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Second-order optimality conditions for problems with C^1 data

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Abstract

In this paper we obtain second-order optimality conditions of Karush–Kuhn–Tucker type and Fritz John one for a problem with inequality constraints and a set constraint in nonsmooth settings using second-order directional derivatives. In the necessary conditions we suppose that the objective function and the active constraints are continuously differentiable, but their gradients are not necessarily locally Lipschitz. In the sufficient conditions for a global minimum $\bar{x}$ we assume that the objective function is differentiable at $\bar{x}$ and second-order pseudoconvex at $\bar{x}$, a notion introduced by the authors [I. Ginchev, V.I. Ivanov, Higher-order pseudoconvex functions, in: I.V. Konnov, D.T. Luc, A.M. Rubinov (Eds.), Generalized Convexity and Related Topics, in: Lecture Notes in Econom. and Math. Systems, vol. 583, Springer, 2007, pp. 247–264], the constraints are both differentiable and quasiconvex at $\bar{x}$. In the sufficient conditions for an isolated local minimum of order two we suppose that the problem belongs to the class $C^{1,1}$. We show that they do not hold for C^1 problems, which are not $C^{1,1}$ ones. At last a new notion parabolic local minimum is defined and it is applied to extend the sufficient conditions for an isolated local minimum from problems with $C^{1,1}$ data to problems with C^1 one.

Keywords: Nonsmooth optimization; Second-order optimality conditions; Second-order directional derivatives; Second-order pseudoconvex functions; Quasiconvex functions

1. Introduction

Karush–Kuhn–Tucker optimality conditions are effective tool for solving various nonlinear programming problems. A lot of papers appeared after the well-known work of Kuhn and Tucker [21]. The first-order necessary conditions are used for finding the eventual candidates for solution, the so called stationary points. The second-order necessary conditions are useful for eliminating the non-optimal stationary points. They help us to determine whether a given point is a minimizer (or maximizer). One finds the optimal points with the help of the sufficient conditions. Especially important are the second-order conditions due to Levitin, Miljutin, Osmolovskii [22] and the one due to Ben-Tal [2], obtained for twice continuously differentiable programs where the multipliers could depend on the direction. Various generalizations of these conditions are obtained in the case of nonsmooth settings. We cite several works concerning the second-order case where C^1 programs are considered: [5–7,15,19,23], and the papers

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[8–11,14,17,27,28] where $C^{1,1}$ programs are investigated. In the most papers where problems with C^1 data are treated the results are obtained in terms of approximate Hessians and their variants. Different approach to the optimality conditions is to formulate the nonlinear programming problem as an equivalent unconstrained convex composite minimization one of the function g. Here $g = h \circ F$ is a composition of a lower semicontinuous convex function h and a function F where $F \in C^2$ or $F \in C^{1,1}$ or $F \in C^1$. The case $F \in C^2$ were investigated by Ioffe [18], and Rockefeller [25]. Burke and Poliquain [4] considered convex composite minimization problems such that F possesses a Hessian matrix, Jeyakumar and Yang [20]—when the function F is from the class $C^{1,1}$. The necessary conditions of Yang [29] concern with the case $F \in C^1$, and its sufficient conditions for global minimum—with twice strictly differentiable functions. First-order sufficient optimality conditions for a global minimum for quasiconvex programming problems with smooth data are discussed in the works [1,13,26] and references therein. Nobody has obtained second-order sufficient optimality conditions. In our opinion Theorem 1 is the first result of this type.

In the present paper we deal with optimality conditions of Karush–Kuhn–Tucker type and Fritz John ones for the following problem

$$\text{Minimize} \quad f_0(x) \quad \text{subject to} \quad x \in X, \quad f_i(x) \leqq 0, \quad i = 1, 2, \dots, m, \tag{P}$$

where $X \subset \mathbf{R}^n$ and the functions f_i, $i = 0, 1, \dots, m$, are defined on X. All results given here are obtained for nonsmooth problems in terms of the second-order directional derivative. To obtain more sensitive conditions, we admit that the Lagrange multipliers depend on the direction. We derive second-order necessary conditions for a local minimum of problems with C^1 (i.e. continuously differentiable) data. We show that the same conditions without constraint qualification are sufficient for a global minimum in some problems of generalized convexity type. The objective function is second-order pseudoconvex, a notion recently introduced by the authors [12]. We derive second-order sufficient conditions for an isolated local minimum of order two for problems with $C^{1,1}$ data. Example 4 shows that the sufficient conditions for an isolated local minimum of order two do not hold if the objective function is continuously differentiable, but it does not belong to the class $C^{1,1}$. With the objective function of this example we give a negative answer to an open question from the paper of Ben-Tal, Zowe [3]. At last, new notions called parabolic local minimum and isolated parabolic local minimum are defined. Every parabolic local minimum satisfies the necessary conditions and the sufficient ones for a global minimum. Second-order sufficient conditions for isolated parabolic local minimum are obtained.

The paper is organized as follows: In Section 2 we derive the sufficient conditions for a global minimum. The necessary conditions for a local minimum are given in Section 3. In Section 4 we discuss the sufficient conditions for an isolated minimum of order two. In Section 5 the notion parabolic local minimum is defined and it is compared with the notion local minimum.

2. Sufficient conditions for a global minimum

We begin this section with some preliminary definitions.

Denote by $\mathbf{R}$ the set of reals and $\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty\} \cup \{+\infty\}$. Let the function $f : X \to \mathbf{R}$ with an open domain $X \subset \mathbf{R}^n$ be differentiable at the point $x \in X$. Then the second-order directional derivative $f''(x, u)$ of f at the point $x \in X$ in direction $u \in \mathbf{R}^n$ is defined as element of $\overline{\mathbf{R}}$ by

$$f''(x, u) = \lim_{t \to +0} \frac{2}{t^2}\big(f(x + tu) - f(x) - t\nabla f(x)u\big).$$

The function f is called second-order directionally differentiable on X if the derivative $f''(x, u)$ exists for each $x \in X$ and any direction $u \in \mathbf{R}^n$.

Recall that a function $f : X \to \mathbf{R}$ is said to be quasiconvex at the point $x \in X$ (with respect to X) [24] if the conditions $y \in X$, $f(y) \leqq f(x)$, $t \in [0, 1]$, $(1 - t)x + ty \in X$ imply $f((1 - t)x + ty) \leqq f(x)$. If the set X is convex, then the function f is called quasiconvex on X when for all $x, y \in X$ and $t \in [0, 1]$ it holds $f((1 - t)x + ty) \leqq \max(f(x), f(y))$.

The following result is known and it could be found, for instance, in the book [24, Theorem 9.1.4].

Lemma 1. *Let X be an open set in $\mathbf{R}^n$, and let f be a real function defined on X which is both differentiable and quasiconvex at the point $x \in X$. Then the following implication holds*:

$$y \in X f(y) \leqq f(x) \quad \Longrightarrow \quad \nabla f(x)(y - x) \leqq 0.$$

Let the function $f : X \to \mathbf{R}$ with an open domain $X \subset \mathbf{R}^n$ be differentiable at the point $x \in X$. Then f is said to be pseudoconvex at $x \in X$ if $y \in X$ and $f(y) < f(x)$ imply $\nabla f(x)(y-x) < 0$. If f is differentiable on X, then it is called pseudoconvex on X when f is pseudoconvex at each $x \in X$.

The following definition is due to the authors [12].

Consider a function $f : X \to \mathbf{R}$ with an open domain X, which is differentiable at $x \in X$ and second-order directionally differentiable at $x \in X$ in every direction $y - x$ such that $y \in X$, $f(y) < f(x)$, $\nabla f(x)(y-x) = 0$. We call f second-order pseudoconvex (for short, 2-pseudoconvex) at $x \in X$ if for all $y \in X$ the following implications hold:

$$f(y) < f(x) \quad \text{implies} \quad \nabla f(x)(y-x) \leqq 0;$$
$$f(y) < f(x), \quad \nabla f(x)(y-x) = 0 \quad \text{imply} \quad f''(x, y-x) < 0.$$

Suppose that f is differentiable on X and second-order directionally differentiable at every $x \in X$ in each direction $y - x$ such that $y \in X$, $f(y) < f(x)$, $\nabla f(x)(y-x) = 0$. We call f 2-pseudoconvex on X if it is 2-pseudoconvex at every $x \in X$. It follows from this definition that every differentiable pseudoconvex function is 2-pseudoconvex. The converse does not hold.

In this section we suppose that f_i, $i = 0, 1, \ldots, m$, are real functions defined on the finite-dimensional Euclidean space $\mathbf{R}^n$. Consider the problem (P). Denote

$$S := \{x \in X \mid f_i(x) \leqslant 0,\ i = 1, 2, \ldots, m\}.$$

For every feasible point $x \in S$ let $I(x)$ be the set of active constraints

$$I(x) := \{i \in \{1, 2, \ldots, m\} \mid f_i(x) = 0\}.$$

A direction d is called critical at the point $x \in S$ if $\nabla f_i(x)d \leqq 0$ for all $i \in \{0\} \cup I(x)$.

The main result of this section is the following theorem establishing Karush–Kuhn–Tucker sufficient optimality conditions.

Theorem 1. *Let the set constraint X be open, and the functions f_i, $i = 0, 1, \ldots, m$, defined on X. Suppose that f_i $(i \in \{0\} \cup I(\bar{x}))$ are differentiable at the feasible point $\bar{x}$ and second-order directionally differentiable at $\bar{x}$ in every critical direction $d \in \mathbf{R}^n$, f_0 is 2-pseudoconvex at $\bar{x}$, f_i $(i \in I(\bar{x}))$ are quasiconvex at $\bar{x}$. If for each critical direction $d \in \mathbf{R}^n$ there exist Lagrange nonnegative multipliers $\lambda_1, \lambda_2, \ldots, \lambda_m$ with*

$$\lambda_i f_i(\bar{x}) = 0, \quad i = 1, \ldots, m, \qquad \nabla L(\bar{x}) = 0$$

where $L = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x)$ is the Lagrange function, and $L''(\bar{x}, d) \geqq 0$, then $\bar{x}$ is a global minimizer of (P).

Proof. Assume the contrary that there exists $x \in S$ with $f_0(x) < f_0(\bar{x})$. We prove that $x - \bar{x}$ is a critical direction. By 2-pseudoconvexity $\nabla f_0(\bar{x})(x - \bar{x}) \leqq 0$. Due to quasiconvexity, and $f_i(x) \leqq f(\bar{x})$, $i \in I(\bar{x})$, by Lemma 1, we have $\nabla f_i(\bar{x})(x - \bar{x}) \leqq 0$ for all $i \in I(\bar{x})$ which implies that $x - \bar{x}$ is critical.

Using the assumptions of the theorem we obtain that there exist nonnegative multipliers $\lambda_1, \lambda_2, \ldots, \lambda_m$ with $\lambda_i f_i(\bar{x}) = 0$, $i = 1, \ldots, m$, and $\nabla L(\bar{x})(x - \bar{x}) = 0$ such that $L''(\bar{x}, x - \bar{x}) \geqq 0$. Therefore $\lambda_i = 0$ when $i \notin I(\bar{x})$. Using that $x - \bar{x}$ is critical we obtain

$$\nabla L(\bar{x})(x - \bar{x}) = \nabla f_0(\bar{x})(x - \bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla f_i(\bar{x})(x - \bar{x}) \leqq 0.$$

Hence $\nabla f_0(\bar{x})(x - \bar{x}) = 0$ and $\lambda_i \nabla f_i(\bar{x})(x - \bar{x}) = 0$ for all $i \in I(\bar{x})$. Then $\nabla f_i(\bar{x})(x - \bar{x}) = 0$ when $\lambda_i > 0$. It follows from 2-pseudoconvexity that $f_0''(\bar{x}, x - \bar{x}) < 0$. Therefore

$$L''(\bar{x}, x - \bar{x}) < \sum_{i \in I(\bar{x})} \lambda_i f_i''(\bar{x}, x - \bar{x}) = \sum_{i \in I(\bar{x}),\ \lambda_i > 0} \lambda_i \lim_{t \to +0} \frac{f_i(\bar{x} + t(x - \bar{x})) - f_i(\bar{x})}{t^2/2}.$$

By quasiconvexity $f_i(\bar{x} + t(x - \bar{x})) \leqq f_i(\bar{x}) = 0$ for all $i \in I(\bar{x})$ and for all sufficiently small $t \in [0, 1]$. We conclude from here that $L''(\bar{x}, x - \bar{x}) < 0$ which is a contradiction. □

Theorem 1 is a generalization of the following result due to Mangasarian [24, Theorem 10.1.2] because the class of 2-pseudoconvex functions contains the class of differentiable pseudoconvex ones.

Theorem 2. *(See [24].) Let the set constraint X be open. The functions f_i $(i = 0, 1, \dots, m)$ are defined on X and $\bar{x}$ is a feasible point. Suppose that f_i $(i \in \{0\} \cup I(\bar{x}))$ are differentiable at $\bar{x}$, f_0 is pseudoconvex at $\bar{x}$, and f_i $(i \in I(\bar{x}))$ are quasiconvex at $\bar{x}$. If there exist Lagrange nonnegative multipliers $\lambda_1, \lambda_2, \dots, \lambda_m$ with $\lambda_i f_i(\bar{x}) = 0$, $i = 1, \dots, m$, $\nabla L(\bar{x}) = 0$ where $L = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x)$, then $\bar{x}$ is a global minimizer of* (P).

Example 1. Consider the following simple problem:

$$\text{Minimize} \quad f_0(x) = \begin{cases} x^2, & x \geqq 0, \\ -x^2, & x < 0 \end{cases} \quad \text{subject to} \quad f_1(x) = -x \leqq 0.$$

In this problem $f_i \in C^1$, $i = 0, 1$. The objective function is 2-pseudoconvex at $\bar{x} = 0$. The constraint function is linear, therefore quasiconvex. The Lagrange function is $L(x) = f_0(x) - \lambda x$. The only stationary point is $\bar{x} = 0$ with a Lagrange multiplier $\lambda = 0$. The constraint is active at $\bar{x} = 0$. The critical directions are all directions $d \in \mathbf{R}$ such that $d \geqq 0$. It is easy to verify that $L''(0, d) = f_0''(0, d) = 2d^2 \geqq 0$. Then Theorem 1 implies that $\bar{x} = 0$ is a global minimizer. This problem cannot be solved with the sufficient conditions given by Mangasarian [24, Theorem 10.1.2] because f_0 is not pseudoconvex.

Example 2. Consider the problem

$$\text{Minimize} \quad f_0(x) = x^3 \quad \text{subject to} \quad f_1(x) = x \leqq 0.$$

The constraint function $f_1 = x$ is quasiconvex. The objective function $f_0 = x^3$ is not 2-pseudoconvex at $\bar{x} = 0$, but it is quasiconvex and third-order pseudoconvex (see [12]). The Lagrangian is $L(x, \lambda) = x^3 + \lambda x$. The set of critical directions is $\{d \in \mathbf{R} \mid d \leqq 0\}$. The unique stationary point is $\bar{x} = 0$ with a Lagrange multiplier $\lambda = 0$. Since $L''(0, 0) = 0$, the second-order sufficient conditions of Theorem 1 are satisfied, but $\bar{x} = 0$ is not a global minimizer.

Let us introduce the following notion.

Definition 1. Let $X \subset \mathbf{R}^n$ be an open set and the function $f : X \to \mathbf{R}$ be differentiable at $x \in X$ and second-order directionally differentiable at $x \in X$ in every direction $y - x$ such that $y \in X$, $f(y) < f(x)$, $\nabla f(x)(y - x) = 0$. We call f strictly 2-pseudoconvex at $x \in X$ if for all $y \in X$, $y \neq x$, the following implications hold:

$$f(y) \leqq f(x) \quad \text{implies} \quad \nabla f(x)(y - x) \leqq 0;$$
$$f(y) \leqq f(x), \quad \nabla f(x)(y - x) = 0 \quad \text{imply} \quad f''(x, y - x) < 0.$$

Each strictly 2-pseudoconvex function is 2-pseudoconvex.

Theorem 3. *If additionally in the hypothesis of Theorem* 1 *we suppose that f_0 is strictly* 2*-pseudoconvex at $\bar{x}$, then $\bar{x}$ is a strict global minimizer.*

Proof. We can prove this theorem using the arguments of Theorem 1. □

Theorem 4. *Let $X \subseteq \mathbf{R}^n$ be open, the functions f_i $(i = 0, 1, \dots, m)$ be defined on X, and $\bar{x}$ a feasible point. Suppose that all functions f_i $(i \in \{0\} \cup I(\bar{x}))$ are differentiable at $\bar{x}$, second-order directionally differentiable at $\bar{x}$ in each critical direction $d \in \mathbf{R}^n$, and strictly* 2*-pseudoconvex at $\bar{x}$. If for each critical direction d there exist nonnegative multipliers $\lambda_0, \lambda_1, \dots, \lambda_m$ with $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_m) \neq 0$ and*

$$\lambda_i f_i(\bar{x}) = 0, \quad i = 1, \dots, m, \qquad \nabla L(\bar{x}) = 0, \qquad L''(\bar{x}, d) \geqq 0,$$

where $L = \sum_{i=0}^m \lambda_i f_i(x)$, then $\bar{x}$ is a strict global minimizer of (P).

Proof. Assume the contrary that there exists $x \in S$, $x \neq \bar{x}$ with $f_0(x) \leqq f_0(\bar{x})$. We prove this theorem following the arguments of Theorem 1. By strict 2-pseudoconvexity we obtain

$$0 = \nabla L(\bar{x})(x - \bar{x}) = \sum_{i \in \{0\} \cup I(\bar{x})} \lambda_i \nabla f_i(\bar{x})(x - \bar{x}) \leqq 0.$$

Therefore $\lambda_i \nabla f_i(\bar{x})(x-\bar{x}) = 0$ for all $i \in \{0\} \cup I(\bar{x})$. For all $i \in \{0\} \cup I(\bar{x})$ with $\lambda_i > 0$ we have $\nabla f_i(\bar{x})(x-\bar{x}) = 0$. Again by strict 2-pseudoconvexity the following holds: $f_i''(\bar{x}, x-\bar{x}) < 0$. We conclude from $\lambda \neq 0$ that $L''(\bar{x}, x-\bar{x}) < 0$ which is a contradiction. $\square$

3. Necessary conditions

In this section we derive necessary optimality conditions for the problem (P) with continuously differentiable data. For a fixed vectors $\bar{x} \in \mathbf{R}^n$ and $d \in \mathbf{R}^n$, let

$$I_0(\bar{x}, d) := \{ i \in \{0\} \cup I(\bar{x}) \mid \nabla f_i(\bar{x}) d = 0 \}.$$

Theorem 5 *(Second-order primal necessary conditions). Let X be an open set in the space $\mathbf{R}^n$, the functions f_i $(i = 0, 1, \ldots, m)$ be defined on X. Suppose that $\bar{x}$ is a local minimizer of the problem* (P)*, the functions f_i $(i \notin I(\bar{x}))$ are continuous at $\bar{x}$, the functions f_i $(i \in \{0\} \cup I(\bar{x}))$ are continuously differentiable, and the functions f_i $(i \in I_0(\bar{x}, d))$ are second-order directionally differentiable at $\bar{x}$ in any critical direction $d \in \mathbf{R}^n$. Then for every critical direction $d \in \mathbf{R}^n$, it follows that there is no $z \in \mathbf{R}^n$ which solves the system*

$$\nabla f_i(\bar{x}) z + f_i''(\bar{x}, d) < 0, \quad i \in I_0(\bar{x}, d). \tag{1}$$

Proof. Let d be an arbitrary fixed critical direction. Obviously the case $I_0(\bar{x}, d) = \emptyset$ is impossible since $\bar{x}$ is a minimum. Suppose the contrary that there exists a critical direction d such that the system (1) has a solution $z \in \mathbf{R}^n$. Let $i \in \{0\} \cup I(\bar{x})$ be arbitrary fixed. Consider the function of one variable $\varphi_i(t) = f_i(\bar{x} + td + 0.5t^2 z)$. Since X is open and $\bar{x}$ is feasible, there exists $\delta > 0$ such that φ_i is defined for $0 \leqq t < \delta$. We have

$$\varphi_i'(t) = \nabla f_i(\bar{x} + td + 0.5t^2 z)(d + tz).$$

Therefore $\varphi_i'(0) = \nabla f_i(\bar{x}) d$. Consider the differential quotient

$$2t^{-2}\big(\varphi_i(t) - \varphi_i(0) - t\varphi_i'(0)\big) = 2t^{-2}\big(f_i(\bar{x} + td + 0.5t^2 z) - f_i(\bar{x}) - t\nabla f_i(\bar{x}) d\big).$$

According to the mean-value theorem there exists $\theta_i \in (0, 1)$ such that

$$f_i(\bar{x} + td + 0.5t^2 z) = f_i(\bar{x} + td) + \nabla f_i(\bar{x} + td + 0.5t^2 \theta_i z)(0.5t^2 z).$$

By $f_i \in \mathrm{C}^1$, we obtain that there exists the second-order directional derivative $\varphi_i''(0, 1)$ and

$$\begin{aligned} \nabla f_i(\bar{x}) z + f_i''(\bar{x}, d) &= \lim_{t \to +0} \nabla f_i(\bar{x} + td + 0.5t^2 \theta_i z) z + \lim_{t \to +0} 2t^{-2}\big(f_i(\bar{x} + td) - f_i(\bar{x}) - t\nabla f_i(\bar{x}) d\big) \\ &= \varphi_i''(0, 1). \end{aligned}$$

Since z is a solution of the system (1) with a direction d we conclude that for every $i \in I_0(\bar{x}, d)$ there exists $\varepsilon_i > 0$ such that $\varphi_i(t) - \varphi_i(0) - t\varphi_i'(0) < 0$ for all $t \in (0, \varepsilon_i)$ that is

$$f_i(\bar{x} + td + 0.5t^2 z) - f_i(\bar{x}) < 0 \quad \text{for all } t \in (0, \varepsilon_i). \tag{2}$$

Consider the following cases:

(1) For every $i \in \{1, 2, \ldots, m\} \setminus I(\bar{x})$ we have $f_i(\bar{x}) < 0$. Hence, by continuity, there exists $\varepsilon_i > 0$ such that $f_i(\bar{x} + td + 0.5t^2 z) < 0$ for all $t \in [0, \varepsilon_i)$.
(2) For every $i \in I(\bar{x}) \setminus I_0(\bar{x}, d)$ we have $\nabla f_i(\bar{x}) d = \varphi_i'(0) < 0$. Therefore there exists $\varepsilon_i > 0$ such that $\varphi_i(t) < \varphi_i(0)$ for all $t \in (0, \varepsilon_i)$. Hence we have $f_i(\bar{x} + td + 0.5t^2 z) < f_i(\bar{x}) = 0$ for all $t \in (0, \varepsilon_i)$.
(3) For all $i \in I_0(\bar{x}, d) \setminus \{0\}$, by $\nabla f_i(\bar{x}) d = 0$, it follows from (2) that there exist $\varepsilon_i > 0$ such that $f_i(\bar{x} + td + 0.5t^2 z) < f_i(\bar{x}) = 0$ for all $t \in (0, \varepsilon_i)$.
(4) If $0 \notin I_0(\bar{x}, d)$, then $\nabla f_0(\bar{x}) d < 0$ that is $\varphi_0'(0) < 0$ and therefore for some $\varepsilon_0 > 0$ it holds $f_0(\bar{x} + td + 0.5t^2 z) < f_0(\bar{x})$ for all $t \in (0, \varepsilon_0)$.
(5) If $0 \in I_0(\bar{x}, d)$, then $\nabla f_0(\bar{x}) d = 0$ and according to (2) there exists $\varepsilon_0 > 0$ such that $f_0(\bar{x} + td + 0.5t^2 z) < f_0(\bar{x})$ for all $t \in (0, \varepsilon_0)$.

It is seen that $\bar{x}$ is not a local minimizer contradicting our hypothesis. $\square$

Consider the problem (P). Let $\bar{x} \in S$. Recall that the Bouligand tangent cone of S at $\bar{x}$ is defined as follows:

$$T(S,\bar{x}) := \left\{ d \in \mathbf{R}^n \;\middle|\; \exists \{x^k\} \subset S,\ \lim_{k\to+\infty} x^k = \bar{x},\ \exists \{t^k\} \subset (0,+\infty)\text{: } d = \lim_{k\to+\infty} t^k (x^k - \bar{x}) \right\}.$$

The closed convex hull of $T(S,\bar{x})$ denoted by $P(S,\bar{x}) := \mathrm{cl}(\mathrm{conv}(T(S,\bar{x})))$ is called the pseudotangent cone of S at $\bar{x}$ [16]. Consider the cone

$$L(\bar{x}) = \{ d \in \mathbf{R}^n \mid \nabla f_i(\bar{x})d \leqslant 0,\ i \in I(\bar{x}) \}.$$

Theorem 6 *(Second-order dual necessary conditions). Suppose that all hypotheses of Theorem* 5 *hold. Then corresponding to any critical direction d there exist nonnegative multipliers $\lambda_0, \lambda_1, \dots, \lambda_m$, with*

$$\begin{aligned} &\lambda_i f_i(\bar{x}) = 0, \quad i = 1,2,\dots,m, \qquad \nabla L(\bar{x}) = 0, \\ &\lambda_i \nabla f_i(\bar{x}) d = 0, \quad i \in \{0\} \cup I(\bar{x}), \\ &L''(\bar{x}, d) = \sum_{i \in \{0\}\cup I(\bar{x})} \lambda_i f_i''(\bar{x}, d) \geqq 0. \end{aligned} \tag{3}$$

Assume further that the Guinard constraint qualification $L(\bar{x}) \subseteq P(S,\bar{x})$ holds [16]. *Then we could suppose that $\lambda_0 = 1$.*

Proof. Consider the matrix A whose rows are $\{\nabla f_i(\bar{x}) \mid i \in I_0(\bar{x}, d)\}$ and the vector b whose components are $\{-f_i''(\bar{x}, d) \mid i \in I_0(\bar{x}, d)\}$.

With these notations Theorem 5 claims that the linear system $Az < b$ has no solution. This is equivalent to say that the linear program $\max\{y \mid Az + \vec{y} \leqq b\}$ has optimal value $\bar{y} \leqq 0$. Here $\vec{y}$ is the vector with all components equal to y. Thus, the dual program

$$\min \left\{ b^T \lambda \;\middle|\; A^T \lambda = 0,\ \sum_{i \in I_0(\bar{x}, d)} \lambda_i = 1,\ \lambda_i \geqq 0 \right\},$$

where λ is a vector with components λ_i such that $i \in I_0(\bar{x}, d)$, has a nonpositive optimal value. Therefore the system

$$A^T \lambda = 0, \quad b^T \lambda \leqq 0, \quad \lambda \geqq 0,\ \lambda \neq 0 \tag{4}$$

has a solution λ. If we define $\lambda_i = 0$ for $i \in (\{0\} \cup I(\bar{x})) \setminus I_0(\bar{x}, d)$ or $i \notin \{0\} \cup I(\bar{x})$, then we see from (4) that $\lambda_0, \lambda_1, \dots, \lambda_m$ satisfy the claim of the theorem. □

We see that the Lagrange multipliers in the second-order necessary conditions depend on the direction. In the following example we compare our result with Theorem 3.1 due to Jeyakumar, Wang [19] where C^1 functions are used and the multipliers do not depend on the direction.

The following definition is introduced in [19]. Consider the space $\mathcal{M}(\mathbf{R}^n, \mathbf{R}^n)$ of all square matrices $n \times n$. Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be continuously differentiable function and $v \in \mathbf{R}^n$. Consider the upper Dini-directional derivative of the function $v\nabla f$

$$(v\nabla f)_+^{(1)}(x,u) := \limsup_{t\to 0} \frac{(v\nabla f)(x+tu) - (v\nabla f)(x)}{t}.$$

It is said that the function f admits approximate Hessian $\partial_*^2 f(x)$ at $x \in \mathbf{R}^n$ if there exists a closed and bounded set $\partial_*^2 f(x) \subseteq \mathcal{M}(\mathbf{R}^n, \mathbf{R}^n)$ and for each $v \in \mathbf{R}^n$ it holds

$$(v\nabla f)_+^{(1)}(x,u) \leqq \max_{M \in \partial_*^2 f(x)} \langle Mv, u \rangle \quad \forall u \in \mathbf{R}^n.$$

The following problem with inequality and equality constraints is considered in [19]:

$$\text{Minimize} \quad f_0(x) \quad \text{subject to} \quad f_i(x) \leqq 0, \quad i = 1,2,\dots,m, \quad h_j(x) = 0, \quad j = 1,2,\dots,q, \tag{P$_1$}$$

where f_i, $i = 0, 1, \dots, m$, and h_j, $j = 1, 2, \dots, q$ are C^1-functions on $\mathbf{R}^n$. The Lagrangian function is given by

$$L = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{q} \mu_j h_j(x).$$

Let $C := \{x \in \mathbf{R}^n \mid f_i(x) \leqq 0,\ i = 1, 2, \dots, m,\ h_j(x) = 0,\ j = 1, 2, \dots, q\}$ be the feasible set and let

$$C(\lambda) := \left\{ x \in C \;\middle|\; \sum_{i=1}^{m} \lambda_i f_i(x) = 0 \right\}. \tag{5}$$

The cone of feasible directions to the set C at $x \in C$ is defined as follows

$$F(C, x) := \{ u \in \mathbf{R}^n \mid \exists \delta > 0,\ \forall \alpha,\ 0 \leqq \alpha \leqq \delta \colon\ x + \alpha u \in C \}.$$

Denote, as usually, by $\mathbf{R}^m_+$ the positive orthant in the space $\mathbf{R}^m$.

Then the following theorem holds:

Theorem 7. *(See [19].) Assume that the problem* (P_1) *attains a local minimum at* $\bar{x}$. *Suppose that for each* $\lambda \in \mathbf{R}^m_+$ *and* $\mu \in \mathbf{R}^q$, $L(\cdot, \lambda, \mu)$ *admits an approximate Hessian* $\partial^2_* L(\bar{x}, \lambda, \mu)$ *at* $\bar{x}$. *If a first-order constraint qualification holds at* $\bar{x}$, *then there exist* $\lambda^*_i \geqq 0$, $\lambda^*_i f_i(\bar{x}) = 0$, *for* $i = 1, 2, \dots, m$, $\mu^* \in \mathbf{R}^q$, $\nabla L(\bar{x}, \lambda^*, \mu^*) = 0$ *and*

$$\left(\forall u \in F\left(C\left(\lambda^*\right), \bar{x}\right)\right) \quad \left(\exists M \in \partial^2_* L\left(\bar{x}, \lambda^*, \mu^*\right)\right) \quad \langle Mu, u \rangle \geqq 0.$$

Example 3. Consider the problem

$$\text{Minimize} \quad f_0 = -x_1^2 - x_2^2 + x_1 - x_2 \quad \text{subject to} \quad f_1 = \sqrt{1 + 2x_2} - x_1 - 1 \leqq 0, \quad f_2 = x_1^2 - x_2 \leqq 0.$$

Here f_0, f_1, f_2 are C^1 functions around $\bar{x} = (0, 0)$. The point $\bar{x} = (0, 0)$ is not optimal. If $\varepsilon > 0$ is arbitrary sufficiently small positive number, then $x(\varepsilon) = (\varepsilon, \varepsilon)$ is feasible and $f_0(\varepsilon, \varepsilon) < f_0(0, 0)$. $\bar{x}$ is a stationary point with $\lambda = (\lambda_1, \lambda_2) = (1, 0)$. The critical directions have the type $d = (u, u)$ where $u \geqq 0$. Since $d^T(\nabla^2 f_0 + \lambda_1 \nabla^2 f_1 + \lambda_2 \nabla^2 f_2)d = -5u^2 < 0$ when $u \neq 0$, then by Theorem 6 $\bar{x}$ is not optimal.

On the other hand $F(C(1, 0), \bar{x}) = (0, 0)$. Therefore Theorem 7 cannot reject $\bar{x}$ as non-optimal because $\partial^2_* L(\bar{x}, \lambda) = \{\nabla^2 L(\bar{x}, \lambda)\}$ and $(0, 0)^T \nabla^2 L(\bar{x}, \lambda)(0, 0) = 0$.

In the next remark we compare Theorem 6 with Theorem 3.2 given by Hiriart-Urruty, Strodiot, Nguyen [17] where $C^{1,1}$ functions are used and the multipliers do not depend on the direction. The following definition is introduced in this paper. Let $f \in C^{1,1}(\mathbf{R}^n)$. The generalized Hessian matrix of f at $\bar{x}$, denoted by $\partial^2 f(\bar{x})$, is the set of matrices defined as the convex hull of the set

$$\{ M \mid \exists x_i \to \bar{x} \text{ with } f \text{ twice differentiable at } x_i \text{ and } \nabla^2 f(x_i) \to M \}.$$

The set $\partial^2 f(\bar{x})$ reduces to $\{\nabla^2 f(\bar{x})\}$ whenever ∇f is strictly differentiable at $\bar{x}$.

Consider the set $C(\lambda)$ defined by Eq. (5) and the Bouligand tangent cone $T(C(\lambda), \bar{x})$ to $C(\lambda)$. Then the following theorem holds.

Theorem 8. *(See [17].) Assume that the problem* (P_1) *with* $C^{1,1}$ *data attains a local minimum at* $\bar{x}$. *If a first-order constraint qualification holds at* $\bar{x}$, *then for each multiplier* $(\lambda, \mu) \in \mathbf{R}^m_+ \times \mathbf{R}^q$ *and for each* $d \in T(C(\lambda), \bar{x})$, *there exists a matrix* $M \in \partial^2_{xx} L(\bar{x}, \lambda, \mu)$ *such that* $\langle Md, d \rangle \geqq 0$.

Remark 1. The difference between Theorems 6 and 8 is not only in the used generalized derivatives. In Theorem 8 is used the set $C(\lambda)$, whereas in Theorem 6 a critical direction d. It is seen from the comments after Remark 3.3 in Ref. [17] that the use of the critical directions is more tractable than the set $C(\lambda)$. Moreover in Ref. [2] is given an example of a twice-continuously differentiable problem where there are no fixed multipliers satisfying the first-order conditions and the second-order ones.

4. Sufficient conditions for an isolated local minimum of order two

In this section we derive sufficient conditions for an isolated local minimum using the following lemmas.

Lemma 2 *(Second-order Taylor expansion). Let $f : X \to \mathbf{R}$ be a function with an open convex domain X which is differentiable on X. Suppose that f is second-order directionally differentiable on X. Then, for every $x, y \in X$ there exists $\xi \in [x, y)$ with*

$$\frac{1}{2} f''(\xi, y-x) \leqq f(y) - f(x) - \nabla f(x)(y-x). \tag{6}$$

Proof. For every fixed $x, y \in X$ consider the function of one variable

$$\varphi(t) = f\big(x + t(y-x)\big) - f(x) - t\nabla f(x)(y-x) + \frac{1}{2}\alpha t^2$$

where α is a constant. By convexity of X it is defined on $[0,1]$ and $\varphi(0) = 0$. We choose α such that $\varphi(1) = 0$. Therefore $\alpha = 2(-f(y) + f(x) + \nabla f(x)(y-x))$. By the Weierstrass theorem φ attains its global maximal value on $[0,1]$ at some point θ. If $\max\{\varphi(t) \mid t \in [0,1]\} > 0$, then $0 < \theta < 1$ and therefore $\varphi'(\theta) = 0$, $\varphi''(\theta, 1) \leqq 0$. If $\max\{\varphi(t) \mid t \in [0,1]\} = 0$, then without loss of generality we assume that $\theta = 0$. We calculate from the definition of φ that $\varphi'(0) = 0$. By maximality we obtain that $\varphi''(\theta, 1) \leqq 0$. In both cases $\varphi''(\theta, 1) \leqq 0$ which implies that $f''(\xi, y-x) + \alpha \leqq 0$ where $\xi = x + \theta(y-x)$. Hence inequality (6) holds. □

Lemma 3. *(See Ginchev, Guerraggio, Rocca [10, Lemma 1].) Let $\varphi : \mathbf{R}^n \to \mathbf{R}$ be a $\mathrm{C}^{1,1}$ function which is second-order directionally differentiable. Assume that $\nabla\varphi$ is Lipschitz with a constant L on $\bar{x} + r\,\mathrm{cl}\,B$ where $\bar{x} \in \mathbf{R}^n$, $r > 0$ and $B := \{x \in \mathbf{R}^n \mid \|x\| < 1\}$ is the unit open ball. Then for $u, v \in \mathbf{R}^n$ and $0 < t < r$ it holds*

$$\left| \frac{2}{t^2}\big(\varphi(\bar{x} + tv) - \varphi(\bar{x}) - t\nabla\varphi(\bar{x})v\big) - \frac{2}{t^2}\big(\varphi(\bar{x} + tu) - \varphi(\bar{x}) - t\nabla\varphi(\bar{x})u\big) \right| \leqq L\big(\|u\| + \|v\|\big)\|v - u\|$$

and $|\varphi''(\bar{x}, u)| \leqq 2L\|u\|^2$.

The feasible point $\bar{x}$ is called an isolated local minimizer of second-order of the problem (P) if there exist a neighbourhood N of $\bar{x}$ and a constant $C > 0$ with $f_0(x) \geqq f_0(\bar{x}) + C\|x - \bar{x}\|^2$ for all $x \in N \cap S$ where

$$S := \big\{x \in X \;\big|\; f_i(x) \leqq 0,\ i = 1, 2, \ldots, m\big\}.$$

Theorem 9 *(Second-order dual sufficient conditions). Suppose that X is an open convex set, and $\bar{x}$ is a feasible point. Let f_i $(i \in \{0\} \cup I(\bar{x}))$ belong to the class $\mathrm{C}^{1,1}(X)$, and they are second-order directionally differentiable. If for every nonzero critical direction d there exist Lagrange multipliers $\lambda_i \geqq 0$, $i = 0, 1, 2, \ldots, m$, with $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_m) \neq 0$ such that*

$$\sum_{i=0}^{m} \lambda_i \nabla f_i(\bar{x}) = 0, \tag{7}$$

$$\lambda_i f_i(\bar{x}) = 0, \quad i = 1, 2, \ldots, m, \tag{8}$$

$$L''(\bar{x}, d) > 0, \tag{9}$$

where L is the Lagrange function, then $\bar{x}$ is an isolated local minimizer of second-order.

Proof. Assume that $\bar{x}$ is not an isolated minimizer of second-order. Therefore, for every sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ of positive numbers converging to zero, there exists a sequence $\{x_k\}$ with

$$\|x_k - \bar{x}\| \leqq \varepsilon_k, \qquad f_0(x_k) < f_0(\bar{x}) + \varepsilon_k\|x_k - \bar{x}\|^2,$$
$$f_i(x_k) \leqq f_i(\bar{x}), \quad i \in I(\bar{x}).$$

Without loss of generality $x_k = \bar{x} + t_k d_k$ where $\|d_k\| = 1$. Passing to a subsequence, we may suppose that $d_k \to \bar{d}$ where $\|\bar{d}\| = 1$.

We prove that $\bar{d}$ is critical. Due to Lemma 2 there exists $\theta_{0,k} \in [0,1)$ such that

$$f_0(x_k) - f_0(\bar{x}) \geqq \nabla f_0(\bar{x})(x_k - \bar{x}) + \frac{1}{2} f_0''(\xi_{0,k}, x_k - \bar{x})$$

where $\xi_{0,k} = \bar{x} + \theta_{0,k} t_k d_k$. Hence

$$\varepsilon_k t_k^2 \geqq t_k \nabla f_0(\bar{x}) d_k + \frac{1}{2} t_k^2 f_0''(\xi_{0,k}, d_k). \tag{10}$$

Since $f_0 \in \mathrm{C}^{1,1}$ there exist a Lipschitz constant L and $r > 0$ such that ∇f_0 satisfies the Lipschitz condition on $\bar{x} + r \operatorname{cl} B$. Let k be a sufficiently large integer such that $\xi_{0,k}$ belongs to $\bar{x} + rB$. There exists $r_k > 0$ with $\xi_{0,k} + r_k B \subset \bar{x} + rB$. Therefore, by Lemma 3, $|f_0''(\xi_{0,k}, d_k)| \leqq 2L$. By canceling t_k in (10) and taking the limits when $k \to \infty$ we obtain $\nabla f_0(\bar{x})\bar{d} \leqq 0$.

In a similar way we prove that $\nabla f_i(\bar{x})\bar{d} \leqq 0$ for $i \in I(\bar{x})$. Thus $\bar{d}$ is critical.

For $i \in \{0\} \cup I(\bar{x})$ consider the differential quotients:

$$y_k^i := \frac{2}{t_k^2}\big(f_i(\bar{x} + t_k d_k) - f_i(\bar{x}) - t_k \nabla f_i(\bar{x}) d_k\big),$$
$$\bar{y}_k^i := \frac{2}{t_k^2}\big(f_i(\bar{x} + t_k \bar{d}) - f_i(\bar{x}) - t_k \nabla f_i(\bar{x}) \bar{d}\big).$$

Due to Lemma 3 the sequence $\{\bar{y}_k^i\}$ is bounded and passing to a subsequence without loss of generality we may assume that it is convergent, in other words $\bar{y}_k^i \to \bar{y}^i$. On the other hand we have

$$\|y_k^i - \bar{y}^i\| \leqq \|y_k^i - \bar{y}_k^i\| + \|\bar{y}_k^i - \bar{y}^i\|.$$

It follows from Lemma 3 that $\|y_k^i - \bar{y}_k^i\| \leqq 2L\|d_k - \bar{d}\|$. Therefore $y_k^i \to \bar{y}^i$, since $d_k \to \bar{d}$.

By inequalities (9), (7) and the choice of the sequence $\{x_k\}$ we get

$$\begin{aligned} 0 < L''(\bar{x}, \bar{d}) &= \lim_{k\to\infty} \sum_{i \in \{0\}\cup I(\bar{x})} \lambda_i \bar{y}_k^i = \sum_{i \in \{0\}\cup I(\bar{x})} \lambda_i \bar{y}^i = \lim_{k\to\infty} \sum_{i \in \{0\}\cup I(\bar{x})} \lambda_i y_k^i \\ &= \lim_{k\to\infty} \Bigg(\sum_{i \in \{0\}\cup I(\bar{x})} \lambda_i \frac{2}{t_k^2}\big(f_i(\bar{x} + t_k d_k) - f_i(\bar{x})\big) - \sum_{i \in \{0\}\cup I(\bar{x})} \lambda_i \frac{2}{t_k} \nabla f_i(\bar{x}) d_k \Bigg) \leqq \lambda_0 \lim_{k\to\infty} \frac{2}{t_k^2} \varepsilon_k t_k^2 = 0 \end{aligned}$$

which is a contradiction. □

Theorem 10 *(Second-order primal sufficient conditions). Let $X \subseteq \mathbf{R}^n$ be an open convex set, and $\bar{x}$ be a feasible point. Suppose that f_i $(i \in \{0\} \cup I(\bar{x}))$ belong to the class* $\mathrm{C}^{1,1}(X)$, and they are second-order directionally differentiable. If for every critical direction $d \in \mathbf{R}^n \setminus \{0\}$ there is no $z \in \mathbf{R}^n$ with

$$\nabla f_i(\bar{x}) z + f_i''(\bar{x}, d) \leqq 0 \quad \text{for all } i \in I_0(\bar{x}, d), \tag{11}$$

then $\bar{x}$ is an isolated local minimizer of order two.

Proof. The proof follows the arguments of Theorem 6. Let $d \neq 0$ be arbitrary critical direction. $I_0(\bar{x}, d) \neq \emptyset$, because by definition every $z \in \mathbf{R}^n$ is a solution of a system with unknown z which does not contain any inequality. Using the same notation we obtain from the inconsistency of the system (11) that the linear system $Az \leqq b$ has no solutions. Therefore the dual of the program $\max\{y \mid Az + \vec{y} \leqq b\}$ has a negative optimal value. We choose $\lambda_i = 0$ for $i \in (\{0\} \cup I(\bar{x})) \setminus I_0(\bar{x}, d)$ or $i \notin I(\bar{x})$. Therefore relations (7)–(9) hold, and the claim is a consequence of Theorem 9. □

The following example shows that Theorem 9 is not true for functions being continuously differentiable only, but not $\mathrm{C}^{1,1}$.

Example 4. Consider the problem

$$\text{Minimize} \quad f_0 = \big(\max\big(0, x_2 - 2\sqrt[3]{x_1^4}\big)\big)^{3/2} + \big(\max\big(0, \sqrt[3]{x_1^4} - x_2\big)\big)^{3/2} \quad \text{subject to} \quad f_1 = -x_1 \leqq 0.$$

Of course, the point $\bar{x}=(0,0)$ is not an isolated minimizer of order two because $f_0(x)=0$ for all $x=(x_1,x_2)$ between the curves $x_2=x_1^{4/3}$ and $x_2=2x_1^{4/3}$. Even it is not a strict local minimizer. The objective function f_0 belongs to the class $\mathrm{C}^1(\mathbf{R}^2)$, but $f_0\notin \mathrm{C}^{1,1}(\mathbf{R}^2)$. For example, ∇f_0 do not satisfy the Lipschitz condition in a neighbourhood of $\bar{x}=(0,0)$. If we take $x_k'=(0,k^{-1})$ and $x_k''=(k^{-3/4},k^{-1})$, then

$$\lim_{k\to+\infty}\big\|\nabla f_0\big(x_k''\big)-\nabla f_0\big(x_k'\big)\big\|/\big\|x_k''-x_k'\big\|=+\infty$$

f_0 is second-order directionally differentiable. Simple calculations show that $\nabla f_0(\bar{x})=(0,0)$, $f_0''(\bar{x},(d_1,d_2))=+\infty$ if $d_2\neq 0$, and $f_0''(\bar{x},(d_1,d_2))=2d_1^2$ if $d_2=0$. On the other hand $\bar{x}$ is stationary point with a Lagrange multiplier $\lambda_1=0$ where $L=f_0-\lambda_1x_1$. The set of critical directions is $\{(d_1,d_2)\mid d_1\geqq 0\}$. $L''(\bar{x},d)>0$ for each $d=(d_1,d_2)\neq(0,0)$ and so the sufficient conditions of Theorem 9 are satisfied.

The following claim is due to Ben-Tal and Zowe [3, Theorem 3.2].

Proposition 1. *Suppose that $f\in\mathrm{C}^{1,1}(\mathbf{R}^n)$, $\nabla f(\bar{x})=0$, and $f''(\bar{x},d)>0$ for all $d\in\mathbf{R}^n\setminus\{0\}$. Then $\bar{x}$ is a strict local minimizer of the function f.*

The following open question remained from [3]. Can we replace the condition $f\in\mathrm{C}^{1,1}(\mathbf{R}^n)$ by $f\in\mathrm{C}^1(\mathbf{R}^n)$? The objective function of Example 4 gives a negative answer, because the point $\bar{x}=(0,0)$ is not a strict local minimizer, but it satisfies all conditions of Proposition 1.

5. Optimality conditions for parabolic local minimum

We received the necessary conditions taking account the variations of the values of the objective function and the constraints over parabolas only. In relevance to this fact we introduce the following definition.

Definition 2. We call a feasible point $\bar{x}$ parabolic local minimum (for short, pl-min) of the problem (P) if for every d, $z\in\mathbf{R}^n$ there exists $\varepsilon=\varepsilon(d,z)>0$ such that

$$f_0\big(\bar{x}+td+0.5t^2z\big)\geqq f_0(\bar{x})\quad\text{for all } t\in[0,\varepsilon)$$

provided that $\bar{x}+td+0.5t^2z$ is a feasible point.

Obviously, each local minimum is a pl-min. Simple examples show that the converse is not true.

Example 5. Consider the function

$$f_0(x)=\begin{cases}x_1^2+x_2^2, & x_2\neq x_1^3,\\ -(x_1^2+x_2^2), & x_2=x_1^3.\end{cases}$$

The point $\bar{x}=(0,0)$ is a pl-minimizer, but it is not a local one.

The introduction of the notion pl-min enlarges the class of functions which have minimizers.

Obviously the necessary conditions of Theorems 5, 6 are necessary for pl-min. Since each global minimizer is a local one, then the sufficient conditions of Theorem 1 are sufficient for pl-min.

Definition 3. We call a feasible point $\bar{x}$ isolated parabolic local minimizer of second-order of the problem (P) if for every $d,z\in\mathbf{R}^n$ there exist positive reals $A=A(d,z)$ and $\varepsilon=\varepsilon(d,z)$ such that

$$f_0\big(\bar{x}+td+0.5t^2z\big)\geqq f_0(\bar{x})+A\big\|td+0.5t^2z\big\|^2\quad\text{for all } t\in[0,\varepsilon)$$

provided that $\bar{x}+td+0.5t^2z$ is a feasible point.

In the following sufficient condition we suppose that the constraints are continuously differentiable, but with not necessarily locally Lipschitz gradients.

Theorem 11. *Let X be an open set, and $\bar{x}$ be a feasible point. Suppose that f_i $(i \in \{0\} \cup I(\bar{x}))$ belong to the class $\mathrm{C}^1(X)$, and they are second-order directionally differentiable at $\bar{x}$ in every direction d. If for every critical direction $d \in \mathbf{R}^n$ there is no $z \in \mathbf{R}^n$ such that $(d, z) \neq (0, 0)$ and*

$$\nabla f_i(\bar{x})z + f_i''(\bar{x}, d) \leqq 0 \quad \text{for all } i \in I_0(\bar{x}, d), \tag{12}$$

then $\bar{x}$ is an isolated parabolic local minimizer of second-order.

Proof. Assume the contrary that there exist directions d and z such that for every sequence of positive numbers $\{\varepsilon_k\}_{k=1}^{\infty}$, converging to 0, there is a sequence of positive numbers $\{t_k\}_{k=1}^{\infty}$, also converging to 0, with

$$f_0\big(\bar{x} + t_k d + 0.5t_k^2 z\big) < f_0(\bar{x}) + \varepsilon_k t_k^2 \|d + 0.5t_k z\|^2, \tag{13}$$

$$f_i\big(\bar{x} + t_k d + 0.5t_k^2 z\big) \leqq 0 \quad \text{for all } i = 0, 1, 2, \ldots, m. \tag{14}$$

It is clear that $(d, z) \neq (0, 0)$. We prove that d is a critical direction. According to the mean-value theorem there exists $\tau_k \in (0, 1)$ such that

$$f_0\big(\bar{x} + t_k d + 0.5t_k^2 z\big) = f_0(\bar{x}) + \nabla f_0\big(\bar{x} + t_k \tau_k d + 0.5t_k^2 \tau_k z\big)\big(t_k d + 0.5t_k^2 z\big).$$

It follows from (13) by canceling t_k and taking the limits when $k \to +\infty$ that $\nabla f_0(\bar{x})d \leqslant 0$. We conclude from (14) using similar arguments that $\nabla f_i(\bar{x})d \leqslant 0$ for all $i \in I(\bar{x})$.

$I_0(\bar{x}, d) \neq \emptyset$ according to the arguments of Theorem 10. Let $i \in I_0(\bar{x}, d)$ be arbitrary fixed. Applying the mean-value theorem we obtain that there exists $\theta_i^k \in (0, 1)$ with

$$f_i\big(\bar{x} + t_k d + 0.5t_k^2 z\big) = f_i(\bar{x} + t_k d) + \nabla f_i\big(\bar{x} + t_k d + 0.5t_k^2 \theta_i^k z\big)\big(0.5t_k^2 z\big).$$

Then it follows from (13), (14) that $\nabla f_i(\bar{x})z + f_i''(\bar{x}, d) \leqq 0$ which contradicts the assumption that the system (12) has no solutions. □

Theorem 12. *Let X be an open set, and $\bar{x}$ be a feasible point. Suppose that f_i $(i \in \{0\} \cup I(\bar{x}))$ belong to the class $\mathrm{C}^1(X)$, and they are second-order directionally differentiable at $\bar{x}$ in every direction d. If for every critical direction $d \in \mathbf{R}^n \setminus \{0\}$ there exists $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_m)$, $\lambda_i \geqq 0$, $i = 0, 1, \ldots, m$, $\lambda \neq 0$ such that conditions* (7)–(9) *and the following one hold*

$$\lambda_i \nabla f_i(\bar{x})d = 0, \quad i \in \{0\} \cup I(\bar{x}), \tag{15}$$

then $\bar{x}$ is an isolated pl-minimizer of second-order.

Proof. Using the notations from the proof of Theorem 6, we conclude from the assumptions of the theorem that the system

$$A^T\lambda = 0, \quad b^T\lambda < 0, \quad \lambda \geqq 0,\ \lambda \neq 0,$$

where λ has components λ_i, $i \in I_0(\bar{x}, d)$, has a solution. It follows from the duality arguments of Theorem 6 that the system $Az \leqq b$ has no solutions. Thus the claim is a consequence of Theorem 11. □

Remark 2. Condition (15) is not among the hypotheses of Theorem 9, but it appears in Theorem 6. Really, it takes implicit part in Theorem 10.

6. Comparison remarks

There is no second-order sufficient conditions for a global minimum of Karush–Kuhn–Tucker type in the literature without gap between the necessary conditions and the sufficient ones. Theorem 1 is the first result of this type. The notion second-order strictly pseudoconvex function is introduced and it is applied in the optimality conditions. In our opinion nobody has obtained necessary and sufficient optimality conditions of the considered type in terms of the second-order directional derivative which is used in this paper. In our opinion the second-order Taylor expansion is new. Nobody has shown that the sufficient conditions for an isolated minimum of order two do not hold when the problem contains C^1 data. Example 4 is the first such result and it gives a negative answer to an open question from [3, Theorem 3.2]. The notion parabolic local minimum is new. This notion helps us to extend the sufficient conditions for an isolated local minimum from problems with $C^{1,1}$ data to C^1 problems.

The nonlinear programming problem could be reformulated as a convex composite minimization problem. We compare our article with the ones concerning this approach [4,18,20,25,29]. Primal conditions are not obtained in these papers. The results there are applicable for problems with C^2 or $C^{1,1}$ data, or second-order differentiable ones. Only Yang consider in his necessary conditions the C^1 case, but sufficient conditions for a local minima are not established in this work. All our results are based on too different approach.

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