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Suborbits of (m, k)-isotropic subspaces under finite singular classical groups

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ABSTRACT

Let $\mathbb{F}_q^{2\nu+\delta+l}$ be one of the $(2\nu+\delta+l)$-dimensional singular classical spaces and let $G_{2\nu+\delta+l,2\nu+\delta}$ be the corresponding singular classical group of degree $2\nu+\delta+l$. All the (m,k)-isotropic subspaces form an orbit under $G_{2\nu+\delta+l,2\nu+\delta}$, denoted by $\mathcal{M}(m,k;2\nu+\delta+l,2\nu+\delta)$. Let Λ be the set of all the orbitals of $(G_{2\nu+\delta+l,2\nu+\delta}, \mathcal{M}(m,k;2\nu+\delta+l,2\nu+\delta))$. Then $(\mathcal{M}(m,k;2\nu+\delta+l,2\nu+\delta),\Lambda)$ is a symmetric association scheme. First, we determine all the orbitals and the rank of $(G_{2\nu+\delta+l,2\nu+\delta}, \mathcal{M}(m,k;2\nu+\delta+l,2\nu+\delta))$, calculate the length of each suborbit. Next, we compute all the intersection numbers of the symmetric association scheme $(\mathcal{M}(\nu+k,k;2\nu+\delta+l,2\nu+\delta),\Lambda)$, where $k=1$ or $k=l-1$. Finally, we construct a family of symmetric graphs with diameter 2 based on $\mathcal{M}(2,0;4+\delta+l,4+\delta)$.

1. Introduction

Let G be a transitive permutation group on a finite set Ω. Then G acts on the set $\Omega \times \Omega$ in a natural way as

$$(a,b)^\sigma = \left(a^\sigma, b^\sigma\right), \quad \forall a,b \in \Omega,\ \forall \sigma \in G.$$

The orbits $\Lambda_0, \Lambda_1, \dots, \Lambda_t$ of $(G, \Omega \times \Omega)$ are said to be *orbitals* of (G,Ω), where $\Lambda_0 = \{(a,a) \mid a \in \Omega\}$. The number of orbitals is called the *rank* of (G,Ω). For $a \in \Omega$, let

$$\Lambda_i(a) = \left\{b \in \Omega \;\middle|\; (a,b) \in \Lambda_i\right\}.$$

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Then $\Lambda_0(a), \Lambda_1(a), \ldots, \Lambda_t(a)$ are just the orbits of (G_a, Ω), where G_a is the stabilizer of a. The orbit $\Lambda_i(a)$ is called a *suborbit* of (G, Ω). The length of $\Lambda_i(a)$ is independent of the choice of a. Let $\Lambda = \{\Lambda_0, \Lambda_1, \ldots, \Lambda_t\}$. Then the configuration (Ω, Λ) forms an association scheme (see [1]). The theory of association schemes may be found in [1,2].

The results on suborbits may be found in Wang and Wei [11], Wei and Wang [12,13]. Applying the matrix method, Wan, Dai, Feng and Yang [8] computed all the intersection numbers of dual polar schemes and Grassmann schemes. As a generalization of dual polar schemes, Rieck [6] constructed association schemes by the subspaces of a given dimension in finite classical polar spaces. As generalizations of bilinear forms schemes and dual polar schemes, Guo, Wang and Li constructed association schemes from singular linear space and singular classical spaces, respectively (see [4,5,9]). As generalizations of above researches, Guo and Wang [3] studied suborbits of all $(m, 0)$-isotropic subspaces under singular classical groups, Wang, Guo and Li [10] studied suborbits of all subspaces of type (m, k) under singular general linear groups. This paper is a generalization of [3].

The rest of this article is organized as follows. In Section 2, we introduce the singular classical spaces. In Section 3, we determine all the orbitals and the rank of $(G_{2\nu+\delta+l,2\nu+\delta}, \mathcal{M}(m, k; 2\nu + \delta + l, 2\nu + \delta))$, calculate the length of each suborbit. In Section 4, we compute all the intersection numbers of the scheme $(\mathcal{M}(\nu + k, k; 2\nu + \delta + l, 2\nu + \delta), \Lambda)$, where $k = 1$ or $k = l - 1$. In Section 5, we construct a family of symmetric graphs with diameter 2 based on $\mathcal{M}(2, 0; 4 + \delta + l, 4 + \delta)$.

2. The singular classical spaces

We always assume that

$$K_l = \begin{pmatrix} 0 & I^{(\nu)} & \\ -I^{(\nu)} & 0 & \\ & & 0^{(l)} \end{pmatrix}, \qquad H_{0;l} = \begin{pmatrix} 0 & I^{(\nu)} & \\ I^{(\nu)} & 0 & \\ & & 0^{(l)} \end{pmatrix}, \qquad H_{1;l} = \begin{pmatrix} 0 & I^{(\nu)} & & \\ I^{(\nu)} & 0 & & \\ & & 1 & \\ & & & 0^{(l)} \end{pmatrix}.$$

Let $\mathbb{F}_q$ be a finite field with q elements, and let E denote the subspace of $\mathbb{F}_q^{2\nu+\delta+l}$ generated by $e_{2\nu+\delta+1}, e_{2\nu+\delta+2}, \ldots, e_{2\nu+\delta+l}$, where e_i is the row vector in $\mathbb{F}_q^{2\nu+\delta+l}$ whose ith coordinate is 1 and all other coordinates are 0s.

The *singular symplectic group* of degree $2\nu + l$ over $\mathbb{F}_q$, denoted by $Sp_{2\nu+l,2\nu}(\mathbb{F}_q)$, consists of all $(2\nu + l) \times (2\nu + l)$ nonsingular matrices T over $\mathbb{F}_q$ satisfying $TK_lT^t = K_l$. The row vector space $\mathbb{F}_q^{2\nu+l}$ together with the right multiplication action of $Sp_{2\nu+l,2\nu}(\mathbb{F}_q)$ is called the $(2\nu + l)$-dimensional *singular symplectic space* over $\mathbb{F}_q$ or SSy for short. An m-dimensional subspace P in the $(2\nu + l)$-dimensional singular symplectic space is said to be of *type* (m, s, k), if PK_lP^t is of rank $2s$ and $\dim(P \cap E) = k$. In particular, subspaces of type $(m, 0, k)$ are called (m, k)*-isotropic subspaces*.

Let $q = q_0^2$, where q_0 is a prime power. Then $\mathbb{F}_q$ has an *involutive automorphism* $a \mapsto \bar{a} = a^{q_0}$. The *singular unitary group* of degree $2\nu + \delta + l$ over $\mathbb{F}_q$, denoted by $U_{2\nu+\delta+l,2\nu+\delta}(\mathbb{F}_q)$, consists of all $(2\nu + \delta + l) \times (2\nu + \delta + l)$ nonsingular matrices T over $\mathbb{F}_q$ satisfying $TH_{\delta;l}\bar{T}^t = H_{\delta;l}$, where $\delta = 0$ or 1. The row vector space $\mathbb{F}_q^{2\nu+\delta+l}$ together with the right multiplication action of $U_{2\nu+\delta+l,2\nu+\delta}(\mathbb{F}_q)$ is called the $(2\nu + \delta + l)$-dimensional *singular unitary space* over $\mathbb{F}_q$ or SUn for short. An m-dimensional subspace P in the $(2\nu + \delta + l)$-dimensional singular unitary space is said to be of *type* (m, r, k), if $PH_{\delta;l}\bar{P}^t$ is of rank r and $\dim(P \cap E) = k$. In particular, subspaces of type $(m, 0, k)$ are called (m, k)-*isotropic subspaces*.

Denote by $\mathcal{K}_{2\nu+\delta+l}$ the set of all $(2\nu+\delta+l)\times(2\nu+\delta+l)$ alternate matrices over $\mathbb{F}_q$, where $\delta = 0, 1$ or 2. Two $(2\nu + \delta + l) \times (2\nu + \delta + l)$ matrices A and B over $\mathbb{F}_q$ are said to be *congruent* mod $\mathcal{K}_{2\nu+\delta+l}$, denoted by $A \equiv B$ (mod $\mathcal{K}_{2\nu+\delta+l}$), if $A - B \in \mathcal{K}_{2\nu+\delta+l}$. Clearly, $\equiv$ is an equivalence relation on the set of all $(2\nu + \delta + l) \times (2\nu + \delta + l)$ matrices. Let $[A]$ denote the equivalence class containing A. Two matrix classes $[A]$ and $[B]$ are said to be *cogredient* if there is a nonsingular $(2\nu + \delta + l) \times (2\nu + \delta + l)$ matrix Q over $\mathbb{F}_q$ such that $[QAQ^t] \equiv [B]$. For q being odd, let

$$S_{2s+\delta,\Delta;l}=\begin{pmatrix}0 & I^{(s)} & & \\ I^{(s)} & 0 & & \\ & & \Delta & \\ & & & 0^{(l)}\end{pmatrix},\quad \text{where } \Delta=\begin{cases}\emptyset, & \text{if } \delta=0,\\ (1) \text{ or } (z), & \text{if } \delta=1,\\ \operatorname{diag}(1,-z), & \text{if } \delta=2,\end{cases}$$

where z is a fixed non-square element of $\mathbb{F}_q$. For q being even, let

$$S_{2s+\delta,\Delta;l}=\begin{pmatrix}0 & I^{(s)} & & \\ & 0 & & \\ & & \Delta & \\ & & & 0^{(l)}\end{pmatrix},\quad \text{where } \Delta=\begin{cases}\emptyset, & \text{if } \delta=0,\\ (1), & \text{if } \delta=1,\\ \begin{pmatrix}\alpha & 1\\ & \alpha\end{pmatrix}, & \text{if } \delta=2,\end{cases}$$

where α is a fixed element of $\mathbb{F}_q$ such that $\alpha \notin \{x^2+x \mid x \in \mathbb{F}_q\}$. The *singular orthogonal group* of degree $2\nu+\delta+l$ over $\mathbb{F}_q$ with respect to $S_{2\nu+\delta,\Delta;l}$, denoted by $O_{2\nu+\delta+l,2\nu+\delta}(\mathbb{F}_q)$, consists of all $(2\nu+\delta+l)\times(2\nu+\delta+l)$ nonsingular matrices T over $\mathbb{F}_q$ satisfying $[TS_{2\nu+\delta,\Delta;l}T^t] \equiv [S_{2\nu+\delta,\Delta;l}]$. The row vector space $\mathbb{F}_q^{2\nu+\delta+l}$ together with the right multiplication action of $O_{2\nu+\delta+l,2\nu+\delta}(\mathbb{F}_q)$ is called the $(2\nu+\delta+l)$-dimensional *singular orthogonal space* over $\mathbb{F}_q$ or SOr for short. An m-dimensional subspace P in the $(2\nu+\delta+l)$-dimensional singular orthogonal space is a subspace of *type* $(m, 2s+\gamma, s, \Gamma, k)$ if $PS_{2\nu+\delta,\Delta;l}P^t$ is cogredient to $S_{2s+\gamma,\Gamma;m-2s-\gamma}$ and $\dim(P\cap E)=k$. In particular, subspaces of type $(m,0,0,0,k)$ are called *(m,k)-isotropic subspaces*.

Let $\mathbb{F}_q^{2\nu+\delta+l}$ be one of the $(2\nu+\delta+l)$-dimensional singular classical spaces and let $G_{2\nu+\delta+l,2\nu+\delta}$ be the corresponding singular classical group of degree $2\nu+\delta+l$. If $l=0$, $\mathbb{F}_q^{2\nu+\delta+l}$ is the $(2\nu+\delta)$-dimensional classical space and $G_{2\nu+\delta+l,2\nu+\delta}$ is the corresponding classical group of degree $2\nu+\delta$. Clearly, each singular classical group $G_{2\nu+\delta+l,2\nu+\delta}$ is transitive on the set of all subspaces of the same type in $\mathbb{F}_q^{2\nu+\delta+l}$, see [7, Theorems 3.22, 5.23, 6.28, 7.30]. Denote by $\mathcal{M}(m,k;2\nu+\delta+l,2\nu+\delta)$ the set of all the (m,k)-isotropic subspaces of $\mathbb{F}_q^{2\nu+\delta+l}$. Denote by $N(m,0;2\nu)$, $N(m,0;2\nu+\delta)$, $N(m,0,0;2\nu+\delta,\Delta)$ and $N(m,0,0;2\nu+\delta)$ the number of subspaces of type $(m,0)$, $(m,0)$, $(m,0,0)$ and $(m,0,0)$ in $(2\nu+\delta)$-dimensional symplectic space, unitary space, orthogonal space with $\operatorname{ch}\mathbb{F}_q \neq 2$ and with $\operatorname{ch}\mathbb{F}_q=2$, respectively. These numbers are given in [7, Corollaries 3.19, 5.20, 6.23 and 7.25].

3. Orbitals and suborbits

In this section, we determine all the orbitals and the rank of $(G_{2\nu+\delta+l,2\nu+\delta}, \mathcal{M}(m,k;2\nu+\delta+l, 2\nu+\delta))$, and calculate the length of each suborbit.

Theorem 3.1. *Let $0 \leqslant k \leqslant l$. For any four elements of $\mathcal{M}(m,k;2\nu+\delta+l,2\nu+\delta)$*

$$U=\begin{array}{cc} \begin{array}{cc} 2\nu+\delta & l \end{array} & \\ \begin{pmatrix} U_{11} & U_{12}\\ 0 & U_{22}\end{pmatrix} & \begin{array}{c} m-k\\ k\end{array}\end{array},\qquad V=\begin{array}{cc} \begin{array}{cc} 2\nu+\delta & l \end{array} & \\ \begin{pmatrix} V_{11} & V_{12}\\ 0 & V_{22}\end{pmatrix} & \begin{array}{c} m-k\\ k\end{array}\end{array},$$

$$P=\begin{array}{cc} \begin{array}{cc} 2\nu+\delta & l \end{array} & \\ \begin{pmatrix} P_{11} & P_{12}\\ 0 & P_{22}\end{pmatrix} & \begin{array}{c} m-k\\ k\end{array}\end{array},\qquad Q=\begin{array}{cc} \begin{array}{cc} 2\nu+\delta & l \end{array} & \\ \begin{pmatrix} Q_{11} & Q_{12}\\ 0 & Q_{22}\end{pmatrix} & \begin{array}{c} m-k\\ k\end{array}\end{array},$$

the two pairs (U,V) and (P,Q) are in the same orbital of $(G_{2\nu+\delta+l,2\nu+\delta}, \mathcal{M}(m,k;2\nu+\delta+l,2\nu+\delta))$ if and only if

$$\dim(U_{11}\cap V_{11})=\dim(P_{11}\cap Q_{11}),\qquad \dim(U_{22}\cap V_{22})=\dim(P_{22}\cap Q_{22}),$$
$$\dim(U\cap V)=\dim(P\cap Q)$$

and

$$\begin{aligned}
&\operatorname{rank} UK_lV^t = \operatorname{rank} PK_lQ^t, && SSy;\\
&\operatorname{rank} UH_{\delta;l}\bar{V}^t = \operatorname{rank} PH_{\delta;l}\bar{Q}^t, && SUn;\\
&\operatorname{rank} US_{2\nu+\delta,\Delta;l}V^t = \operatorname{rank} PS_{2\nu+\delta,\Delta;l}Q^t, && SOr,\ \operatorname{ch}\mathbb{F}_q \neq 2;\\
&\operatorname{rank} U\big(S_{2\nu+\delta,\Delta;l} + S^t_{2\nu+\delta,\Delta;l}\big)V^t = \operatorname{rank} P\big(S_{2\nu+\delta,\Delta;l} + S^t_{2\nu+\delta,\Delta;l}\big)Q^t, && SOr,\ \operatorname{ch}\mathbb{F}_q = 2.
\end{aligned}$$

Proof. If (U, V) and (P, Q) are in the same orbital of $(Sp_{2\nu+l,2\nu}(\mathbb{F}_q), \mathcal{M}(m,k;2\nu+l,2\nu))$, there exists

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \in Sp_{2\nu+l,2\nu}(\mathbb{F}_q)$$

such that

$$UT = \begin{pmatrix} U_{11}T_{11} & U_{11}T_{12} + U_{12}T_{22} \\ 0 & U_{22}T_{22} \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix} = P,$$
$$VT = \begin{pmatrix} V_{11}T_{11} & V_{11}T_{12} + V_{12}T_{22} \\ 0 & V_{22}T_{22} \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} \\ 0 & Q_{22} \end{pmatrix} = Q.$$

Then

$$(U \cap V)T = P \cap Q, \qquad U_{11}T_{11} = P_{11}, \qquad U_{22}T_{22} = P_{22}, \qquad V_{11}T_{11} = Q_{11}$$

and

$$V_{22}T_{22} = Q_{22}.$$

By [12, Theorem 2.1],

$$\dim(U_{11} \cap V_{11}) = \dim(P_{11} \cap Q_{11}) \quad \text{and} \quad \operatorname{rank}\big(U_{11}K_0V_{11}^t\big) = \operatorname{rank}\big(P_{11}K_0Q_{11}^t\big).$$

It follows that

$$\dim(U_{11} \cap V_{11}) = \dim(P_{11} \cap Q_{11}), \qquad \dim(U_{22} \cap V_{22}) = \dim(P_{22} \cap Q_{22}),$$
$$\dim(U \cap V) = \dim(P \cap Q), \qquad \operatorname{rank} UK_lV^t = \operatorname{rank} PK_lQ^t.$$

Conversely, let

$$\dim(U_{11} \cap V_{11}) = m-k-i, \qquad \dim(U_{22} \cap V_{22}) = k-a,$$
$$\dim(U \cap V) = m-j, \qquad \operatorname{rank} UK_lV^t = r. \tag{1}$$

Then U and V have the matrix representations of the forms

$$U = \begin{array}{cc} \begin{array}{cc} 2\nu & l \end{array} & \\ \begin{pmatrix} U_{111} & U_{121} \\ U_{112} & U_{122} \\ U_{113} & U_{123} \\ 0 & U_{221} \\ 0 & U_{222} \end{pmatrix} & \begin{array}{c} i \\ m-k-j+a \\ j-i-a \\ a \\ k-a \end{array} \end{array} \quad \text{and} \quad V = \begin{array}{cc} \begin{array}{cc} 2\nu & l \end{array} & \\ \begin{pmatrix} V_{111} & V_{121} \\ U_{112} & U_{122} \\ U_{113} & V_{123} \\ 0 & V_{221} \\ 0 & U_{222} \end{pmatrix} & \begin{array}{c} i \\ m-k-j+a \\ j-i-a \\ a \\ k-a \end{array} \end{array}, \tag{2}$$

where $\operatorname{rank}(U_{123} - V_{123}) = j - i - a$. Then $U + V$ is a subspace of type $(m + j, r, j - i + k)$ with a matrix representation of the form

$$\begin{pmatrix} U_{111} & U_{121} \\ U_{112} & U_{122} \\ U_{113} & U_{123} \\ V_{111} & V_{121} \\ 0 & U_{221} \\ 0 & U_{222} \\ 0 & V_{221} \\ 0 & U_{123} - V_{123} \end{pmatrix}.$$

Similarly, $P + Q$ is also a subspace of type $(m + j, r, j - i + k)$ with a matrix representation just like that of $U + V$. Since $Sp_{2\nu+l,2\nu}(\mathbb{F}_q)$ is transitive on the set of all subspaces of the same type in $\mathbb{F}_q^{2\nu+l}$, there exists a $T \in Sp_{2\nu+l,2\nu}(\mathbb{F}_q)$ such that $(P + Q)T = U + V$. It follows that $PT = U$ and $QT = V$. Hence both (U, V) and (P, Q) are in the same orbital of $(Sp_{2\nu+l,2\nu}(\mathbb{F}_q), \mathcal{M}(m, k; 2\nu + l, 2\nu))$. $\square$

For any U and V of the form (2), let $\Lambda_{(i,a,j-i-a,r)}$ denote the orbital of $(Sp_{2\nu+l,2\nu}(\mathbb{F}_q), \mathcal{M}(m, k; 2\nu + l, 2\nu))$ containing (U, V) satisfying (1). Note that $\operatorname{rank}(UK_lV^t) = r$ if and only if $\operatorname{rank}(U_{11}K_0V_{11}^t) = r$. By [7, Theorem 3.22] and the proof of Theorem 3.1 we have

$$0 \leqslant r \leqslant i, \qquad 0 \leqslant i \leqslant m-k, \qquad 0 \leqslant a \leqslant \min\{k, l-k\}, \qquad 2r \leqslant m-k+i \leqslant \nu + r$$

and

$$k-a \leqslant m-j \leqslant (m-k-i) + (k-a), \qquad (k+a) + (j-i-a) \leqslant l,$$

we obtain

$$\max\{k-a, m+k-i-l\} \leqslant m-j \leqslant m-i-a, \qquad \max\{0, m-k+i-\nu\} \leqslant r \leqslant i.$$

Hence

$$\begin{gathered} 0 \leqslant i \leqslant m-k, \qquad 0 \leqslant a \leqslant \min\{k, l-k\}, \\ 0 \leqslant j-i-a \leqslant \min\{m-k-i, l-k-a\}, \qquad \max\{0, m-k+i-\nu\} \leqslant r \leqslant i. \end{gathered} \tag{3}$$

Conversely, for any given integers i, a, j and r satisfying (3), by [7] there exists a subspace $\widetilde{U}$ of type $(m-k+i, r)$ in the symplectic space $\mathbb{F}_q^{2\nu}$ such that

$$\widetilde{U} K_0 \widetilde{U}^t = \begin{pmatrix} 0 & I^{(r)} & & & \\ -I^{(r)} & 0 & & & \\ & & 0^{(m-k-i)} & & \\ & & & 0^{(i-r)} & \\ & & & & 0^{(i-r)} \end{pmatrix}.$$

Write

$$\widetilde{U} = \begin{array}{c} \begin{pmatrix} U'_{11} \\ U'_{21} \\ U'_{31} \\ U'_{41} \\ U'_{51} \end{pmatrix} \end{array} \begin{array}{c} r \\ r \\ m-k-i \\ i-r \\ i-r \end{array}, \qquad U_{11} = \begin{pmatrix} U'_{11} \\ U'_{31} \\ U'_{41} \end{pmatrix} \quad \text{and} \quad V_{11} = \begin{pmatrix} U'_{21} \\ U'_{31} \\ U'_{51} \end{pmatrix}.$$

Let

$$U_{12} = \begin{pmatrix} 0 & 0^{(r,j-i-a)} \\ 0 & I^{(j-i-a)} \\ 0 & 0 \end{pmatrix}, \qquad U_{22} = \left(I^{(k)} 0^{(k,l-k)} \right), \qquad V_{22} = \left(0^{(k,a)} I^{(k)} 0^{(k,l-k-a)} \right).$$

Take

$$U = \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix}, \qquad V = \begin{pmatrix} V_{11} & 0 \\ 0 & V_{22} \end{pmatrix}.$$

Then $(U, V) \in \Lambda_{(i,a,j-i-a,r)}$; and so the orbital $\Lambda_{(i,a,j-i-a,r)}$ exists. It follows that the orbitals of $(Sp_{2\nu+l,2\nu}(\mathbb{F}_q), \mathcal{M}(m,k; 2\nu+l, 2\nu))$ are completely determined by $(i, a, j-i-a, r)$ satisfying (3).

Theorem 3.2. *Let $0 \leqslant k \leqslant l$. Then the number of orbitals of $(G_{2\nu+\delta+l,2\nu+\delta}, \mathcal{M}(m,k; 2\nu+\delta+l, 2\nu+\delta))$ is*

$$\sum_{i=0}^{m-k} \sum_{a=0}^{\min\{k,l-k\}} \min\{m-k-i+1, l-k-a+1\} \cdot \min\{i+1, \nu+k-m+1\}.$$

Proof. By above discussion, the number of orbitals is equal to the number of $(i, a, j-i-a, r)$ satisfying (3). For a fixed pair (i, a) satisfying

$$0 \leqslant i \leqslant m-k \quad \text{and} \quad 0 \leqslant a \leqslant \min\{k, l-k\},$$

$j-i-a$ may take $\min\{m-k-i+1, l-k-a+1\}$ values $0, \ldots, \min\{m-k-i, l-k-a\}$ and r may take $\min\{i+1, \nu+k-m+1\}$ values $\max\{0, m-k+i-\nu\}, \ldots, i$. Hence, the desired result follows. □

In order to compute the length of suborbits of $(G_{2\nu+\delta+l,2\nu+\delta}, \mathcal{M}(m,k; 2\nu+\delta+l, 2\nu+\delta))$, we need the following results.

Proposition 3.3. *(See [8, Chapter 1, Theorem 5].) The number of $m \times n$ matrices with rank i over $\mathbb{F}_q$ is*

$$N(i; m \times n) = q^{i(i-1)/2} \begin{bmatrix} m \\ i \end{bmatrix}_q \prod_{t=n-i+1}^{n} \left(q^t - 1\right).$$

Proposition 3.4. *(See [9, Proposition 2.3].) Let $1 \leqslant k \leqslant l$. For a given k-dimensional subspace P of $\mathbb{F}_q^l$, the number of k-dimensional subspaces intersecting P at $(k-a)$-dimensional subspaces of $\mathbb{F}_q^l$ is*

$$q^{a^2}\begin{bmatrix} l-k \\ a \end{bmatrix}_q \begin{bmatrix} k \\ a \end{bmatrix}_q.$$

Theorem 3.5. *Suppose* (3) *holds. For each $P \in \mathcal{M}(m,k;2\nu+\delta+l,2\nu+\delta)$, the length $n_{(i,a,j-i-a,r)}$ of the suborbit $\Lambda_{(i,a,j-i-a,r)}(P)$ of $(G_{2\nu+\delta+l,2\nu+\delta}, \mathcal{M}(m,k;2\nu+\delta+l,2\nu+\delta))$ is*

$$\begin{aligned} &q^{a^2+r(2(\nu-m+k)+\delta)+(i-r)^2+a(m-k-i)+i(l-k)}\begin{bmatrix} m-k \\ i \end{bmatrix}_q \begin{bmatrix} i \\ r \end{bmatrix}_q \\ &\times \begin{bmatrix} l-k \\ a \end{bmatrix}_q \begin{bmatrix} k \\ a \end{bmatrix}_q N\big(j-i-a;(m-k-i)\times(l-k-a)\big) \\ &\times \begin{cases} q^{r(r+1)/2}N(i-r,0;2(\nu-m+k)), & SSy; \\ q^{r^2/2}N(i-r,0;2(\nu-m+k)+\delta), & SUn; \\ q^{r(r-1)/2}N(i-r,0,0;2(\nu-m+k)+\delta,\Delta), & SOr,\ \mathrm{ch}\,\mathbb{F}_q \neq 2; \\ q^{r(r-1)/2}N(i-r,0,0;2(\nu-m+k)+\delta), & SOr,\ \mathrm{ch}\,\mathbb{F}_q = 2. \end{cases} \end{aligned}$$

Proof. Let

$$P_{11} = \big(I^{(m-k)}0^{(m-k,2\nu-m+k)}\big), \qquad P_{22} = \big(I^{(k)}0^{(k,l-k)}\big), \qquad P = \begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix}.$$

Then $n_{(i,a,j-i-a,r)}$ is the number of subspaces U satisfying $(P,U) \in \Lambda_{(i,a,j-i-a,r)}$. Write

$$U = \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix},$$

where U_{11} is an $(m-k)\times 2\nu$ matrix of rank $m-k$, U_{12} is an $(m-k)\times l$ matrix and U_{22} is a $k \times l$ matrix of rank k. Then U_{11} is an m-dimensional totally isotropic subspace of the symplectic space $\mathbb{F}_q^{2\nu}$ such that $\dim(P_{11}\cap U_{11}) = m-k-i$ and $\operatorname{rank}(P_{11}K_0U_{11}^t) = r$, and U_{22} is a k-dimensional subspace of $\mathbb{F}_q^l$ such that $\dim(P_{22}\cap U_{22}) = k-a$. By [12, Theorem 2.7] and Proposition 3.4, there are

$$\begin{aligned} \Omega = {} & q^{a^2+2r(\nu-m+k)+(i-r)^2+r(r+1)/2}\begin{bmatrix} m-k \\ i \end{bmatrix}_q \begin{bmatrix} i \\ r \end{bmatrix}_q \\ &\times \begin{bmatrix} l-k \\ a \end{bmatrix}_q \begin{bmatrix} k \\ a \end{bmatrix}_q N\big(i-r,0;2(\nu-m+k)\big) \end{aligned}$$

choices for (U_{11},U_{22}). By the transitivity of $Sp_{2\nu+l,2\nu}(\mathbb{F}_q)$, we may take

$$U_{11} = \begin{array}{c} \begin{array}{cccccc} i & m-k-i & i-r & \nu+k+r-m-i & r & \nu-r \end{array} \\ \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & I & 0 & 0 & 0 \end{pmatrix} \begin{array}{c} m-k-i \\ r \\ i-r \end{array} \end{array}, \qquad U_{22} = \big(0^{(k,a)}I^{(k)}0^{(k,l-k-a)}\big).$$

Then U_{12} has the matrix representation of the form

$$\begin{array}{cccc} a & k & l-k-a & \\ \left(\begin{matrix} A_{11} & 0 & A_{12} \\ A_{21} & 0 & A_{22} \end{matrix}\right. & & \left.\right) & \begin{matrix} m-k-i \\ i \end{matrix} \end{array},$$

where rank $A_{12} = j-i-a$. By Proposition 3.3, there are $N(j-i-a; (m-k-i)\times(l-k-a))$ choices for A_{12}; and so

$$n_{(i,a,j-i-a,r)} = \Omega q^{a(m-k-i)+i(l-k)} N\big(j-i-a; (m-k-i)\times(l-k-a)\big).$$

Hence the desired result follows. □

4. Association schemes

Let Λ be the set of orbitals of $(G_{2\nu+\delta+l,2\nu+\delta}, \mathcal{M}(m,k; 2\nu+\delta+l, 2\nu+\delta))$. Then $(\mathcal{M}(m,k; 2\nu+\delta+l, 2\nu+\delta), \Lambda)$ is a symmetric association scheme. If $k=0$, all the intersection numbers of $(\mathcal{M}(\nu,0; 2\nu+\delta+l, 2\nu+\delta), \Lambda)$ were given by Guo, Wang and Li [4]. In this section, we compute all the intersection numbers of $(\mathcal{M}(\nu+1,1; 2\nu+\delta+l, 2\nu+\delta), \Lambda)$ and $(\mathcal{M}(\nu+l-1, l-1; 2\nu+\delta+l, 2\nu+\delta), \Lambda)$. We begin with two useful propositions.

Proposition 4.1. *(See [8].) For $1 \leqslant \nu$, let P_{11} and Q_{11} be two fixed maximal totally isotropic subspaces of $\mathbb{F}_q^{2\nu+\delta}$ with* $\dim(P_{11}\cap Q_{11}) = \nu - i$. *Then the number of maximal totally isotropic subspaces S_{11} of $\mathbb{F}_q^{2\nu+\delta}$ satisfying* $\dim(P_{11}\cap S_{11}) = \nu - s$ *and* $\dim(S_{11}\cap Q_{11}) = \nu - u$, *denoted by $p^i_{s,u}(\nu; 2\nu+\delta)$, is given by* [4, *Proposition* 2.2].

Proposition 4.2. *(See [4, Theorem 1.1].) The intersection numbers of the scheme $(\mathcal{M}(\nu,0; 2\nu+\delta+l, 2\nu+\delta), \Lambda)$, denoted by $p^{(i,j-i)}_{(s,t-s)(u,\nu-u)}(\nu; 2\nu+\delta+l, 2\nu+\delta)$, are given by* [4, *Theorem* 1.1].

Suppose $k=1$ or $k=l-1$, and (3) holds. Since $m=\nu+k$, $r=i$. Now we compute the intersection numbers of $(\mathcal{M}(\nu+k,k; 2\nu+\delta+l, 2\nu+\delta), \Lambda)$. By the transitivity of $G_{2\nu+\delta+l,2\nu+\delta}$ on $\Lambda_{(i,a,j-i-a,i)}$, we may choose two fixed $(\nu+k,k)$-isotropic subspaces

$$P = \begin{array}{c} \begin{matrix} \nu & \nu+\delta & k & l-k \end{matrix} \\ \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{pmatrix} \end{array} \begin{matrix} \nu \\ k \end{matrix}$$

and

$$Q = \begin{array}{c} \begin{matrix} i & \nu-i & i & \nu+\delta-i & a & k-a & a & l-k-a \end{matrix} \\ \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 & A \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \end{pmatrix} \end{array} \begin{matrix} \nu-i \\ i \\ k-a \\ a \end{matrix},$$

where

$$A = \begin{pmatrix} I^{(j-i-a)} & 0^{(j-i-a,i+l-j-k)} \\ 0 & 0 \end{pmatrix}.$$

Then $(P,Q) \in \Lambda_{(i,a,j-i-a,i)}$, and $p^{(i,a,j-i-a,i)}_{(s,b,t-s-b,s)(u,c,\nu-u-c,u)}$ is the number of subspaces S satisfying $(P,S) \in \Lambda_{(s,b,t-s-b,s)}$ and $(S,Q) \in \Lambda_{(u,c,\nu-u-c,u)}$. Write

$$S = \begin{array}{c} \begin{array}{cc} \scriptstyle 2\nu+\delta & \scriptstyle l \end{array} \\ \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} \end{array} \begin{array}{l} \scriptstyle \nu \\ \scriptstyle k \end{array},$$

where rank $S_{11} = \nu$ and rank $S_{22} = k$. Since $0 \leqslant a, b, c \leqslant 1$, we first discuss the following four cases:

Case 1: $a = b = c = 0$. Then

$$S = \begin{array}{c} \begin{array}{ccc} \scriptstyle 2\nu+\delta & \scriptstyle k & \scriptstyle l-k \end{array} \\ \begin{pmatrix} S_{11} & 0 & S_{122} \\ 0 & I & 0 \end{pmatrix} \end{array} \begin{array}{l} \scriptstyle \nu \\ \scriptstyle k \end{array},$$

where rank $S_{11} = \nu$. Proposition 4.2 implies that

$$p^{(i,0,j-i,i)}_{(s,0,t-s,s)(u,0,\nu-u,u)} = p^{(i,j-i)}_{(s,t-s)(u,\nu-u)}(\nu; 2\nu+\delta+l-k, 2\nu+\delta). \tag{4}$$

Case 2: $a = b = 0, c = 1$. Then

$$p^{(i,0,j-i,i)}_{(s,0,t-s,s)(u,1,\nu-u-1,u)} = 0. \tag{5}$$

Case 3: $a = 0, b = c = 1$. Proposition 4.1 implies that

$$p^{(i,0,j-i,i)}_{(s,1,t-s-1,s)(u,1,\nu-u-1,u)} + p^{(i,0,j-i,i)}_{(s,0,t-s,s)(u,0,\nu-u,u)} = q^{\nu(l-k)} \begin{bmatrix} l \\ 1 \end{bmatrix}_q p^{i}_{s,u}(\nu; 2\nu+\delta). \tag{6}$$

Case 4: $a = b = c = 1$. Proposition 4.1 implies that

$$\begin{aligned} &p^{(i,1,j-i-1,i)}_{(s,1,t-s-1,s)(u,1,\nu-u-1,u)} + p^{(i,1,j-i-1,i)}_{(s,1,t-s-1,s)(u,0,\nu-u,u)} + p^{(i,1,j-i-1,i)}_{(s,0,t-s,s)(u,1,\nu-u-1,u)} \\ &\quad = q^{\nu(l-k)} \begin{bmatrix} l \\ 1 \end{bmatrix}_q p^{i}_{s,u}(\nu; 2\nu+\delta). \end{aligned} \tag{7}$$

By [1, Proposition 2.2] and (4)–(7), all the intersection numbers of $(\mathcal{M}(\nu+k,k; 2\nu+\delta+l, 2\nu+\delta), \Lambda)$ can be given.

5. Symmetric graphs

In this section we construct a family of symmetric graphs with diameter 2.

Let $\nu = 2$, $l \geqslant 1$ and let $\mathbb{F}_q^{4+\delta+l}$ be one of the $(4+\delta+l)$-dimensional singular classical spaces except SOr with $\delta = 0$. Define a graph Γ^δ with the vertex set $\mathcal{M}(2,0; 4+\delta+l, 4+\delta)$, and two vertices P and Q are adjacent if and only if $P+Q$ is a subspace of type ϑ, where $\vartheta = (4,1,1), (4,2,1)$ or $(4,2,1,0,1)$ according to SSy, SUn or SOr, respectively. For any $P, Q \in \Gamma^\delta$, $\partial(P,Q)$ means the distance between vertices P and Q.

Theorem 5.1. *Γ^δ is a symmetric graph with diameter* 2.

Proof. By [7, Theorems 3.22, 5.23, 6.28, 7.30] and [4, Lemma 2.1], Γ^δ is symmetric. To prove the theorem it suffices to show $\partial(P,Q)=2$ for any two non-adjacent vertices P and Q of Γ^δ. We distinguish the following two cases:

Case 1: $l=1$. Then $P+Q$ is of type Θ, Δ or Λ, where

$$\Theta=\begin{cases}(3,1,0), & \text{SSy},\\(3,2,0), & \text{SUn},\\(3,2,1,0,0), & \text{SOr},\end{cases}\qquad \Delta=\begin{cases}(3,0,1), & \text{SSy},\\(3,0,1), & \text{SUn},\\(3,0,0,0,1), & \text{SOr},\end{cases}\qquad \Lambda=\begin{cases}(4,2,0), & \text{SSy},\\(4,4,0), & \text{SUn},\\(4,4,2,0,0), & \text{SOr}.\end{cases}$$

If $P+Q$ is of type Θ, by [4, Lemma 2.1] we may assume that

$$P=\left(I^{(2)}0^{(2,3+\delta)}\right),\qquad Q=\left(0^{(2,1)}I^{(2)}0^{(2,2+\delta)}\right). \tag{8}$$

Take

$$U=\begin{array}{c}\begin{matrix}1&1&1&1&1\end{matrix}\\ \begin{pmatrix}0&1&0&0&1\\1&0&1&0&0\end{pmatrix}\end{array},\quad \begin{array}{c}\begin{matrix}1&1&1&1&\delta&1\end{matrix}\\ \begin{pmatrix}0&1&0&0&0&1\\a&0&1&0&0&0\end{pmatrix}\end{array},\quad \begin{array}{c}\begin{matrix}1&1&1&1&1&\delta-1&1\end{matrix}\\ \begin{pmatrix}0&1&0&0&0&0&1\\1/2&0&1&0&1&0&0\end{pmatrix}\end{array}$$

$$\text{or}\quad \begin{array}{c}\begin{matrix}1&1&1&1&1&\delta-1&1\end{matrix}\\ \begin{pmatrix}0&1&0&0&0&0&1\\1&0&1&0&1&0&0\end{pmatrix}\end{array}$$

according to SSy, SUn, SOr with $\mathrm{ch}\,\mathbb{F}_q\neq 2$ or with $\mathrm{ch}\,\mathbb{F}_q=2$, respectively, where $a+\bar{a}=0$ and $a\neq 0$. Then $\partial(P,U)=\partial(Q,U)=1$, which implies $\partial(P,Q)=2$. If $P+Q$ is of type Δ, by [4, Lemma 2.1] we may assume that P as in (8), and

$$Q=\begin{array}{c}\begin{matrix}1&1&2+\delta&1\end{matrix}\\ \begin{pmatrix}1&0&0&0\\0&1&0&1\end{pmatrix}\end{array}.$$

Take

$$U=\begin{array}{c}\begin{matrix}1&1&1&1&\delta&1\end{matrix}\\ \begin{pmatrix}1&0&0&0&0&1\\0&0&0&1&0&0\end{pmatrix}\end{array},$$

then $\partial(P,U)=\partial(Q,U)=1$, which implies $\partial(P,Q)=2$. If $P+Q$ is of type Λ, by [4, Lemma 2.1] we may assume that P as in (8), and

$$Q=\begin{array}{c}\begin{matrix}2&2&\delta&1\end{matrix}\\ \begin{pmatrix}0&I&0&0\end{pmatrix}\end{array}.$$

Take

$$U=\begin{array}{c}\begin{matrix}1&1&1&1+\delta&1\end{matrix}\\ \begin{pmatrix}0&1&0&0&1\\0&0&1&0&1\end{pmatrix}\end{array},$$

then $\partial(P,U)=\partial(Q,U)=1$, which implies $\partial(P,Q)=2$.

Case 2: $l > 1$. Then $P + Q$ is of type Θ, Δ, Λ or Σ, where $\Sigma = (4, 0, 2), (4, 0, 2)$ or $(4, 0, 0, 0, 2)$ according to SSy, SUn or SOr, respectively. If $P + Q$ is of type Θ, Δ or Λ, similar to the proof of Case 1, we have $\partial(P, Q) = 2$. If $P + Q$ is of type Σ, by [4, Lemma 2.1] we may assume that P as in (8), and

$$Q = \begin{array}{c} \begin{matrix} 2 & 2 & \delta & 2 & l-2 \end{matrix} \\ \begin{pmatrix} I & 0 & 0 & I & 0 \end{pmatrix} \end{array}.$$

Take

$$U = \begin{array}{c} \begin{matrix} 1 & 1 & 1 & 1+\delta & 1 & l-1 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array},$$

then $\partial(P, U) = \partial(Q, U) = 1$, which implies $\partial(P, Q) = 2$.

Hence the desired result follows. $\square$

Remarks. Let $\Lambda = \{\Lambda_{(i,j-i)} \mid 0 \leqslant i \leqslant 2,\ 0 \leqslant j - i \leqslant \min\{2 - i, l\}\}$ be the set of all the orbitals of $(G_{4+\delta+l,4+\delta}, \mathcal{M}(2, 0; 4 + \delta + l, 4 + \delta))$. Then the relation graph $(\mathcal{M}(2, 0; 4 + \delta + l, 4 + \delta), \Lambda_{(1,1)})$ is the graph Γ^{δ} (see [1,4]).

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