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A Hyperalgebraic Proof of the Isomorphism and Isogeny Theorems for Reductive Groups

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INTRODUCTION

We present a quite elementary proof of the isomorphism and isogeny theorems for reductive algebraic groups over an algebraically closed field. We do not require a special discussion of rank 2 groups. Our method is to construct directly the graph of an isomorphism or an isogeny as a closed subgroup of the direct product by preassigning the hyperalgebra. We need only standard results on reductive groups as summarized in Section 1. Let k be an algebraically closed field of arbitrary characteristic, and let G and G' be connected reductive algebraic groups over k with maximal tori T and T', respectively. We begin with the isomorphism theorem. Assume the root datum (for which see [6, p. 189]) of G with respect to T is isomorphic to the root datum of G' with respect to T'. We are going to construct an isomorphism of algebraic groups $G \to G'$ which induces the isomorphism of root data. To avoid the notational complexity, we think G and G' have the same maximal torus T and assume that the root data of G and G' with respect to T are the same $(X, X^\vee, \Phi, \Phi^\vee)$. This abuse of notation makes the argument simple. Let $\{\alpha_1,..., \alpha_l\}$ be a base of Φ. We construct some universal hyperalgebra $\mathscr{U}(A)$ associated with the Cartan matrix $A = (\langle \alpha_i^\vee, \alpha_j \rangle)$, which is defined by some generators and relations similar to the condition for Kac–Moody Lie algebras. There is a hyperalgebra map $\mathscr{U}(A) \to \mathrm{hy}(G \times G')$ whose image is the hyperalgebra of some connected closed subgroup H of $G \times G'$ which is normalised by $\Delta(T) = \{(t, t) \mid t \text{ in } T\}$, and $\tilde{G} = H \cdot \Delta(T)$ is seen reductive with maximal torus $\Delta(T)$. We can prove that the projections $\tilde{G} \to G$ and $\tilde{G} \to G'$ are isomorphisms of algebraic groups. Hence $\tilde{G}$ is the graph of an isomorphism $G \to G'$ which is the identity on T. The isogeny theorem can be proved quite similarly. In fact we need not prove the isomorphism theorem since it is included in the isogeny theorem. We make the isomorphism theorem precede the isogeny one because of its notational simplicity.

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In case of semisimple groups, we can avoid use of the coroots. Since the coroots $\Phi^\vee$ are determined by the root system (X, Φ), two semisimple algebraic groups are isomorphic if their root systems are isomorphic. We will use coroots and root data to deal with reductive groups according to [2, 6].

The main theorems of the paper are formulated in terms of linear algebraic groups [1, 4, 6]. The viewpoint of groups schemes [3, 10], however, is required as a tool. A linear algebraic group is identified with the group of rational points at some fixed algebraically closed field of some smooth algebraic affine group scheme. Our main tool, the theory of hyperalgebras of algebraic affine group schemes, is summarized in Section 2. The kernel of a map of algebraic groups always means the *group scheme kernel*, but not the set-theoretic kernel.

G_a and G_m denote the one-dimensional additive and multiplicative groups, respectively.

1. Some Standard Results on Reductive Groups

We need know nothing about groups of semisimple rank 2 in order to prove the isomorphism or isogeny theorem. What we assume about reductive groups is very standard and relatively small as summerized in the following. Mostly it concerns commutator relations of root groups. The main references of this section are [1, 2, 4, 6].

Let k be an algebraically closed field. Every algebraic group is linear and defined over k. Let G be a reductive group, T a maximal torus of G, Φ the set of roots of G with respect to T, and $\{\alpha_1,..., \alpha_l\}$ a base of Φ. For each $\alpha \in \Phi$, there is a connected T-stable subgroup U_α of G together with an isomorphism $x_\alpha : G_a \to U_\alpha$ such that $tx_\alpha(a)\, t^{-1} = x_\alpha(\alpha(t)a)$ for $t \in T$, $a \in G_a$.

1.1. Theorem [1, p. 341; 4, p. 166]. *G is generated by T and U_α for $\alpha = \pm \alpha_i$ $(i = 1,..., l)$.*

1.2. Theorem [1, p. 352; 4, p. 174]. *Let U^+ (resp. U^-) be the subgroup generated by U_α for all positive (resp. negative) roots α. The product map defines an open immersion of varieties*

$$U^- \times T \times U^+ \hookrightarrow G.$$

Preassign any order on the positive (resp. negative) roots. Then the product map induces isomorphisms of varieties

$$\prod_{\alpha>0} U_\alpha \to U^+, \qquad \prod_{\alpha<0} U_\alpha \to U^-.$$

1.3. PROPOSITION [1, p. 332; 4, p. 203; 6, p. 207]. *Let* $\alpha, \beta \in \Phi$, $\alpha \neq \pm\beta$. U_α *commutes with* U_β *if there is no root of the form* $m\alpha + n\beta$ *with* $m, n > 0$. *In particular,* U_{α_i} *commutes with* $U_{-\alpha_j}$ *for* $1 \leqslant i \neq j \leqslant l$.

Let $X = X(T)$ be the character group of T. It is a finite free abelian group and its dual $X^\vee$ is identified with $\mathrm{Hom}(G_m, T)$. We denote by $\langle \lambda, \alpha \rangle$ the value of $\lambda \in X^\vee$ at $\alpha \in X$. We follow [2] for the definition of coroots. (The definition of [6] is a bit different but equivalent.) In what follows, we view G as a *group functor* on commutative k-algebras.

1.4. DEFINITION OF COROOTS [2, pp. 156–158]. For each $\alpha \in \Phi$, there is a unique homomorphism $\alpha^\vee \colon G_m \to T$ together with an invertible element $c = c(\alpha) \in k$ such that

$$x_\alpha(t)\, x_{-\alpha}(u) = x_{-\alpha}\left(\frac{u}{1 + ctu}\right) \alpha^\vee(1 + ctu)\, x_\alpha\left(\frac{t}{1 + ctu}\right)$$

in the group $G(k[t, u, (1 + ctu)^{-1}])$. The map $\alpha^\vee$ is called the coroot associated with α. It does not depend on the choice of $(x_\alpha, x_{-\alpha})$. The set of all coroots is denoted by $\Phi^\vee$. The set $(X, X^\vee, \Phi, \Phi^\vee)$ satisfies the axiom of a root datum and called the root datum of G relative to T [6, pp. 189–190]. The pair $(x_\alpha, x_{-\alpha})$ is called normal (*appariée* [2, ibid.]) if $c(\alpha) = 1$. If we define $x'_\alpha(t) = x_\alpha(c(\alpha)^{-1} t)$, then $(x'_\alpha, x_{-\alpha})$ is normal.

2. HYPERALGEBRAIC INTERPRETATION OF RESULTS IN SECTION 1

A *hyperalgebra* means an irreducible cocommutative Hopf algebra. Let G be an algebraic affine group scheme over k represented by a commutative Hopf algebra A with the augmentation ideal M. For each integer $n > 0$, A/M^n is a finite dimensional algebra; hence

$$\mathrm{hy}(G) = \bigcup_n (A/M^n)^*$$

has the structure of a cocommutative (irreducible) coalgebra. It is at the same time a subalgebra of the dual algebra A^*, and seen to become a hyperalgebra. It is called *the hyperalgebra of* G. The general theory of hyperalgebras is developed in [8] and summarized in [9, (0.3), p. 258]. We begin with reviewing the required properties of hyperalgebras.

Let J be a subhyperalgebra of $\mathrm{hy}(G)$. J is *closed* if there is a closed subgroup scheme H of G such that $J = \mathrm{hy}(H)$. J is *dense* if $\mathrm{hy}(G)$ is the only closed subhyperalgebra containing J. There is a one-to-one correspondence between closed subhyperalgebras of $\mathrm{hy}(G)$ and closed *connected* subgroup schemes of G given by $\mathrm{hy}(H) \leftrightarrow H$, and this correspondence preserves and reflects the inclusion [8, (3.3.9), p. 117].

For any integer $n \geqslant 0$, there is some cocommutative coalgebra B_n, called the n-dimensional *Birkhoff–Witt* coalgebra [8, (1.6), p. 39] such that the dual algebra B_n^* is isomorphic to $k[[t_1,..., t_n]]$ the algebra of formal power series in n indeterminates. An n-dimensional algebraic affine group scheme G is *smooth* if and only if $\mathrm{hy}(G) \simeq B_n$ as coalgebras [8, (1.9.5), p. 56]. It follows from [7, (4.1.9) and (4.2.7)] that a hyperalgebra is a Birkhoff–Witt coalgebra if (and only if) it is generated by sequences of divided powers.

For two elements x, y in any cocommutative Hopf algebra, we put [8, (1.10.5), p. 64]

$$[x, y] = \sum x_{(1)} y_{(1)} S(x_{(2)}) S(y_{(2)})$$

with the sigma notation and the antipode S. A subhyperalgebra H of a hyperalgebra *normalizes* another subhyperalgebra K if $[x, y] \in K$ for all $x \in H$ and $y \in K$. The above one-to-one correspondence preserves and reflects the normalization [8, (3.4.15), p. 131]. The subalgebra generated by all commutators $[x, y]$ in a hyperalgebra J is a subhyperalgebra, denoted $[J, J]$ and called the *derived* subhyperalgebra of J. We have the following closedness criterion:

2.0.1. THEOREM. *Let G be a connected algebraic affine group scheme, and let J be a subhyperalgebra of* $\mathrm{hy}(G)$.

(a) [8, (3.6.3), p. 140]: *If J is of the Birkhoff–Witt type, then $[J, J]$ is a closed subhyperalgebra of* $\mathrm{hy}(G)$.

(b) [8, ibid. and (3.5.6), p. 138]: *If G is smooth and J is dense in* $\mathrm{hy}(G)$, *then $[J, J]$ is the hyperalgebra of the derived subgroup scheme* $[G, G]$.

Let G_1 and G_2 be connected algebraic affine group schemes, and let $f: G_1 \to G_2$ be a homomorphism. It induces a hyperalgebra map $\mathrm{hy}(f): \mathrm{hy}(G_1) \to \mathrm{hy}(G_2)$. f is *faithfully flat* if and only if $\mathrm{hy}(f)$ is *surjective* [8, (3.3.7), p. 116]. $\mathrm{hy}(f)$ is *injective* if and only if the kernel of f is etale [8, (3.3.3), p. 114].

We will use the functorial characterization of the hyperalgebra: Let G be an algebraic affine group scheme. For any connected cocommutative coalgebra C, there is a group isomorphism natural in C

$$\mathrm{Ker}(G(C^*) \to G(k)) \simeq \mathrm{Coalg}_k(C, \mathrm{hy}(G))$$

where the left-hand side means the kernel of the homomorphism induced by the canonical map $C^* \to k$, and the right-hand side means the group of all coalgebra maps $C \to \mathrm{hy}(G)$ [8, (3.1.2), p. 101]. This characterization has the following important consequence (2.0.3).

Let $\{t_1,..., t_n\}$ be the set of n indeterminates. For any algebra A, let $A[[t_1,..., t_n]]$ be the algebra of all formal sums

$$\sum_{(e)} t_1^{e_1} \cdots t_n^{e_n} \cdot a_{(e)}$$

with $a_{(e)}$ in A for all $(e) = (e_1,..., e_n)$ with $e_i \geqslant 0$.

Let H be a hyperalgebra with comultiplication Δ and counit ε.

2.0.2. DEFINITION. An element $\sum_{(e)} t^{(e)} g_{(e)}$ in $H[[t_1,..., t_n]]$ with $t^{(e)} = t_1^{e_1} \cdots t_n^{e_n}$ is called *group-like* if we have

$$\sum_{(e)} t^{(e)} \Delta(g_{(e)}) = \sum_{(e)} t^{(e)} \left\{ \sum_{(e)=(c)+(d)} g_{(c)} \otimes g_{(d)} \right\}$$

in the algebra $(H \otimes H)[[t_1,..., t_n]]$ and $\varepsilon(g_{(e)}) = \delta_{(e),(0)}$. The set of all group-like elements in $H[[t_1,..., t_n]]$ forms a subgroup of units denoted by

$$\mathrm{gr}(H[[t_1,..., t_n]]).$$

2.0.3. THEOREM. *Let G be an algebraic affine group scheme over k and let $H = \mathrm{hy}(G)$. There is a canonical isomorphism of groups*

$$\mathrm{Ker}(G(k[[t_1,..., t_n]]) \xrightarrow{\pi} G(k)) \to \mathrm{gr}(H[[t_1,..., t_n]])$$

where π denotes the map induced from the augmentation. This isomorphism is natural in $k[[t_1,..., t_n]]$ in the following sense: Let $f: k[[t_1,..., t_n]] \to k[[u_1,..., u_m]]$ be an algebra map commuting with the augmentation. We then get a commutative diagram:

$$\begin{array}{ccc} \mathrm{Ker}(G(k[[t_1,..., t_n]]) \to G(k)) & \to & \mathrm{gr}(H[[t_1,..., t_n]]) \\ \downarrow{\scriptstyle G(f)} & & \downarrow{\scriptstyle f} \\ \mathrm{Ker}(G(k[[u_1,..., u_m]]) \to G(k)) & \to & \mathrm{gr}(H[[u_1,..., u_m]]) \end{array}$$

Proof. We have $k[[t_1,..., t_n]] = B_n^*$ and a natural group isomorphism

$$\mathrm{Ker}(G(B_n^*) \to G(k)) \simeq \mathrm{Coalg}_k(B_n, H).$$

By a simple calculation, we see the subgroup $\mathrm{Coalg}_k(B_n, H)$ corresponds to the subgroup $\mathrm{gr}(H[[t_1,..., t_n]])$ under the canonical isomorphism of algebras

$$\mathrm{Hom}_k(B_n, H) \simeq H[[t_1,..., t_n]].$$

The claim follows from this. Q.E.D.

Let us see what the above isomorphism becomes when $G = G_a$ and G_m.

2.0.4. *Fact.* Let $k[T]$ and $k[U, U^{-1}]$ be the affine Hopf algebra of G_a and G_m, respectively. (T is primitive and U is group-like.) Define linear maps

$$X^{(n)}: k[T] \to k,\ H^{(n)}: k[U, U^{-1}] \to k \qquad (n = 0, 1, \ldots)$$

by

$$\langle X^{(n)}, T^i \rangle = \delta_{n,i} \qquad \text{for} \quad i \geqslant 0,$$

$$\langle H^{(n)}, U^i \rangle = \binom{i}{n} = \frac{i(i-1)\cdots(i-n+1)}{1 \cdot 2 \cdots n} \qquad \text{for} \quad i \in \mathbb{Z}.$$

(Note that $\binom{i}{n}$ are integers, hence meaningful in k.) Then $\{X^{(n)}\}_{n \geqslant 0}$ and $\{H^{(n)}\}_{n \geqslant 0}$ form linear bases for $\mathrm{hy}(G_a)$ and $\mathrm{hy}(G_m)$, respectively. They are sequences of divided powers, i.e.,

$$\Delta(X^{(n)}) = \sum_{i+j=n} X^{(i)} \otimes X^{(j)}, \qquad \Delta(H^{(n)}) = \sum_{i+j=n} H^{(i)} \otimes H^{(j)}$$

for all n. We put

$$X(t) = \sum_{n=0}^{\infty} t^n X^{(n)}, \qquad H(t) = \sum_{n=0}^{\infty} t^n H^{(n)}$$

in $\mathrm{hy}(G_a)[[t]]$ and $\mathrm{hy}(G_m)[[t]]$, respectively. They are group-like elements. Let $k[[t_1, \ldots, t_n]]_0$ be the augmentation ideal of $k[[t_1, \ldots, t_n]]$. The isomorphism in (2.0.3) for G_a and G_m is given by the following formulas:

$$G_a(k[[t_1, \ldots, t_n]]_0) \simeq \mathrm{gr}(\mathrm{hy}(G_a)[[t_1, \ldots, t_n]])$$

$$P(t_1, \ldots, t_n) \leftrightarrow X(P(t_1, \ldots, t_n)),$$

$$G_m(1 + k[[t_1, \ldots, t_n]]_0) \simeq \mathrm{gr}(\mathrm{hy}(G_m)[[t_1, \ldots, t_n]])$$

$$1 + P(t_1, \ldots, t_n) \leftrightarrow H(P(t_1, \ldots, t_n)).$$

In particular we have

$$X(t)\, X(u) = X(t+u) \qquad \text{in} \quad \mathrm{hy}(G_a)[[t, u]],$$

$$H(t)\, H(u) = H(t + u + tu) \qquad \text{in} \quad \mathrm{hy}(G_m)[[t, u]].$$

This fact can be verified easily. We leave it to the reader (cf. [8, (1.5.8), p. 35]). In characteristic 0, $\mathrm{hy}(G_a)$ and $\mathrm{hy}(G_m)$ are the polynomial algebras $k[X^{(1)}]$ and $k[H^{(1)}]$, respectively, and we have

$$X^{(n)} = \frac{X^{(1)n}}{n!}, \qquad H^{(n)} = \binom{H^{(1)}}{n}.$$

We give the hyperalgebraic meaning of (1.1)–(1.4). Let the notation be as in Section 1.

2.1. THEOREM. *The hyperalgebra* $\mathrm{hy}(G)$ *is generated as an algebra by* $\mathrm{hy}(T)$ *and* $\mathrm{hy}(U_\alpha)$ *for* $\alpha = \pm\alpha_i$ $(i = 1,..., l)$.

Proof. The subalgebra J generated by $\mathrm{hy}(T)$ and $\mathrm{hy}(U_\alpha)$ for $\alpha = \pm\alpha_i$ is dense in $\mathrm{hy}(G)$ by (1.1). Hence the commutator subhyperalgebra $[J, J]$ equals $\mathrm{hy}([G, G])$ by (2.0.1)(b). Since $G = [G, G] \cdot T$, we have $\mathrm{hy}(G) = \mathrm{hy}([G, G]) \cdot \mathrm{hy}(T) = J$. Q.E.D.

The next two items are direct consequences of (1.2) and (1.3).

2.2. THEOREM. *The product map defines an isomorphism of coalgebras*

$$\mathrm{hy}(U^-) \otimes \mathrm{hy}(T) \otimes \mathrm{hy}(U^+) \to \mathrm{hy}(G).$$

Let $\{\gamma_1,..., \gamma_m\}$ *be the positive roots in any order. The product map induces isomorphisms of coalgebras*

$$\mathrm{hy}(U_{\gamma_1}) \otimes \cdots \otimes \mathrm{hy}(U_{\gamma_m}) \to \mathrm{hy}(U^+),$$
$$\mathrm{hy}(U_{-\gamma_1}) \otimes \cdots \otimes \mathrm{hy}(U_{-\gamma_m}) \to \mathrm{hy}(U^-).$$

2.3. PROPOSITION. $\mathrm{hy}(U_{\alpha_i})$ *commutes with* $\mathrm{hy}(U_{-\alpha_j})$ *for* $1 \leqslant i \neq j \leqslant l$.

For each $\alpha \in \Phi$ and $n \geqslant 0$, let $H_\alpha^{(n)}$ and $X_\alpha^{(n)}$ be the images of $H^{(n)}$ and $X^{(n)}$ (2.0.4) by the hyperalgebra maps

$$\mathrm{hy}(\alpha^\vee)\colon \mathrm{hy}(G_m) \to \mathrm{hy}(T), \mathrm{hy}(x_\alpha)\colon \mathrm{hy}(G_a) \to \mathrm{hy}(U_\alpha)$$

respectively. Put

$$H_\alpha(t) = \sum_{n=0}^{\infty} t^n H_\alpha^{(n)}, \qquad X_\alpha(t) = \sum_{n=0}^{\infty} t^n X_\alpha^{(n)}$$

which are group-like elements in $\mathrm{hy}(G)[[t]]$.

2.4. THEOREM. *For each* $\alpha \in \Phi$, *we have*

$$X_\alpha(t)\, X_{-\alpha}(u) = X_{-\alpha}\left(\frac{u}{1 + ctu}\right) H_\alpha(ctu)\, X_\alpha\left(\frac{t}{1 + ctu}\right)$$

in $\mathrm{hy}(G)[[t, u]]$ *with constant* $c = c(\alpha)$ *of* (1.4).

Proof. There is a canonical algebra map

$$k[t, u, (1 + ctu)^{-1}] \to k[[t, u]].$$

Hence the formula of (1.4) is valid in the group $G(k[[t,u]])$, and both terms of the formula belong to $\mathrm{Ker}(G(k[[t,u]]) \to G(k))$. We have an associated formula in $\mathrm{hy}(G)[[t,u]]$ by (2.0.3). That is precisely the formula to be proved by (2.0.4). Q.E.D.

We get the following proposition in the same way from

$$\alpha^\vee(t)\, x_\beta(u)\, \alpha^\vee(t^{-1}) = x_\beta(t^{\langle \alpha^\vee, \beta \rangle} u)$$

for $\alpha, \beta \in \Phi$, $t \in G_m$, $u \in G_a$.

2.5. PROPOSITION. *For $\alpha, \beta \in \Phi$ we have*

$$H_\alpha(t)\, X_\beta(u)\, H_\alpha(t)^{-1} = X_\beta((1+t)^{\langle \alpha^\vee, \beta \rangle} u)$$

in $\mathrm{hy}(G)[[t,u]]$.

3. SOME UNIVERSAL HYPERALGEBRA

Let $A = (A_{ij})$, $1 \leqslant i, j \leqslant l$, be a *generalized Cartan matrix* [5], i.e., (i) $A_{ii} = 2$ for all i, (ii) A_{ij} is an integer $\leqslant 0$ if $i \neq j$, (iii) $A_{ij} = 0$ if $A_{ji} = 0$. We will construct a hyperalgebra $\mathscr{U}(A)$.

Let F_l be the free associative k-algebra with the unit generated by the set of symbols:

$$\{H_i^{(n)}, X_i^{(n)}, Y_i^{(n)} \mid i = 1, \dots, l, n = 1, 2, \dots\}.$$

We put in $F_l[[t]]$

$$H_i(t) = \sum_{n=0}^{\infty} t^n H_i^{(n)}, \qquad X_i(t) = \sum_{n=0}^{\infty} t^n X_i^{(n)}, \qquad Y_i(t) = \sum_{n=0}^{\infty} t^n Y_i^{(n)}$$

with $H_i^{(0)} = X_i^{(0)} = Y_i^{(0)} = 1$ for $i = 1, \dots, l$.

3.1. DEFINITION. Let $\mathbf{a}$ be the smallest ideal of F_l such that the following identities hold in $(F_l/\mathbf{a})[[t,u]]$:

(1) $H_i(t)\, H_i(u) = H_i(t + u + tu)$ for all i,

(2) $H_i(t)\, H_j(u) = H_j(u)\, H_i(t)$ for all i, j,

(3) $X_i(t)\, X_i(u) = X_i(t+u)$ for all i,

(4) $Y_i(t)\, Y_i(u) = Y_i(t+u)$ for all i,

(5) $X_i(t)\, Y_j(u) = Y_j(u)\, X_i(t)$ if $i \neq j$,

(6) $H_i(t)\, X_j(u)\, H_i(t)^{-1} = X_j((1+t)^{A_{ij}} u)$ for all i, j,

(7) $H_i(t)\, Y_j(u)\, H_i(t)^{-1} = Y_j((1+t)^{-A_{ij}}u)$ for all i, j,

(8) $X_i(t)\, Y_i(u) = Y_i\left(\frac{u}{1+tu}\right) H_i(tu)\, X_i\left(\frac{t}{1+tu}\right)$ for all i.

The quotient algebra $F_l/\mathbf{a}$ is denoted by $\mathscr{U}(A)$.

In other words, $\mathbf{a}$ is the ideal of F_l generated by the coefficients in F_l of the differences of both sides of the above identities. Let $\mathscr{U}(A)_i^+$ (resp. $\mathscr{U}(A)_i^-$) be the subalgebra of $\mathscr{U}(A)$ generated by $\{X_i^{(n)}\}_n$ (resp. $\{Y_i^{(n)}\}_n$) for $i = 1,\dots l$.

3.2. Lemma. *The algebra $\mathscr{U}(A)$ is generated by $\mathscr{U}(A)_i^+$ and $\mathscr{U}(A)_i^-$ for $i = 1,\dots, l$.*

Proof. We have by (8)

$$H_i(tu) = Y_i\left(\frac{-u}{1+tu}\right) X_i(t)\, Y_i(u)\, X_i\left(\frac{-t}{1+tu}\right)$$

in $\mathscr{U}(A)[[t,u]]$. Hence $H_i^{(n)}$ belong to the subalgebra generated by $\mathscr{U}(A)_i^+$ and $\mathscr{U}(A)_i^-$.

3.3. Proposition. *$\mathscr{U}(A)$ has a unique coalgebra structure such that* (i) *$\mathscr{U}(A)$ becomes a hyperalgebra, and* (ii) *$\{H_i^{(n)}\}_n$, $\{X_i^{(n)}\}_n$, $\{Y_i^{(n)}\}_n$ are sequences of divided powers for $i = 1,\dots, l$.*

Proof. We can make F_l into a hyperplane by condition (ii) above. The condition is equivalent to saying that $H_i(t)$, $X_i(t)$, $Y_i(t)$ are group-like elements in $F_l[[t]]$. Hence both sides of all identities in (3.1) are group-like elements in $F_l[[t,u]]$. It follows easily from this that $\mathbf{a}$ is a Hopf ideal of F_l.

Q.E.D.

3.4. Proposition. $\mathscr{U}(A) = [\mathscr{U}(A), \mathscr{U}(A)]$.

Proof. It is enough to show $X_i^{(n)}$, $Y_i^{(n)} \in [\mathscr{U}(A), \mathscr{U}(A)]$ by (3.2). We have by (6) in $\mathscr{U}(A)[[t,u]]$

$$X_i((1+t)^2 - 1)u) = H_i(t)\, X_i(u)\, H_i(t)^{-1} X_i(u)^{-1}.$$

The right-hand side is equal to

$$\sum_{m,n \geqslant 0} t^m u^n [H_i^{(m)}, X_i^{(n)}].$$

By comparing the coefficients of $t^{2n}u^n$, we have

$$X_i^{(n)} = [H_i^{(2n)}, X_i^{(n)}] \in [\mathscr{U}(A), \mathscr{U}(A)].$$

The same is true for $Y_i^{(n)}$. Q.E.D.

Let $\mathscr{U}(A)^0$, $\mathscr{U}(A)^+$, $\mathscr{U}(A)^-$ be the subalgebra of $\mathscr{U}(A)$ generated by $\{H_i^{(n)}\}_{n,i}$, $\{X_i^{(n)}\}_{n,i}$, $\{Y_i^{(n)}\}_{n,i}$, respectively.

3.5. PROPOSITION. $\mathscr{U}(A)=\mathscr{U}(A)^-\mathscr{U}(A)^0\mathscr{U}(A)^+$.

Proof. It is enough to show that $\mathscr{U}(A)^-\mathscr{U}(A)^0\mathscr{U}(A)^+$ is a subalgebra. Since $\mathscr{U}(A)^0$ normalizes $\mathscr{U}(A)^\pm$, we have only to show

$$\mathscr{U}(A)^+\mathscr{U}(A)^-\subset\mathscr{U}(A)^-\mathscr{U}(A)^0\mathscr{U}(A)^+.$$

It follows from (5) and (8) that

$$\mathscr{U}(A)_i^+\mathscr{U}(A)_j^-\subset\mathscr{U}(A)_j^-\mathscr{U}(A)^0\mathscr{U}(A)_i^+$$

for any i, j. The claim follows easily from this. Q.E.D.

In fact, we have the tensor product decomposition which will not be used to prove the main results.

3.5′. THEOREM. *The multiplication induces a coalgebra isomorphism*

$$\mathscr{U}(A)^-\otimes\mathscr{U}(A)^0\otimes\mathscr{U}(A)^+\to\mathscr{U}(A).$$

Proof. The hyperalgebra $\mathscr{U}(A)_k$ (with base field specified) is defined even when k is a commutative ring. In particular we have a hyperalgebra $\mathscr{U}(A)_{\mathbb{Z}}$, and $\mathscr{U}(A)_k$ is precisely the scalar extension $\mathscr{U}(A)_{\mathbb{Z}}\otimes k$. Assume k is a field of characteristic 0 (or more generally a commutative $\mathbb{Q}$-algebra). The identities of (3.1) reduce to the following:

(1) $H_i^{(n)}=\binom{H_i^{(1)}}{n}$ for all n, i,

(2) $[H_i^{(1)}, H_j^{(1)}]=0$ for all i, j,

(3) $X_i^{(n)}=\dfrac{(X_i^{(1)})^n}{n!}$ for all n, i,

(4) $Y_i^{(n)}=\dfrac{(Y_i^{(1)})^n}{n!}$ for all n, i,

(5) $[X_i^{(1)}, Y_j^{(1)}]=0$ if $i\neq j$,

(6) $[H_i^{(1)}, X_j^{(1)}]=A_{ij}X_j^{(1)}$ for all i, j,

(7) $[H_i^{(1)}, Y_j^{(1)}]=-A_{ij}Y_j^{(1)}$ for all i, j,

(8) $[X_i^{(1)}, Y_i^{(1)}]=H_i^{(1)}$ for all i.

Hence $\mathscr{U}(A)_k$ is precisely the universal enveloping algebra of the Lie algebra generated by symbols

$$\{H_1^{(1)},..., H_l^{(1)}, X_1^{(1)},..., X_l^{(1)}, Y_1^{(1)},..., Y_l^{(1)}\}$$

subject to relations (2), (5), (6), (7), (8). It follows from [5, (1.1)] that the product map induces a coalgebra isomorphism

$$\mathscr{U}(A)_k^- \otimes \mathscr{U}(A)_k^0 \otimes \mathscr{U}(A)_k^+ \to \mathscr{U}(A)_k$$

and that $\mathscr{U}(A)_k^{\pm}$ is the free associative algebra generated by $\{X_1^{(1)},..., X_l^{(1)}\}$ (resp. $\{Y_1^{(1)},..., Y_l^{(1)}\}$). This is true for $k = \mathbb{Q}$ in particular. It is easy to prove that $\mathscr{U}(A)_{\mathbb{Z}}$ is identified with the $\mathbb{Z}$-subalgebra of $\mathscr{U}(A)_{\mathbb{Q}}$ generated by all $H_i^{(n)}$, $X_i^{(n)}$, $Y_i^{(n)}$. It follows that the above isomorphism is valid for $k = \mathbb{Z}$, hence for any field or commutative ring k. Q.E.D.

We return to reductive groups. Let the notation be as before. We use the isomorphisms x_α for $\alpha = \pm\alpha_i$ $(i = 1,..., l)$.

3.6. Proposition. *Assume that* $(x_{\alpha_i}, x_{-\alpha_i})$ *is normal for* $i = 1,..., l$. *Let* $A = (\langle\alpha_i^\vee, \alpha_j\rangle)$ *be the Cartan matrix of* $(X, X^\vee, \Phi, \Phi^\vee)$ *with respect to the base* $\{\alpha_1,..., \alpha_l\}$. *With the notation above* (2.4) *we have a hyperalgebra map*

$$\phi: \mathscr{U}(A) \to \mathrm{hy}(G)$$

such that $\phi(H_i^{(n)}) = H_{\alpha_i}^{(n)}$, $\phi(X_i^{(n)}) = X_{\alpha_i}^{(n)}$, $\phi(Y_i^{(n)}) = X_{-\alpha_i}^{(n)}$. *We have*

$$\phi(\mathscr{U}(A)^0) \subset \mathrm{hy}(T), \quad \phi(\mathscr{U}(A)^\pm) \subset \mathrm{hy}(U^\pm), \qquad \phi(\mathscr{U}(A)) \cdot \mathrm{hy}(T) = \mathrm{hy}(G).$$

Proof. For the existence of ϕ, it is enough to verify the identities (1)–(8) in $\mathrm{hy}(G)[[t, u]]$ by replacing $(H_i(t), X_i(t), \cdot Y_i(t)) \to (H_{\alpha_i}(t), X_{\alpha_i}(t), X_{-\alpha_i}(t))$. Identities (1)–(4) are trivially true, (5) follows from (2.3), (8) from (2.4), and (6) and (7) from (2.5). It is clear that

$$\phi(\mathscr{U}(A)^0) \subset \mathrm{hy}(T), \qquad \phi(\mathscr{U}(A)^\pm) \subset \mathrm{hy}(U^\pm).$$

$\mathrm{hy}(T)$ normalizes $\phi(\mathscr{U}(A))$ which is the subalgebra generated by $\mathrm{hy}(U_\alpha)$ for $\alpha = \pm\alpha_i$ $(i = 1,..., l)$. Hence $\phi(\mathscr{U}(A)) \cdot \mathrm{hy}(T)$ is a subalgebra containing $\mathrm{hy}(T)$ and $\mathrm{hy}(U_\alpha)$ for $\alpha = \pm\alpha_i$ $(i = 1,..., l)$. The last identity follows from this by (2.1). Q.E.D.

We emphasize that the hyperalgebra map is constructed by using only $(x_{\alpha_1},..., x_{\alpha_l}, x_{-\alpha_1},..., x_{-\alpha_l})$. By the normality condition, $(x_{-\alpha_1},..., x_{-\alpha_l})$ is determined by $(x_{\alpha_1},..., x_{\alpha_l})$. Hence the hyperalgebra map is determined by $(x_{\alpha_1},..., x_{\alpha_l})$ which may be chosen arbitrarily.

4. The Isomorphism Theorem

We prove that if two reductive algebraic groups have isomorphic root data with respect to some maximal tori, then there is an isomorphism of algebraic groups inducing the isomorphism of root data. Some isomorphism of the maximal tori is associated with the isomorphism of the root data. Let us identify the maximal tori through that isomorphism. Strictly speaking we are considering two inclusions of the same torus into the reductive groups. Then, the two root data become the same. Hence we may begin with the following convention.

Let G and G' be reductive algebraic groups having a common maximal torus T. Assume that the root data of G and G' with respect to T are the same $(X, X^\vee, \Phi, \Phi^\vee)$. Let U_α and U'_α, $\alpha \in \Phi$, be the root groups of G and G'. We prove that there is an isomorphism $G \simeq G'$ which is the identity on T. Such an isomorphism will induce $U_\alpha \simeq U'_\alpha$ for all $\alpha \in \Phi$.

Let $\{\alpha_1, \dots, \alpha_l\}$ be a base of Φ, and $A = (\langle \alpha_i^\vee, \alpha_j \rangle)$ the Cartan matrix. Choose arbitrary sets of admissible isomorphisms

$$x_i\colon G_a \to U_{\alpha_i}, \qquad x'_i\colon G_a \to U'_{\alpha_i} \qquad (i = 1, \dots, l)$$

and determine the admissible isomorphisms

$$x_{-i}\colon G_a \to U_{-\alpha_i}, \qquad x'_{-i}\colon G_a \to U'_{-\alpha_i} \qquad (i = 1, \dots, l)$$

in such a way that (x_i, x_{-i}) and (x'_i, x'_{-i}) are normal. Let $\tilde{U}_{\pm i}$ be the image of the inclusion

$$\tilde{x}_{\pm i} = (x_{\pm i}, x'_{\pm i})\colon G_a \to U_{\pm\alpha_i} \times U'_{\pm\alpha_i}.$$

Put $\Delta(T) = \{(t, t) \mid t \in T\}$. Let

$$\phi\colon \mathscr{U}(A) \to \mathrm{hy}(G) \qquad \text{and} \qquad \phi'\colon \mathscr{U}(A) \to \mathrm{hy}(G')$$

be the hyperalgebra maps determined by $(x_1, \dots, x_l)$ and $(x'_1, \dots, x'_l)$ respectively (3.6). Put

$$\tilde{\phi}\colon \mathscr{U}(A) \xrightarrow{\Delta} \mathscr{U}(A) \otimes \mathscr{U}(A) \xrightarrow{\phi \otimes \phi'} \mathrm{hy}(G) \otimes \mathrm{hy}(G') = \mathrm{hy}(G \times G').$$

4.1. LEMMA. (a) *There is a unique connected closed subgroup* H *of* $G \times G'$ *such that* $\mathrm{hy}(H) = \mathrm{Im}(\tilde{\phi})$.

(b) H *contains all* $\tilde{U}_i$ *for* $i = \pm 1, \dots, \pm l$.

(c) $\Delta(T)$ *normalizes* H.

Proof. (a) The hyperalgebra $\mathrm{Im}(\tilde{\phi})$ is generated by sequences of divided powers and equal to the commutator subhyperalgebra by (3.3) and (3.4). Hence $\mathrm{Im}(\tilde{\phi}) = \mathrm{hy}(H)$ for a uniquely determined connected reduced closed subgroup scheme H of $G \times G'$ by (2.0.1)(a). H is identified with a closed subgroup of $G \times G'$. (b) We have $\tilde{\phi}(\mathscr{U}(A)_i^{\pm}) = \mathrm{hy}(\tilde{U}_{\pm i})$ by definition. Hence H contains $\tilde{U}_{\pm i}$ for $i = 1, \dots, l$. (c) $\Delta(T)$ normalizes $\tilde{U}_{\pm i}$ for all i since $(t, t)\, \tilde{x}_{\pm i}(a)(t, t)^{-1} = \tilde{x}_{\pm i}(\alpha_i(t^{\pm 1})a)$ for $t \in T$, $a \in G_a$. Hence $\mathrm{hy}(\Delta(T))$

normalizes $\mathrm{hy}(H)$ which is the subalgebra generated by $\mathrm{hy}(\tilde{U}_i)$ for $i = \pm 1, \ldots, \pm l$ (3.2). Therefore $\Delta(T)$ normalizes H. Q.E.D.

It follows from (c) that $\tilde{G} = H \cdot \Delta(T)$ is a connected closed subgroup of $G \times G'$ having the hyperalgebra $\mathrm{Im}(\tilde{\phi}) \cdot \mathrm{hy}(\Delta(T))$. We show that $\tilde{G}$ is the graph of a desired isomorphism $G \simeq G'$. This is done in a sequence of lemmas.

4.2. Lemma. (a) *The projections* $pr_1 : \tilde{G} \to G$ *and* $pr_2 : \tilde{G} \to G'$ *are surjective.*

(b) $\tilde{G}$ *is reductive.*

Proof. (a) $\mathrm{hy}(pr_1)$ and $\mathrm{hy}(pr_2)$ are surjective by (3.6). Hence pr_1 and pr_2 are surjective (or faithfully flat) (see below (2.0.1)). (b) Let R_u be the unipotent radical of $\tilde{G}$. Then $pr_1(R_u)$ and $pr_2(R_u)$ are trivial by (a). Hence $R_u = (1)$.

4.3. Lemma. $\mathrm{hy}(\tilde{G}) \cap \mathrm{hy}(T \times T) = \mathrm{hy}(\Delta(T))$.

Proof. We have

$$\mathrm{hy}(\tilde{G}) = \tilde{\phi}(\mathscr{U}(A)^-)\, \mathrm{hy}(\Delta(T))\, \tilde{\phi}(\mathscr{U}(A)^+),$$
$$\mathrm{hy}(G \times G') = \mathrm{hy}(U^- \times U'^-) \otimes \mathrm{hy}(T \times T) \otimes \mathrm{hy}(U^+ \times U'^+)$$

by (3.5) and (2.2). Since $\tilde{\phi}(\mathscr{U}(A)^\pm) \subset \mathrm{hy}(U^\pm \times U'^\pm)$, it follows that the multiplication induces a coalgebra isomorphism

$$\tilde{\phi}(\mathscr{U}(A)^-) \otimes \mathrm{hy}(\Delta(T)) \otimes \tilde{\phi}(\mathscr{U}(A)^+) \to \mathrm{hy}(\tilde{G}).$$

Let C be a cocommutative coalgebra and let $f: C \to \mathrm{hy}(\tilde{G})$ be a coalgebra map. There is a unique decomposition

$$f(c) = \sum f^-(c_{(1)})\, f^0(c_{(2)})\, f^+(c_{(3)}) \qquad (c \in C)$$

with coalgebra maps $f^0: C \to \mathrm{hy}(\Delta(T))$, $f^\pm : C \to \tilde{\phi}(\mathscr{U}(A)^\pm)$. (We are using the sigma notation.) A similar decomposition is valid for coalgebra maps $C \to \mathrm{hy}(G \times G')$, and the inclusion $\mathrm{hy}(\tilde{G}) \to \mathrm{hy}(G \times G')$ preserves the decomposition. Hence, if $\mathrm{Im}(f)$ is contained in $\mathrm{hy}(\tilde{G}) \cap \mathrm{hy}(T \times T)$, then $f^\pm$ should be trivial. Thus $f = f^0$. This means that $\mathrm{hy}(\tilde{G}) \cap \mathrm{hy}(T \times T) = \mathrm{hy}(\Delta(T))$.

4.4. Lemma. (a) $\Delta(T)$ *is a maximal torus of* $\tilde{G}$.

(b) $\tilde{U}_{\pm i}$ *is a root group of* $\tilde{G}$ *with respect to* $\Delta(T)$ *for* $i = 1, \ldots, l$. *The projections induce isomorphisms*

$$\tilde{U}_{\pm i} \to U_{\pm \alpha_i}, \quad \tilde{U}_{\pm i} \to U'_{\pm \alpha_i} \qquad (i = 1, \ldots, l)$$

Proof. (a) Let S be a maximal torus of $\tilde{G}$ containing $\Delta(T)$. Applying pr_1 and pr_2, we see $S \subset T \times T$. Thus $\mathrm{hy}(S) \subset \mathrm{hy}(\tilde{G}) \cap \mathrm{hy}(T \times T) = \mathrm{hy}(\Delta(T))$. Hence $S = \Delta(T)$. (b) Follows from the proof of (4.1)(c). Q.E.D.

Let $\tilde{\Phi}$ be the set of roots of $\tilde{G}$ with respect to $\Delta(T)$, and let $\tilde{X}$ be the character group of $\Delta(T)$. The projection $\Delta(T) \to T$ induces an isomorphism $\eta: X \to \tilde{X}$. Let $\tilde{U}_{\tilde{\alpha}}$, $\tilde{\alpha} \in \tilde{\Phi}$, be the root groups of $\tilde{G}$ with respect to $\Delta(T)$.

4.5. LEMMA. (a) *The projections pr_1 and pr_2 induce isomorphisms of Weyl groups*

$$W(\Delta(T), \tilde{G}) \to W(T, G), \qquad W(\Delta(T), \tilde{G}) \to W(T, G').$$

(b) $\{\eta(\alpha_1),..., \eta(\alpha_l)\}$ *is a base for* $\tilde{\Phi}$.

(c) η *induces a bijection* $\Phi \to \tilde{\Phi}$.

(d) pr_1 *and* pr_2 *induce isomorphisms of root groups*

$$\tilde{U}_{\eta(\alpha)} \to U_\alpha, \qquad \tilde{U}_{\eta(\alpha)} \to U'_\alpha$$

for all $\alpha \in \Phi$.

Proof. (a) Follows from [1, p. 282]. (b) For $i = 1,..., l$, we have $\tilde{U}_{\pm i} = \tilde{U}_{\pm \eta(\alpha_i)}$, hence $\pm\eta(\alpha_i)$ are roots in $\tilde{\Phi}$. We show that every root in $\tilde{\Phi}$ is an integral linear combination of $\{\eta(\alpha_1),..., \eta(\alpha_l)\}$ of like sign. By the tensor product decomposition $\mathrm{hy}(\tilde{G}) = \tilde{\phi}(\mathscr{U}(A)^-) \otimes \mathrm{hy}(\Delta(T)) \otimes \tilde{\phi}(\mathscr{U}(A)^+)$, the Lie algebra $\mathrm{Lie}(\tilde{G})$ is the direct sum $P(\tilde{\phi}(\mathscr{U}(A)^-)) \oplus \mathrm{Lie}(\Delta(T)) \oplus P(\tilde{\phi}(\mathscr{U}(A)^+))$, where $P(-)$ denotes the primitive elements. Since the algebra $\tilde{\phi}(\mathscr{U}(A)^+)$ is generated by $\mathrm{hy}(\tilde{U}_1),..., \mathrm{hy}(\tilde{U}_l)$, every weight of the adjoint representation of $\Delta(T)$ on it is a non-negative integral linear combination of $\{\eta(\alpha_1),..., \eta(\alpha_l)\}$. Every root of the representation on $P(\tilde{\phi}(\mathscr{U}(A)^+))$ is so a fortiori. The same is true for the $(-)$ part. This proves the claim. (c) Every root is conjugate to a simple root under the operation of the Weyl group. Hence the claim follows from (a) and (b). (d) Consider conjugacy by elements in $N_{\tilde{G}}(\Delta(T))$. We see that the set of those $\alpha \in \Phi$ for which the statement is true is closed under the operation of the Weyl group. Since the set contains a base, it should be the whole. Q.E.D.

We are now in a position to prove the main theorem.

4.6. THEOREM. *Let G and G' be reductive algebraic groups over an algebraically closed field. Assume they have a common maximal torus T and that the root data of G and G' with respect to T are the same $(X, X^\vee, \Phi, \Phi^\vee)$. There is an isomorphism of algebraic groups $G \simeq G'$ which is the identity on T.*

Proof. It is enough to show $pr_1: \tilde{G} \to G$ and $pr_2: \tilde{G} \to G'$ are isomorphisms. It follows from (4.5) and (2.2) that $\mathrm{hy}(pr_1)$ and $\mathrm{hy}(pr_2)$ are isomorphisms. Hence pr_1 and pr_2 are etale coverings (see below (2.0.1)). Since the base field is algebraically closed, this means that the kernels of pr_1 and pr_2 (see the Introduction) are finite constant, hence identified with some finite subgroups of rational points. Obviously, they are contained in the center, hence in the maximal torus $\Delta(T)$. Since pr_i are monomorphisms on $\Delta(T)$, it follows that $\mathrm{Ker}(pr_i)$ are trivial. Hence pr_i are isomorphisms.

Q.E.D.

5. The Isogeny Theorem

We shall now use the above hyperalgebra method, slightly modified, to prove the isogeny theorem. Let G and G' be reductive algebraic groups over k and let T and T' be maximal tori of G and G'. Assume there is an isogeny $f: T \to T'$. We want to extend it to an isogeny $G \to G'$. Let $(X, X^\vee, \Phi, \Phi^\vee)$ and $(X', X'^\vee, \Phi', \Phi'^\vee)$ be the root data of (G, T) and (G', T'), respectively. The isogeny f induces injective maps $X' \to X$ and $X^\vee \to X'^\vee$, through which we identify X' and $X^\vee$ as subgroups of X and $X'^\vee$, respectively. In order that f extend to an isogeny $G \to G'$, it is necessary [6, p. 268] that there is a bijection $\Phi \simeq \Phi'$ given by $\alpha \leftrightarrow \alpha'$, together with a family $\{q(\alpha)\}_{\alpha \in \Phi}$ of powers of $\mathrm{Max}(1, \mathrm{char}(k))$ such that

$$\alpha' = q(\alpha)\alpha \text{ in } X, \qquad \alpha^\vee = q(\alpha)\, \alpha'^\vee \text{ in } X'^\vee$$

for $\alpha \in \Phi$. We assume this is the case. Let $\{\alpha_1, \dots, \alpha_l\}$ be a base of Φ, and let $A = (\langle \alpha_i^\vee, \alpha_j \rangle)_{i,j=1,\dots,l}$, and $A' = (\langle \alpha_i'^\vee, \alpha_j' \rangle)_{i,j=1,\dots,l}$.

5.1. Lemma. (a) *$q(\alpha_i) A'_{ij} = A_{ij} q(\alpha_j)$ for $i, j = 1, \dots, l$.*

(b) *$\{\alpha'_1, \dots, \alpha'_l\}$ is a base of Φ'.*

Proof. (a) Follows from [6, (11.4.8), p. 269]. (b) It follows from [6, *ibid.*] that $s_\alpha(\beta)' = s_{\alpha'}(\beta')$ for $\alpha, \beta \in \Phi$. For every root $\alpha \in \Phi$, there are $x_1, \dots, x_r \in \{\alpha_1, \dots, \alpha_l\}$ such that $\alpha = s_{x_1} s_{x_2} \cdots s_{x_{r-1}}(x_r)$. We then have $\alpha' = s_{x'_1} s_{x'_2} \cdots s_{x'_{r-1}}(x'_r)$ which is an integral linear combination of $\{\alpha'_1, \dots, \alpha'_l\}$. It is clear that the coefficients have the same sign. Q.E.D.

Let $\{H_i^{(n)}, X_i^{(n)}, Y_i^{(n)}\}_{n,i}$ and $\{H_i'^{(n)}, X_i'^{(n)}, Y_i'^{(n)}\}_{n,i}$ be the canonical sets of generators of $\mathscr{U}(A)$ and $\mathscr{U}(A')$, respectively.

5.2. Lemma. *There is a hyperalgebra map $\sigma: \mathscr{U}(A) \to \mathscr{U}(A')$ such that the induced $k[[t]]$-algebra map*

$$\mathscr{U}(A)[[t]] \to \mathscr{U}(A')[[t]]$$

maps $H_i(t)$, $X_i(t)$, $Y_i(t)$ *to* $H_i'(t^{q(\alpha_i)})$, $X_i'(t^{q(\alpha_i)})$, $Y_i'(t^{q(\alpha_i)})$ *respectively for* $i = 1, \ldots, l$.

Proof. It is enough to verify that the conditions (1)–(8) of (3.1) are preserved by these substitutions. There is no problem about (1)–(5) and (8). We have by (5.1)(a) that

$$\begin{aligned} &H_i'(t^{q(\alpha_i)})\, X_j'(u^{q(\alpha_j)})\, H_i'(t^{q(\alpha_i)})^{-1} \\ &\quad = X_j'((1 + t^{q(\alpha_i)})^{A'_{ij}} u^{q(\alpha_j)}) = X_j'(\{(1+t)^{A_{ij}} u\}^{q(\alpha_j)}). \end{aligned}$$

Hence condition (6) is preserved. The same is true for (7). Q.E.D.

Choose arbitrary sets of admissible isomorphisms

$$x_i : G_a \to U_{\alpha_i}, \quad x_i' : G_a \to U'_{\alpha_i'} \qquad (i = 1, \ldots, l)$$

and determine the admissible isomorphisms

$$x_{-i} : G_a \to U_{-\alpha_i}, \qquad x'_{-i} : G_a \to U'_{-\alpha_i'} \qquad (i = 1, \ldots, l)$$

in such a way that (x_i, x_{-i}) and (x_i', x'_{-i}) are normal. Let $\tilde{U}_{\pm i}$ be the image of the inclusion

$$\tilde{x}_{\pm i} : G_a \to U_{\pm \alpha_i} \times U'_{\pm \alpha_i'}, \quad \tilde{x}_{\pm i}(a) = (x_{\pm i}(a), x'_{\pm i}(a^{q(\alpha_i)})).$$

(Note that $q(\alpha_i) = q(-\alpha_i)$.) Put $\tilde{T} = \{(t, f(t)) \mid t \in T\}$. Let

$$\phi : \mathscr{U}(A) \to \mathrm{hy}(G), \qquad \phi' : \mathscr{U}(A') \to \mathrm{hy}(G')$$

be the hyperalgebra maps related to $(x_1, \ldots, x_l)$ and $(x_1', \ldots, x_l')$, respectively, and put

$$\tilde{\phi} : \mathscr{U}(A) \to \mathscr{U}(A) \otimes \mathscr{U}(A) \xrightarrow{\phi \otimes \phi' \sigma} \mathrm{hy}(G) \otimes \mathrm{hy}(G') = \mathrm{hy}(G \times G').$$

With the modification, we have almost the same lemmas as in Section 4. Thus:

5.3. Proposition. (a) *There is a unique connected closed subgroup* H *of* $G \times G'$ *such that* $\mathrm{hy}(H) = \mathrm{Im}(\tilde{\phi})$.

(b) H *contains all* $\tilde{U}_i$ *for* $i = \pm 1, \ldots, \pm l$.

(c) $\tilde{T}$ *normalizes* H.

Let $\tilde{G} = H \cdot \tilde{T}$.

(d) *The projections* $pr_1 : \tilde{G} \to G$ *and* $pr_2 : \tilde{G} \to G'$ *are surjective.*

(e) $\tilde{G}$ *is reductive.*

(f) $\mathrm{hy}(\tilde{G}) \cap \mathrm{hy}(T \times T') = \mathrm{hy}(\tilde{T})$.

(g) *$\tilde{T}$ is a maximal torus of $\tilde{G}$.*

(h) *$\tilde{U}_{\pm i}$ is a root group of $\tilde{G}$ with respect to $\tilde{T}$ for $i = 1,\ldots, l$. The projection pr_1 induces isomorphisms*

$$\tilde{U}_{\pm i} \to U_{\pm \alpha_i}$$

while the projection pr_2 igogenies

$$\tilde{U}_{\pm i} \to U'_{\pm \alpha'_i}$$

for $i = 1,\ldots, l$.

(i) *pr_1 and pr_2 induce isomorphisms of Weyl groups*

$$W(\tilde{T}, \tilde{G}) \to W(T, G), \qquad W(\tilde{T}, \tilde{G}) \to W(T', G').$$

(j) *Let $\eta: X \to X(\tilde{T})$ be the isomorphism corresponding to $pr_1: \tilde{T} \to T$. Then $\{\eta(\alpha_1),\ldots, \eta(\alpha_l)\}$ is a base of $\tilde{\Phi}$ the roots of $\tilde{G}$ with respect to $\tilde{T}$.*

(k) *η induces a bijection $\Phi \to \tilde{\Phi}$.*

(l) *Let $\tilde{U}_{\tilde{\alpha}}$, $\tilde{\alpha} \in \tilde{\Phi}$, be the root groups of $\tilde{G}$ related to $\tilde{T}$. The projection pr_1 induces isomorphisms*

$$\tilde{U}_{\eta(\alpha)} \to U_\alpha$$

while the projection pr_2 isogenies

$$\tilde{U}_{\eta(\alpha)} \to U'_{\alpha'}$$

for $\alpha \in \Phi$.

This is proved in the same way as in Section 4. We leave the reader to verify details. As a consequence we have the following isogeny theorem:

5.4. THEOREM. *Let G and G' be reductive algebraic groups over an algebraically closed field, and let T and T' be maximal tori of G and G', respectively. Let $(X, X^\vee, \Phi, \Phi^\vee)$ and $(X', X'^\vee, \Phi', \Phi'^\vee)$ be the root data of (G, T) and (G', T'), respectively. Let $f: T \to T'$ be an isogeny, and let $u: X' \to X$ and $u^\vee: X^\vee \to X'^\vee$ be the induced injective maps. If there is a bijection $\Phi \simeq \Phi'$, $\alpha \leftrightarrow \alpha'$, together with a family $q(\alpha)$, $\alpha \in \Phi$, of powers of* $\mathrm{Max}(1, \mathrm{char}(k))$ *such that*

$$u(\alpha') = q(\alpha)\alpha, \qquad u^\vee(\alpha^\vee) = q(\alpha)\, \alpha'^\vee, \qquad \alpha \in \Phi$$

then f extends to an isogeny $G \to G'$.

Proof. $\mathrm{pr}_1 : \tilde{G} \to G$ is an isomorphism just as in (4.6). We claim that

$pr_2 : \tilde{G} \to G'$ is an isogeny. The (group scheme) kernel $\mathrm{Ker}(pr_2)$ is finite (i.e., a finite group scheme) if its connected component is. A connected algebraic group scheme is finite (or infinitesimal) if and only if the hyperalgebra is finite dimensional. We have only to prove that the Hopf algebra kernel of $\mathrm{hy}(pr_2)$: $\mathrm{hy}(\tilde{G}) \to \mathrm{hy}(G')$ is finite dimensional since it gives the hyperalgebra of $\mathrm{Ker}(pr_2)$ [8, (3.1.5), p. 103]. By (2.2) we have tensor product decompositions

$$\begin{aligned} \mathrm{hy}(\tilde{G}) &= \mathrm{hy}(\tilde{U}_{-\gamma_1}) \otimes \cdots \otimes \mathrm{hy}(\tilde{U}_{-\gamma_m}) \otimes \mathrm{hy}(\tilde{T}) \\ &\quad \otimes \mathrm{hy}(\tilde{U}_{\gamma_1}) \otimes \cdots \otimes \mathrm{hy}(\tilde{U}_{\gamma_m}) \\ \mathrm{hy}(G') &= \mathrm{hy}(U'_{-\gamma'_1}) \otimes \cdots \otimes \mathrm{hy}(U'_{-\gamma'_m}) \otimes \mathrm{hy}(\mathrm{T}') \\ &\quad \otimes \mathrm{hy}(U'_{\gamma'_1}) \otimes \cdots \otimes \mathrm{hy}(U'_{\gamma'_m}) \end{aligned}$$

where $\{\gamma_1, \ldots, \gamma_m\}$ are the positive roots in Φ in some order and we denote by $\tilde{U}_\gamma$ the root group $\tilde{U}_{n(\gamma)}$, $\gamma \in \Phi$. By (5.3)(1), the hyperalgebra map $\mathrm{hy}(pr_2)$ has a finite dimensional kernel on each factor. Since the Hopf kernel is the tensor product of the intersections with each factor, it is finite dimensional. Hence pr_2 is an isogeny, and $\tilde{G}$ is the graph of a desired isogeny $G \to G'$.

Q.E.D.

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