax-1$. This can be seen from the corresponding continued fraction representation

$$\int_{\mathbb{R}}\frac{1}{z-y}\mu_{a,b}(dy)=\cfrac{1}{z-\cfrac{1}{z-a-\cfrac{1+b}{z-a-\cfrac{1+b}{\ddots}}}}.$$

For ease of reference we state the corresponding R-transform

$$r_{\mu_{a,b}}(z)=\frac{2z}{1-az+\sqrt{(1-az)^2-4z^2b}}. \tag{4}$$

Since (2) defines measures with mean zero and variance 1, it is convenient to enlarge this class by allowing translations by t and dilations by r; that is, to consider probability laws up to their type. Recall that a μ-type law is a law of an arbitrary affine function of a random variable with the law μ. In other words, the μ-type laws consist of all probability laws $\{D_r(\mu)\boxplus\delta_t\colon r,t\in\mathbb{R}\}$, where the dilation D_r is defined as $D_r(\mu)(A)=\mu(A/r)$ if $r\neq 0$ and $D_0(\mu)=\delta_0$.

The laws of free Meixner type correspond to various reparametrizations of (2) and occurred in many places in the literature, see [3,12,14], [18, Corollary 7.2], [22, Example 5.6], [26,33, 40]. In particular, recursion (3) for $b\geqslant 0$ is a reparametrization of five recursions that appear in Anshelevich [3, Theorem 4].

Saitoh and Yoshida [40, Theorem 3.2] transcribed into our notation says that for $b\geqslant 0$ (standardized) free Meixner law $\mu_{a,b}$ is $\boxplus$-infinitely divisible, i.e., for every integer n there exists a measure ν_n such that $\mu_{a,b}=\nu_n^{\boxplus n}$, and that the corresponding Lévy–Khinchin measure (5) is the Wigner semicircle law $\omega_{a,b}$ of mean a and variance b, i.e. the R transform of $\mu_{a,b}$ is given by

$$r_{\mu_{a,b}}(z)=\int\frac{z}{1-xz}\omega_{a,b}(dx); \tag{5}$$

see also [3, p. 242]. (Here we follow the Lévy–Khinchin representation from [24, Theorem 3.3.6]; other authors state the Lévy–Khinchin formula in another form, see [5,9]. The

“Lévy–Khinchin” measures that enter these two representations differ by a factor of $(1+x^2)$, see [6, Theorem 2.7] or [2, Section 6.1] who also discusses the q-interpolation of (5).) From (5) it follows that the free cumulants of $\mu_{a,b}$ are $R_1(\mu_{a,b})=0$ and

$$R_{n+2}(\mu_{a,b})=\int x^n\omega_{a,b}(dx),\quad n\geqslant 0. \tag{6}$$

As observed in [40, Theorem 3.2(i)], the free Meixner type laws with $-1\leqslant b<0$ are not $\boxplus$-infinitely divisible. In fact, they are the free counterpart of the classical binomial laws

$$\big((1-p)\delta_0+p\delta_1\big)^{*n}.$$

This was first noticed for symmetric free Meixner laws in [13], where the authors prove that

$$\mu_{0,b}=D_{1/\sqrt{t}}\left(\frac{1}{2}\delta_{-1}+\frac{1}{2}\delta_1\right)^{\boxplus t}$$

with $t=-1/b\geqslant 1$. Compare also [40, Example 3.4]. (The additive t-fold free convolution $\mu^{\boxplus t}$ is well defined for the continuous range of values $t\geqslant 1$, see [38].) Since

$$\mu_{a,-1}=\frac{1}{2}\left(1+\frac{a}{\sqrt{1+a^2}}\right)\delta_{\frac{1}{2}(a-\sqrt{4+a^2})}+\frac{1}{2}\left(1-\frac{a}{\sqrt{1+a^2}}\right)\delta_{\frac{1}{2}(a+\sqrt{4+a^2})}$$

is supported on two-points, the following is a generalization of the above result.

Proposition 2.1. *If $\mu_{a,b}$ is a free Meixner measure* (2) *with parameters $a\in\mathbb{R}$, $-1\leqslant b<0$ and $t=-1/b$ then*

$$\mu_{a,b}=D_{\sqrt{|b|}}\big(\mu_{a/\sqrt{|b|},-1}^{\boxplus t}\big). \tag{7}$$

Proof. We observe in more generality that if $\mu_{a,b}$ is given by (2) then

$$D_{1/\lambda}\big(\mu_{a,b}^{\boxplus\lambda^2}\big)=\mu_{a/\lambda,b/\lambda^2},\quad \lambda\neq 0.$$

This follows from (4) and the fact that the R transform of the dilatation $D_\lambda(\mu)$ is $\lambda r_\mu(\lambda z)$. $\square$

The similarity of $\mu_{a,b}$ with $b<0$ to the classical binomial law is further strengthened by comparing Theorem 3.1(vi) with Theorem 3.2(vi), and by random projection asymptotic of [20, Section 3.2].

Next we show that the free cumulants of the free Meixner law can be expressed as sums over the non-crossing partitions $\mathcal{NC}_{\leqslant 2}(n)$ with blocks of size at most 2.

Proposition 2.2. *If $\mu_{a,b}$ is a* (*standardized*) *free Meixner law with parameters $a\in\mathbb{R}$, $b\geqslant -1$, then the free cumulants are $R_1(\mu_{a,b})=0$ and*

$$R_{n+2}(\mu_{a,b})=\sum_{\mathcal{V}\in\mathcal{NC}_{\leqslant 2}(n)}a^{s(\mathcal{V})}b^{|\mathcal{V}|-s(\mathcal{V})},\quad n\geqslant 0, \tag{8}$$

where $s(\mathcal{V})$ is the number of blocks of size 1 (*singletons*) *of a partition $\mathcal{V}\in\mathcal{NC}_{\leqslant 2}(n)$.*

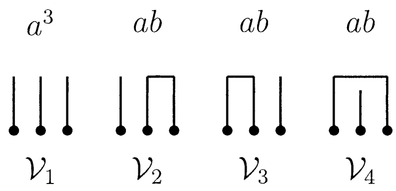

Fig. 1. $\mathcal{NC}_{\leqslant 2}(3) = \{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4\}$.

For example, $R_5(\mu_{a,b}) = a^3 + 3ab$, see Fig. 1.

Proof. For $b \geqslant 0$ the monic orthogonal polynomials $q_n(x)$ corresponding to the semicircle law $\omega_{a,b}$ of mean a and variance b satisfy the three-step recursion

$$(x-a)q_n(x) = q_{n+1}(x) + bq_{n-1}(x).$$

Since by [40, Theorem 3.2], semicircle law $\omega_{a,b}$ is the Lévy–Khinchin measure (6) for $\mu_{a,b}$, from [1, Corollary 5.1] we deduce (8). Since the cumulants are given by algebraic expressions, formula (8) extends to all $-1 \leqslant b < \infty$. $\square$

3. Conditional moments and free Meixner laws

We begin with the classical characterization which, as observed in [47], follows from the argument in [27].

Theorem 3.1. *Suppose X, Y are non-degenerate independent square-integrable classical random variables, and there are numbers $\alpha, \alpha_0, C, a, b \in \mathbb{R}$ such that*

$$E(X|X+Y) = \alpha(X+Y) + \alpha_0 \tag{9}$$

and

$$\mathrm{Var}(X|X+Y) = C\big(1 + a(X+Y) + b(X+Y)^2\big). \tag{10}$$

Then X and Y have the classical Meixner type laws. In particular, if $E(X) = E(Y) = 0$, $E(X^2) + E(Y^2) = 1$ then $\alpha = E(X^2)$, $\alpha_0 = 0$, $C = \alpha(1-\alpha)/(1+b)$, and the law of X is:

(i) *normal* (*Gaussian*), *if $a = b = 0$*;
(ii) *Poisson type, if $b = 0$ and $a \neq 0$*;
(iii) *Pascal type, if $b > 0$ and $a^2 > 4b$*;
(iv) *gamma type, if $b > 0$ and $a^2 = 4b$*;
(v) *Meixner type, if $b > 0$ and $a^2 < 4b$*;
(vi) *binomial type, if $b = -\alpha/n = -(1-\alpha)/m$ and m, n are integers.*

To state the free version of this theorem, recall that if $\mathcal{B} \subset \mathcal{A}$ is a von Neumann subalgebra of a von Neumann algebra $\mathcal{A}$ with a normalized trace τ, then there exists a unique conditional expectation from $\mathcal{A}$ to $\mathcal{B}$ with respect to τ, see [44, vol. I, p. 332], which we denote by $\tau(\cdot|\mathcal{B})$; the conditional expectation of a self-adjoint element $\mathbb{X} \in \mathcal{A}$ is a unique self-adjoint element of $\mathcal{B}$.

The conditional variance is as usual $\mathrm{Var}(\mathbb{X}|\mathcal{B}) = \tau(\mathbb{X}^2|\mathcal{B}) - (\tau(\mathbb{X}|\mathcal{B}))^2$. If $\mathbb{Y} = \tau(\mathbb{X}|\mathcal{B})$ then $\mathbb{Y} \in \mathcal{B}$ so $\tau(\mathbb{X}\mathbb{Y}|\mathcal{B}) = \tau(\mathbb{X}|\mathcal{B})\mathbb{Y} = \mathbb{Y}^2$ and similarly $\tau(\mathbb{Y}\mathbb{X}|\mathcal{B}) = \mathbb{Y}^2$. Thus

$$\mathrm{Var}(\mathbb{X}|\mathcal{B}) = \tau\big((\mathbb{X} - \mathbb{Y})^2|\mathcal{B}\big). \tag{11}$$

For fixed $\mathbb{X} \in \mathcal{A}$ by $\tau(\cdot\,|\mathbb{X})$ we denote the conditional expectation corresponding to the von Neumann algebra $\mathcal{B}$ generated by $\mathbb{X}$. A random variable $\mathbb{X}$ is non-degenerate if it is not a multiple of identity; under faithful state this is equivalent to $\mathrm{Var}(\mathbb{X}) \neq 0$.

Theorem 3.2. *Suppose $\mathbb{X}, \mathbb{Y} \in \mathcal{A}$ are free, self-adjoint, non-degenerate and there are numbers $\alpha, \alpha_0, C, a, b \in \mathbb{R}$ such that*

$$\tau(\mathbb{X}|\mathbb{X} + \mathbb{Y}) = \alpha(\mathbb{X} + \mathbb{Y}) + \alpha_0\mathbb{I} \tag{12}$$

and

$$\mathrm{Var}(\mathbb{X}|\mathbb{X} + \mathbb{Y}) = C\big(\mathbb{I} + a(\mathbb{X} + \mathbb{Y}) + b(\mathbb{X} + \mathbb{Y})^2\big). \tag{13}$$

Then $\mathbb{X}$ and $\mathbb{Y}$ have the free Meixner type laws. In particular, if $\tau(\mathbb{X}) = \tau(\mathbb{Y}) = 0$, $\tau(\mathbb{X}^2) + \tau(\mathbb{Y}^2) = 1$ then $\alpha = \tau(\mathbb{X}^2)$, $\alpha_0 = 0$, $C = \alpha(1-\alpha)/(1+b)$, $b \geqslant -\min\{\alpha, 1-\alpha\}$, and $\mathbb{X}/\sqrt{\alpha}$ has the $\mu_{a/\sqrt{\alpha},b/\alpha}$ law with the Cauchy–Stieltjes transform (2). *In particular, the law of $\mathbb{X}$ is*:

(i) *the Wigner's semicircle law if $a = b = 0$;*
(ii) *the free Poisson type law if $b = 0$ and $a \neq 0$;*
(iii) *the free Pascal* (*negative binomial*) *type law if $b > 0$ and $a^2 > 4b$;*
(iv) *the free gamma type law if $b > 0$ and $a^2 = 4b$;*
(v) *the pure free Meixner type law if $b > 0$ and $a^2 < 4b$;*
(vi) *the free binomial type law* (7) *if $-\min\{\alpha, 1-\alpha\} \leqslant b < 0$.*

The proof of this theorem is given in Section 4. We now list some consequences.

We need the following properties of conditional expectations.

Lemma 3.3.

(i) *If $\mathbb{X} \in \mathcal{A}$, $\mathbb{Y} \in \mathcal{B}$, then*

$$\tau(\mathbb{X}\mathbb{Y}) = \tau\big(\tau(\mathbb{X}|\mathcal{B})\mathbb{Y}\big). \tag{14}$$

(ii) *If random variables $\mathbb{U}, \mathbb{V} \in \mathcal{A}$ are free then $\tau(\mathbb{U}|\mathbb{V}) = \tau(\mathbb{U})\mathbb{I}$.*
(iii) *Let $\mathbb{W}$ be a* (*self-adjoint*) *element of the von Neumann algebra generated by a self-adjoint $\mathbb{V} \in \mathcal{A}$. If for all $n \geqslant 1$ we have $\tau(\mathbb{U}\mathbb{V}^n) = \tau(\mathbb{W}\mathbb{V}^n)$ then $\tau(\mathbb{U}|\mathbb{V}) = \mathbb{W}$.*
(iv) *If $\tau(\mathbb{U}_1\mathbb{V}^n) = \tau(\mathbb{U}_2\mathbb{V}^n)$ for all $n \geqslant 1$, then $\tau(\mathbb{U}_1|\mathbb{V}) = \tau(\mathbb{U}_2|\mathbb{V})$.*
(v) *If $\mathbb{U}$ is self-adjoint then $\tau((\tau(\mathbb{U}|\mathcal{B}))^2) \leqslant \tau(\mathbb{U}^2)$.*

Proof. (i) This follows from the definition, see [43] or [44, vol. II, p. 211].

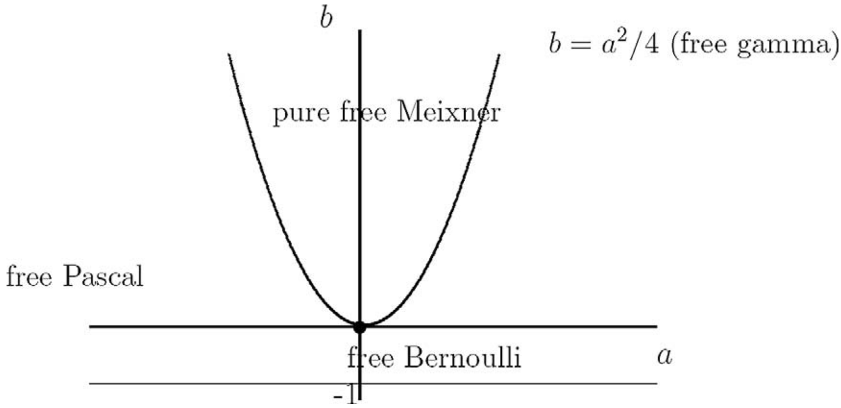

Fig. 2. Types of $\mu_{a,b}$-measures classified as in Theorem 3.2. The free-Poisson laws are at the boundary line $b = 0$ of the "free Bernoulli" laws with $b < 0$. The other boundary of this region is the line $b = -1$ which consists of the two-point discrete measures. The semicircle law is a single point $(0, 0)$. The free-gamma parabola $4b = a^2$ separates the free Pascal and the pure free Meixner laws.

(ii) If $\mathbb{Z}$ is in the von Neumann algebra generated by $\mathbb{V}$, then $\tau((\mathbb{U} - c\mathbb{I})\mathbb{Z}) = \tau(\mathbb{U} - c\mathbb{I})\tau(\mathbb{Z})$. Applying this to $\mathbb{Z} = \tau(\mathbb{U}|\mathbb{V}) - \tau(\mathbb{U})\mathbb{I}$ and $c = \tau(\mathbb{U})$ after taking into account (14) we get

$$\tau(\mathbb{Z}^2) = \tau\big(\mathbb{Z}\big(\tau(\mathbb{U}|\mathbb{V}) - c\mathbb{I}\big)\big) = \tau\big(\mathbb{Z}\tau(\mathbb{U} - c\mathbb{I}|\mathbb{V})\big) = \tau\big(\mathbb{Z}(\mathbb{U} - c\mathbb{I})\big) = \tau(\mathbb{Z})\tau(\mathbb{U} - c\mathbb{I}) = 0.$$

Thus $\tau(\mathbb{U}|\mathbb{V}) = \tau(\mathbb{U})\mathbb{I}$.

(iii) Let $\mathbb{W}' = \tau(\mathbb{U}|\mathbb{V})$. Then $\tau((\mathbb{W} - \mathbb{W}')p(\mathbb{V})) = 0$ for all polynomials p. Since polynomials $p(\mathbb{V})$ are dense in the von Neumann algebra generated by $\mathbb{V}$, and $\tau(\cdot)$ is normal, this implies that $\tau((\mathbb{W} - \mathbb{W}')(\mathbb{W} - \mathbb{W}')^*) = 0$; by faithfulness of $\tau(\cdot)$ we deduce that $\mathbb{W}' = \mathbb{W}$.

(iv) Apply (iii).

(v) See [44, vol. II, p. 211]. □

Recall that a non-commutative stochastic process $(\mathbb{X}_t)_{t\geqslant 0}$ on a probability space $(\mathcal{A}, \tau)$ is a mapping $[0, \infty) \to \mathcal{A}$. A stochastic process $(\mathbb{X}_t)$ has (additive) free increments if for every $t_1 < t_2 < \cdots < t_k$ random variables

$$\mathbb{X}_{t_1}, \quad \mathbb{X}_{t_2} - \mathbb{X}_{t_1}, \quad \ldots, \quad \mathbb{X}_{t_k} - \mathbb{X}_{t_{k-1}}$$

are free. A stochastic process $(\mathbb{X}_t)$ is a free Lévy process if the law of $\mathbb{X}_t$ converges to δ_0 as $t \to 0$ and it has free and stationary increments, i.e. for $s < t$, the law of $\mathbb{X}_t - \mathbb{X}_s$ is the same as the law of $\mathbb{X}_{t-s}$. (Compare [6, Section 5].)

We remark that martingale properties of free Lévy processes follow from the classical arguments. In particular, if $(\mathbb{X}_t)$ is a free Lévy process such that $\tau(\mathbb{X}_t) = 0$ and $\tau(\mathbb{X}_t^2) = t$ for all $t > 0$ then

$$\tau(\mathbb{X}_s|\mathbb{X}_u) = \frac{s}{u}\mathbb{X}_u \tag{15}$$

for all $s < u$. For completeness we include the proof. Suppose $s = \frac{m}{n}u$ for some integers $m < n$. Let

$$\mathbb{S}_k = \sum_{j=1}^{k} (\mathbb{X}_{ju/n} - \mathbb{X}_{(j-1)u/n}).$$

Then $\mathbb{S}_k$ is the sum of free random variables with the same distribution, and by exchangeability (Lemma 3.3(iii)) we have

$$\tau(\mathbb{X}_s|\mathbb{X}_u) = \tau(\mathbb{S}_m|\mathbb{S}_n) = m\tau(\mathbb{S}_1|\mathbb{S}_n) = \frac{m}{n}\mathbb{S}_n = \frac{s}{u}\mathbb{X}_u.$$

Suppose now $s < u$ is arbitrary. Take a sequence $s_q \to s$ of the previous form. Since $\tau(\mathbb{X}_s - \mathbb{X}_{s_q})^2 = |s - s_q| \to 0$, by Lemma 3.3(v) we get (15).

The following is a free version of two different classical probability results: [47, Theorem 2.1] which characterizes classical Lévy processes with quadratic conditional variances, and [16, Theorem 4.3] which characterizes intrinsically the classical versions of the free Lévy processes that satisfy (16).

Proposition 3.4. *Suppose $(\mathbb{X}_t)_{t\geqslant 0}$ is a free Lévy process such that $\tau(\mathbb{X}_t) = 0$, $\tau(\mathbb{X}_t^2) = t$, and that there are constants $\alpha, \beta \in \mathbb{R}$ and a normalizing function $C_{t,u}$ such that for all $t < u$*

$$\operatorname{Var}(\mathbb{X}_t|\mathbb{X}_u) = C_{t,u}\left(1 + \frac{\eta}{u}\mathbb{X}_u + \frac{\sigma}{u^2}\mathbb{X}_u^2\right). \tag{16}$$

Then for $s, t > 0$ the increment $\mathbb{X}_{s+t} - \mathbb{X}_s$ has the free Meixner type law $\mu_{a,b}$ with parameters $a = \eta/\sqrt{t} \in \mathbb{R}$, $b = \sigma/t \geqslant 0$. Moreover, the R transform of $\mathbb{X}_t$ is

$$r_{\mathbb{X}_t}(z) = \frac{2zt}{1 - \eta z + \sqrt{(1 - \eta z)^2 - 4z^2\sigma}}.$$

Proof. From (15) we get $\tau(\tau(\mathbb{X}_t|\mathbb{X}_u)^2) = t^2/u$, so applying $\tau(\cdot)$ to (16) we see that $C_{t,u} = t(u - t)/(u + \sigma)$.

Since $\mathbb{X}_t - \mathbb{X}_s$ has the same distribution as $\mathbb{X}_{t-s}$, it suffices to determine the distribution of $\mathbb{X}_t$. Fix $t > 0$ and let $\mathbb{X} = \mathbb{X}_t/\sqrt{2t}$ and $\mathbb{Y} = (\mathbb{X}_{2t} - \mathbb{X}_t)/\sqrt{2t}$. Then $\mathbb{X}, \mathbb{Y}$ are free, centered, and identically distributed. From (15), it follows that $\tau(\mathbb{X}|\mathbb{X} + \mathbb{Y}) = (\mathbb{X} + \mathbb{Y})/2$, which implies (12).

Assumption (16) gives

$$\operatorname{Var}(\mathbb{X}|\mathbb{X} + \mathbb{Y}) = \frac{1}{4 + 2\sigma/t}\left(1 + \frac{\eta}{\sqrt{2t}}(\mathbb{X} + \mathbb{Y}) + \frac{\sigma}{2t}(\mathbb{X} + \mathbb{Y})^2\right).$$

Thus (13) holds with parameters $a = \eta/\sqrt{2t}$, $b = \sigma/(2t)$. With $\alpha = 1/2$, Theorem 3.2 says that $\mathbb{X}_t/\sqrt{t}$ has free Meixner $\mu_{\eta/\sqrt{t},\sigma/t}$ law, so a dilation of (4) gives the R-transform of $\mathbb{X}_t$.

Since the law of $\mathbb{X}_t$ is a non-negative measure, from (3) we get $1 + \sigma/t \geqslant 0$. As $t > 0$ can be arbitrarily small, we deduce that $\sigma \geqslant 0$. □

The next result gives a converse to [18, Corollary 7.2], and was inspired by the characterization of Wishart matrices in [11, Theorem 4]. Our proof relies on conditional moments and

Theorem 3.2. This method does not work for Wishart matrices, see [30, p. 582], who characterize Wishart matrices by conditional moments of other quadratic expressions.

Recall that a random variable $\mathbb{S} \in \mathcal{A}$ is strictly positive if its law is supported on $[a, b]$ for some $a > 0$; in this case $\tau(\mathbb{S}) > 0$, and from the functional calculus the inverse $\mathbb{S}^{-1}$ exists and is also strictly positive (see [44, vol. I]).

Proposition 3.5. *Suppose random variables $\mathbb{X}, \mathbb{Y} \in \mathcal{A}$ are non-degenerate, free, and such that $\mathbb{S} = \mathbb{X} + \mathbb{Y}$ is strictly positive. Let $\mathbb{Z} = \mathbb{S}^{-1/2}\mathbb{X}\mathbb{S}^{-1/2}$. If $\mathbb{Z}$ and $\mathbb{S}$ are free, then $\mathbb{X}$ and $\mathbb{Y}$ have free-Poisson type laws. Moreover, $\tau(\mathbb{X}) > 0$ and the centered standardized $\mathbb{X}$ has the free Meixner law $\mu_{a,0}$ with $a = \sqrt{\operatorname{Var}(\mathbb{X})}/\tau(\mathbb{X})$.*

Proof. We verify that the assumptions of Theorem 3.2 are satisfied. For this proof, we denote the centering operation by $\mathbb{U}^\circ = \mathbb{U} - \tau(\mathbb{U})\mathbb{I}$. Denote $\sigma_{\mathbb{X}}^2 = \operatorname{Var}(\mathbb{X})$, $\sigma_{\mathbb{Y}}^2 = \operatorname{Var}(\mathbb{Y})$, $m_{\mathbb{X}} = \tau(\mathbb{X})$, $m_{\mathbb{Y}} = \tau(\mathbb{Y})$. Since $\mathbb{S}$ is strictly positive, $m_{\mathbb{X}} + m_{\mathbb{Y}} > 0$.

We have

$$\tau(\mathbb{X}|\mathbb{S}) = \tau(\mathbb{Z})\mathbb{S},$$

as by tracial property and freeness

$$\tau\big(\mathbb{X}\mathbb{S}^n\big) = \tau\big(\mathbb{S}^{1/2}\mathbb{Z}\mathbb{S}^{n+1/2}\big) = \tau\big(\mathbb{Z}\mathbb{S}^{n+1}\big) = \tau(Z)\tau\big(\mathbb{S}^{n+1}\big).$$

This verifies (12).

Applying $\tau(\,)$ we get $\tau(\mathbb{Z}) = m_{\mathbb{X}}/(m_{\mathbb{X}} + m_{\mathbb{Y}}) = \alpha$, so after centering

$$\tau\big(\mathbb{X}^\circ|\mathbb{S}^\circ\big) = \frac{m_{\mathbb{X}}}{m_{\mathbb{X}} + m_{\mathbb{Y}}}\mathbb{S}^\circ.$$

From (12) we get

$$\sigma_{\mathbb{X}}^2 = \frac{m_{\mathbb{X}}}{m_{\mathbb{X}} + m_{\mathbb{Y}}}\big(\sigma_{\mathbb{X}}^2 + \sigma_{\mathbb{Y}}^2\big). \tag{17}$$

By non-degeneracy assumption $\sigma_{\mathbb{X}}^2 > 0$, this implies that $m_{\mathbb{X}} > 0$. (By symmetry, $m_{\mathbb{Y}} > 0$, too.)

We now verify that $\operatorname{Var}(\mathbb{X}|\mathbb{S})$ is a linear function of $\mathbb{S}$. Using tracial property and freeness of $\mathbb{S}, \mathbb{Z}$, we have

$$\begin{aligned}
\tau\big(\mathbb{X}^2\mathbb{S}^m\big) &= \tau\big(\mathbb{Z}\mathbb{S}\mathbb{Z}\mathbb{S}^{m+1}\big) = \tau\big(\mathbb{Z}\mathbb{S}\big(\mathbb{Z}^\circ + \alpha\mathbb{I}\big)\mathbb{S}^{m+1}\big) \\
&= \alpha\tau\big(\mathbb{Z}\mathbb{S}^{m+2}\big) + \tau\big(\mathbb{Z}\big(\mathbb{S}^\circ + (m_{\mathbb{X}} + m_{\mathbb{Y}})\mathbb{I}\big)\mathbb{Z}^\circ\mathbb{S}^{m+1}\big) \\
&= \alpha^2\tau\big(\mathbb{S}^{m+2}\big) + (m_{\mathbb{X}} + m_{\mathbb{Y}})\tau\big(\mathbb{Z}\mathbb{Z}^\circ\mathbb{S}^{m+1}\big) + \tau\big(\mathbb{Z}\mathbb{S}^\circ\mathbb{Z}^\circ\mathbb{S}^{m+1}\big) \\
&= \alpha^2\tau\big(\mathbb{S}^{m+2}\big) + (m_{\mathbb{X}} + m_{\mathbb{Y}})\operatorname{Var}(\mathbb{Z})\tau\big(\mathbb{S}^{m+1}\big) + \tau\big(\mathbb{Z}\mathbb{S}^\circ\mathbb{Z}^\circ\mathbb{S}^{m+1}\big).
\end{aligned}$$

Continuing in the same manner, we use freeness to verify that the last term vanishes:

$$\begin{aligned}\tau\big(\mathbb{Z}\mathbb{S}^\circ\mathbb{Z}^\circ\mathbb{S}^{m+1}\big) &= \tau\big(\big(\mathbb{Z}^\circ+\alpha\mathbb{I}\big)\mathbb{S}^\circ\mathbb{Z}^\circ\mathbb{S}^{m+1}\big)\\ &= \alpha\tau\big(\mathbb{Z}^\circ\big)\tau\big(\mathbb{S}^{m+2}\big)+\tau\big(\mathbb{Z}^\circ\mathbb{S}^\circ\mathbb{Z}^\circ\mathbb{S}^{m+1}\big)\\ &= 0+\tau\big(\mathbb{Z}^\circ\mathbb{S}^\circ\mathbb{Z}^\circ\big)\tau\big(\mathbb{S}^{m+1}\big)=0.\end{aligned}$$

Therefore,

$$\tau\big(\mathbb{X}^2\mathbb{S}^m\big)=\tau\big(\big(\alpha^2\mathbb{S}^2+(m_\mathbb{X}+m_\mathbb{Y})\operatorname{Var}(\mathbb{Z})\mathbb{S}\big)\mathbb{S}^m\big),$$

which by Lemma 3.3(iii) implies that

$$\tau\big(\mathbb{X}^2\big|\mathbb{S}\big)=\alpha^2\mathbb{S}^2+(m_\mathbb{X}+m_\mathbb{Y})\operatorname{Var}(\mathbb{Z})\big(\mathbb{S}^\circ\big)+(m_\mathbb{X}+m_\mathbb{Y})^2\operatorname{Var}(\mathbb{Z})\mathbb{I}.$$

Normalizing the variables we get

$$\operatorname{Var}\bigg(\frac{1}{\sqrt{\sigma_\mathbb{X}^2+\sigma_\mathbb{Y}^2}}\mathbb{X}\bigg|\mathbb{S}\bigg)=\frac{(m_\mathbb{X}+m_\mathbb{Y})^2}{\sigma_\mathbb{X}^2+\sigma_\mathbb{Y}^2}\operatorname{Var}(\mathbb{Z})\Bigg(1+\frac{\sqrt{\sigma_\mathbb{X}^2+\sigma_\mathbb{Y}^2}}{m_\mathbb{X}+m_\mathbb{Y}}\frac{\mathbb{S}^\circ}{\sqrt{\sigma_\mathbb{X}^2+\sigma_\mathbb{Y}^2}}\Bigg).$$

Thus (13) holds with

$$a=\frac{\sqrt{\sigma_\mathbb{X}^2+\sigma_\mathbb{Y}^2}}{m_\mathbb{X}+m_\mathbb{Y}}\quad\text{and}\quad b=0.$$

By Theorem 3.2(ii) we see that $\mathbb{X}$ is free Poisson type, and $\mathbb{X}^\circ/\sigma_\mathbb{X}$ is free Meixner $\mu_{a/\sqrt{\alpha},0}$ with parameter

$$\frac{a}{\sqrt{\alpha}}=\frac{\sigma_\mathbb{X}^2+\sigma_\mathbb{Y}^2}{\sigma_\mathbb{X}(m_\mathbb{X}+m_\mathbb{Y})}=\frac{\sigma_\mathbb{X}}{m_\mathbb{X}},$$

see (17). This also determines

$$\operatorname{Var}(\mathbb{Z})=\frac{\sigma_\mathbb{X}^2\sigma_\mathbb{Y}^2}{(\sigma_\mathbb{X}^2+\sigma_\mathbb{Y}^2)(m_\mathbb{X}+m_\mathbb{Y})^2}.\qquad\square$$

Similar reasoning gives the following free analog of Lukacs' theorem [31]. We do not know whether the property we assume in fact holds for the free gamma law.

Proposition 3.6. *Suppose random variables $\mathbb{X},\mathbb{Y}\in\mathcal{A}$ are non-degenerate, i.e.*

$$\sigma=\sqrt{\operatorname{Var}(\mathbb{X})}>0,$$

free, identically distributed, and strictly positive; *in particular, $m=\tau(\mathbb{X})>0$. For $\mathbb{S}=\mathbb{X}+\mathbb{Y}$ let $\mathbb{Z}=\mathbb{S}^{-1}\mathbb{X}^2\mathbb{S}^{-1}$. If $\mathbb{Z}$ and $\mathbb{S}$ are free, then $\mathbb{X}$ has free-gamma type law μ_{2a,a^2} with $a=\sigma/m$.*

Proof. By exchangeability, $\tau(\mathbb{X}|\mathbb{S}) = \mathbb{S}/2$, which implies (12) with $\alpha = 1/2$. By freeness, $\tau(\mathbb{X}^2|\mathbb{S}) = \mathbb{S}\tau(\mathbb{Z}|\mathbb{S})\mathbb{S} = \tau(\mathbb{Z})\mathbb{S}^2$. This shows that

$$\mathrm{Var}(\mathbb{X}|\mathbb{S}) = c\mathbb{S}^2,$$

where $c = \tau(\mathbb{Z}) - 1/4 \geqslant 0$. After centering and normalizing by $\sqrt{2}\sigma$, this implies (13) with $a/\sqrt{\alpha} = 2\sigma/m$, $b/\alpha = \sigma^2/m^2$. (The latter also determines $c = \sigma^2/(m^2 + 2\sigma^2)$.) □

Next we deduce from Theorem 3.2 a simple variant of [37, Theorem 5.3]. (For related characterizations under more general concepts of non-commutative independence, see [23] and [28, Proposition 2.5 and Section 4].)

Corollary 3.7 *(Nica). Suppose random variables $\mathbb{X}, \mathbb{Y} \in \mathcal{A}$ are free, non-degenerate, and such that $\mathbb{X} + \mathbb{Y}$ and $\mathbb{X} - \mathbb{Y}$ are free. Then $\mathbb{X}$ has the semicircle law.*

Proof. Changing the random variables to $\mathbb{X} - \tau(\mathbb{X})\mathbb{I}$ and $\mathbb{Y} - \tau(\mathbb{Y})\mathbb{I}$ preserves the assumptions and the conclusion. Therefore, without loss of generality we may assume that $\tau(\mathbb{X}) = \tau(\mathbb{Y}) = 0$. Since $\mathbb{X} + \mathbb{Y}$ and $\mathbb{X} - \mathbb{Y}$ are free, by Lemma 3.3(ii) we have $\tau(\mathbb{Y} - \mathbb{X}|\mathbb{X} + \mathbb{Y}) = 0$, so (12) holds with $\alpha = 1/2$. Moreover, $\tau(\mathbb{X}(\mathbb{X} + \mathbb{Y})^n) = \tau(\mathbb{Y}(\mathbb{X} + \mathbb{Y})^n)$ for all n, which by (1) and freeness implies that

$$\begin{aligned} R_n(\mathbb{X}) &= R_n(\mathbb{X}, \mathbb{X} + \mathbb{Y}, \mathbb{X} + \mathbb{Y}, \ldots, \mathbb{X} + \mathbb{Y}) \\ &= R_n(\mathbb{Y}, \mathbb{X} + \mathbb{Y}, \mathbb{X} + \mathbb{Y}, \ldots, \mathbb{X} + \mathbb{Y}) = R_n(\mathbb{Y}), \end{aligned}$$

so $\mathbb{X}, \mathbb{Y}$ have the same law. Therefore, we can standardize $\mathbb{X}, \mathbb{Y}$, dividing both by the same number $\sqrt{\mathrm{Var}(\mathbb{X})} > 0$. This operation preserves the freeness of $\mathbb{X} + \mathbb{Y}$, $\mathbb{X} - \mathbb{Y}$ and shows that without loss of generality we may assume that $\mathbb{X}, \mathbb{Y}$ are standardized with mean 0 and variance 1.

Using freeness of $\mathbb{Y} - \mathbb{X}$ and $\mathbb{X} + \mathbb{Y}$ again, we get $\tau((\mathbb{Y} - \mathbb{X})^2|\mathbb{X} + \mathbb{Y}) = \tau((\mathbb{Y} - \mathbb{X})^2)\mathbb{I} = 2\mathbb{I}$. Thus $\mathrm{Var}(\mathbb{Y}|\mathbb{X} + \mathbb{Y}) = 2$ by (11) and Theorem 3.2 says that $\mathbb{X}$ has the Meixner-type law $\mu_{0,0}$, which is the semicircle law. □

We now deduce a version of [25, Corollary 2.4], with the assumption of the freeness of the sample mean and the sample variance slightly relaxed.

Corollary 3.8. *Let $\mathbb{X}_1, \mathbb{X}_2, \ldots, \mathbb{X}_n$ be free identically distributed random variables, and we put $\overline{\mathbb{X}} = \frac{1}{n}\sum_{j=1}^n \mathbb{X}_j$ (sample mean), $\mathbb{V} = \frac{1}{n}\sum_{j=1}^n (\mathbb{X}_j - \overline{\mathbb{X}})^2$ (sample variance). If $n \geqslant 2$ and $\tau(\mathbb{V}|\overline{\mathbb{X}})$ is a multiple of identity* (*in particular, if $\overline{\mathbb{X}}$ and $\mathbb{V}$ are free*)*, then $\mathbb{X}_1$ has the semicircle law.*

Proof. Subtracting $\tau(\mathbb{X}_1)\mathbb{I}$ from all random variables, without loss of generality we may assume that $\tau(\mathbb{X}_1) = 0$. If $\tau(\mathbb{X}_1^2) = 0$, then $\mathbb{X}_1 = 0$ has the (degenerate) semicircle law. Otherwise, we rescale the random variables, and reduce the problem to the case $\tau(\mathbb{X}_1) = 0$, $\tau(\mathbb{X}_1^2) = 1$.

We now verify that the assumptions of Theorem 3.2(i) hold with free random variables $\mathbb{X} = \mathbb{X}_1/\sqrt{n}$ and $\mathbb{Y} = (\mathbb{X}_2 + \mathbb{X}_3 + \cdots + \mathbb{X}_n)/\sqrt{n}$. By exchangeability, compare proof of (15), we have

$$\tau(\mathbb{X}_1|\overline{\mathbb{X}}) = \frac{1}{n}\sum_{j=1}^n \tau\left(\mathbb{X}_j \,\middle|\, \sum_{j=1}^n \mathbb{X}_j\right) = \tau\left(\frac{1}{n}\sum_{j=1}^n \mathbb{X}_j \,\middle|\, \sum_{j=1}^n \mathbb{X}_j\right) = \overline{\mathbb{X}}.$$

Therefore, (12) holds with $\alpha = 1/n$. Using (11) and exchangeability we get

$$\mathrm{Var}(\mathbb{X}|\mathbb{X}+\mathbb{Y}) = \tau\big((\mathbb{X}_1-\overline{\mathbb{X}})^2\big|\overline{\mathbb{X}}\big) = \frac{1}{n}\sum_{j=1}^{n}\tau\big((\mathbb{X}_j-\overline{\mathbb{X}})^2\big|\overline{\mathbb{X}}\big) = \tau(\mathbb{V}|\overline{\mathbb{X}}) = C\mathbb{I},$$

verifying (13) with $a = b = 0$. Since $\mathbb{Y}$ is non-degenerate for $n \geqslant 2$, Theorem 3.2(i) implies that $\mathbb{X}_1$ has the semicircle law $\mu_{0,0} = \omega_{0,1}$. □

4. Proof of Theorem 3.2

Since $\mathbb{S} := \mathbb{X}+\mathbb{Y}$ is non-degenerate, without loss of generality we may assume that $\tau(\mathbb{X}) = \tau(\mathbb{Y}) = 0$, $\tau(\mathbb{X}^2)+\tau(\mathbb{Y}^2) = 1$. Applying $\tau(\cdot)$ to both sides of (12) we see that $\alpha_0 = 0$. Multiplying both sides of (12) by $\mathbb{S}$ and applying $\tau(\cdot)$ we get $\alpha = \tau(\mathbb{X}^2)$.

Denote $\beta = \tau(\mathbb{Y}^2) = 1-\alpha$ and let $\mathbb{V} = \beta\mathbb{X} - \alpha\mathbb{Y}$. From (12) it follows that $\tau(\mathbb{Y}|\mathbb{S}) = \beta\mathbb{S}$ and (11) gives $\mathrm{Var}(\mathbb{Y}|\mathbb{S}) = \tau(\mathbb{V}^2|\mathbb{S}) = \mathrm{Var}(\mathbb{X}|\mathbb{S})$. Thus the assumptions are symmetric with respect to $\mathbb{X}$, $\mathbb{Y}$, and we only need to prove that $\mathbb{X}$ has a free Meixner type law.

Applying $\tau(\cdot)$ to both sides of (13), we get $\tau(\mathbb{V}^2) = C(1+b)$, so the normalizing constant is $C = (\beta^2\alpha + \alpha^2\beta)/(1+b) = \alpha\beta/(1+b)$ as claimed.

Next, we establish the following identity:

$$R_n(\mathbb{X}) = \alpha R_n(\mathbb{S}) \quad \text{and} \quad R_n(\mathbb{Y}) = \beta R_n(\mathbb{S}). \tag{18}$$

We prove this by induction. Since the variables are centered, $R_1(\mathbb{X}) = R_1(\mathbb{Y}) = R_1(\mathbb{S}) = 0$. Suppose (18) holds true for some $n \geqslant 1$. Since (12) implies that $\tau(\mathbb{X}\mathbb{S}^n) = \alpha\tau(\mathbb{S}^{n+1})$, expanding both sides of this identity into free cumulants we get

$$\begin{aligned}
&R_{n+1}(\mathbb{X},\mathbb{S},\mathbb{S},\dots,\mathbb{S}) + \sum_{k=1}^{n}\sum_{\mathcal{V}=\{B_0,B_1,\dots,B_k\}} R_{B_0}(\mathbb{X},\mathbb{S},\mathbb{S},\dots,\mathbb{S})\prod_{j=1}^{k}R_{|B_j|}(\mathbb{S})\\
&\quad= \alpha R_{n+1}(\mathbb{S}) + \alpha\sum_{k=1}^{n}\sum_{\mathcal{V}=\{B_0,B_1,\dots,B_k\}}\prod_{j=0}^{k}R_{|B_j|}(\mathbb{S}).
\end{aligned}$$

Then (18) for $n+1$ follows from induction assumption, as $R_{n+1}(\mathbb{X},\mathbb{S},\mathbb{S},\dots,\mathbb{S}) = R_{n+1}(\mathbb{X},\mathbb{X}+\mathbb{Y},\mathbb{X}+\mathbb{Y},\dots,\mathbb{X}+\mathbb{Y}) = R_{n+1}(\mathbb{X})$ by the freeness of $\mathbb{X}$, $\mathbb{Y}$.

In particular, $R_n(\mathbb{V},\mathbb{S},\mathbb{S},\dots,\mathbb{S}) = R_n(\beta\mathbb{X}-\alpha\mathbb{Y},\mathbb{S},\mathbb{S},\dots,\mathbb{S}) = \beta R_n(\mathbb{X}) - \alpha R_n(\mathbb{Y})$, so (18) implies

$$R_n(\mathbb{V},\mathbb{S},\mathbb{S},\dots,\mathbb{S}) = 0, \quad n \geqslant 1. \tag{19}$$

Similarly, $R_n(\mathbb{V},\mathbb{V},\mathbb{S},\mathbb{S},\dots,\mathbb{S}) = \beta^2R_n(\mathbb{X}) + \alpha^2R_n(\mathbb{Y})$, so (18) implies

$$R_n(\mathbb{V},\mathbb{V},\mathbb{S},\mathbb{S},\dots,\mathbb{S}) = \alpha\beta R_n(\mathbb{S}), \quad n \geqslant 2. \tag{20}$$

Denote $m_n = \tau(\mathbb{S}^n)$ and let

$$M(z) = \sum_{n=0}^{\infty} z^n m_n$$

be the moment generating function.

Lemma 4.1. *$M(z)$ satisfies the quadratic equation*

$$\left(z^2+az+b\right)M^2-(1+az+2b)M+1+b=0. \tag{21}$$

Proof. Multiplying (13) by $\mathbb{S}^n$ for $n \geqslant 0$ and applying $\tau(\cdot)$ we obtain

$$\tau\left(\mathbb{V}^2\mathbb{S}^n\right)=\frac{\alpha\beta}{1+b}(m_n+am_{n+1}+bm_{n+2}). \tag{22}$$

Expanding the left-hand side into the free cumulants we see that

$$\tau\left(\mathbb{V}^2\mathbb{S}^n\right)=\sum_{\mathcal{V}\in\mathcal{NC}(n+2)} R_{\mathcal{V}}(\mathbb{V},\mathbb{V},\mathbb{S},\mathbb{S},\dots,\mathbb{S}). \tag{23}$$

Since $R_1(\mathbb{S})=\tau(\mathbb{S})=0$, the sum in (23) can be restricted to partitions that have no singleton blocks.

Let $\widetilde{\mathcal{NC}}(n+2)$ be the set of all non-crossing partitions of $\{1,2,\dots,n+2\}$ which separate 1 and 2 and have no singleton blocks. Let $\widetilde{\widetilde{\mathcal{NC}}}(n+2)$ denote the set of all non-crossing partitions of $\{1,2,\dots,n+2\}$ with the first two elements in the same block and which have no singleton blocks. By (19), if a partition $\mathcal{V}$ separates the first two elements of $\{1,2,\dots,n+2\}$, then $R_{\mathcal{V}}(\mathbb{V},\mathbb{V},\mathbb{S},\mathbb{S},\dots,\mathbb{S})=0$. Thus the sum in (23) can be taken over $\widetilde{\widetilde{\mathcal{NC}}}(n+2)$.

If $\mathcal{V}\in\widetilde{\widetilde{\mathcal{NC}}}(n+2)$ is such a partition, then from (20) we have

$$R_{\mathcal{V}}(\mathbb{V},\mathbb{V},\mathbb{S},\mathbb{S},\dots,\mathbb{S})=\alpha\beta R_{\mathcal{V}}(\mathbb{S},\mathbb{S},\dots,\mathbb{S}).$$

This shows that we can eliminate $\mathbb{V}$ from the right-hand side of (23). Thus

$$\begin{aligned}\tau\left(\mathbb{V}^2\mathbb{S}^n\right)&=\alpha\beta\sum_{\mathcal{V}\in\mathcal{NC}(n+2)\setminus\widetilde{\mathcal{NC}}(n+2)} R_{\mathcal{V}}(\mathbb{S})\\&=\alpha\beta\sum_{\mathcal{V}\in\mathcal{NC}(n+2)}R_{\mathcal{V}}(\mathbb{S})-\alpha\beta\sum_{\mathcal{V}\in\widetilde{\mathcal{NC}}(n+2)}R_{\mathcal{V}}(\mathbb{S})=\alpha\beta m_{n+2}-s,\end{aligned}$$

where

$$s=\alpha\beta\sum_{\mathcal{V}\in\widetilde{\mathcal{NC}}(n+2)}R_{\mathcal{V}}(\mathbb{S}).$$

Since $\widetilde{\mathcal{NC}}(n+2)$ has no singleton blocks, for every $\mathcal{V}\in\widetilde{\mathcal{NC}}(n+2)$ there is an index $k=k(\mathcal{V})\in\{3,4,\dots,n+2\}$ such that k is the second left-most element of the block containing 1; for example, in the partition shown in Fig. 3, $k_{\mathcal{V}}=r$. This decomposes $\widetilde{\mathcal{NC}}(n+2)$ into the n classes $\widetilde{\mathcal{NC}}_j=\{\mathcal{V}\in\widetilde{\mathcal{NC}}(n+2)\colon k(\mathcal{V})=j+2\}$, $j=1,2,\dots,n$.

Each of the sets $\widetilde{\mathcal{NC}}_j$ is in one-to-one correspondence with the product

$$\mathcal{NC}(j)\times\widetilde{\widetilde{\mathcal{NC}}}(n+2-j).$$

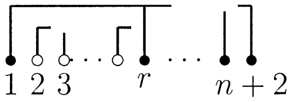

Fig. 3. $\mathcal{V} \in \widetilde{\mathcal{NC}}_j$ with $j = r - 2$ is decomposed into two partitions, the first one partitioning the white circles and the second one partitioning the black circles.

Indeed, the blocks of each partition in $\widetilde{\mathcal{NC}}_j$ consist of the partition of $\{2, 3, \ldots, j+1\}$ which can be uniquely identified with the appropriate partition in $\mathcal{NC}(j)$, and the remaining blocks which partition the $(n+2-j)$-element set $\{1, j+2, j+3, \ldots, n+2\}$ under the additional constraint that the first two elements $1, j+2$ are in the same block, see Fig. 3. These remaining blocks can therefore be uniquely identified with the partition in $\widetilde{\widetilde{\mathcal{NC}}}(n+2-j)$. This gives

$$s = \alpha\beta \sum_{j=1}^{n} \sum_{\mathcal{V} \in \widetilde{\mathcal{NC}}_j} R_{\mathcal{V}}(\mathbb{S}) = \sum_{j=1}^{n} \sum_{\mathcal{V} \in \mathcal{NC}(j)} R_{\mathcal{V}}(\mathbb{S}) \sum_{\mathcal{V} \in \widetilde{\widetilde{\mathcal{NC}}}(n+2-j)} \alpha\beta R_{\mathcal{V}}(\mathbb{S})$$
$$= \sum_{j=1}^{n} m_j \tau\left(\mathbb{V}^2 \mathbb{S}^{n-j}\right),$$

which establishes the identity

$$\tau\left(\mathbb{V}^2 \mathbb{S}^n\right) = \alpha\beta m_{n+2} - \sum_{j=1}^{n} m_j \tau\left(\mathbb{V}^2 \mathbb{S}^{n-j}\right).$$

Since $m_0 = 1$ and $\alpha\beta > 0$ as $\mathbb{X}, \mathbb{Y}$ are non-degenerate, combining the above formula with (22) we get

$$m_{n+2} = \frac{1}{1+b} \sum_{j=0}^{n} m_j (m_{n-j} + a m_{n+1-j} + b m_{n+2-j}). \tag{24}$$

Using the fact that $m_1 = 0$, from (24) we obtain

$$M(z) - 1 = \frac{z^2}{1+b} M^2(z) + \frac{az}{1+b} M(z)\big(M(z) - 1\big) + \frac{b}{1+b} M(z)\big(M(z) - 1\big),$$

which is equivalent to (21). $\square$

From (21) we see that

$$M(z) = \frac{1 + 2b + az - \sqrt{(1-az)^2 - 4z^2(1+b)}}{2(z^2 + az + b)}.$$

From this we calculate the corresponding Cauchy–Stieltjes transform $G_{\mathbb{S}}(z) = M(1/z)/z$ and the R-transform of $\mathbb{S}$ which for $b \neq 0$ takes the form

$$r_{\mathbb{S}}(z) = \frac{1 - az - \sqrt{(1-az)^2 - 4z^2 b}}{2zb}.$$

Thus the distribution of $\mathbb{S}$ is the free Meixner measure $\mu_{a,b}$ given by (4). The R-transform of $\mathbb{X}$ is

$$r_{\mathbb{X}}(z) = \alpha \frac{1 - az - \sqrt{(1-az)^2 - 4z^2 b}}{2zb},$$

see (18). After standardization,

$$r_{\mathbb{X}/\sqrt{\alpha}}(z) = \frac{1 - az/\sqrt{\alpha} - \sqrt{(1-az/\sqrt{\alpha})^2 - 4z^2 b/\alpha}}{2zb/\alpha}$$

so $\mathbb{X}$ has the free Meixner type law $\mu_{a/\sqrt{\alpha},b/\alpha}$.

To end the proof, we notice that (3) applied to the law of $\mathbb{X}$ implies $b/\alpha \geqslant -1$, see [19, p. 21]. Since the distribution of $\mathbb{Y}$ must also be of free Meixner type and well defined, we get $b/\beta \geqslant -1$. Thus $b \geqslant -\min\{\alpha, 1-\alpha\}$ as claimed.

5. Remarks

Remark 5.1. If $\mathbb{X}$ is free Poisson with mean $m > 0$ then $\sigma^2 = m$, thus $a = 1/\sqrt{m}$ and the law of $\mathbb{X}$ is $\delta_m \boxplus D_{\sqrt{m}}(\mu_{1/\sqrt{m},0})$. By [18, Corollary 7.2], when $m > 1$ the random variables $\mathbb{Z}, \mathbb{S}$ are indeed free as assumed in Proposition 3.5. But Proposition 3.5 is not a characterization of all free-Poisson type laws as the free-Poisson laws with $m \leqslant 1$ fail to be strictly positive, and when $m < 1$ have an atom at 0.

Remark 5.2. Proposition 3.5 extends with the same proof to the case when $\mathbb{S} = \mathbb{X} + \mathbb{Y}$ and $\mathbb{Z} = \mathbb{S}_1^{-1}\mathbb{X}\mathbb{S}_2^{-1}$ are free for any decomposition of $\mathbb{S} = \mathbb{S}_1\mathbb{S}_2$, which is the setting of the original Olkin–Rubin [39] characterization of the Wishart matrices.

The fact that (13) should hold true with $b = 0$ is to be expected from the expression for the conditional moment of the square of a Wishart matrix given after [30, Corollary 2.3].

Remark 5.3. Our proof of Theorem 3.2 does not rely to a significant degree on $*$-operation of $\mathcal{A}$ and could apply to any abstract probability space, see [21]. One exception is the argument that shows $b \geqslant -\min\{\alpha, 1-\alpha\}$; the setting of von Neumann algebras helps with conditional expectations, existence of which would have to be assumed in the more general setting.

In the tracial von Neumann setting, the proof seems to cover free random variables $\mathbb{X}, \mathbb{Y} \in \bigcap_{p>1} L_p(\mathcal{A}, \tau)$. It would be nice to extend Theorem 3.2 to a complete analog of the classical setting, with random variables in $L_2(\mathcal{A}, \tau)$ only.

Remark 5.4 *(q-interpolation).* For $-1 < q \leqslant 1$ consider the following recurrence:

$$R_{n+1} = aR_n + b\sum_{j=2}^{n-1} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_q R_j R_{n+1-j}, \quad n \geqslant 2, \tag{25}$$

with the initial values $R_1 = 0$, $R_2 = 1$. Here we use the standard notation

$$[n]_q = 1 + q + \cdots + q^{n-1}, \qquad [n]_q! = [1]_q[2]_q \ldots [n]_q, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!},$$

with the usual conventions $[0]_q = 0$, $[0]_q! = 1$.

The proof of Theorem 3.1 relies on the differential equation for the derivative of the logarithm of the characteristic function, i.e. for the R_1-transform of Nica [36]. This differential equation is equivalent to (25) holding with $q = 1$ for the classical cumulants $c_n(X + Y)$.

One can check that (4) is equivalent to the quadratic equation

$$zbr^2 - (1 - az)r + z = 0$$

for the R transform $r = r(z)$. Since $r(z) = \sum_{k=0}^{\infty} R_{k+1}(\mu)z^k$, this implies that when $q = 0$, the recurrence (25) holds for the free cumulants $R_n(\mathbb{X} + \mathbb{Y})$. For a related observation see [4, Propositions 1 and 2].

Remark 5.5. Recall that a classical version of a non-commutative process $(\mathbb{X}_t)$ is a classical process (X_t) on some probability space $(\Omega, \mathcal{F}, P)$ such that

$$\tau(\mathbb{X}_{t_1}\mathbb{X}_{t_2}\dots\mathbb{X}_{t_n}) = E(X_{t_1}X_{t_2}\dots X_{t_n})$$

for all $0 \leqslant t_1 \leqslant t_2 \leqslant \dots \leqslant t_n$. In [15, Section 4], the authors show that classical versions exist for all q-Gaussian Markov processes. From [10, Theorem 4.2] it follows that the classical version of the process $(\mathbb{X}_t)$ from our Proposition 3.4 exists and is a classical Markov process. This is the same process that appears in [16, Theorem 3.5] when the parameters are $q = 0$, $\theta = \eta$, $\tau = \sigma$. It is interesting to note that the conditional variances of the classical versions are also quadratic, see [16, Theorem 4.3] and that in the classical case there is a family of Markov processes with the laws that for $0 \leqslant q \leqslant 1$ interpolate between the free Meixner laws of Theorem 3.2 and the classical Meixner laws of Theorem 3.1.

Remark 5.6. The free Meixner laws with $-1 \leqslant b < 0$ are not $\boxplus$-infinitely divisible [40], but they are infinitely divisible with respect to the c-convolution [14], and appear in generalized limit theorems [13].

Remark 5.7. The Catalan numbers show up as cumulants of the free Meixner laws in several different situations. Firstly, for the symmetric free Meixner law $\mu_{0,b}$, the cumulants are

$$R_{2k+1} = 0 \quad \text{and} \quad R_{2k+2} = \frac{1}{k+1}\binom{2k}{k}b^k, \quad k \geqslant 0;$$

in particular, the Catalan numbers appear as cumulants of the $\boxplus$-infinitely divisible law $\mu_{0,1}$ and, with alternating signs, as cumulants of the two-point law $\mu_{0,-1}$. Secondly, the cumulants of the standardized free Gamma type law μ_{2a,a^2} are

$$R_1 = 0 \quad \text{and} \quad R_{k+1} = \frac{1}{k+1}\binom{2k}{k}a^{k-1}, \quad k \geqslant 1.$$

Compare the Delaney triangle in [13].

Remark 5.8. It would be interesting to know whether Theorem 3.2 admits random matrix models, i.e. whether there are pairs of independent random matrices $\mathbf{X}, \mathbf{Y}$ which have the same law that is invariant under orthogonal transformations, i.e. $U\mathbf{X}U^T$ has the same law as $\mathbf{X}$ for any deterministic orthogonal matrix U, satisfy $E(\mathbf{X}^2|\mathbf{X}+\mathbf{Y}) = C(4+2a(\mathbf{X}+\mathbf{Y})+b^2(\mathbf{X}+\mathbf{Y})^2)$ and are asymptotically free.

All $\boxplus$-infinitely divisible laws have matrix models, see [7,17], see also [24, Section 4.4], but it is not clear whether one can preserve the quadratic form of the conditional variance.

Remark 5.9. Regarding Proposition 3.6, it would be interesting to know whether there are i.i.d. symmetric random matrices $\mathbf{X}, \mathbf{Y}$ with independent $\mathbf{S} = \mathbf{X} + \mathbf{Y}$ and $\mathbf{Z} = \mathbf{S}^{-1}\mathbf{X}^2\mathbf{S}^{-1}$ and with the law of $\mathbf{X}$ that is invariant under orthogonal transformations. Wishart $n \times n$ matrices with scale parameter $\mathbf{I}$ cannot have the above property for large n, as their asymptotic distribution is different.

Note added late. According to Létac [29], positive-definite $n \times n$ matrices with the above property do not exist for $n > 1$.

Acknowledgments

We thank M. Anshelevich, G. Létac, and J. Wesołowski for helpful comments and references. We thank the anonymous referee for very careful reading of the submitted manuscript. The first author would like to thank for fantastic working conditions at the Department of Mathematical Sciences of the University of Cincinnati during his visit to Cincinnati in September–October 2004. The second author would like to thank M. Anshelevich for raising the question of q-generalizations of [27].

References

[1] L. Accardi, M. Bożejko, Interacting Fock spaces and Gaussianization of probability measures, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 1 (1998) 663–670.
[2] M. Anshelevich, Partition-dependent stochastic measures and q-deformed cumulants, Doc. Math. 6 (2001) 343–384 (electronic).
[3] M. Anshelevich, Free martingale polynomials, J. Funct. Anal. 201 (2003) 228–261, math.CO/0112194.
[4] M. Anshelevich, Orthogonal polynomials with a resolvent-type generating function, math.CO/0410482, 2004; Trans. Amer. Math. Soc., in press.
[5] O.E. Barndorff-Nielsen, S. Thorbjørnsen, Lévy processes in free probability, Proc. Natl. Acad. Sci. USA 99 (2002) 16576–16580 (electronic).
[6] O.E. Barndorff-Nielsen, S. Thorbjørnsen, Self-decomposability and Lévy processes in free probability, Bernoulli 8 (2002) 323–366.
[7] F. Benaych-Georges, Classical and free infinitely divisible distributions and random matrices, Ann. Probab. 33 (2005) 1134–1170, math.PR/0406082.
[8] H. Bercovici, V. Pata, Stable laws and domains of attraction in free probability theory, Ann. of Math. (2) 149 (1999) 1023–1060, with an appendix by P. Biane.
[9] H. Bercovici, D. Voiculescu, Lévy–Hinčin type theorems for multiplicative and additive free convolution, Pacific J. Math. 153 (1992) 217–248.
[10] P. Biane, Processes with free increments, Math. Z. 227 (1998) 143–174.
[11] K. Bobecka, J. Wesołowski, The Lukacs–Olkin–Rubin theorem without invariance of the "quotient", Studia Math. 152 (2002) 147–160.
[12] M. Bożejko, R. Speicher, ψ-independent and symmetrized white noises, in: Quantum Probability and Related Topics, QP–PQ, VI, World Sci. Publ., River Edge, NJ, 1991, pp. 219–236.
[13] M. Bożejko, J. Wysoczański, Remarks on t-transformations of measures and convolutions, Ann. Inst. H. Poincaré Probab. Statist. 37 (2001) 737–761.
[14] M. Bożejko, M. Leinert, R. Speicher, Convolution and limit theorems for conditionally free random variables, Pacific J. Math. 175 (1996) 357–388.
[15] M. Bożejko, B. Kümmerer, R. Speicher, q-Gaussian processes: non-commutative and classical aspects, Comm. Math. Phys. 185 (1997) 129–154.

[16] W. Bryc, J. Wesołowski, Conditional moments of q-Meixner processes, Prob. Theory Related Fields 131 (2005) 415–441, arxiv.org/abs/math.PR/0403016.

[17] T. Cabanal-Duvillard, A matrix representation of the Bercovici–Pata bijection, Electron. J. Probab. 10 (2005) 632–661, paper 18.

[18] M. Capitaine, M. Casalis, Asymptotic freeness by generalized moments for Gaussian and Wishart matrices. Application to beta random matrices, Indiana Univ. Math. J. 53 (2004) 397–431.

[19] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon & Breach, New York, 1978.

[20] B. Collins, Product of random projections, Jacobi ensembles and universality problems arising from free probability, math.PR/0406560.

[21] U. Haagerup, On Voiculescu's R- and S-transforms for free non-commuting random variables, in: Free Probability Theory, Waterloo, ON, 1995, in: Fields Inst. Commun., vol. 12, Amer. Math. Soc., Providence, RI, 1997, pp. 127–148.

[22] U. Haagerup, F. Larsen, Brown's spectral distribution measure for R-diagonal elements in finite von Neumann algebras, J. Funct. Anal. 176 (2000) 331–367.

[23] G.C. Hegerfeldt, A quantum characterization of Gaussianness, in: Quantum Probability and Related Topics QP–PQ, VII, World Sci. Publ., River Edge, NJ, 1992, pp. 165–173.

[24] F. Hiai, D. Petz, The semicircle law, free random variables and entropy, Math. Surveys Monogr., vol. 77, Amer. Math. Soc., Providence, RI, 2000.

[25] O. Hiwatashi, M. Nagisa, H. Yoshida, The characterizations of a semicircle law by the certain freeness in a C^*-probability space, Probab. Theory Related Fields 113 (1999) 115–133.

[26] H. Kesten, Symmetric random walks on groups, Trans. Amer. Math. Soc. 92 (1959) 336–354.

[27] R.G. Laha, E. Lukacs, On a problem connected with quadratic regression, Biometrika 47 (1960) 335–343.

[28] F. Lehner, Cumulants in noncommutative probability theory. II. Generalized Gaussian random variables, Probab. Theory Related Fields 127 (2003) 407–422.

[29] G. Létac, Remarks on the Bozejko and Bryc problem, private communication, LaTeX file: July 13, 2005.

[30] G. Létac, H. Massam, Quadratic and inverse regressions for Wishart distributions, Ann. Statist. 26 (1998) 573–595.

[31] E. Lukacs, A characterization of the gamma distribution, Ann. Math. Statist. 26 (1955) 319–324.

[32] V.A. Marchenko, L.A. Pastur, Distribution of eigenvalues for some sets of random matrices, USSR Sb. 1 (1967) 457–483.

[33] B.D. McKay, The expected eigenvalue distribution of a large regular graph, Linear Algebra Appl. 40 (1981) 203–216.

[34] J. Meixner, Orthogonale Polynomsysteme mit einer besonderen Gestalt der erzeugenden Funktion, J. London Math. Soc. 9 (1934) 6–13.

[35] C.N. Morris, Natural exponential families with quadratic variance functions, Ann. Statist. 10 (1982) 65–80.

[36] A. Nica, A one-parameter family of transforms, linearizing convolution laws for probability distributions, Comm. Math. Phys. 168 (1995) 187–207.

[37] A. Nica, R-transforms of free joint distributions and non-crossing partitions, J. Funct. Anal. 135 (1996) 271–296.

[38] A. Nica, R. Speicher, On the multiplication of free N-tuples of noncommutative random variables, Amer. J. Math. 118 (1996) 799–837.

[39] I. Olkin, H. Rubin, A characterization of the Wishart distribution, Ann. Math. Statist. 33 (1962) 1272–1280.

[40] N. Saitoh, H. Yoshida, The infinite divisibility and orthogonal polynomials with a constant recursion formula in free probability theory, Probab. Math. Statist. 21 (2001) 159–170.

[41] W. Schoutens, Stochastic Processes and Orthogonal Polynomials, Lecture Notes in Statist., vol. 146, Springer-Verlag, New York, 2000.

[42] R. Speicher, Free probability theory and non-crossing partitions, Sém. Lothar. Combin. 39 (1997), Art. B39c, 38 pp. (electronic).

[43] M. Takesaki, Conditional expectations in von Neumann algebras, J. Funct. Anal. 9 (1972) 306–321.

[44] M. Takesaki, Theory of Operator Algebras I–III, Encyclopaedia Math. Sci., vol. 124, Springer-Verlag, Berlin, 2002, reprint of the first ed. in: Oper. Algebras and Non-Commut. Geom., vols. 5, 6, 8, Springer-Verlag, Berlin, 1979.

[45] D. Voiculescu, Lectures on free probability theory, in: Lectures on Probability Theory and Statistics, Saint-Flour, 1998, in: Lecture Notes in Math., vol. 1738, Springer-Verlag, Berlin, 2000, pp. 279–349.

[46] D.V. Voiculescu, K.J. Dykema, A. Nica, Free Random Variables, Amer. Math. Soc., Providence, RI, 1992.

[47] J. Wesołowski, Characterizations of some processes by properties of conditional moments, Demonstratio Math. 22 (1989) 537–556.

[48] J. Wesołowski, Stochastic processes with linear conditional expectation and quadratic conditional variance, Probab. Math. Statist. 14 (1993) 33–44.

[49] E.P. Wigner, On the distribution of the roots of certain symmetric matrices, Ann. of Math. (2) 67 (1958) 325–327.