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Rank Jumps in Codimension 2 A-hypergeometric Systems

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The holonomic rank of the A-hypergeometric system $H_A(\beta)$ is shown to depend on the parameter vector β when the underlying toric ideal I_A is a non-Cohen–Macaulay codimension 2 toric ideal. The set of exceptional parameters is usually infinite.

1. Introduction

A-hypergeometric systems are systems of linear partial differential equations with polynomial coefficients. In other words, they are left ideals in the Weyl algebra D, which is the free associative algebra with generators $x_1, \dots, x_n, \partial_1, \dots, \partial_n$ modulo the relations:

$$x_i x_j = x_j x_i; \qquad \partial_i \partial_j = \partial_j \partial_i; \qquad \partial_i x_j = x_j \partial_i + \delta_{ij}, \qquad \forall 1 \le i, j \le n,$$

where δ_{ij} is the Kronecker delta.

Given a configuration of n distinct points $A := \{a_1, \dots, a_n\} \subset \{1\} \times \mathbb{Z}^{d-1}$ that spans the lattice $\mathbb{Z}^d$ (we also think of $A = (a_{ij})$ as a $d \times n$ integer matrix of rank d), and a complex vector $\beta \in \mathbb{C}^d$, let $H_A(\beta)$ denote the left ideal in the Weyl algebra generated by:

$$\partial^u - \partial^v, \qquad u, v \in \mathbb{N}^n \text{ such that } A \cdot u = A \cdot v, \tag{1}$$

$$\sum_{j=1}^{n} a_{ij} x_j \partial_j - \beta_i, \qquad i = 1, \dots, d. \tag{2}$$

Operators (1) are called *toric operators*, and operators (2) are called *homogeneities*. If we set $\theta_i = x_i \partial_i$ and θ the column vector whose entries are θ_i, then the homogeneities are simply the coordinates of the vector of operators $A \cdot \theta - \beta$.

The D-ideal $H_A(\beta)$ is called the *A-hypergeometric system with parameter β*. These systems, which are the object of study of this paper, were first introduced and studied by Gel'fand *et al.* (1989). Solutions to particular instances of $H_A(\beta)$ generalize the classical hypergeometric functions.

The commutative ideal of $\mathbb{C}[\partial_1, \dots, \partial_n]$ generated by the toric operators will be denoted I_A; it is called *toric ideal* or *lattice ideal*. The convex hull $\operatorname{conv}(A)$ of the configuration A is a polytope of dimension $d-1$. We denote its normalized volume by $\operatorname{vol}(A)$. Under these hypotheses, $H_A(\beta)$ is a regular holonomic D-ideal; its *holonomic rank* is, by definition, the common dimension of the spaces of holomorphic solutions of $H_A(\beta)$ around nonsingular points. This number is finite.

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Theorem 1.1. *If I_A is Cohen–Macaulay, then* $\operatorname{rank}(H_A(\beta)) = \operatorname{vol}(A)$ *for all parameter vectors $\beta \in \mathbb{C}^d$.*

A proof of this result, originally due to Gel'fand, Kapranov and Zelevinsky, can be found in Saito *et al.* (1999, Section 4.3). The equality in the theorem can fail if I_A is not Cohen–Macaulay. The following example is thoroughly analyzed in Sturmfels and Takayama (1998).

Example 1.1. Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$. Then $\operatorname{vol}(A) = 4$, but if we set $\beta = \binom{1}{2}$, we have $\operatorname{rank}(H_A(\beta)) = 5$.

However, the rank of $H_A(\beta)$ is almost everywhere equal to $\operatorname{vol}(A)$, as the following result shows (see Adolphson, 1994; Saito *et al.*, 1999, Theorem 3.5.1, equation (4.3)).

Theorem 1.2. *If β is generic, then* $\operatorname{rank}(H_A(\beta)) = \operatorname{vol}(A)$. *The inequality* $\operatorname{rank}(H_A(\beta)) \geq \operatorname{vol}(A)$ *always holds.*

Definition 1.2. The **exceptional set** of A is

$$\mathcal{E}(A) = \{\beta \in \mathbb{C}^d : \operatorname{rank}(H_A(\beta)) > \operatorname{vol}(A)\}.$$

In the case $d = 2$, the exceptional set is completely understood by the following result due to Cattani *et al.* (1999):

Theorem 1.3. *If* $A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & a_2 & \dots & a_n \end{pmatrix}$ *with* $0 < a_2 < \dots < a_n$, *then*

$$\mathcal{E}(A) = \left(\left(\mathbb{N}A + \mathbb{Z}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \cap \left(\mathbb{N}A + \mathbb{Z}\begin{pmatrix} 1 \\ a_n \end{pmatrix} \right) \right) \Big\backslash \mathbb{N}A.$$

This set is non-empty if and only if I_A is not Cohen–Macaulay. Moreover, $\mathcal{E}(A)$ coincides with the set of parameters that maximize the dimension of the space of Laurent polynomial solutions of $H_A(\beta)$. This maximum dimension is 2.

Theorem 1.3 and experimental evidence suggest the following conjecture.

Conjecture 1.3. *The exceptional set $\mathcal{E}(A)$ of a matrix A is empty if and only if the toric ideal I_A is Cohen–Macaulay.*

The purpose of this paper is to prove Conjecture 1.3 in the codimension 2 case, that is, when $n - d = 2$. We do this by explicitly constructing exceptional parameters for any codimension 2 non-Cohen–Macaulay toric ideal (see Construction 3.2). Our main result, which is a direct consequence of Theorems 4.3 and 4.4, is the following:

Theorem 1.4. *Given a codimension* 2 *configuration A whose toric ideal I_A is not Cohen–Macaulay, there exist exceptional parameters β, provided by Construction* 3.2. *Moreover, if $n > 4$ the exceptional set contains an affine space of dimension $n - 4$.*

Saito has recently announced (see Saito, 2000, and also Saito, 1999) that Conjecture 1.3 also holds when $\operatorname{conv}(A)$ is a simplex.

This paper is organized as follows. Section 2 contains background material about canonical series solutions of regular holonomic systems, and in particular, canonical A-hypergeometric series. The main reference is Saito *et al.* (1999). In Section 3 we construct our candidates for exceptional parameters, and develop some useful technical tools. Section 4 contains the proofs of Theorems 4.3 and 4.4. In Section 5 we apply our methods to a concrete example, and point out some open questions.

2. Canonical Hypergeometric Series

In this section we review material concerning the series solutions of hypergeometric systems. We follow Saito *et al.* (1999, Sections 2.5, 3.1, 3.2, 3.4).

DEFINITION 2.1. If I is a left ideal in the Weyl algebra D, its **distraction** $\tilde{I}$ is defined to be

$$\tilde{I} := RI \cap \mathbb{C}[\theta],$$

where $R = \mathbb{C}(x_1, \ldots, x_n)\langle \partial_1, \ldots, \partial_n \rangle$ is the ring of differential operators with rational function coefficients, and $\mathbb{C}[\theta] = C[\theta_1, \ldots, \theta_n]$ is the (commutative) subring of D generated by the operators $\theta_i = x_i \partial_i$.

The concept of distraction will allow us to define the indicial and fake indicial ideals of a hypergeometric system.

DEFINITION 2.2. Let $w \in \mathbb{R}^n$ be a weight vector. If I is a holonomic left D-ideal, its **indicial** ideal is

$$\operatorname{ind}_w(I) := \widetilde{\operatorname{in}_{(-w,w)}(I)} = R \operatorname{in}_{(-w,w)}(I) \cap \mathbb{C}[\theta].$$

The indicial ideal of a regular holonomic D-ideal is a zero-dimensional ideal of the polynomial ring $\mathbb{C}[\theta]$. Its solutions, called *exponents*, give the starting monomials (in a term order induced by w) of the solutions of I. By a *monomial* here we mean a product $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ such that $\alpha_i \in \mathbb{C}$ and $x_i^{\alpha_i} = \exp(\alpha_i \log(x_i))$.

For hypergeometric ideals, there is another ideal that is closely related to $\operatorname{ind}_w(H_A(\beta))$, but is easier to compute and understand. Its definition is motivated by the following facts.

PROPOSITION 2.3. (SAITO *et al.*, 1999, COROLLARY 3.1.6, EXAMPLE 3.1.8)
For generic parameters β we have

$$\operatorname{ind}_w(H_A(\beta)) = \widetilde{\operatorname{in}_w(I_A)} + \langle A \cdot \theta - \beta \rangle.$$

The containment $\supseteq$ always holds, but $\subseteq$ can fail for non-generic parameters.

The ideal $\widetilde{\operatorname{in}_w(I_A)} + \langle A \cdot \theta - \beta \rangle$ is an ideal of the polynomial ring $\mathbb{C}[\theta]$, called the *fake indicial ideal* of $H_A(\beta)$. Its roots in affine n-space are called the *fake exponents* of $H_A(\beta)$ with respect to w. Exponents are always fake exponents, and, though the converse is not true, fake exponents are easier to describe. In order to do this we need to define standard pairs.

Definition 2.4. Let M be a monomial ideal of $\mathbb{C}[\partial_1, \ldots, \partial_n]$. A **standard pair** of M is a pair (∂^η, σ), where $\eta \in \mathbb{N}^n$ and $\sigma \subset \{1, \ldots, n\}$ are subject to the following three conditions:

1. $\eta_i = 0$ for $i \in \sigma$;
2. for all choices of integers $\mu_j \geq 0$, $j \in \sigma$, the monomial $\partial^\eta \cdot \prod_{i\in\sigma} \partial_i^{\mu_i}$ is not in M;
3. for all $l \notin \sigma$, there exist $\mu_j \geq 0$, $j \in \sigma \cup \{l\}$, such that $\partial^\eta \cdot \partial_l^{\mu_l} \cdot \prod_{i\in\sigma} \partial_i^{\mu_i}$ is in M.

The set of standard pairs of M is denoted $S(M)$.

Now we can describe the radical of the fake indicial ideal, and therefore, the fake exponents.

Lemma 2.5. (Saito *et al.*, 1999, Lemma 4.1.3) *For any parameter vector β and weight vector w such that $\mathrm{in}_w(I_A)$ is a monomial ideal, the radical of the fake indicial ideal is zero dimensional and equals the following intersection*

$$\bigcap_{(\partial^\eta,\sigma)\in S(\mathrm{in}_w(I_A))} (\langle \theta_i - \eta_i : i \notin \sigma\rangle + \langle A\cdot\theta - \beta\rangle).$$

This means that, in order to compute the fake exponents, one need only compute the standard pairs of $\mathrm{in}_w(I_A)$, and then do linear algebra. Given a standard pair (∂^η, σ), the vector $v \in \mathbb{C}^n$ such that $v_i = \eta_i$, $i \notin \sigma$ and $A \cdot v = \beta$ is called the fake exponent with respect to w corresponding to that standard pair. If v exists, it is unique.

Since $H_A(\beta)$ is a regular holonomic ideal, we can find a basis of canonical solutions of $H_A(\beta)$ with respect to a weight vector w (see Saito *et al.*, 1999, Section 2.5). The elements of that basis are logarithmic series of the form:

$$x^v \sum c_{v',\gamma} x^{v'} \log(x)^\gamma,$$

where v is an exponent, $v' \in \ker_{\mathbb{Z}}(A)$, $c_{v',\gamma} \in \mathbb{C}$, $\gamma \in \{0, 1, \ldots, \nu - 1\}^n$, and $\nu = \mathrm{rank}\,(H_A(\beta))$.

Our goal now is to describe more explicitly a basis of the space of logarithm-free solutions of $H_A(\beta)$. The elements of this basis will also be canonical series.

Let v be any vector in $\mathbb{R}^n$. Its *negative support* $\mathrm{nsupp}(v)$ is defined by:

$$\mathrm{nsupp}(v) = \{i \in \{1, \ldots, n\} : v_i \in \mathbb{Z}_{<0}\}.$$

The vector v is said to have *minimal negative support* if

$$v' \in \ker_{\mathbb{Z}}(A) \text{ and } \mathrm{nsupp}(v + v') \subseteq \mathrm{nsupp}(v) \text{ imply}$$
$$\mathrm{nsupp}(v + v') = \mathrm{nsupp}(v).$$

In that case, let

$$N_v = \{v' \in \ker_{\mathbb{Z}}(A) : \mathrm{nsupp}(v + v') = \mathrm{nsupp}(v)\},$$

and define the following formal power series:

$$\phi_v = \sum_{v'\in N_v} \frac{[v]_{v'_-}}{[v' + v]_{v'_+}} x^{v+v'}, \tag{3}$$

where

$$[v]_{v'_-} = \prod_{i:v'_i<0} \prod_{j=1}^{-v'_i} (v_i - j + 1) \qquad \text{and} \qquad [v'+v]_{v'_+} = \prod_{i:v'_i>0} \prod_{j=1}^{v'_i} (v_i + j).$$

THEOREM 2.1. (SAITO *et al.*, 1999, THEOREM 3.4.14, COROLLARY 3.4.15) *Let* $v \in \mathbb{C}^n$ *be a fake exponent of* $H_A(\beta)$ *with minimal negative support. Then* v *is an exponent and the series* ϕ_v *defined in* (3) *is a canonical solution of the* A*-hypergeometric system* $H_A(\beta)$*. In particular,* ϕ_v *converges in a region of* $\mathbb{C}^n$*. The set:*

$$\{\phi_v : v \textit{ is a fake exponent with minimal negative support}\}$$

is a basis of the space of logarithm-free solutions of $H_A(\beta)$*.*

Note that if $v \in \mathbb{C}^n$ satisfies $A \cdot v = \beta$ and has minimal negative support, it makes sense to define N_v and ϕ_v. In this case, ϕ_v is a formal power series, formally annihilated by $H_A(\beta)$. In order for ϕ_v to be a canonical solution of $H_A(\beta)$ with respect to w (that is, in order for it to have a common domain of convergence with the other canonical solutions), one of the vectors $v + v'$ with $v' \in N_v$ must be a fake exponent with respect to w. An equivalent way to say this is that the linear functional given by w must be minimized in the set $\{v + v' : v' \in N_v\}$, and the minimum must be attained uniquely.

3. Construction of Exceptional Parameters in Codimension 2

We assume from now on that $n - d = 2$. Then $\ker_{\mathbb{Z}}(A)$, the integer kernel of A, is a two-dimensional sublattice of $\mathbb{Z}^n$. Let $\{B_1, B_2\}$ be a $\mathbb{Z}$-basis of $\ker_{\mathbb{Z}}(A)$. We think of the B_i as columns of an $n \times 2$ integer matrix $B = (b_{ji})$. The rows of B form a configuration of n points in $\mathbb{Z}^2$. This configuration is called a *Gale diagram* of A, and it is unique up to the action of $\mathrm{GL}_2(\mathbb{Z})$. The following result is contained in Peeva and Sturmfels (1997).

THEOREM 3.1. *A toric ideal* I_A *is* <u>*not*</u> *Cohen–Macaulay if and only if* A *has a Gale diagram that meets the four open quadrants of* $\mathbb{Z}^2$*.*

The goal of this section is to construct exceptional parameters for A when I_A is a non-Cohen–Macaulay (codimension 2) toric ideal. In what follows I_A will always denote such an ideal, with a Gale diagram $B = (b_{ij})$ that meets the four open quadrants of $\mathbb{Z}^2$. By interchanging columns of A (and the corresponding rows of B) we may assume that the first four rows of B meet each of the four open quadrants of $\mathbb{Z}^2$, that is, B is of the following form:

$$B = \begin{pmatrix} + & + \\ - & + \\ - & - \\ + & - \\ \vdots & \vdots \end{pmatrix}.$$

We will need an extra assumption that will only be used in Lemma 4.2. If the second and fourth row of B are linearly independent, we will assume that the cone $\{z \in \mathbb{R}^2 : (B \cdot z)_2 \geq 0,\ (B \cdot z)_4 \geq 0\}$ is contained in the first quadrant. This is possible since, if this cone is not contained in the first quadrant, it will be contained in the third. In this case replace B by $-B$. Interchanging the necessary rows of the new B, we obtain our desired configuration.

Also, notice that a non-Cohen–Macaulay codimension 2 toric ideal I_A cannot have a Gale diagram contained in two lines. If this were the case, then all the rows of B would be integer multiples of either $\frac{1}{\gcd(b_{11},b_{12})}(b_{11}, b_{12})$ or $\frac{1}{\gcd(b_{21},b_{22})}(b_{21}, b_{22})$, and all maximal minors of B would be integer multiples of the determinant of the 2×2 matrix whose rows are those vectors. Since the columns of B form a $\mathbb{Z}$-basis of a lattice, the greatest common divisor of all maximal minors of B must be 1. This means that the previous determinant must be 1. But the determinant of a 2×2 integer matrix whose first row lies in the first quadrant and whose second row lies in the second quadrant cannot be 1 unless one of those rows lies on a coordinate axis, and this is a contradiction.

Thus, there must be a row of B that is neither a multiple of the first nor of the second row of B. By applying a suitable element of $\mathrm{GL}_2(\mathbb{Z})$, we may assume this row lies on one of the open quadrants of $\mathbb{Z}^2$, say the first. Replacing the first row of B by this one, we see that we may assume that either the first and third or the second and fourth row of B are linearly independent.

In the sequel, it will be very useful to compute canonical series solutions with respect to the weight vector $-e_3$. However, this cannot be done if $\mathrm{in}_{-e_3}(I_A)$ is not a monomial ideal. To solve this problem while keeping all the good properties of $-e_3$ as a weight vector, we will use the following lemma.

Lemma 3.1. *There exist $\epsilon_0 > 0$ and a generic vector $w \in \mathbb{R}^n$ such that, for $0 < \epsilon < \epsilon_0$, the ideal $\mathrm{in}_{-e_3+\epsilon w}(I_A) = \mathrm{in}_w(\mathrm{in}_{-e_3}(I_A))$ is a monomial ideal, and all standard pairs of $\mathrm{in}_{-e_3+\epsilon w}(I_A)$ are of the form $(\partial^\eta, \sigma = \{3\} \cup \tau)$.*

Proof. The first assertion is proved using Sturmfels (1995, Proposition 1.13) and the fact that the full-dimensional cones of the Gröbner fan of I_A are exactly the cones corresponding to monomial initial ideals of I_A. The second assertion is easily proved by noticing that $\mathrm{in}_{-e_3+\epsilon w}(I_A)$ is a monomial ideal none of whose generators contain the variable ∂_3. □

Now we are ready to construct our candidates for exceptional parameters.

Construction 3.2. *Pick non-rational numbers $\alpha_5, \ldots, \alpha_n \in \mathbb{C}$ such that $\alpha_i \notin \mathbb{Q}(\alpha_5, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n)$ for $5 \le i \le n$. Remember B_1 and B_2 are the columns of B. Let*

$$v = B_{1+} + B_{2+} - e_1 - e_2 - e_4 + \sum_{i=5}^{n} \alpha_i e_i,$$

$$\beta = A \cdot (v - e_3) = A \cdot (v - e_3 - B_1) = A \cdot (v - e_3 - B_2) = A \cdot (v - e_3 - B_1 - B_2).$$

Here we denote by u_+ the vector such that $(u_+)_l = u_l$ if $u_l \ge 0$, or $(u_+)_l = 0$ otherwise, where $u \in \mathbb{Z}^n$. The vector u_- is defined so that $u = u_+ - u_-$. Notice that $\mathrm{nsupp}(v - e_3) = \{3\}$, $\mathrm{nsupp}(v - e_3 - B_1) = \{4\}$, $\mathrm{nsupp}(v - e_3 - B_2) = \{2\}$, and $\mathrm{nsupp}(v - e_3 - B_1 - B_2) = \{1\}$. Further, $(v - e_3)_3 = (v - e_3 - B_1)_4 = (v - e_3 - B_2)_2 = (v - e_3 - B_1 - B_2)_1 = -1$.

First notice that this construction is valid when $n = 4$, and it produces an exceptional parameter by Theorem 1.3. In this case, Construction 3.2 yields only those exceptional degrees for which there exist four expressions with minimal support. In Section 5 we provide an example (kindly suggested by one of the referees) where there are exceptional parameters with $n = 4$ that do not come from Construction 3.2.

Now, it follows from results in Peeva and Sturmfels (1997) that the vector $A \cdot (B_{1+} + B_{2+})$ is the multidegree of a high syzygy of I_A. It is natural, then, that our potential exceptional parameter should be closely related to this multidegree. Actually, the syzygy in degree $A \cdot (B_{1+} + B_{2+})$ arises because of the fact that the first four rows of B meet each of the open quadrants of $\mathbb{Z}^2$. The numbers α_i are used to make v "generic" except in the crucial first four coordinates. This will simplify the proofs later on.

We now construct some distinguished solutions of $H_A(\beta)$ and $H_A(A \cdot v)$.

Lemma 3.3. *The vectors $v - e_3$, $v - e_3 - B_1$, $v - e_3 - B_2$ and $v - e_3 - B_1 - B_2$ have minimal negative support. In particular, if $n = 4$, $\beta \notin \mathbb{N}A$. It follows that there are four logarithm-free formal power series solutions of $H_A(\beta)$, ϕ_{v-e_3}, $\phi_{v-e_3-B_1}$, $\phi_{v-e_3-B_2}$ and $\phi_{v-e_3-B_1-B_2}$, which, for convenience in the notation, we call ϕ_3, ϕ_4, ϕ_2 and ϕ_1, respectively. Here the subscripts refer to the corresponding negative supports. Then ϕ_3 is a canonical solution of $H_A(\beta)$ with respect to $-e_3 + \epsilon w$, and at least one of the other ϕ_i is also a canonical solution.*

Proof. First suppose that $v - e_3$ does not have minimal negative support. Then there is $z \in \mathbb{Z}^2$ such that $\mathrm{nsupp}(v - e_3 - B \cdot z)$ is strictly contained in $\mathrm{nsupp}(v - e_3)$. This means that $(B \cdot z)_i \leq v_i$ for $i = 1, 2, 4$, and $(B \cdot z)_3 < 0$. Say $z = (z_1, z_2)^t$. If $z_1, z_2 > 0$, then $(B \cdot z)_1 \leq v_1$ does not hold. If $z_1 \leq 0, z_2 > 0$, then $(B \cdot z)_2 \leq v_2$ does not hold. If $z_1 \leq 0, z_2 \leq 0$, then $(B \cdot z)_3 < 0$ does not hold, and if $z_1 > 0, z_2 \leq 0$ then $(B \cdot z)_4 \leq v_4$ does not hold. All of this means that such a $z \in \mathbb{Z}^2$ cannot exist, and thus $v - e_3$ has minimal negative support.

We show that $v - e_3 - B_1$ has minimal negative support by contradiction. Assume it does not have minimal negative support. Then there is $z \in \mathbb{Z}^2$ such that $\mathrm{nsupp}(v - e_3 - B_1 - B \cdot z) = \emptyset$. But then the negative support of $v - e_3 - B \cdot (z + (1, 0)^t)$ is strictly contained in the negative support of $v - e_3$, a contradiction. The proofs for the other two vectors are similar.

The formal power series ϕ_3 is an actual holomorphic function since it is clear that $(-e_3 + \epsilon w) \cdot x$ will be minimized for $x \in \{v - e_3 + v' : v' \in N_{v-e_3}\}$.

Now look at, say, ϕ_4. Since we are choosing ϵ very small, we see that $(-e_3 + \epsilon w) \cdot x$ will not be minimized unless $N_{v-e_3-B_1}$ is bounded. So all that is left to prove is that at least one of the sets $N_{v-e_3-B_1}$, $N_{v-e_3-B_2}$, $N_{v-e_3-B_1-B_2}$ is bounded. But looking at the inequalities that define these sets, the only way all of them can be unbounded is if the first and third rows of B are linearly dependent, and the second and fourth rows of B are linearly dependent, which is not the case by construction. □

We have found some exponents with minimal negative support of $H_A(\beta)$. Our construction also gives an exponent with minimal negative support for $H_A(A \cdot v)$.

Lemma 3.4. *There is only one vector with minimal negative support in the set $\{v + B \cdot z : z \in \mathbb{Z}^2\}$. This vector is v, and the corresponding logarithm-free solution of $H_A(A \cdot v)$ is $\phi_v = x^v$ (which is everywhere convergent).*

Proof. This follows from the same arguments that proved Lemma 3.3. The fact that v is a fake exponent of $H_A(A \cdot v)$ with respect to $-e_3 + \epsilon w$ is a consequence of the following lemma. □

Lemma 3.5. *The pair*

$$(\partial_1^{v_1}\partial_2^{v_2}\partial_4^{v_4}, \{3, 5, 6, \dots, n\})$$

is a standard pair of $\mathrm{in}_{-e_3}(I_A)$*, if this is a monomial ideal. It is a standard pair of* $\mathrm{in}_{-e_3+\epsilon w}(I_A)$ *otherwise. Here* ϵ *and* w *come from Lemma* 3.1*. Notice that* v *is the fake exponent of* $H_A(A\cdot v)$ *corresponding to this standard pair.*

Proof. We use the Hoşten and Thomas (see Theorem 2.5 in Hoşten and Thomas, 1999) description of standard pairs of initial ideals of toric ideals.

Let ω *be a weight vector such that* $\mathrm{in}_\omega(I_A)$ *is a monomial ideal. A pair* (∂^η, σ) *is a standard pair of* $\mathrm{in}_\omega(I_A)$ *if and only if the origin is the unique lattice point in the polytope*

$$P_\eta^{\bar\sigma}(0) := \{z \in \mathbb{R}^2 : (B\cdot z)_i \leq \eta_i, i \notin \sigma,\ -\omega B\cdot z \leq 0\}$$

and each of the inequalities $(B\cdot z)_i \leq \eta_i$ *is essential, in the sense that removing it introduces a new lattice point into* $P_\eta^{\bar\sigma}(0)$.

To see that our candidate for standard pair satisfies this criterion, we have to show that the only integer point in $P_\eta^{\bar\sigma}(0)$ is the origin. This follows exactly from the same arguments of Lemma 3.3 if $\mathrm{in}_{-e_3}(I_A)$ is a monomial ideal. Otherwise, we shrink ϵ so that the same arguments will work when we use the weight vector $-e_3+\epsilon w$.

We also need to find elements in $\mathbb{Z}^2$ that belong to that polytope when one of the defining inequalities is removed. Those elements will be $(1,0)^t$, $(0,1)^t$ and $(1,1)^t$. □

We want to show that $\mathrm{rank}\,(H_A(\beta)) > \mathrm{vol}\,(A)$. In view of Theorem 1.2, one way to do this is to show that $\mathrm{rank}\,(H_A(\beta))$ is strictly greater than $\mathrm{rank}\,(H_A(A\cdot v))$. In order to compare these two numbers, we need a link between $H_A(\beta)$ and $H_A(A\cdot v)$. This is provided by the following D-module map (see Saito *et al.*, 1999, Section 4.5):

$$\partial_3 : D/H_A(\beta) \longrightarrow D/H_A(A\cdot v).$$

This D-module map induces a vector space homomorphism in the opposite direction between the solution spaces of the corresponding hypergeometric ideals, namely, if φ is a solution of $H_A(A\cdot v)$, then $\partial_3\varphi$ is a solution of $H_A(\beta)$.

Our strategy is to show that $\mathrm{rank}\,(H_A(\beta)) > \mathrm{rank}\,(H_A(A\cdot v))$ will be as follows. First, characterize the kernel of ∂_3 (as a map between solution spaces). There is an obvious element of this kernel, namely the function $\phi_v = x^v$. After we have done that, we will construct, for each element of a vector space basis of $\ker(\partial_3)$, a non-zero function in the cokernel of ∂_3. However, for the function ϕ_v (which will belong to that generating set) we will construct at least two functions in $\mathrm{coker}\,(\partial_3)$. After showing all of the functions thus constructed are linearly independent, we will conclude $\dim(\mathrm{coker}\,(\partial_3)) \geq \dim(\ker(\partial_3)) + 1$. This will imply the desired result (that is, that $\mathrm{rank}\,(H_A(\beta)) > \mathrm{rank}\,(H_A(A\cdot v))$ using elementary linear algebra.

Before we can look at the kernel and cokernel of ∂_3, we need a couple of technical facts.

Observation 3.6. Let ψ be a solution of $H_A(A\cdot v)$. This function is of the form:

$$\psi = \sum c_{\alpha,\gamma} x^\alpha \log(x)^\gamma,$$

where $A\alpha = A\cdot v$, $\nu = \mathrm{rank}\,(H_A(A\cdot v))$, $\gamma \in \{0, 1, \dots, \nu-1\}^n$, and $\log(x)^\gamma = \log(x_1)^{\gamma_1}\cdots\log(x_n)^{\gamma_n}$.

The set $\mathcal{S} := \{\gamma \in [0, \nu-1]^n \cap \mathbb{N}^n : \exists\alpha \in \mathbb{C}^n \text{ such that } c_{\alpha,\gamma} \neq 0\}$ is partially ordered with respect to:

$$(\gamma_1, \ldots, \gamma_n) \leq (\gamma'_1, \ldots, \gamma'_n) \Leftrightarrow \gamma_i \leq \gamma'_i,\ i = 1, \ldots, n.$$

Denote by $\mathcal{S}_{\max}$ the set of maximal elements of $\mathcal{S}$. Let $\delta \in \mathcal{S}_{\max}$ and $f = \sum_{\alpha \in \mathbb{C}^n} c_{\alpha,\delta} x^\alpha$. Write

$$\psi = \psi_\delta + \log(x)^\delta f,$$

so that the logarithmic terms in ψ_δ are either less than or incomparable to δ. If P is a differential operator that annihilates ψ, we have:

$$0 = P\psi = P\psi_\delta + \log(x)^\delta Pf + \text{terms whose log factor is lower than } \delta.$$

Since $P\psi_\delta$ is a sum of terms whose log factor is either lower than δ or incomparable to δ, we conclude that Pf must be 0. This implies that f is a logarithm-free A-hypergeometric function of degree $A \cdot v$. *Moreover, if $\partial_3 \psi$ is logarithm-free, then $\partial_3 f$ must vanish.*

The following lemma is used to analyze the kernel and cokernel of the map ∂_3, and it will be used repeatedly in the sequel. Its proof is inspired by the proofs of Theorems 2.5 and 3.1 in Hoşten and Thomas (1999).

Lemma 3.7. *Let u be a fake exponent of $H_A(A \cdot v)$ such that $u_3 = 0$, corresponding to a standard pair $(\partial^\eta, \sigma = \{3\} \cup \tau)$ of $\text{in}_{-e_3+\epsilon w}(I_A)$. Here ϵ and w are chosen so that $\text{in}_{-e_3+\epsilon' w} = \text{in}_w(\text{in}_{-e_3}(I_A))$ for all $0 < \epsilon' \leq \epsilon$ and this is a monomial ideal. Then there exists a set $\mathcal{I} \subseteq \{1, 2, 4, \ldots, n\} \backslash \tau$ of cardinality 2 such that, for each $i \in \mathcal{I}$, we can find a vector $z^{(i)} \in \mathbb{Z}^2$ that satisfies the following three properties:*

1. $(B \cdot z^{(i)})_i > \eta_i$;
2. $(B \cdot z^{(i)})_j \leq u_j$ *for all $j \neq 3, i$ such that $u_j \in \mathbb{N}$;*
3. $(B \cdot z^{(i)})_3 < 0$.

Moreover, $\mathcal{I}$ can be chosen so that the rows of B indexed by $\mathcal{I}$ are linearly independent.

Proof. Fix $l \notin \sigma = \{3\} \cup \tau$. Let $\mu \in \mathbb{N}^n$ such that $\mu_j = u_j$ if $u_j \in \mathbb{N}$; $\mu_j = 0$ otherwise. Observe that $\mu_j = \eta_j$ for $j \notin \sigma$. For $v' \in \mathbb{N}^n$ we define, following Hoşten and Thomas (1999):

$$P_{v'}(0) = \{y \in \mathbb{R}^2 : B \cdot y \leq v'; -(-e_3 + \epsilon w)^t (B \cdot y) \leq 0\}.$$

Following the proof of Theorem 2.5 in Hoşten and Thomas (1999), we see that, for $l \notin \sigma$ there exists a positive integer m_l such that $P_{\mu+m_l e_l}(0)$ contains a non-zero integer vector $z^{(l)} \in \mathbb{Z}^2$. It must satisfy $-(-e_3 + \epsilon w)^t (B \cdot z^{(l)}) < 0$. The reason for this is that, since $\text{in}_{-e_3+\epsilon w}(I_A)$ is a monomial ideal, there exists a unique solution of the integer program

$$\text{minimize } -(-e_3 + \epsilon w)^t (B \cdot z) \text{ subject to } z \in P_{\mu+m_l e_l} \cap \mathbb{Z}^2,$$

where $P_{\mu+m_l e_l} := \{y \in \mathbb{R}^2 : B \cdot y \leq \mu + m_l e_l\}$ (see Hoşten and Thomas, 1999, Section 2), and we can choose $z^{(l)}$ as that solution.

The vectors $z^{(l)}$ are almost what we want, except that we cannot a priori guarantee that $(B \cdot z^{(l)})_3 < 0$, even if we look at all the polytopes $P_{\mu+me_l}(0)$ for $m \geq m_l$, that is, even if we look at the (possibly unbounded) polyhedron:

$$R^l := \{y \in \mathbb{R}^2 : (B \cdot y)_j \leq \mu_j,\ j \neq l; -(-e_3 + \epsilon w)^t (B \cdot y) \leq 0\}.$$

However, we may assume $(B \cdot z^{(l)})_3 \leq 0$ since we can always choose ϵ small enough so that a feasible point that satisfies $(B \cdot z)_3 \leq 0$ is better than one that satisfies $(B \cdot z)_3 > 0$.

The following notation is very convenient:

$$P^{\bar{\sigma}}_{\eta}(0) := \{z \in \mathbb{R}^2 : (B \cdot z)_i \leq \eta_i, i \notin \sigma; -(-e_3 + \epsilon w)^t (B \cdot z) \leq 0\},$$

$$E^l := \left\{ \quad z \in \mathbb{R}^2 : \quad \begin{array}{l} (B \cdot z)_j \leq \mu_j,\ j \neq l; (B \cdot z)_l > \eta_l; \\ -(-e_3 + \epsilon w)^t (B \cdot z) \leq 0 \end{array} \quad \right\}.$$

Notice that $E^l = R^l \backslash P^{\bar{\sigma}}_{\eta}(0)$.

Let us first deal with the case when the hyperplane $\{(B \cdot z)_l = 0\}$ is the same as $\{(B \cdot z)_3 = 0\}$, that is, when there exists $\lambda \in \mathbb{Q}$ such that $e_l^t B = \lambda e_3^t B$. We know that $u = v - B \cdot y$ for some $y \in \mathbb{C}^2$. Since $u_3 = 0 = v_3$, we have $(B \cdot y)_3 = 0$ which implies $(B \cdot y)_l = 0$, so that $u_l = v_l$. But u_l is an integer. By construction of v, this implies $l = 1$ and $\lambda < 0$ (remember that the only integer coordinates of v are the first four). Now $v_1 < (B \cdot z^{(1)})_1 = \lambda (B \cdot z^{(1)})_3$ and $\lambda < 0$ imply that $(B \cdot z^{(1)})_3 < v_l / \lambda < 0$.

Now fix $l \notin \sigma$ such that the lth row of B is not a multiple of the third one, and suppose that the integer program

$$\text{minimize } - (-e_3 + \epsilon w)^t (B \cdot z) \text{ subject to } z \in R^l \cap \mathbb{Z}^2$$

is unbounded, and every bounded subprogram has its solution on the hyperplane $\{(B \cdot z)_3 = 0\}$. Then $R^l \cap \mathbb{Z}^2 \cap \{(B \cdot z)_3 = 0\}$ is an infinite set. Notice that R^l is not contained in the half-space $\{(B \cdot z)_3 \geq 0\}$, since the defining inequalities of R^l given by rows that are multiples of the first row of B are of the form $(B \cdot z)_3 \leq 0$. This follows from similar arguments as those in the preceding paragraph. But now the set $R^l \cap \{(B \cdot z)_3 \leq 0\}$ contains infinitely many lattice points on the hyperplane $\{(B \cdot z)_3 = 0\}$, is not itself contained in this hyperplane, but is a subset of $\{z \in \mathbb{R}^2 : -1 < (B \cdot z)_3 \leq 0\}$. This is impossible.

Thus, if $z^{(l)}$ satisfies $(B \cdot z)_3 = 0$, the integer program:

$$\text{minimize } - (-e_3 + \epsilon w)^t (B \cdot z) \text{ subject to } z \in R^l \cap \mathbb{Z}^2$$

must be bounded. Let $\mathcal{J} \subseteq \{1, 2, 4, \ldots, n\} \backslash \tau$ be the set of all such indices l, with $z^{(l)}$ the (unique) solution to the corresponding integer program. We can now follow the proof of Theorem 3.1 in Hoşten and Thomas (1999) to show that $\langle \partial_i : i \notin \sigma \cup \mathcal{J} \rangle$ is an associated prime of $\text{in}_{-e_3 + \epsilon w}(I_A)$.

Now let $\mathcal{I}$ be such that $\langle \partial_i : i \in \mathcal{I} \rangle$ is a minimal prime of $\text{in}_{e_3 + \epsilon w}(I_A)$ containing $\langle \partial_i : i \notin \sigma \cup \mathcal{J} \rangle$. Then the vectors $z^{(l)}$ for $l \in \mathcal{I}$ satisfy all the desired properties, and the cardinality of $\mathcal{I}$ is 2.

The only thing we still have to show is that the rows of B indexed by $\mathcal{I}$ are linearly independent. To see this, let $(\partial^{\eta'}, \sigma' := \{1, \ldots, n\} \backslash \mathcal{I})$ be a standard pair of $\text{in}_{-e_3 + \epsilon w}(I_A)$, and look at the set:

$$P^{\bar{\sigma'}}_{\eta'}(0) = \{z \in \mathbb{R}^2 : (B \cdot z)_i \leq \eta'_i, i \notin \sigma'; -(-e_3 + \epsilon w)^t (B \cdot z) \leq 0\},$$

which, by Theorem 2.5 in Hoşten and Thomas (1999), is a polytope. If the rows of B indexed by $\mathcal{I}$ are not linearly independent, then the 2×2 matrix whose rows are those rows of B has a non-trivial kernel. Hence there exists $y \in \mathbb{R}^2$ such that $(B \cdot y)_i = 0$ for all $i \notin \sigma'$. Since all the η'_i are non-negative, this means that $P^{\bar{\sigma'}}_{\eta'}(0)$ contains at least half of the line $\{sy : s \in \mathbb{R}\}$, contradicting the fact that $P^{\bar{\sigma'}}_{\eta'}(0)$ is a bounded set. This concludes the proof. $\square$

Notice that a stronger result holds for the fake exponent v corresponding to the standard pair $(\partial_1^{v_1}\partial_2^{v_2}\partial_3^{v_3}, \{3, 5, \ldots, n\})$, namely the three vectors $(1,1)^t$,$(0,1)^t$ and $(1,0)^t$ satisfy the properties required of the vectors $z^{(l)}$ in Lemma 3.7.

4. The Structure of the Map ∂_3

In this section we study the kernel and cokernel of the map ∂_3 between the solution spaces of $H_A(A \cdot v)$ and $H_A(\beta)$. The following proposition is the first step towards describing its kernel.

PROPOSITION 4.1. *If φ is a canonical logarithm-free series solution of $H_A(A \cdot v)$ and $\partial_3\varphi$ belongs to Span$\{\phi_1, \phi_2, \phi_3, \phi_4\}$, where the functions ϕ_i are the logarithm-free formal power series from Lemma* 3.3*, then φ is a monomial and $\partial_3\varphi = 0$. The exponents of two monomials arising this way do not differ by an integer vector.*

PROOF. We compute canonical series with respect to the weight vector $-e_3$, as in Saito *et al.* (1999, Sections 2.5, 3.4), assuming that $\mathrm{in}_{-e_3}(I_A)$ is a monomial ideal. We will deal with the case when $\mathrm{in}_{-e_3}(I_A)$ is not monomial later.

The logarithm-free canonical solutions of $H_A(A \cdot v)$ are of the form

$$\phi_u = \sum_{v' \in N_u} \frac{[u]_{v'_-}}{[v'+u]_{v'_+}} x^{u+v'},$$

where u is a fake exponent with minimal negative support. The fact that u is a fake exponent means that there exists a standard pair (∂^η, σ) of $\mathrm{in}_{-e_3}(I_A)$, with $\sigma = \{3\} \cup \tau$, such that u is the unique vector satisfying $A \cdot u = A \cdot v$ and $u_i = \eta_i,\ i \notin \sigma$.

The only fake exponent with minimal negative support in $\{v + B \cdot z : z \in \mathbb{Z}^2\}$ is v, whose canonical solution is x^v, and this function satisfies $\partial_3 x^v = 0$. Let u be a fake exponent with minimal negative support that does not differ from v by an integer vector. Call φ its canonical solution. If $\partial_3\varphi$ belongs to Span$\{\phi_1, \phi_2, \phi_3, \phi_4\}$, it is clear that we must have $\partial_3\varphi = 0$, that is, φ must be a constant function with respect to x_3. In particular, we need $u_3 = 0$.

If $v' = B \cdot z$ is an element of N_u, then it must satisfy the inequalities

$$(B \cdot z)_i \geq -\eta_i, \qquad i \notin \sigma; \qquad (B \cdot z)_3 \geq 0.$$

But the set

$$P_\eta^{\bar{\sigma}}(0) := \{z \in \mathbb{R}^2 : (B \cdot z)_i \leq \eta_i,\ i \notin \sigma;\ \ (B \cdot z)_3 \leq 0\}$$

intersects the lattice $\mathbb{Z}^2$ only at 0 (see Hoşten and Thomas, 1999, Theorem 2.5). Switching the inequality signs, we conclude $N_u = \{0\}$, so that $\varphi = x^u$.

Now, if $\mathrm{in}_{-e_3}(I_A)$ is not a monomial ideal, take w and ϵ_0 as in Lemma 3.1. We can choose $0 < \epsilon < \epsilon_0$ so that the polytopes

$$P_\eta^{\bar{\sigma}}(0) := \{z \in \mathbb{R}^2 : (B \cdot z)_i \leq \eta_i,\ i \notin \sigma;\ \ -(-e_3 + \epsilon w)^t (B \cdot z) \leq 0\}$$

and

$$\{z \in \mathbb{R}^2 : (B \cdot z)_i \leq \eta_i,\ i \notin \sigma;\ \ (B \cdot z)_3 \leq 0\}$$

have the same integer points. Now the previous reasoning applies when we compute canonical series with respect to $-e_3 + \epsilon w$ instead of $-e_3$.

All that is left to show is the last assertion. Pick u and $\tilde{u}$ arising as before. We want to show that $u-\tilde{u}$ is not an integer vector. This is clear if they have the same negative support, or if the negative support of one is contained in the negative support of the other.

Thus we may assume that both negative supports are non-empty, in particular, u and $\tilde{u}$ have at least four integer coordinates. Write $u = v - B \cdot y$ for some $y \in \mathbb{C}^2$. We claim that u has exactly four integer coordinates, and they are the first four. To show this claim, we will use the numbers α_i from Construction 3.2. We know $u_3 = 0$, so that $(B\cdot y)_3 = 0$. Suppose that $u_j \in \mathbb{Z}$ for some $j > 4$. Then the jth column of B and the third column of B are linearly independent, because otherwise, we would have $(B \cdot y)_j = 0$ so that $u_j = v_j \notin \mathbb{Z}$. This means that $y \in (\mathbb{Q}(\alpha_j)\backslash\mathbb{Q})^2$. But now the construction of the numbers α_i implies that the only integer coordinates of u must be the third one, the jth one, and maybe the first one (if the first row of B is a multiple of the third). We obtain a contradiction. Thus, the only integer coordinates of u are the first four.

Similarly, $\tilde{u}$ has exactly four integer coordinates, the first four. Write $\tilde{u} = v - B\cdot\tilde{y}$. Since both the negative supports of u and $\tilde{u}$ are non-empty (and they are disjoint), we conclude that $\operatorname{nsupp}(u) = \{l\}$, $\operatorname{nsupp}(\tilde{u}) = \{\tilde{l}\}$, with $l, \tilde{l} \in \{1,2,4\}$ and $l \neq \tilde{l}$.

The non-integer vectors y and $\tilde{y}$ belong to the regions obtained from:

$$\{z \in \mathbb{R}^2 : (B\cdot z)_1 \leq v_1, (B\cdot z)_2 \leq v_2, (B\cdot z)_4 \leq v_4, (-e_3 + \epsilon w)^t (B \cdot z) \leq 0\}$$

by removing the lth and $\tilde{l}$th inequality, respectively.

Suppose one of these regions is unbounded, say the one corresponding to u. Then we can find vectors in this region whose difference from y is an integer vector. This contradicts the fact that the canonical solution corresponding to u is a monomial.

Therefore, both regions are bounded. But it is clear then that only one of them intersects the line $\{z \in \mathbb{R}^2 : (B\cdot z)_3 = 0\}$. This means that u and $\tilde{u}$ cannot have different non-empty negative supports, which implies that they cannot differ by an integer vector. □

We are now ready to characterize the kernel of ∂_3 as a map between solution spaces.

Theorem 4.1. *The kernel of the map*

$$\partial_3 : \{\, \textit{Solutions of}\, H_A(A\cdot v)\} \longrightarrow \{\, \textit{Solutions of}\, H_A(\beta)\}$$

is spanned by

$$\left\{ x^u : \begin{array}{c} u \textit{ is (a fake) exponent with minimal} \\ \textit{negative support such that } u_3 = 0 \end{array} \right\}.$$

Proof. It is clear that the functions previously described belong to the kernel of ∂_3. Suppose first that φ is a logarithm-free solution of $H_A(A\cdot v)$ that is constant with respect to x_3. We compute canonical series with respect to the weight vector $-e_3$. If this cannot be done (that is, if $\operatorname{in}_{-e_3}(I_A)$ is not a monomial ideal) we replace this weight by $-e_3 + \epsilon w$ from Lemma 3.1 with ϵ small enough so that the ideas still work.

Now φ is a linear combination of logarithm-free canonical series (with respect to the weight $-e_3$), each corresponding to a fake exponent with minimal negative support. Say $\varphi = \sum c_{u^{(i)}} \phi_{u^{(i)}}$, where $c_{u^{(i)}} \in \mathbb{C}$ and $u^{(i)}$ are the exponents with minimal negative support.

By taking initials, we see that at least one of those exponents must have its third coordinate equal to 0. Call that exponent u. But then, by the proof of Proposition 4.1, the canonical series corresponding to u is x^u, and this function belongs to our candidate spanning set. Subtracting $c_u x^u$ to φ and repeating the process, we conclude that φ is a linear combination of the functions in our candidate spanning set.

Our task now is to show that no logarithmic solution of $H_A(A \cdot v)$ can be constant with respect to x_3.

Let ψ be a (possibly logarithmic) solution of $H_A(A \cdot v)$ and suppose that $\partial_3 \psi = 0$. The function ψ is a linear combination of canonical series. We write $\psi = \varphi_1 + \cdots + \varphi_k$ where in each φ_i we collect all canonical series appearing as summands in ψ whose corresponding exponents differ by integer vectors. Then there exist exponents $u^{(i)}$ with minimal negative support and third coordinate equal to 0, such that $\varphi_i = \sum c_{\gamma,\alpha} x^\alpha \log(x)^\gamma$, where $c_{\alpha,\gamma} \neq 0 \Rightarrow \alpha - u^{(i)} \in \mathbb{Z}^n$. Also notice that each φ_i must be constant with respect to x_3.

We must show that each function φ_i must be logarithm-free. Pick one of those functions φ_i and the exponent $u^{(i)}$. We will now drop the index i for convenience in the notation. Write φ in the form of Observation 3.6. In this case $f = x^u$ for any $\delta \in \mathcal{S}_{\max}$ by construction of φ. To see this, remember that f is a linear combination of logarithm-free canonical series that are constant with respect to x_3 and whose fake exponents differ from u by an integer vector. Using the last part of Proposition 4.1, we see that there is only one such logarithm-free canonical series, namely x^u.

Now we apply Lemma 3.7 to the exponent u. Let $j \in \mathcal{I}$, write z for the vector $z^{(j)}$ and let $\delta \in \mathcal{S}_{\max}$ be maximal with respect to the jth coordinate. Remember $\varphi = \varphi_\delta + c_\delta x^u \log(x)^\delta$, where φ_δ contains only terms in log that are either less than δ or incomparable to δ. We know that $\partial^{(B\cdot z)_-} \varphi = 0$, since $(B \cdot z)_3 < 0$. Then

$$\begin{aligned} 0 &= \partial^{(B\cdot z)_-} \varphi \\ &= \partial^{(B\cdot z)_+} \varphi \\ &= \partial^{(B\cdot z)_+} \varphi_\delta + \partial^{(B\cdot z)_+} x^u \log(x)^\delta. \end{aligned}$$

All the terms that come from $\partial^{(B\cdot z)_+} x^u \log(x)^\delta$ by applying the product rule are either 0 or must be cancelled by something from $\partial^{(B\cdot z)_+} \varphi_\delta$. As a matter of fact, $\partial^{(B\cdot z)_+} x^u \log(x)^\delta$ has a non-zero term which is a multiple of

$$\frac{(\partial^{(B\cdot z)_+ + (-(B\cdot z)_j + \eta_j)e_j} x^u) \log(x)^{\delta - ((B\cdot z)_j - \eta_j)e_j}}{x_j^{(B\cdot z)_j - \eta_j}}$$

if $(B \cdot z)_j - \eta_j \leq \delta_j$, or of

$$\frac{(\partial^{(B\cdot z)_+ + (-(B\cdot z)_j + \eta_j)e_j} x^u) \log(x)^{\delta - \delta_j e_j}}{x_j^{(B\cdot z)_j - \eta_j}}$$

otherwise. The numerators of these fractions are non-zero by construction of z. Then we have a sub-series g of φ_δ such that

$$\partial^{(B\cdot z)_+} (g - x^u \log(x)^\delta) = 0.$$

This means that $g - x^u \log(x)^\delta$ is a polynomial in the variable x_j, which contradicts the fact that φ_δ contains no term in $\log(x)^\delta$. This implies that $\delta_j = 0$, so that φ contains no $\log(x_j)$, and this is true for all $j \in \mathcal{I}$.

Now pick any $l \notin \mathcal{I}$, and $\delta \in \mathcal{S}_{\max}$ maximal with respect to the lth coordinate. As before, $\varphi = \varphi_\delta + c_\delta x^u \log(x)^\delta$. Of course, since x^u is itself a hypergeometric function constant with respect to x_3, we may assume that φ has no term in x^u. This and the homogeneity equations (2) imply that there is a subsum of φ of the form $x^u \sum_{k=1}^n c_k \log(x)^{\delta - e_l + e_k}$, such that there are no other terms in $x^u \log(x)^{\delta - e_l + e_k}$ in φ, and $A \cdot (c_1, \dots, c_n)^t = c_\delta v$.

From our previous reasoning, we know that $c_j = 0$ for all $j \in \mathcal{I} \cup \{3\}$ so that $\frac{1}{c_\delta}(c_1, \dots, c_n)^t - v$ is of the form $B \cdot z$ for some $z \in \mathbb{R}^2$, and $(B \cdot z)_j = v_j$ for $j \in \mathcal{I} \cup \{3\}$. From the reasoning in Lemma 3.7, we conclude that the set given by the inequalities $(B \cdot z)_j \leq 0$ for $j \in \mathcal{I} \cup \{3\}$ is bounded. This implies that we can write 0 as a positive linear combination of the rows of B indexed by $j \in \mathcal{I} \cup \{3\}$. Hence the only vector $z \in \mathbb{R}^2$ such that $(B \cdot z)_j \geq 0$ for $j \in \mathcal{I} \cup \{3\}$ is the origin 0. From this contradiction it follows that $c_\delta = 0$, and thus φ is logarithm-free. □

Remark. Currently, all the examples where we have computed the map ∂_3 have a one-dimensional kernel. However, all these examples are small, so we believe that there will be examples where ∂_3 has a higher-dimensional kernel.

We want to compute the dimension of the solution space of $H_A(\beta)$ using information about the dimension of the kernel and cokernel of the map ∂_3. In particular, our goal is to show that the sum of the dimension of the image of ∂_3 and the dimension of the cokernel of ∂_3 is at least the dimension of the solution space of $H_A(A \cdot v)$ plus one. The next step in this direction is to find linearly independent solutions of $H_A(\beta)$ not lying in the image of ∂_3 corresponding to the elements of the kernel of ∂_3.

Lemma 4.2. *Let u be a fake exponent of $H_A(A \cdot v)$ with minimal negative support corresponding to a standard pair $(\partial^\eta, \sigma = \{3\} \cup \tau)$, and assume that $u_3 = 0$. Then $u - e_3$ is the fake exponent of $H_A(\beta)$ corresponding to $(\partial^\eta, \sigma = \{3\} \cup \tau)$, and it has minimal negative support.*

Proof. That $u - e_3$ is the fake exponent of $H_A(\beta)$ corresponding to $(\partial^\eta, \sigma = \{3\} \cup \tau)$ follows from the fact that $3 \in \sigma$ (and that we have only modified the third coordinate of u).

Now we have to show that $u - e_3$ has minimal negative support. We know that $u_i \in \mathbb{N}$ for $i \notin \tau$, so that u has at least three integer coordinates. If it has exactly those integer coordinates, or if its integer coordinates are all greater than or equal to 0, then $\mathrm{nsupp}(u - e_3) = \{3\}$. It follows that it has minimal negative support. To see this, suppose $\mathrm{nsupp}(u - e_3 - (B \cdot z))$ is strictly contained in $\mathrm{nsupp}(u - e_3)$ for some $z \in \mathbb{Z}^2$. This means that $(B \cdot z)_i \leq \eta_i$, for $i \notin \sigma$, and $(B \cdot z)_3 < 0$. Then $z \in P_\eta^{\bar{\sigma}}(0) \cap \mathbb{Z}^2 = \{0\}$, a contradiction.

Now assume that u has some negative integer coordinates, and write $u = v - B \cdot y$ for some $y \in \mathbb{C}^2$. Then u has at least four integer coordinates. We claim that in that case, u has exactly four integer coordinates, and they are the first four. This follows from our choice of the numbers α_i, as in the last part of the proof of Proposition 4.1.

Moreover, u has some negative integer coordinates. This can only happen if the pair $(\partial^\eta, \sigma = \{3\} \cup \tau)$ is a top-dimensional standard pair, $\{1, \dots, n\} \backslash \sigma$ is strictly contained in $\{1, 2, 4\}$, and u_j is a negative integer, where j is the only element of $\{1, 2, 4\} \cap \sigma$.

Assume that $u - e_3$ does not have minimal negative support, and pick $z \in \mathbb{Z}^2$ such that $\mathrm{nsupp}(u - e_3 - (B \cdot z))$ is strictly contained in $\mathrm{nsupp}(u - e_3)$. Looking at $P_\eta^{\bar{\sigma}}(0)$, we conclude that we cannot have $(B \cdot z)_3 \leq 0$. Then $(B \cdot z)_3 > 0$ and $(B \cdot z)_j \leq u_j < 0$. It follows that $u - B \cdot z$ has minimal negative support $\{3\}$ (and is thus an exponent of $H_A(A \cdot v)$). We will show that $u - B \cdot z$ actually does not have minimal negative support. This contradiction will imply the desired conclusion about $u - e_3$.

In order to show that $u - B \cdot z$ does not have minimal negative support, we need to find a vector $\tilde{z} \in \mathbb{Z}^2$ such that the negative support of $u - B \cdot z - B \cdot \tilde{z}$ is empty. We know that $u - B \cdot z = v - B \cdot (y+z)$, $(B \cdot (y+z))_3 > 0$ and $(y+z) \neq 0$. We have the following cases:

1. $(B \cdot (y+z))_1 < 0, (B \cdot (y+z))_2 \geq 0, (B \cdot (y+z))_4 < 0$;
2. $(B \cdot (y+z))_1 < 0, (B \cdot (y+z))_2 < 0, (B \cdot (y+z))_4 \geq 0$;
3. $(B \cdot (y+z))_1 < 0, (B \cdot (y+z))_2 < 0, (B \cdot (y+z))_4 < 0$;
4. $(B \cdot (y+z))_1 < 0, (B \cdot (y+z))_2 \geq 0, (B \cdot (y+z))_4 \geq 0$;
5. $(B \cdot (y+z))_1 \geq 0, (B \cdot (y+z))_2 \geq 0, (B \cdot (y+z))_4 < 0$;
6. $(B \cdot (y+z))_1 \geq 0, (B \cdot (y+z))_2 < 0, (B \cdot (y+z))_4 \geq 0$.

In case 1, $\text{nsupp}(v - B \cdot (y+z) - B_1)$ is contained in $\text{nsupp}(v - B \cdot (y+z))$. In case 2, $\text{nsupp}(v - B \cdot (y+z) - B_2)$ is contained in $\text{nsupp}(v - B \cdot (y+z))$. In case 3, $\text{nsupp}(v - B \cdot (y+z) - (B_1 + B_2))$ is contained in $\text{nsupp}(v - B \cdot (y+z))$. In case 4, $\text{nsupp}(v - B \cdot (y+z) - (B_1 + B_2))$ is contained in $\text{nsupp}(v - B \cdot (y+z))$. In case 5 $\text{nsupp}(v - B \cdot (y+z) - B_1)$ is contained in $\text{nsupp}(v - B \cdot (y+z))$. In case 6, $\text{nsupp}(v - B \cdot (y+z) - B_2)$ is contained in $\text{nsupp}(v - B \cdot (y+z))$. Cases 1, 2 and 3 follow directly from the construction of v. For case 4, remember that we assumed at the beginning of Section 3 that either the second and fourth rows of B are linearly dependent, or the cone $\{z \in \mathbb{R}^2 : (B \cdot z)_2 \geq 0,\ (B \cdot z)_4 \geq 0\}$ is contained in the first quadrant. This means that the only way case 4 could happen is if the second and fourth rows of B are linearly dependent, and $(B \cdot z)_2 = (B \cdot z)_4 = 0$. Then our assertion about negative supports follows by direct verification. Finally, let us do case 5. Case 6 will be similar.

Since $b_{21} < 0$ and $(v - B \cdot (y+z))_2 \geq 0$, we have $(v - B \cdot (y+z) - B_1)_2 = (v - B \cdot (y+z))_2 - b_{21} \geq 0$. Since $(B \cdot (y+z))_4$ is a negative integer, we have $(v - B \cdot (y+z) - B_1)_4 = b_{41} - 1 - (B \cdot (y+z))_4 - b_{41} \geq 0$. The inequalities that $y+z$ satisfies imply that this vector belongs to the second quadrant of $\mathbb{Z}^2$. Its second coordinate is strictly less than one. To see this, remember that $(B \cdot (y+z))_2 \leq v_2 = b_{22} - 1$. The line $\{(s_1, s_2) \in \mathbb{R}^2 : (B \cdot (s_1, s_2)^t)_2 = v_2\}$ cuts the vertical axis of $\mathbb{R}^2$ above 0 and strictly below 1 (this is because the line $\{(s_1, s_2) : (B \cdot (s_1, s_2)^t)_2 = v_2 + 1\}$ cuts the vertical axis at height 1). It follows that $0 \leq y_2 + z_2 < 1$. We have:

$$\begin{aligned}(v - B \cdot (y+z) - B_1)_1 &= b_{11} + b_{12} - 1 - b_{11}(z_1 + y_1) \\ &\quad - b_{12}(z_2 + y_2) - b_{11} \\ &= -b_{11}(z_1 + y_1) + b_{12} - b_{12}(z_2 + y_2) - 1.\end{aligned}$$

We know $-b_{11}(z_1 + y_1) \geq 0$, since $z_1 + y_1 \leq 0$. We also know $b_{12} - b_{12}(z_2 + y_2) \geq 0$. The sum of these two numbers is an integer (since $(v - B \cdot (y+z) - B_1)_1$ is an integer), so it must be greater than or equal to 1. This implies $(v - B \cdot (y+z) - B_1)_1 \geq 0$, and concludes the proof that $\text{nsupp}(v - B \cdot (y+z) - B_1)$ is contained in $\text{nsupp}(v - B \cdot (y+z))$.

This containment might not be strict, but certainly $(v - B \cdot (y+z) - B_1)_3 > (v - B \cdot (y+z))_3$ (or $(v - B \cdot (y+z) - B_2)_3 > (v - B \cdot (y+z))_3$ in the other cases). Moreover, we can repeat this process, and keep adding columns of B until the third coordinate is a non-negative integer, while keeping the first, second and fourth coordinates also non-negative. This shows that $u - B \cdot z = v - B \cdot (y+z)$ does not have minimal negative support, which is the contradiction we wanted. □

Now we can look at the logarithm-free canonical series solution ϕ_{u-e_3} of $H_A(\beta)$ corresponding to the fake exponent $u-e_3$. We claim that this function does not lie in the image of the map ∂_3 between the solution spaces of $H_A(A\cdot v)$ and $H_A(\beta)$.

Proposition 4.3. *If ψ is a solution of $H_A(A\cdot v)$ and u is as in Lemma 4.2, then $\partial_3\psi \neq \phi_{u-e_3}$.*

Proof. Suppose there is a solution ψ of $H_A(A\cdot v)$ such that $\partial_3\psi = \phi_{u-e_3}$. We will obtain a contradiction.

We proceed as in the part of the proof of Theorem 4.1 where we show that the functions φ_i are logarithm-free. The first step is to use Observation 3.6 to write $\psi = \psi_\delta + c_\delta x^u \log(x)^\delta$ for every $\delta \in \mathcal{S}_{\max}$. We apply Lemma 3.7, with the goal of showing that ψ has no terms in $\log(x_j)$ for $j \in \mathcal{I}$. Let $j\in\mathcal{I}$, pick $\delta\in\mathcal{S}_{\max}$ maximal with respect to the jth coordinate, and let $z = z^{(j)}$ from Lemma 3.7. Then

$$\partial^{(B\cdot z)_-}\psi = \partial^{(B\cdot z)_+}\psi_\delta + \partial^{(B\cdot z)_+}c_\delta x^u \log(x)^\delta.$$

As in the proof of Theorem 4.1, there are non-zero terms when we compute $\partial^{(B\cdot z)_+}c_\delta x^u$ $\log(x)^\delta$ using the product rule. Now, all these terms have either logarithms or denominator a positive integer power of x_j.

By construction, $u_j = \eta_j \geq 0$, so that $j \notin \mathrm{nsupp}(u-e_3)$. Now, since $(B\cdot z)_3 < 0$, $\partial^{(B\cdot z)_-}\psi$ is a further derivative of ϕ_{u-e_3}. But ϕ_{u-e_3} has no terms with denominator x_j^k with $0<k\in\mathbb{N}$. This means that $\partial^{(B\cdot z)_+}c_\delta x^u\log(x)^\delta$ must be cancelled with terms coming from $\partial^{(B\cdot z)_+}\psi_\delta$, and this implies (again, as in Theorem 4.1) that $\delta_j = 0$ or, equivalently, that ψ has no terms in $\log(x_j)$ for $j\in\mathcal{I}$. From this we can show that ψ is logarithm-free.

Now, ϕ_{u-e_3} has a term x^{u-e_3}, and the only way this term matches with a term of $\partial_3\psi$ is if ψ has a term $x^u\log(x_3)$. But ψ is logarithm-free, and we obtain a contradiction. □

It is now time to deal with logarithmic solutions of $H_A(A\cdot v)$ corresponding to exponents that differ by an integer vector from v.

Proposition 4.4. *If ψ is a solution of $H_A(A\cdot v)$ such that the function $\partial_3\psi$ lies in Span$\{\phi_1,\phi_2,\phi_3,\phi_4\}$, where the functions ϕ_i are the formal power series solutions of $H_A(\beta)$ we introduced in Lemma 3.3, then (modulo the kernel of ∂_3),*

$$\psi = \tilde{\psi} + x^v \sum_{i=1}^n c_i \log(x_i),$$

where $\tilde{\psi}$ is a logarithm free series with exponents that differ by integer vectors from v, that has no term in x^v, and the vector $(c_1,\dots,c_n)^t$ belongs to the kernel of A.

Proof. Pick any solution ψ of $H_A(A\cdot v)$ whose derivative with respect to x_3 lies in Span$\{\phi_1,\phi_2,\phi_3,\phi_4\}$. Write ψ as in Observation 3.6:

$$\psi = \psi_\delta + \log(x)^\delta f,$$

for $\delta\in\mathcal{S}_{\max}$.

Here we must have $f = c_\delta x^v$, since $\partial_3 \psi$ lies in $\mathrm{Span}\,\{\phi_1, \phi_2, \phi_3, \phi_4\}$. Suppose that $\delta_3 \neq 0$. Look at the logarithm-free function

$$\partial_3 \, \psi = \partial_3 \, \psi_\delta + c_\delta \log(x)^{\delta - e_3} \frac{x^v}{x_3}.$$

If $\delta \neq e_3$, the term $c_\delta \log(x)^{\delta - e_3} \frac{x^v}{x_3}$ has logarithms, so it must be cancelled with terms coming from $\partial_3 \, \psi_\delta$. Thus ψ_δ must have a subseries g such that $\partial_3 g = c_\delta \log(x)^{\delta - e_3} \frac{x^v}{x_3}$. Then $g - c_\delta x^v \log(x)^\delta$ is constant with respect to x_3, which contradicts the construction of ψ_δ (all its logarithmic terms are either less than or incomparable to δ). Therefore, $\delta_3 = 0$ or $\delta = e_3$.

Now choose $\delta \in \mathcal{S}_{\max}$ with δ_2 maximal. Suppose $\delta_2 \geq 1$. Consider the function:

$$\begin{aligned} \partial^{(B_2)_-} \psi &= \partial^{(B_2)_+} \psi \\ &= \partial^{(B_2)_+} \psi_\delta + c_\delta \log(x)^{\delta - e_2} \frac{\partial^{(B_2)_+ - e_2} x^v}{x_2} \\ &\quad + \text{other terms coming from } c_\delta \partial^{(B_2)_+} x^v \log(x)^\delta. \end{aligned}$$

If $\delta \neq e_2$ the non-zero summand $c_\delta \log(x)^{\delta - e_2} \frac{\partial^{(B_2)_+ - e_2} x^v}{x_2}$ has logarithms, hence it must be cancelled by some other term of the right-hand side sum. Since the numerator $\partial^{(B_2)_+ - e_2} x^v$ is constant with respect to x_2, this is impossible. Thus $\delta = e_2$ or $\delta_2 = 0$.

Similar arguments using $B_1 + B_2$ show that $\delta \in \mathcal{S}_{\max}$ maximal with respect to the first coordinate must be either e_1 or have $\delta_1 = 0$, and the same using B_1 will give the analogous conclusion when δ is maximal with respect to the fourth coordinate.

Now pick $i > 4$ and choose $\delta \in \mathcal{S}_{\max}$ with δ_i maximal. Suppose $\delta_i > 1$. Then $\delta_l = 0$ for $1 \leq l \leq 4$.

We know (looking at homogeneities (2)) that

$$(A \cdot v)_j \psi = \sum_{k=1}^{n} a_{jk} \theta_k \psi, \qquad j = 1, \ldots, d.$$

Call $\tilde{c}$ the coefficient of $\log(x)^{\delta - e_i} x^v$ in ψ. Comparing both sides of the previous equalities, we conclude that

$$\tilde{c}(A \cdot v)_j = \sum_{\gamma \in \mathcal{S}_{\max} : \gamma - e_k = \delta - e_i} a_{jk} c_\gamma, \qquad j = 1, \ldots, d.$$

Let v' be the vector whose kth coordinate is c_γ if $\gamma - e_k = \delta - e_i$, and the rest are zeros. If $\tilde{c} = 0$, v' is a non-zero element of the kernel of A, whose first four coordinates are 0. Such an element does not exist. If $\tilde{c} \neq 0$, $A(1/\tilde{c})v' = A \cdot v$, and the first four coordinates of $(1/\tilde{c})v'$ are 0. Thus, $v - (1/\tilde{c})v'$ is a vector in the kernel of A and is therefore of the form $B \cdot z$, for $z \in \mathbb{R}^2$. Since the first four rows of B lie in different quadrants of $\mathbb{Z}^2$ and the first four entries of $v - (1/\tilde{c})v'$ are non-negative, this is impossible.

Hence

$$\psi = \tilde{\psi} + x^v \sum_{i=1}^{n} c_i \log(x_i),$$

where $\tilde{\psi}$ is logarithm-free. If we assume that $\tilde{\psi}$ has no term x^v (perfectly legal, since this is a solution of $H_A(A \cdot v)$ that is constant with respect to x_3), the fact that the vector $(c_1, \ldots, c_n)^t$ belongs to the kernel of A follows from homogeneities (2). $\square$

THEOREM 4.2. *There are at least two linearly independent convergent functions in Span $\{\phi_1, \phi_2, \phi_3, \phi_4\}$ which span a two-dimensional subspace of the cokernel of ∂_3.*

PROOF. By Proposition 4.4, an element ψ of the solution space of $H_A(A \cdot v)$ such that $\partial_3 \psi$ lies in Span $\{\phi_1, \phi_2, \phi_3, \phi_4\}$ is of the form

$$\psi = \tilde{\psi} + x^v \sum_{i=1}^{n} c_i \log(x_i),$$

with $\tilde{\psi}$ a logarithm-free function with integer exponents, no term x^v, and the vector $(c_1, \ldots, c_n)^t$ belongs to the kernel of A.

Notice that once the c_i are fixed, ψ is unique with those c_i, since the difference of two such functions would be a logarithm-free solution of $H_A(A \cdot v)$ with no term in the kernel of ∂_3, whose derivative with respect to x_3 belongs to Span $\{\phi_1, \phi_2, \phi_3, \phi_4\}$. It follows from Proposition 4.1 that this difference must be 0.

Since the vector of the c_i is in the kernel of A, the previous remark implies that the space of solutions of $H_A(A \cdot v)$ whose derivative with respect to x_3 lies in Span $\{\phi_1, \phi_2, \phi_3, \phi_4\}$ has dimension at most 2, the dimension of the kernel of A. Therefore, if all four of the series ϕ_j are convergent, the statement follows.

If three are convergent, and say ϕ_l is not, this imposes a condition on the acceptable vectors $(c_1, \ldots, c_n)^t$, which makes the dimension of the space of solutions of $H_A(A \cdot v)$ whose derivative with respect to x_3 lies in Span $\{\phi_1, \phi_2, \phi_3, \phi_4\} \backslash \{\phi_j\}$ drop by one.

If only two of the series are convergent, this imposes two conditions on the acceptable vectors $(c_1, \ldots, c_n)^t$. These conditions must be independent, so that $(c_1, \ldots, c_n)^t = 0$, and the result follows. □

LEMMA 4.5. *The functions ϕ_{u-e_3} constructed in Proposition* 4.3 *and the functions from Theorem* 4.2 *that span a two-dimensional subspace of the cokernel of ∂_3 are linearly independent modulo the image of the map ∂_3.*

PROOF. By contradiction, suppose there is a solution ψ of $H_A(A \cdot v)$ such that

$$\partial_3 \psi = L + \left(\sum_{u \notin v + \mathbb{Z}^n : x^u \in \ker(\partial_3)} c_u \phi_{u-e_3} \right),$$

where L is a linear combination of the functions from Theorem 4.2. We can write

$$\psi = \psi_L + \left(\sum_{u \notin v + \mathbb{Z}^n : x^u \in \ker(\partial_3)} \psi_u \right),$$

where ψ_u is the sum of the terms in ψ whose exponents and u differ by an integer vector, and ψ_L is the sum of the terms in ψ whose exponents and v differ by an integer vector. Clearly, $\partial_3 \psi_u = c_u \phi_{u-e_3}$ and $\partial_3 \psi_L = L$. But the functions ψ_u and ψ_L must be solutions of $H_A(A \cdot v)$. To see this, notice that if two monomials x^{α_1} and x^{α_2} are such that $\alpha_1 - \alpha_2 \notin \mathbb{Z}^n$, then the intersection of the D-modules obtained by acting with D on x^{α_1} and x^{α_2} is either empty or $\{0\}$.

Therefore all the c_u must be 0 (by Proposition 4.3) and also L must be 0 (by Theorem 4.2). □

We now have all the ingredients to show that the parameter β from Construction 3.2 is indeed an exceptional parameter.

THEOREM 4.3. *Let β be the parameter from Construction* 3.2. *Then*

$$\operatorname{rank}(H_A(\beta)) \geq \operatorname{vol}(A) + 1.$$

PROOF. In Proposition 4.3 and Theorem 4.2 we built one function in $\operatorname{coker}(\partial_3)$ for each function in a basis of $\ker(\partial_3)$ (which we knew from Theorem 4.1). Moreover, Theorem 4.2 provided at least two linearly independent functions for x^v. Lemma 4.5 shows that all of these functions are linearly independent. Therefore

$$\dim(\operatorname{coker}(\partial_3)) \geq \dim(\ker(\partial_3)) + 1,$$

and this implies that:

$$\dim(\operatorname{coker}(\partial_3)) + \dim(\operatorname{im}(\partial_3)) \geq \dim(\ker(\partial_3)) + \dim(\operatorname{im}(\partial_3)) + 1, \tag{4}$$

where $\operatorname{im}(\partial_3)$ is the image of ∂_3. The left-hand side of (4) equals the dimension of the solution space of $H_A(\beta)$. The right-hand side equals 1 plus the dimension of the solution space of $H_A(A \cdot v)$. This concludes the proof. □

When $n > 4$, we can use Theorem 4.3 to reach a stronger conclusion about the exceptional set of A.

THEOREM 4.4. *Let A be such that I_A is a non-Cohen–Macaulay toric ideal, with a Gale diagram whose first four rows meet each of the open quadrants of $\mathbb{Z}^2$. Let v_1, v_2, v_4 be as in Construction* 3.2. *If $n > 4$, the $(n-4)$-dimensional affine space parametrized by:*

$$(s_5, \ldots, s_n) \longmapsto A \cdot \left(v_1 e_1 + v_2 e_2 - e_3 + v_4 e_4 + \sum_{i=5}^{n} s_i e_i \right)$$

is contained in the exceptional set $\mathcal{E}(A)$ In particular, $\mathcal{E}(A)$ is an infinite set.

PROOF. Pick $(s_5, \ldots, s_n) \in \mathbb{C}^{n-4}$, and $\alpha_5, \ldots, \alpha_n$ as in Construction 3.2. We can choose κ_0 small enough so that the numbers $\tilde{\alpha}_i = s_i + \kappa\alpha_i$ satisfy the conditions of Construction 3.2 for all $0 < \kappa < \kappa_0$. Call

$$\beta_\kappa := A \cdot \left(v_1 e_1 + v_2 e_2 - e_3 + v_4 e_4 + \sum_{i=5}^{n} \tilde{\alpha}_i e_i \right),$$

and

$$\beta := A \cdot \left(v_1 e_1 + v_2 e_2 - e_3 + v_4 e_4 + \sum_{i=5}^{n} s_i e_i \right).$$

Then Theorem 4.3 implies that $\operatorname{rank}(H_A(\beta_\kappa)) \geq \operatorname{vol}(A) + 1$ for all $0 < \kappa < \kappa_0$. Now the proof of Theorem 3.5.1 in Saito *et al.* (1999) implies that $\operatorname{rank}(H_A(\beta)) \geq \operatorname{vol}(A) + 1$. This concludes the proof. □

5. Examples and Final Remarks

To conclude, we illustrate our ideas in an example, and point out some open questions about rank jumps, even in codimension 2. We choose the following matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & -2 & 0 & 3 \end{pmatrix}.$$

In this case, $\mathrm{vol}\,(A) = 9$. Since A has the Gale diagram

$$B = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ -2 & -1 \\ 3 & -1 \\ -1 & -1 \end{pmatrix},$$

which meets the four open quadrants of $\mathbb{Z}^2$, we conclude that the toric ideal I_A is not Cohen–Macaulay.

Theorem 4.4 produces the line $\{A \cdot (2, 0, -1, 2, s)^t : s \in \mathbb{C}\}$, which is contained in the exceptional set of A. Let us take $s = 0$, and analyze the map ∂_3. In this case, $v = (2, 0, 0, 2, 0)^t$, $A \cdot v = (4, 2, 0)^t$ and $\beta = (3, 3, 2)^t$. Using the `Macaulay2` implementation for the Weyl algebra, which can be downloaded from Leykin *et al.* (2000), we check that indeed, $\mathrm{rank}\,(H_A(\beta)) = 10 > 9 = \mathrm{vol}\,(A)$. Here, the distinguished monomial solution of $H_A(A \cdot v)$ is $x^v = x_1^2 x_4^2$. The toric ideal I_A is:

$$I_A = \langle \partial_1^2\partial_2 - \partial_3\partial_4\partial_5, \partial_1\partial_4^3 - \partial_2\partial_3^2\partial_5, \partial_1\partial_2^2\partial_3 - \partial_4^4, \partial_1^3\partial_4^2 - \partial_3^3\partial_5^2, \partial_4^7 - \partial_2^3\partial_3^3\partial_5 \rangle.$$

The fake exponents of $H_A(A \cdot v)$ with respect to the degree reverse lexicographic term order (with $\partial_1 < \cdots < \partial_5$) are:

$$\begin{array}{ll} (0, -1, 1, 3, 1)^t, & (0, 2/7, 16/7, 0, 10/7)^t, \\ (0, -1/7, 13/7, 1, 9/7)^t, & (1/2, 0, 3/2, 1, 1)^t, \\ (0, -13/7, 1/7, 5, 5/7)^t, & (2, 0, 0, 2, 0)^t, \\ (0, -10/7, 4/7, 4, 6/7)^t, & (0, -16/7, -2/7, 6, 4/7)^t, \\ (0, -4/7, 10/7, 2, 8/7)^t, & (-1, 0, 3, 0, 2)^t. \end{array}$$

The fake exponents with minimum negative support are the initial terms of the logarithm-free solutions of $H_A(A \cdot v)$. None of these has zero third coordinate, except the one that corresponds to x^v. Thus, this is the only (up to a constant factor) logarithm-free solution to $H_A(A \cdot v)$ that is constant with respect to x_3.

The distinguished solutions of $H_A(\beta)$ from Lemma 3.3 are

$$\phi_1 = \frac{x_3^2 x_5^2}{x_1}, \qquad \phi_2 = \frac{x_4^3 x_5}{x_2}, \qquad \phi_3 = \frac{x_1^2 x_4^2}{x_3}.$$

There is one solution of $H_A(A \cdot v)$ whose third derivative belongs to the span of the functions ϕ_i, namely

$$\begin{aligned} \psi = {} & x_1^2 x_4^2 (21 \log(x_1) + 6 \log(x_2) - 15 \log(x_3) - 12 \log(x_5)) \\ & + 7 \frac{x_3^3 x_5^2}{x_1} + 4 \frac{x_3 x_4^3 x_5}{x_2}. \end{aligned}$$

Thus, we see that the intersection of the cokernel of ∂_3 with the span of the functions ϕ_i has dimension 2, in particular, the dimension of the cokernel of ∂_3 is at least 2. We know that the kernel of ∂_3 is one dimensional, so that $\dim(\ker(\partial_3)) < \dim(\operatorname{coker}(\partial_3))$. Adding the dimension of the image of ∂_3 to both sides of this inequality, we obtain that $\operatorname{rank}(H_A(A \cdot v)) < \operatorname{rank}(H_A(\beta))$.

The following example, provided by one of the referees, illustrates the fact that, in general, not all exceptional parameters come from Construction 3.2. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 5 & 11 \end{pmatrix}.$$

In this case,

$$B = \begin{pmatrix} 2 & 2 \\ -1 & -4 \\ -2 & 3 \\ 1 & -1 \end{pmatrix},$$

and Construction 3.2 produces $\beta = (4, 9)^t$. Since there is only one high syzygy of I_A, this is the only exceptional parameter we can obtain using our construction. However, it is easily checked using Theorem 1.3 that $(3, 4)^t$ belongs to the exceptional set.

Despite the previous example, there are cases in which Construction 3.2 does give rise to all exceptional parameters. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 \end{pmatrix}.$$

Here we have $\operatorname{vol}(A) = 4$. Since A has the Gale diagram

$$B = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ -1 & -1 \\ 1 & -1 \\ 0 & -1 \end{pmatrix},$$

which meets the four open quadrants of $\mathbb{Z}^2$, we conclude that the toric ideal I_A is not Cohen–Macaulay.

We will show that

$$\mathcal{E}(A) = \{(1, 0, -1)^t + s(1, 0, -2)^t : s \in \mathbb{C}\}.$$

From Theorem 4.4 we conclude that the line $\{A \cdot (2, 0, -1, 0, s)^t : s \in \mathbb{C}\} = \{(1, 0, -1)^t + s(1, 0, -2)^t : s \in \mathbb{C}\}$ is contained in the exceptional set $\mathcal{E}(A)$.

In Section 4.6 of Saito *et al.* (1999), we see that, for each initial monomial ideal of I_A, we can construct a finite arrangement of planes in $\mathbb{C}^d$ that contains the exceptional set. It is therefore informative to compute all initial monomial ideals of I_A, form the corresponding arrangement for each initial ideal, and intersect all of them. In our example, I_A has nine initial monomial ideals (computed using `TiGERS`, Huber and Thomas, 2000). The intersection of all the arrangements coming from these initial ideals is the zero set of the ideal:

$$\begin{gathered} \langle \beta_2, 2\beta_1 + \beta_3 - 1 \rangle \cap \langle \beta_3 - 1, \beta_2 - 3, \beta_1 - 4 \rangle \\ \cap \langle \beta_3 - 1, \beta_2 - 1, \beta_1 - 1 \rangle \cap \langle \beta_3^2, \beta_2 - 3\beta_3, \beta_1 - 4\beta_3 \rangle \end{gathered}$$

that is, the union of the points:

$$(0,0,0)^t, (4,3,1)^t, (1,1,1)^t$$

and the line:

$$\{(1,0,-1)^t + s(1,0,-2)^t : s \in \mathbb{C}\}.$$

With the help of `Macaulay2` for the Weyl algebra, we find that the points $(0,0,0)^t$, $(4,3,1)^t$, $(1,1,1)^t$ do not belong to $\mathcal{E}(A)$, so that the line $\{(1,0,-1)^t+s(1,0,-2)^t : s \in \mathbb{C}\}$ contains $\mathcal{E}(A)$. We conclude that

$$\{(1,0,-1)^t + s(1,0,-2)^t : s \in \mathbb{C}\} = \mathcal{E}(A).$$

It is an open question to give a sharp bound for the maximum possible magnitude of rank jumps, even in codimension 2. Corollary 4.1.2 in Saito *et al.* (1999) gives the only known upper bound for the rank of A-hypergeometric systems, but most likely, it is far from optimal. Also, we can find examples in codimension 2 of exceptional parameters where the rank jump is more than 1. For instance, let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -2 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 1 \end{pmatrix}.$$

Then we have $\mathrm{vol}\,(A) = 9$. Computing a Gale diagram, we see that I_A is not Cohen–Macaulay. In fact, Theorem 4.4 produces, for instance, $\beta = (4,2,0,5)^t$, with rank $(H_A(\beta)) = 10$. However, for $\beta = (2,1,0,2)^t$, $\mathrm{rank}\,(H_A(\beta)) = 11$. This parameter vector also comes from Theorem 4.4.

There is hope that the construction in this article can be extended to provide exceptional parameters for A-hypergeometric systems such that certain reverse lexicographic initial ideals of I_A have embedded primes. Work in that direction is ongoing.

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Erratum

After this paper was revised following the referee reports, the author realised that there is an error in Proposition 4.1, namely, the canonical series φ need not be a monomial. However, it is a finite combination of monomials, and with the appropriate minor modifications, all subsequent proofs go through.

References

Adolphson, A. (1994). Hypergeometric functions and rings generated by monomials. *Duke Math. J.*, **73**, 269–290.

Cattani, E., D'Andrea, C., Dickenstein, A. (1999). The $\mathcal{A}$-Hypergeometric System associated with a monomial curve. *Duke Math. J.*, **99**, 179–207.

Gel'fand, I., Kapranov, M., Zelevinsky, A. (1989). Hypergeometric functions and toral manifolds. *Funct. Anal. Appl.*, **23**, 94–106.

Hoşten, S., Thomas, R. (1999). The associated primes of initial ideals of lattice ideals. *Math. Res. Lett.*, **6**, 83–97.

Huber, B., Thomas, R. (2000). Computing Gröbner fans of toric ideals. *Exp. Math.*, **9**, 321–331. `TiGERS` is available at:`http://www.math.tamu.edu/~rekha.thomas/programs.html`.

Leykin, A., Stillman, M., Tsai, H. (2000). D-module package for `Macaulay2`. Available at: `http://www.math.cornell.edu/~htsai`

Peeva, I., Sturmfels, B. (1997). Syzygies of codimension 2 lattice ideals. *Math Z.*, **229**, 163–194.

Saito, M. (1999). Isomorphism classes of A-Hypergeometric systems. To appear in *Compositio Mathematica*.

Saito, M. (2000). Logarithm-free A-hypergeometric Series. Preprint.

Saito, M., Sturmfels, B., Takayama, N. (1999). *Gröbner Deformations of Hypergeometric Differential Equations*, volume 6 of Algorithms and Computation in Mathematics, Berlin, Springer.

Sturmfels, B. (1995). *Gröbner Bases and Convex Polytopes*, volume 8 of University Lecture Series, Providence, American Mathematical Society.

Sturmfels, B., Takayama, N. (1998). Gröbner bases and hypergeometric functions. In Buchberger, B., Winkler, F. eds, *Gröbner Bases and Applications*, volume 251 of London Mathematical Society Lecture Note Series, pp. 246–258. New York, Cambridge University Press.

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