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On the positivity of certain trigonometric sums and their applications

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ABSTRACT

In this paper, we find conditions on the coefficients $\{b_k\}_{k=1}^n$ such that the corresponding trigonometric (cosine and sine) sums given respectively by $\sum_{k=1}^{n} b_k \sin k\theta > 0$ and $\sum_{k=1}^{n} b_k \cos k\theta > 0$ for all $n \in \mathbb{N}$ are positive. Using these results, we find that the functions f that are in the class of analytic functions $\mathcal{A}$ are starlike of certain order in the unit disc $\mathbb{D}$ by means of conditions on the Taylor coefficients of f. As an application, we also find conditions such that the Cesáro means of order β of $f(z)$ are close-to-convex and starlike in $\mathbb{D}$.

1. Introduction

Trigonometric series have been an important and interesting part of mathematics over the last centuries and have been used as an important tool in pure mathematics, particularly after Fourier series and harmonic functions. Among the contributions made by various mathematicians such as Fejér et al., in the aspect of positivity of trigonometric sums, the most familiar one is the Fejér–Jackson–Gronwall inequality,

$$\sum_{k=1}^{n} \frac{\sin k\theta}{k} > 0, \quad \text{for all } n \in \mathbb{N} \text{ and } 0 < \theta < \pi, \tag{1.1}$$

conjectured by Fejér in 1910 and proved independently by Jackson [1] in 1911 and by Gronwall [2] in 1912. Since then, several other proofs were given and the shortest proof is due to Landau [3]. In 1953, Turán [4] established that if

$$\sum_{k=1}^{n} a_k \sin(2k-1)\theta \geq 0, \quad 0 < \theta < \pi,$$

for some n, then

$$\sum_{k=1}^{n} \frac{a_k}{k} \sin k\theta > 0, \quad 0 < \theta < \pi,$$

for the same n, unless all a_k are zero. This exhibits (1.1), as a consequence of the basic inequality

$$\sum_{k=1}^{n} \sin(2k-1)\theta \geq 0, \quad 0 < \theta < \pi.$$

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A short proof of this result is given in [5]. In 1997, Brown and Wang [6] considered the positivity of trigonometric sums of the form

$$T_n^{\alpha}(\theta) = \sum_{k=1}^{n} \frac{\sin k\theta}{k+\alpha}, \quad 0 < \theta < \pi \tag{1.2}$$

and they have shown that when n is odd, $T_n^{\alpha}(\theta)$ are positive throughout the interval $0 < \theta < \pi$ whenever $-1 < \alpha \le \alpha_0$ and that $\alpha_0 = 2.1102\ldots$ is the best possible. In the case of n being even, the positivity of $T_n^{\alpha}(\theta)$ fails to hold for all $\theta \in (0, \pi)$.

In 1913, analogous to (1.1), Young [7] established the inequality

$$1 + \sum_{k=1}^{n} \frac{\cos k\theta}{k} > 0, \quad \text{for all } n \in \mathbb{N} \text{ and } 0 < \theta < \pi. \tag{1.3}$$

Rogosinski and Szegö [8] considered inequalities of the form

$$\frac{1}{1+\alpha} + \sum_{k=1}^{n} \frac{\cos k\theta}{k+\alpha} > 0, \quad \text{for all } n \in \mathbb{N} \text{ and } 0 < \theta < \pi. \tag{1.4}$$

and observed the existence of a constant A, $1 \le A \le 2(1+\sqrt{2})$, such that (1.4) hold for every α, $-1 < \alpha \le A$. In 1969 Gasper [9] determined the exact value of A as follows:

Lemma 1.1 ([9]). *Let A be the positive root of the equation*

$$9x^7 + 55x^6 - 14x^5 - 948x^4 - 3247x^3 - 5013x^2 - 3780x - 1134 = 0.$$

If $-1 < \alpha \le A$, then the sum

$$T_n^{\alpha}(\phi) = \frac{1}{1+\alpha} + \frac{\cos\phi}{1+\alpha} + \frac{\cos 2\phi}{2+\alpha} + \frac{\cos 3\phi}{3+\alpha} + \cdots + \frac{\cos n\phi}{n+\alpha}, \tag{1.5}$$

is non-negative for $0 \le \phi \le 2\pi$ and for all $n \in \mathbb{N}$. However, if $\alpha > A$, then $T_3^{\alpha} < 0$ for some ϕ. The value of A is $4.5678018\ldots$ approximately.

In 1958, a surprising and quite deep result about the simultaneous positivity of a general class of cosine and sine sum was published by Vietoris [10], which can be stated as follows:

Theorem A. *Let $\{a_k\}_{k=0}^{\infty}$ be any non-increasing sequence of non-negative real numbers such that $a_0 > 0$ and*

$$2ka_{2k} \le (2k-1)a_{2k-1}, \quad k \ge 1. \tag{1.6}$$

Then for all positive integers n, we have

$$\sum_{k=0}^{n} a_k \cos k\theta > 0, \quad 0 \le \theta < \pi, \tag{1.7}$$

and

$$\sum_{k=1}^{n} a_k \sin k\theta > 0, \quad 0 < \theta < \pi. \tag{1.8}$$

Vietoris himself observed that (1.7) and (1.8) satisfy the special case $a_k = c_k$, where

$$c_{2k} = c_{2k+1} = \frac{(1/2)_k}{k!}, \quad k = 0, 1, 2, \ldots. \tag{1.9}$$

Here, by $(a)_k$ we mean the Pochhammer symbol, defined by

$$(a)_0 = 1, \quad \text{and} \quad (a)_k = a(a+1)\ldots(a+k-1) = \frac{\Gamma(k+a)}{\Gamma(a)}, \quad k = 1, 2, \ldots.$$

Inequalities (1.7) and (1.8) of Vietoris extend both (1.1) and (1.3). The significance of Theorem A was unknown till the appearance of the work of Askey and Steinig [11], where a simplified proof of Theorem A is given and further shown that this result has some nice applications in estimating the zeros of certain trigonometric polynomials. They also observed that these inequalities are better viewed in the context of more general inequalities concerning positive sums of Jacobi polynomials and they play a role in problems dealing with quadrature methods.

The cosine inequality (1.7) received considerable improvement in the last few years. Brown and Hewitt [12] have shown that (1.7) remains true if the condition (1.6) is replaced by $(2k+1)a_{2k} \le (2k)a_{2k-1}$, $k \ge 1$. In [13], Brown and Yin gave a further generalization of (1.7) by showing that it remains valid under the condition

$$(2k+\beta)a_{2k} \le (2k+\beta+1)a_{2k-1}, \quad k \ge 1, \text{ and } \beta \in (-1, 2]. \tag{1.10}$$

They also suggested two different (unrelated) directions of possibility of additional sharpening of their result, by raising the following two questions:

(1) Determine the maximum range of β in (1.10), for which (1.7) remains true.
(2) Modify the sequence c_k of (1.9), by taking

$$c_{2k} = c_{2k+1} = \frac{(1-\alpha)_k}{k!}, \quad k = 0, 1, 2, \ldots,$$

and determine the best possible range of α for which all the cosine sums in (1.7) are positive.

In fact, it is expected in [13] that the upper bound for β in (1.10) will be less than 2.34. The complete answers for the above two questions were given in [14]. A similar result with an independent proof can also be found in [15]. In [16], a systematic account of these new results which ensure the positivity and boundedness of partial sums of cosine or sine series were discussed.

On the other hand, the inequality (1.8) does not have much more improvement. It turns out that inequality (1.8) is the best possible in the sense that, if the condition (1.6) is weakened then the corresponding sine sums are not everywhere positive in $(0, \pi)$. In a remarkable result, Belov [17] obtained a necessary and sufficient condition on the coefficients $\{a_k\}$ which extend both Vietoris' and Brown–Hewitt's [12] results. We state this result as a lemma because of its importance in the present work.

Lemma 1.2. *Let a_k, $k = 0, 1, 2, \ldots$ be any decreasing sequence of positive real numbers, then the condition*

$$\sum_{k=1}^{n} (-1)^{k-1} k a_k \ge 0, \quad \forall n \ge 2, a_1 > 0, \tag{1.11}$$

is necessary and sufficient for the validity of the inequality

$$\sum_{k=1}^{n} a_k \sin k\theta > 0, \quad \forall n \in \mathbb{N},\ 0 < \theta < \pi.$$

Moreover, condition (1.11) *implies that*

$$\sum_{k=1}^{n} a_k \cos k\theta > 0, \quad \forall n \in \mathbb{N},\ 0 < \theta < \pi.$$

Note that the sequence of the coefficients of the sums (1.2) does not satisfy the condition (1.11).

Several applications that have used the generalizations of (1.1) and (1.3) are available in the literature and have led to a deeper understanding of these results. Similarly, a variety of problems can be reduced to positivity results for trigonometric or other orthogonal sums of this type. Indeed, these inequalities have remarkable applications in the theory of Fourier series, summability theory, approximation theory, positive quadrature methods, the theory of univalent functions and many others. We refer the reader to the recently published research articles [18,19,14,20–22] and the references therein for some new results on positive trigonometric sums including refinements and extensions of (1.1) and (1.3) and various applications. We also note that positivity results for trigonometric sums and geometric function theory have been closely related subjects over the past century. Both areas have contributed to each other and this paper intends to present few more results of this interplay.

This paper is organized as follows. In Section 2 key lemmas required for the work given in the paper, main results which are generalizations of previously obtained results and their deductions are given. In Section 3 applications of our results to the results in geometric function theory (GFT) are given. In Section 4, using results from earlier sections, we find the geometric properties of certain type of Cesáro means and compare with earlier known results.

2. Preliminaries and main results

In this section, we deal with the partial sums of two important trigonometric (cosine and sine) series. Among various results in this section, we generalize the following result given in [23].

Lemma 2.1 (*[23]*). *Let $\alpha \geq 0$, $b_0 = 2$, $b_1 = 1$ and $b_k = \frac{1}{k+\alpha}$ for $k \in \mathbb{N}$, $k \geq 2$. Then for all $0 < \phi < \pi$ and for all $n \in \mathbb{N}$, the following inequalities hold:*

$$\frac{b_0}{2} + \sum_{k=1}^{n} b_k \cos k\phi > 0 \quad \textit{and} \quad \sum_{k=1}^{n} b_k \sin k\phi > 0.$$

Though an independent proof of Lemma 2.1 is given in [23], one can easily prove that

$$1 + \sum_{k=2}^{n} (-1)^{k-1} \frac{k}{k+\alpha} > 0, \quad \forall \alpha \geq 0,\ n \in \mathbb{N}$$

and therefore Lemma 2.1 directly follows from Lemma 1.2.

Abel's transformation or summation by parts is a standard technique in obtaining positivity results for trigonometric sums. We state this as a lemma.

Lemma 2.2. *Let $\{b_k\}_{k=0}^{\infty}$ and $\{c_k\}_{k=0}^{\infty}$ be two sequences of real numbers, then*

$$\sum_{k=0}^{n} b_k c_k = \sum_{k=0}^{n-1} \left(\Delta b_k \sum_{j=0}^{k} c_j \right) + b_n \sum_{k=0}^{n} c_k$$

where $\Delta b_k = b_k - b_{k+1}$.

Now we state a generalization of Lemma 2.1, which can be obtained from a careful manipulation of Lemma 2.1 and the technique given in Lemma 2.2.

Theorem 2.1. *Let $\alpha \geq 0$, $\gamma \geq 1$, $b_0 = 2$, $b_1 = 1$ and $b_k = \frac{1}{(k+\alpha)^{\gamma}}$ for $k \in \mathbb{N}$, $k \geq 2$. Then for all $0 < \phi < \pi$ and for all $n \in \mathbb{N}$, the following inequalities hold:*

$$\frac{b_0}{2} + \sum_{k=1}^{n} b_k \cos k\phi > 0 \quad \textit{and} \quad \sum_{k=1}^{n} b_k \sin k\phi > 0.$$

Proof. Since

$$\begin{aligned}\frac{b_0}{2} + \sum_{k=1}^{n} b_k \cos k\phi &= \left(1 - \frac{1}{(2+\alpha)^{\gamma-1}}\right)(1+\cos\phi) + \frac{1}{(n+\alpha)^{(\gamma-1)}} \left(1 + \cos\phi + \sum_{k=2}^{n} \frac{\cos k\phi}{(k+\alpha)}\right) \\ &\quad + \sum_{k=2}^{n-1} \left[\left(\frac{1}{(k+\alpha)^{(\gamma-1)}} - \frac{1}{(k+1+\alpha)^{(\gamma-1)}} \right) \left(1 + \cos\phi + \sum_{j=2}^{k} \frac{\cos j\phi}{(j+\alpha)}\right) \right]\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{n} b_k \sin k\phi &= \left(1 - \frac{1}{(2+\alpha)^{\gamma-1}}\right)\sin\phi + \frac{1}{(n+\alpha)^{(\gamma-1)}} \left(\sin\phi + \sum_{k=2}^{n} \frac{\sin k\theta}{(k+\alpha)}\right) \\ &\quad + \sum_{k=2}^{n-1} \left[\left(\frac{1}{(k+\alpha)^{(\gamma-1)}} - \frac{1}{(k+1+\alpha)^{(\gamma-1)}} \right) \left(\sin\phi + \sum_{j=2}^{k} \frac{\sin j\phi}{(j+\alpha)}\right) \right],\end{aligned}$$

both are positive by the given hypothesis and Lemma 2.1. □

The following result which is a consequence of Theorem 2.1 can be obtained by applying Lemma 2.2. This result has many interesting applications, some of them are given in Sections 3 and 4.

Corollary 2.1. *Let $\alpha \geq 0$, $\gamma \geq 1$ and $a_0, a_1, \ldots$ be a sequence of positive numbers such that, for all $k \geq 2$,*

$$(k+1+\alpha)^{\gamma} a_{k+1} \leq (k+\alpha)^{\gamma} a_k \leq \cdots \leq (2+\alpha)^{\gamma} a_2 \leq a_1 \leq \frac{a_0}{2},$$

then for all $0 < \phi < \pi$ and for all $n \in \mathbb{N}$, the following inequalities hold:

1. $\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos k\phi > 0$.
2. $\sum_{k=1}^{n} a_k \sin k\phi > 0$.

Our next result is a generalization of the following Lemma 2.3.

Lemma 2.3 ([24]). *For every positive integer n and for $0 < \theta < \pi$, we have*

$$\frac{d}{d\theta}\left[\cos\frac{\theta}{2}\left(1+\sum_{k=1}^{n}\frac{\cos k\theta}{k^{\gamma}}\right)\right] < 0, \tag{2.1}$$

when $\gamma \geq 1$. This inequality fails to hold for appropriate n and θ, when $0 < \gamma < 1$.

Theorem 2.2. *Let $\alpha \geq 0$, $\gamma \geq 0$. Then, for every positive integer n and for $0 < \theta < \pi$, we have*

$$\frac{d}{d\theta}\left[\cos\frac{\theta}{2}\left(1+\cos\theta+\sum_{k=2}^{n}\frac{\cos k\theta}{k(k+\alpha)^{\gamma}}\right)\right] < 0. \tag{2.2}$$

Proof. Inequality (2.2) is equivalent to

$$\frac{1}{2}\sin(\theta/2)\left(1+\cos\theta+\sum_{k=2}^{n}\frac{\cos k\theta}{k(k+\alpha)^{\gamma}}\right)+\cos\frac{\theta}{2}\left(\sin\theta+\sum_{k=2}^{n}\frac{\sin k\theta}{(k+\alpha)^{\gamma}}\right) > 0. \tag{2.3}$$

By Theorem 2.1, $\sin\theta + \sum_{k=2}^{n}\frac{\sin k\theta}{(k+\alpha)^{\gamma}} > 0$ when $\alpha \geq 0$ and $\gamma \geq 1$. Hence the inequality (2.3) will be true if we show that

$$1+\cos\theta+\sum_{k=2}^{n}\frac{\cos k\theta}{k(k+\alpha)^{\gamma}} > 0.$$

The left hand side of the above inequality can also be written as

$$\begin{aligned}1+\cos\theta+\sum_{k=2}^{n}\frac{\cos k\theta}{k(k+\alpha)^{\gamma}} &= \left(1-\frac{1}{(1+\alpha)^{\gamma}}\right)(1+\cos\theta)+\frac{1}{(n+\alpha)^{\gamma}}\left(1+\sum_{j=1}^{n}\frac{\cos j\theta}{j}\right)\\ &\quad+\sum_{k=1}^{n-1}\left[\left(\frac{1}{(k+\alpha)^{\gamma}}-\frac{1}{(k+1+\alpha)^{\gamma}}\right)\left(1+\sum_{j=1}^{k}\frac{\cos j\theta}{j}\right)\right] > 0.\end{aligned}$$

Since $1+\sum_{k=1}^{n}\frac{\cos k\theta}{k} > 0$ by Theorem 2.1 and $\frac{1}{(k+\alpha)^{\gamma}} > \frac{1}{(k+1+\alpha)^{\gamma}}$ for $\alpha \geq 0$ and $\gamma \geq 1$, we have the positivity of (2.3).

For $\gamma = 0$, (2.2) is an immediate consequence of (2.1). Let $0 < \gamma < 1$. Write $a_0 = a_1 = 1$ and $a_k = \frac{1}{(k+\alpha)^{\gamma}}$, $k \geq 2$. By Lemma 2.2 we have

$$\sigma_n(\theta) := 1+\cos\theta+\sum_{k=2}^{n}a_k\frac{\cos k\theta}{k} = \sum_{k=1}^{n-1}(a_k-a_{k+1})S_k(\theta)+a_nS_n(\theta),$$

where $S_1(\theta) = 1+\cos\theta$, $S_k(\theta) = 1+\sum_{j=1}^{k}\frac{\cos j\theta}{j}$, $k \geq 2$.

Now applying Lemma 2.3 we obtain

$$\frac{d}{d\theta}\left(\sigma_n(\theta)\cos\frac{\theta}{2}\right) = \sum_{k=1}^{n-1}(a_k-a_{k+1})\frac{d}{d\theta}\left(S_k(\theta)\cos\frac{\theta}{2}\right)+a_n\frac{d}{d\theta}\left(S_n(\theta)\cos\frac{\theta}{2}\right) < 0,$$

for $0 < \theta < \pi$ and the proof is complete. □

3. Application in GFT

As usual, by $\mathcal{A}$ we mean the class of analytic functions f in the unit disk $\mathbb{D} = \{z : |z| < 1\}$, normalized by the condition $f(0) = 0 = f'(0) - 1$ and $\mathcal{S} = \{f \in \mathcal{A} : f \text{ is univalent in } \mathbb{D}\}$. A function $f \in \mathcal{S}$ is said to be starlike and convex of order μ $(0 \leq \mu < 1)$, respectively, if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \mu \quad \text{and} \quad \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \mu.$$

These classes are denoted by $\mathcal{S}^*(\mu)$ and $\mathcal{C}(\mu)$ respectively. Note that, $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ and $\mathcal{C}(0) \equiv \mathcal{C}$, respectively, are the well-known subclasses of S that map $\mathbb{D}$ onto domains that are starlike with respect to origin and convex. Let $\mathcal{T}_{\mathcal{R}}$ be the subclass of $\mathcal{S}$, consisting of all typically real functions, i.e, all $f \in \mathcal{S}$ such that $\operatorname{Im} f(z)\operatorname{Im}(z) > 0$. Another important class required for our discussion is the class of close-to-convex functions of order μ with respect to a fixed starlike function $g(z)$ and given by the analytic condition

$$\operatorname{Re} e^{i\eta}\left(\frac{zf'(z)}{g(z)}-\mu\right) > 0, \quad g \in \mathcal{S}^*, \ z \in \mathbb{D}, \tag{3.1}$$

for some real $\eta \in (-\pi/2, \pi/2)$. The family of all close-to-convex functions of order μ, relative to $g \in \mathcal{S}^*$ is denoted by $\mathcal{K}_g(\mu)$. If there is no specification about the function g is given, then $\mathcal{K}_g(\mu)$ is denoted as $\mathcal{K}$. Note that for $0 \le \mu < 1$, each function in $\mathcal{K}_g(\mu)$ is univalent in $\mathbb{D}$. For a particular choice of g, we get particular classes of $\mathcal{K}_g(\mu)$. We list here only the classes that are needed for our results.

(i) $g(z) = z \Longrightarrow \mathcal{K}_z(\mu) =: \mathcal{R}(\mu) = \{f \in \mathcal{A} : \operatorname{Re} f'(z) > \mu\}$
(ii) $g(z) = \frac{z}{(1-z)} \Longrightarrow \mathcal{K}_1(\mu) := \{f \in \mathcal{A} : \operatorname{Re}((1-z)f'(z)) > \mu\}$
(iii) $g(z) = \frac{z}{(1-z^2)} \Longrightarrow \mathcal{K}_2(\mu) := \{f \in \mathcal{A} : \operatorname{Re}((1-z^2)f'(z)) > \mu\}$

where η given in (3.1) is taken as zero. Further, we have $\mathcal{R}(0) := \mathcal{R}$, $\mathcal{K}_1(0) := \mathcal{K}_1$ and $\mathcal{K}_2(0) := \mathcal{K}_2$.

The following are the necessary and sufficient conditions for f to be in $\mathcal{K}_2$.

Lemma 3.1. *$f \in \mathcal{A}$ has real coefficients and $f(z) \in \mathcal{K}_2$ if, and only if, $zf'(z)$ is typically real.*

Proof. A function $f \in \mathcal{A}$ is said to be convex in the direction of the imaginary axis [25] if every line parallel to the imaginary axis either intersects $f(\mathbb{D})$ in an interval or does not intersect it at all. From the well-known result given in [26], it is clear that $f \in \mathcal{A}$ has real coefficients and is convex in the direction of imaginary axis if, and only if, $zf'(z)$ is typically real. It is also known that $f \in \mathcal{A}$ has real coefficients, then f is convex in the direction of imaginary axis if, and only if,

$$\operatorname{Re}((1-z^2)f'(z)) > 0 \quad \forall z \in \mathbb{D}$$

which means that $f \in \mathcal{K}_2$ and the proof is complete. □

Remark 3.1. The functions

$$z, \qquad \frac{z}{1 \pm z}, \qquad \frac{z}{1 \pm z^2}, \qquad \frac{z}{(1 \pm z)^2}, \qquad \frac{z}{1 \pm z + z^2} \tag{3.2}$$

are the only nine functions which are starlike univalent and have integer coefficients in $\mathbb{D}$, (see [27] for details). We note that, it is easy to give sufficient conditions for f to be close-to-convex, in terms of the Taylor coefficients of f, at least when the corresponding starlike function $g(z)$ takes one of the above forms.

For the interested reader on details regarding these classes and the corresponding results, we refer to [25,28–30].

Theorem 3.1. *Let $0 \le \mu < 1$ and $f \in \mathcal{A}$ be such that $f'(z)$ and $f'(z) - \mu\frac{f(z)}{z}$ are typically real in $\mathbb{D}$. Further, if $\operatorname{Re} f'(z) > 0$ and $\operatorname{Re}(f'(z) - \mu\frac{f(z)}{z}) > 0$, then $f \in \mathcal{S}^*(\mu)$.*

Proof. The result for $\mu = 0$ is given in [31]. It remains to prove the result for the case $0 < \mu < 1$. It is enough to prove that $\operatorname{Re}\frac{zf'(z)/f(z)-\mu}{(1-\mu)} > 0$. Consider

$$\frac{(1-\mu)f(z)}{(zf'(z) - \mu f(z))} = \int_0^1 \frac{(1-\mu)f'(tz)}{\left(f'(z) - \mu\frac{f(z)}{z}\right)}dt = \int_0^1 \frac{(1-\mu)f'(tz)}{g(z)}dt \tag{3.3}$$

where,

$$g(z) = f'(z) - \mu\frac{f(z)}{z}.$$

Note that both $f'(tz)$ and $g(z)$ are in same half plane. For, if $\operatorname{Im} z > 0 (< 0)$, then both functions $f'(tz)$ and $g(z)$ being typically real, their values will lie in the same plane, viz., upper half (lower half) plane. Further, as $\operatorname{Re} f'(z) > 0$ and $\operatorname{Re}(f'(z) - \mu\frac{f(z)}{z}) > 0$, both $f'(tz)$ and $g(z)$ are also in the right half plane. Therefore,

$$\operatorname{Re}\frac{(1-\mu)f(z)}{(zf'(z) - \mu f(z))} > 0 \Longrightarrow \operatorname{Re}\frac{zf'(z)/f(z) - \mu}{(1-\mu)} > 0,$$

and the proof is complete. □

Remark 3.2. $\operatorname{Re} f'(z) > 0$ is the condition for $f(z) \in \mathcal{R}$. Further, by the definition of typically real function we have $\operatorname{Im} f(z)\operatorname{Im}(z) > 0$. Hence, under the hypothesis of Theorem 3.1,

$$\operatorname{Re}(1-z)f'(z) = \operatorname{Re}(1-z)\operatorname{Re} f'(z) + \operatorname{Im} f(z)\operatorname{Im}(z) > 0$$

which implies $f(z) \in \mathcal{K}_1$. Therefore, the result of Theorem 3.1 is true for $f(z) \in \mathcal{S}^*(\mu) \cap \mathcal{R} \cap \mathcal{K}_1$. In fact, the same remark also holds good for the following theorem.

Theorem 3.2. *Let $\alpha \geq 0$, $\gamma \geq 1$, $a_1 = 1$ and $a_k > 0$ for $k \geq 2$. If*

$$(2-\mu)a_2 \leq (1-\mu)a_1, \quad (3-\mu)a_3 \leq \frac{1}{(2+\alpha)^\gamma}(2-\mu)a_2, \quad \text{and}$$
$$(k+1-\mu)a_{k+1} \leq \frac{(k-1+\alpha)^\gamma}{(k+\alpha)^\gamma}(k-\mu)a_k, \quad \text{for } k \geq 3.$$

Then, for $0 \leq \mu < 1$, $f(z) = \lim_{n\to\infty} f_n(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is starlike of order μ, where $f_n(z)$ is the n-th partial sum of $f(z)$.

Proof. Let

$$g_n(z) = f_n'(z) - \mu\frac{f_n(z)}{z} = (1-\mu) + \sum_{k=1}^{\infty}(k+1-\mu)a_k z^k = \frac{b_0}{2} + \sum_{k=1}^{\infty} b_k z^k$$

where, $b_0 = 2(1-\mu)$ and $b_k = (k+1-\mu)a_k$, $\forall k \geq 1$. Now, by means of a simple calculation, we can establish the fact that, with the given hypothesis, $\{b_k\}$ satisfies the conditions of Theorem 2.1, which implies $\operatorname{Re} g_n(z) > 0$ in $\mathbb{D}$ and $\operatorname{Im} g_n(z) > 0$, if $\operatorname{Im} z > 0$. Again by reflection principle $\operatorname{Im} g_n(z) < 0$, if $\operatorname{Im} z < 0$. Hence $g_n(z)$ is typically real function. Using the fact,

$$\frac{1-\mu}{2-\mu} \leq \frac{1}{2}, \qquad \frac{k-\mu}{k+1-\mu} < 1, \quad \forall k \geq 2,$$

and Theorem 2.1, one can easily show that, under a given hypothesis, $\operatorname{Re} f_n'(z) > 0$ and $f_n'(z)$ is typically real in $\mathbb{D}$. We conclude the proof by Theorem 3.1 and using the fact that the family of starlike functions is normal. $\square$

Similarly by proving $zf'(z)$ is typically real (or in other sense $f(z) \in \mathcal{K}_2$) and using Lemma 3.1, the following result can be obtained.

Theorem 3.3. *Let $\alpha \geq 0$, $\gamma \geq 1$, $a_1 = 1$ and $a_k > 0$ for $k \geq 2$. If*

$$(k+1+\alpha)^\gamma (k+1)a_{k+1} \leq (k+\alpha)^\gamma k a_k \leq \cdots \leq (2+\alpha)^\gamma 2a_2 \leq 1, \tag{3.4}$$

is true for all $k \geq 2$, then $f_n(z)$ and $f(z)$ belongs to $\mathcal{K}_2$, where $f_n(z) = z + \sum_{k=2}^{n} a_k z^k$ and $f(z) = \lim_{n\to\infty} f_n(z) = z + \sum_{k=2}^{\infty} a_k z^k$.

Corollary 3.1. *Let $\alpha = 0$, $\gamma = 1$, $a_1 = 1$ and $a_k > 0$ for $k \geq 2$. If*

$$(k+1)^2 a_{k+1} \leq k^2 a_k \leq \cdots \leq 4a_2 \leq 1,$$

is true for all $k \geq 2$, then $f_n(z)$ and $f(z)$ belongs to $\mathcal{K}_2$, where $f_n(z) = z + \sum_{k=2}^{n} a_k z^k$ and $f(z) = \lim_{n\to\infty} f_n(z) = z + \sum_{k=2}^{\infty} a_k z^k$.

Remark 3.3. Corollary 3.1 is an immediate consequence of Theorem 3.3. But, even by considering (3.4) as a decreasing sequence, it is not possible to get Theorem 3.3 from Corollary 3.1. We support our claim by the following example.

Example 3.1. let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, with

$$2(2+\alpha)^\gamma a_2 \leq 1, \qquad a_{k+1} = \frac{k(k+\alpha)^\gamma}{(k+1)(k+1+\alpha)^\gamma} a_k, \quad \forall k \geq 2,\ \alpha > 0,\ \gamma \geq 1.$$

Then by Theorem 3.3, $f(z)$ belongs to $\mathcal{K}_2$. But for all $\alpha > 0$, $\gamma \geq 1$, this fact cannot be deduced from Corollary 3.1. Because

1. For $\alpha > 0$ and $\gamma = 1$, and $k \geq 2$, we have

$$\begin{aligned}(k+1)^2 a_{k+1} - k^2 a_k &= \frac{k(k+1)(k+\alpha)}{(k+1+\alpha)} a_k - k^2 a_k \\ &= [(k+1)(k+\alpha) - k(k+1+\alpha)]\frac{k a_k}{(k+1+\alpha)} \\ &= \frac{\alpha k a_k}{(k+1+\alpha)} > 0.\end{aligned}$$

2. For $\alpha = 2$ and $\gamma = \frac{3}{2}$, and for $k \geq 2$, we have

$$\begin{aligned}(k+1)^2 a_{k+1} - k^2 a_k &= \frac{k(k+1)(k+2)^{\frac{3}{2}}}{(k+3)^{\frac{3}{2}}} a_k - k^2 a_k \\ &= [(k+1)(k+2)^{\frac{3}{2}} - k(k+3)^{\frac{3}{2}}] \frac{k a_k}{(k+3)^{\frac{3}{2}}}\end{aligned}$$

which is positive for at least some k. For example, take $k = 2, 3$.

4. Application to Cesàro means

The n-th Cesàro means of order β of $f(z) \in \mathcal{A}$ is given by

$$\sigma_n^\beta(z, f) = \sum_{k=1}^{n} \frac{A_{n-k}^\beta}{A_n^\beta} a_k z^k \tag{4.1}$$

for all $n \in \mathbb{N}$ and $\beta > -1$, where

$$A_0^\beta = 1 \quad \text{and} \quad A_k^\beta = \frac{(k+\beta)}{k} A_{k-1}^\beta, \quad \forall k \geq 1.$$

In particular, we have

$$\sigma_n^\beta(z) = \sum_{k=1}^{n} \frac{A_{n-k}^\beta}{A_n^\beta} z^k. \tag{4.2}$$

Note that $\sigma_n^\beta(z, f) = \sigma_n^\beta(z) * f(z)$, where $*$ denotes the Hadamard product or convolution, defined as $(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k$, where $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ for $z \in \mathbb{D}$. For details about these convolution techniques and the corresponding properties related to the class $\mathcal{S}$, we refer the interested reader to [25,32].

Among the results available in the literature regarding the univalence of $\sigma_n^\beta(z)$, the following result due to Lewis [33] is the most general one.

Lemma 4.1 ([33]). *For $\beta \geq 1$ and $n \in N$ we have $\sigma_n^\beta(z) \in \mathcal{K}$.*

By the convolution property of convex functions and close-to-convex functions [32], we immediately have

Corollary 4.1. *For $\beta \geq 1$, $n \in N$ and $f \in \mathcal{C}$ we have $\sigma_n^\beta(z, f) \in \mathcal{K}$.*

In [34], the following result is established which describes the convexity of Cesàro means.

Lemma 4.2 ([34]). *Let $\beta \geq \alpha > 1$, $f \in \mathcal{C}_{(3-\alpha)/2}$. Then for all $n \in N$:*

$$\frac{n+\beta}{n} \sigma_n^\beta(z, f) \in \mathcal{C}_{(3-\alpha)/2}.$$

The corresponding result holds if $\mathcal{C}_{(3-\alpha)/2}$ is replaced by $\mathcal{S}^*_{(3-\alpha)/2}$ or $\mathcal{K}_{(3-\alpha)/2}$.

The following result is immediate.

Corollary 4.2. *Let $\beta \geq 3$, $f \in \mathcal{C}/\mathcal{S}^*/\mathcal{K}$. Then for all $n \in N$:*

$$\frac{n+\beta}{n} \sigma_n^\beta(z, f) \in \mathcal{C}/\mathcal{S}^*/\mathcal{K}.$$

Note that, $\frac{n+\beta}{n} \sigma_n^\beta(z, f)$ has the representation

$$\frac{n+\beta}{n} \sigma_n^\beta(z, f) = zF(1, -n; -n-\beta; z) * f(z)$$

where $F(a, b; c; z) :=_2 F_1(a, b; c; z)$ is the well-known Gaussian hypergeometric function. We refer the interested reader to [35,36] for background information on hypergeometric functions. The geometric properties such as univalency, close-to-convexity, starlikeness and convexity of $F(a, b; c; z)$ and $zF(a, b; c; z)$ are interesting questions at present for many researchers. For example, see [28] and references therein. Hence in this section, we are mainly interested in the following problems.

Problem 4.1. Is it possible that $f(z) \notin \mathcal{K}(\mathcal{S}^*)$, but $\frac{n+\beta}{n}\sigma_n^{\beta}(z,f) \in \mathcal{K}(\mathcal{S}^*)$?

Problem 4.2. Under what condition (*s*) Corollary 4.2 is true for some $\beta < 3$?

Theorem 4.1. *Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of positive real numbers such that $a_1 = 1$ and $(n-1+\beta)a_1 \geq 2(n-1)a_2$, $n \in \mathbb{N}$. Suppose that, for $1 \leq \gamma < 2$ and $0 \leq \alpha \leq \frac{6}{\gamma+4}$,*

1. $(2-\gamma\alpha)(n-2+\beta)a_2 \geq 2^{\gamma}(n-2)3a_3$ *and*
2. $k(k-1+\alpha-\gamma)(n-k+\beta)a_k \geq (k-1+\alpha)(k+1)(n-k)a_{k+1}, \ \forall k \geq 3.$

Then, $\frac{n+\beta}{n}\sigma_n^{\beta}(z,f)$ is close-to-convex with respect to both the starlike functions z and $z/(1-z)$, where $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Further that, for the same condition, $\frac{n+\beta}{n}\sigma_n^{\beta}(z,f)$ is starlike univalent.

Proof. Let

$$g_n(z) = \frac{n+\beta}{n}\sigma_n^{\beta}(z,f) = z + \sum_{k=2}^{n} \frac{n+\beta}{n}\frac{A_{n-k}^{\beta}}{A_n^{\beta}} a_k z^k. \tag{4.3}$$

Now, for $0 \leq r < 1$ and $0 \leq \theta \leq 2\pi$,

$$\operatorname{Re} g_n'(z) = \frac{b_0}{2} + \sum_{k=1}^{n-1} r^k b_k \cos k\theta \quad \text{and} \quad \operatorname{Im} g_n'(z) = \sum_{k=1}^{n-1} r^k b_k \sin k\theta,$$

where, $b_0 = 2$ and for all $k \geq 1$,

$$b_k = \frac{n+\beta}{n}\frac{A_{n-k-1}^{\beta}}{A_n^{\beta}}(k+1)a_{k+1} \Longrightarrow b_{k+1} = \frac{(n-k-1)}{(n-k-1+\beta)}\frac{(k+2)a_{k+2}}{(k+1)a_{k+1}} b_k.$$

For a given α, a straightforward computation gives

$$\begin{aligned}(2+\alpha)^{-\gamma} &= \frac{2-\alpha\gamma}{2^{(\gamma+1)}} + \frac{(\gamma,2)\alpha^2}{2^{(\gamma+3)}}\left[1-\frac{1}{6}(2+\gamma)\alpha\right] + \frac{(\gamma,4)\alpha^4}{3.2^{(\gamma+7)}}\left[1-\frac{1}{10}(4+\gamma)\alpha\right] \\ &\quad + \frac{(\gamma,6)\alpha^6}{45\cdot 2^{(\gamma+10)}}\left[1-\frac{1}{14}(6+\gamma)\alpha\right] + \cdots \\ &\geq \frac{1}{2^{(\gamma+1)}}[2-\alpha\gamma]\end{aligned}$$

and

$$\begin{aligned}\left(1+\frac{1}{k+\alpha}\right)^{-\gamma} &= \left[1-\frac{\gamma}{k+\alpha}\right] + \frac{(\gamma,2)}{2(k+\alpha)^2}\left[1-\frac{(2+\gamma)}{3(k+\alpha)}\right] + \frac{(\gamma,4)}{24(k+\alpha)^4}\left[1-\frac{(4+\gamma)}{5(k+\alpha)}\right] \\ &\quad + \frac{(\gamma,6)}{720(k+\alpha)^6}\left[1-\frac{(6+\gamma)}{7(k+\alpha)}\right] + \cdots \\ &\geq \left[1-\frac{\gamma}{k+\alpha}\right], \quad \forall k \geq 2.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{(2+\alpha)^{\gamma}}b_1 - b_2 &\geq \frac{1}{2^{(\gamma+1)}}[2-\alpha\gamma]b_1 - b_2 \\ &= \frac{b_1}{2^{(\gamma+1)}}\left[(2-\alpha\gamma) - 2^{\gamma}\frac{(n-2)}{(n-2+\beta)}\frac{3a_3}{a_2}\right] \geq 0.\end{aligned}$$

Similarly, for all $k \geq 3$,

$$\begin{aligned}\frac{(k+\alpha)^{\gamma}}{(k+1+\alpha)^{\gamma}}b_k - b_{k+1} &\geq \left[1-\frac{\gamma}{k+\alpha}\right]b_k - b_{k+1} \\ &= \frac{b_{k-1}}{k-1+\alpha}\left[(k-1+\alpha-\gamma) - (k-1+\alpha)\frac{(n-k)(k+1)a_{k+1}}{(n-k+\beta)ka_k}\right] \geq 0.\end{aligned}$$

Therefore, $\{b_k\}_{k=0}^{n-1}$ satisfies the hypothesis of Theorem 2.1. This clearly means that, from the minimum principle for harmonic functions,

$$\mathrm{Re}\, g_n'(z) > 0.$$

Similarly, we have either

$$\mathrm{Im}\, g_n'(z) \equiv 0 \quad \text{in } -1 < z = x + i0 < 1$$
$$\text{or} \quad \mathrm{Im}\, g_n'(z) > 0 \quad \text{in } \mathbb{D} \cap \{z : \mathrm{Im}\, z > 0\}.$$

The first case implies $g_n(z) = z$, and hence the conclusion. For the second case, using the reflection principle, we have

$$\mathrm{Im}\, g_n'(z) < 0 \quad \text{in } \mathbb{D} \cap \{z : \mathrm{Im}\, z < 0\}.$$

Now $g_n(z)$ is close-to-convex with respect to z, follows from $\mathrm{Re}\, g_n'(z) > 0$ in $\mathbb{D}$. On the other hand,

$$\mathrm{Re}\,(1-z)g_n'(z) = \mathrm{Re}\,(1-z)\mathrm{Re}\, g_n'(z) + \mathrm{Im}\, z\mathrm{Im}\, g_n'(z) > 0$$

implies that $g_n(z)$ is close-to-convex with respect to the starlike function $z/(1-z)$. By virtue of Theorem 3.1 (with $\mu = 0$), it is easy to conclude that $g_n(z)$ is starlike and this completes the proof. $\square$

Example 4.1. Let $1 \le \gamma < 2$, $0 \le \alpha \le \frac{6}{\gamma+4}$ and

$$\beta \ge \max_{n\ge 1} \left\{0, (n-2)\left(\frac{2^\gamma}{2-\alpha\gamma} - 1\right), \frac{\gamma(n-3)}{2+\alpha-\gamma}\right\}.$$

Then $\frac{n+\beta}{n}\sigma_n^\beta(z, -\log(1-z))$ is close-to-convex with respect to z and $z/(1-z)$. Under the same conditions, it is also starlike univalent.

Remark 4.1. It is well known that $f(z) = -\log(1-z)$ is close-to-convex with respect to starlike function $z/(1-z)$. Now taking $\alpha = 0$, $\gamma = 1$ in Example 4.1, we can say that for $1 \le n \le 5$ and $\beta \ge \beta'$, where $0 \le \beta' < 3$,

$$\frac{n+\beta}{n}\sigma_n^\beta(z, -\log(1-z))$$

is close-to-convex with respect to the starlike function $z/(1-z)$. Note that for this particular f and $1 \le n \le 5$, the same conclusion cannot be obtained by Corollary 4.2. But we have no information for other values of f in general. On the other hand, for $n \ge 6$, the order of Cesàro mean of $f(z) = -\log(1-z)$ given in Corollary 4.2 is better.

The following two examples will provide a partial answer to the Problem 4.1.

Example 4.2. For $1 \le \gamma < 2$ and $0 \le \alpha \le \frac{6}{\gamma+4}$, let

$$\beta \ge \max_{n\ge 1} \left\{3(n-1), (n-2)\left(\frac{2^\gamma}{2-\alpha\gamma} - 1\right), \frac{\gamma(n-3)}{2+\alpha-\gamma}\right\}.$$

Consider the function

$$f(z) = -\log(1-z) + \frac{3}{2}z^2 = z + 2z^2 + \sum_{k=3}^{\infty} \frac{z^k}{k}.$$

Clearly, $f'(z) = \frac{1}{1-z} + 3z$ and $(1-z)f'(z) = 1 + 3z(1-z)$. By easy computations, we have

$$(\mathrm{Re}\, f'(z))_{z=-2/3} < 0, \quad \text{and} \quad (\mathrm{Re}\,(1-z)f'(z))_{z=-2/3} < 0.$$

Hence $f(z)$ is not close-to-convex with respect to the starlike functions z and $z/(1-z)$, $z \in \mathbb{D}$. In fact, $f(z)$ is not even univalent in $\mathbb{D}$ as $f'(z) = 0$ at $z = \frac{3-\sqrt{21}}{6} \in \mathbb{D}$. But, with given β, the coefficient of $f(z)$ satisfies the hypothesis of Theorem 4.1. Hence $\frac{n+\beta}{n}\sigma_n^\beta(z, -\log(1-z) + \frac{3}{2}z^2)$ is close-to-convex with respect to z and $z/(1-z)$. It is also starlike univalent.

Example 4.3. For $1 \le \gamma < 2$ and $0 \le \alpha \le \frac{6}{\gamma+4}$,

$$\frac{n+\beta}{n}\sigma_n^\beta\left(z, -\log(1-z) + \frac{1}{2}z^4\right)$$

is close-to-convex with respect to z and $z/(1-z)$ $z \in \mathbb{D}$, where

$$\beta \ge \max_{n\ge 1} \left\{1, (n-2)\left(\frac{2^{\gamma+1}}{2-\alpha\gamma} - 1\right), (n-3)\left(\frac{3(2+\alpha)}{2+\alpha-\gamma} - 1\right), (n-4)\left(\frac{(3+\alpha)}{3(3+\alpha-\gamma)} - 1\right), \frac{\gamma(n-5)}{4+\alpha-\gamma}\right\}.$$

It is also starlike univalent.
Let $f(z) = -\log(1-z) + \frac{1}{2}z^4 = z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \frac{3}{4}z^4 + \sum_{k=5}^{\infty} \frac{z^k}{k}, z \in \mathbb{D}$.
Clearly, $f'(z) = \frac{1}{1-z} + 2z^3$ and $(1-z)f'(z) = 1 + 2z^3(1-z)$. By easy computations, we have

$$(\mathrm{Re}\, f'(z))_{z=-2/3} < 0 \quad \text{and} \quad (\mathrm{Re}\,(1-z)f'(z))_{z=-3/4} < 0.$$

Hence $f(z)$ is not close-to-convex with respect to both the starlike functions z and $z/(1-z)$, $z \in \mathbb{D}$. But with the given β, the coefficient of $f(z)$ satisfies the hypothesis of Theorem 4.1. Hence

$$\frac{n+\beta}{n}\sigma_n^{\beta}\left(z, -\log(1-z) + \frac{1}{2}z^4\right)$$

is close-to-convex with respect to z and $z/(1-z)$. It is also starlike univalent.

Theorem 4.2. *Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of positive real numbers with $a_1 = 1$ and satisfy the hypothesis of Theorem 4.1. Then $\frac{n+\beta}{n}\sigma_n^{\beta}(z, f) \in \mathcal{R}(\mu_1)$, where*

$$\mu_1 \le 1 - \frac{2(n-1)a_2}{n-1+\beta}.$$

Proof. Let for $0 \le r < 1$ and $0 \le \theta \le 2\pi$,

$$\mathrm{Re}\, \frac{g_n'(z) - \mu_1}{1-\mu_1} = \frac{b_0}{2} + \sum_{k=1}^{\infty} r^k b_k \cos k\theta,$$

where $b_0 = 2$ and $b_k = \frac{n+\beta}{n(1-\mu_1)} \frac{A_{n-k-1}^{\beta}}{A_n^{\beta}}(k+1)a_{k+1}$, $\forall k \ge 1$. Now

$$\mu_1 \le 1 - \frac{2(n-1)a_2}{n-1+\beta} \Longrightarrow b_0 \ge 2b_1.$$

We note that,

$$b_{k+1} = \frac{(n-k-1)}{(n-k-1+\beta)} \frac{(k+2)a_{k+2}}{(k+1)a_{k+1}} b_k, \quad \forall k \ge 1.$$

Using this, we obtain the remaining part of the proof, similar to the proof of Theorem 4.1. We omit details. Hence, by the virtue of Theorem 2.1, we have $\mathrm{Re}\, \frac{g_n'(z)-\mu_1}{1-\mu_1} > 0 \Longrightarrow \mathrm{Re}\, g_n'(z) > \mu_1$. $\square$

Theorem 4.3. *Let $\{a_k\}_{k=1}^{\infty}$ be sequence of positive real numbers such that $a_1 = 1$. If, for $1 \le \gamma < 2$ and $0 \le \alpha \le \frac{6}{\gamma+4}$,*

1. $(2-\gamma)(n-1+\beta)a_1 \ge (3-\gamma)(n-1)a_2$,
2. $(2-\alpha\gamma)(n-3+\beta)(3-\gamma)a_2 \ge 2^{\gamma+1}(n-3)(4-\gamma)a_3$, *and*
3. $(k+\alpha-\gamma)(n-k+\beta)(k+1-\gamma)a_k \ge (k+\alpha)(n-k)(k+2-\gamma)a_{k+1}$, $\forall k \ge 3$,

then, $\frac{n+\beta}{n}\sigma_n^{\beta}(z, f) \in S^*(\gamma-1)$.

Proof.

$$g_n(z) = \frac{n+\beta}{n}\sigma_n^{\beta}(z, f) = z + \sum_{k=2}^{n} d_k z^k, \tag{4.4}$$

where,

$$d_1 = 0, \quad \text{and} \quad d_k = \frac{n+\beta}{n} \frac{A_{n-k}^{\beta}}{A_n^{\beta}} a_k, \quad k \ge 2.$$

It is enough to prove that under the given hypothesis, $\{d_k\}$ satisfies the conditions of Theorem 3.2. By simple calculation, we have $(2-\gamma)d_1 \ge (3-\gamma)d_2$. Now

$$\begin{aligned}\frac{1}{(2+\alpha)^{\gamma+1}}(3-\gamma)d_2 - (4-\gamma)d_3 &\ge \frac{(2-\alpha\gamma)(3-\gamma)}{2^{\gamma+1}}d_2 - (4-\gamma)d_3 = \frac{d_2}{2^{\gamma+1}}\left[(2-\alpha\gamma)(3-\gamma) - (4-\gamma)\frac{d_3}{d_2}\right] \\ &= \frac{d_2}{2^{\gamma+1}}\left[(2-\alpha\gamma)(3-\gamma)(n-3+\beta) - (4-\gamma)(n-3)\frac{a_3}{a_2}\right] \ge 0.\end{aligned}$$

Again, for $k \geq 3$, by a simple calculation using the given hypothesis of the theorem, we have

$$(k+2-\gamma)d_{k+1} \leq \frac{(k-1+\alpha)^\gamma}{(k+\alpha)^\gamma}(k+1-\gamma)d_k, \quad \text{for } k \geq 3.$$

Hence, by Theorem 3.2, the result follows and the proof is complete. □

Note that for γ close to 2, the hypothesis of the above theorem restricts the coefficients and hence the coefficients $\{a_k\}$ are too small. Numerical experiments suggest that, for $1 \leq \gamma \leq 3/2$, the coefficients $\{a_k\}$ are comparatively bigger and can have further applications.

Theorem 4.4. *Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of positive real number such that $a_1 = 1$. Suppose that, for $1 \leq \gamma < 2$ and $0 \leq \alpha \leq \frac{6}{\gamma+4}$,*

1. $(n-1+\beta)a_1 \geq 2^{\gamma+2}(n-1)a_2$,
2. $k(k+\alpha-\gamma)(n-k+\beta)a_k \geq (k+\alpha)(n-k)(k+1)a_{k+1}, \ \forall k \geq 2.$

Then, $\frac{n+\beta}{n}\sigma_n^{\beta}(z,f)$ is close-to-convex with respect to the starlike function $\frac{z}{1-z^2}$.

Proof. Consider $g_n(z)$ given in (4.3). Then for $0 \leq r < 1$ and $0 \leq \theta \leq 2\pi$, we have

$$\operatorname{Im} zg_n'(z) = \sum_{k=1}^{n} r^k b_k \sin k\theta \tag{4.5}$$

where,

$$b_k = \frac{n+\beta}{n}\frac{A_{n-k}^{\beta}}{A_n^{\beta}}ka_k \Longrightarrow b_{k+1} = \frac{n-k}{n-k+\beta}\frac{(k+1)a_{k+1}}{ka_k}b_k; \quad \forall k \geq 1.$$

Now,

$$\begin{aligned}\frac{1}{(2+\alpha)^\gamma}b_1 - b_2 \geq \frac{1}{2^{(\gamma+1)}}[2-\alpha\gamma]b_1 - b_2 &= \frac{b_1}{2^{(\gamma+1)}}\left[(2-\alpha\gamma) - 2^{(\gamma+1)}\frac{b_2}{b_1}\right] \\ &= \frac{b_1}{2^{(\gamma+1)}}\left[(2-\alpha\gamma) - 2^{(\gamma+2)}\frac{(n-1)}{(n-1+\beta)}\frac{2a_2}{a_1}\right] \geq 0.\end{aligned}$$

Similarly, for $k \geq 2$,

$$\begin{aligned}\frac{(k+\alpha)^\gamma}{(k+1+\alpha)^\gamma}b_k - b_{k+1} \geq \left[1 - \frac{\gamma}{k+\alpha}\right]b_k - b_{k+1} &= \frac{b_k}{k+\alpha}\left[(k+\alpha-\gamma) - (k+\alpha)\frac{b_{k+1}}{b_k}\right] \\ = \frac{b_{k-1}}{k+\alpha}\left[(k+\alpha-\gamma) - (k+\alpha)\frac{(n-k)}{(n-k+\beta)}\frac{(k+1)a_{k+1}}{ka_k}\right],&\end{aligned}$$

which is non-negative. Now by the same argument as Theorem 4.1, $zg_n'(z)$ is typically real in $\mathbb{D}$. Hence by Lemma 3.1, we have the result. □

Remark 4.2. Since, for $\beta = 0$, $\frac{n+\beta}{n}\sigma_n^{\beta}(z,f) = f_n(z)$, and as the class of all close-to-convex functions with respect to a particular starlike function is a Normal family, $f(z) = \lim_{n\to\infty} f_n(z)$ is also close-to-convex with respect to the same starlike function. By the same argument $f(z)$ is also starlike when $f_n(z)$ is starlike.

Note that, with reference to Remark 3.1, we have no result for the close-to-convexity of $\frac{n+\beta}{n}\sigma_n^{\beta}(z,f)$ with respect to the starlike functions $z/(1-z)^2$ and $z/(1-z+z^2)$. Hence it will be interesting if one can find results in this direction. In particular, with respect to the starlike function $z/(1-z+z^2)$, there are not many results on close-to-convexity of functions $f \in \mathcal{A}$ in the literature.

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