The L^p boundary value problems on Lipschitz domains

Zhongwei Shen [1]

Department of Mathematics, University of Kentucky, Lexington, KY 40506, USA

Abstract

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. We develop a new approach to the invertibility on $L^p(\partial\Omega)$ of the layer potentials associated with elliptic equations and systems in Ω. As a consequence, for $n \geqslant 4$ and $2(n-1)/(n+1) - \varepsilon < p < 2$ where $\varepsilon > 0$ depends on Ω, we obtain the solvability of the L^p Neumann type boundary value problems for second order elliptic systems. The analogous results for the biharmonic equation are also established.

© 2007 Elsevier Inc. All rights reserved.

1. Introduction and statement of main results

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. The Dirichlet and Neumann problems for Laplace's equation in Ω with boundary data in $L^p(\partial\Omega)$ had been well understood more than twenty years ago. Indeed it is known that the L^p Dirichlet problem is uniquely solvable for $2 - \varepsilon < p \leqslant \infty$, while the L^p Neumann problem is uniquely solvable for $1 < p < 2 + \varepsilon$, where $\varepsilon > 0$ depends on n and Ω. Furthermore, the ranges of p's are sharp; and the solutions may be represented by the classical layer potentials [5,6,15,30]. Due to the lack of maximum principles and De Giogi–Nash Hölder estimates, the attempts to extend these results to second order elliptic systems as well as to higher order elliptic equations had been successful only in the case $n \geqslant 2$

E-mail address: shenz@ms.uky.edu.

[1] The author is supported in part by the NSF (DMS-0500257).

for p close to 2 [8,9,11–13,16,20,31,32], and in the lower dimensional case $n = 2$ or 3 for the sharp ranges of p's [7,18,19,21]. Recently in [25,26], we introduced a new approach to the L^p Dirichlet problem via L^2 estimates, reverse Hölder inequalities and a real variable argument. For second order elliptic systems as well as higher order elliptic equations, this led to the solvability of the L^p Dirichlet problem for $n \geqslant 4$ and $2 < p < 2(n-1)/(n-3) + \varepsilon$. In the case of elliptic equations of order 2ℓ, the upper bound of p is known to be sharp for $4 \leqslant n \leqslant 2\ell + 1$ and $\ell \geqslant 2$ [20,21].

The main purpose of this paper is to study the solvability of the L^p Neumann type boundary value problems for elliptic systems and higher order equations. We develop a new approach that can be used to establish the L^p invertibility of the trace operators $\pm (1/2)I + \mathcal{K}^*$ of the double layer potentials for a limited range of p's. This limited-range approach is essential to the higher order elliptic equations, as the L^p invertibility of $\pm (1/2)I + \mathcal{K}^*$ fails in general for large p in higher dimensions. By duality, the invertibility of $\pm (1/2)I + \mathcal{K}^*$ on L^p implies the invertibility of the Neumann trace operators $\pm (1/2)I + \mathcal{K}$ on $L^{p'}$ of the single layer potentials. As a consequence, we are able to solve the L^p Neumann type problems for p in the dual range $2(n-1)/(n+1) - \varepsilon_1 < p < 2$. We remark that in the lower dimensional case $n = 2$ or 3, our approach recovers, without the use of the Hardy spaces, the L^p solvability of the Neumann problem for $1 < p < 2$ obtained in [7] for elliptic systems. The analogous results for the biharmonic equation, however, are new even in the case $n = 2$ or 3. It is also interesting to point out that the approach we use here is in contrast with the method used in [6], where the operators $\pm (1/2)I + \mathcal{K}$ for the Neumann problem are shown to be invertible first and the invertibility of $\pm (1/2)I + \mathcal{K}^*$ for the Dirichlet problem is then established by duality.

This paper may be divided into three parts: elliptic systems, the biharmonic equation, and Laplace's equation. In the first part we consider the system of second order elliptic operators $(\mathcal{L}(\mathbf{u}))^k = -a_{ij}^{k\ell} D_i D_j u^\ell$ in Ω, where $D_i = \partial/\partial x_i$ and $k, \ell = 1, \ldots, m$. Let $N = (N_1, N_2, \ldots, N_n)$ be the unit outward normal to Ω and

$$\left(\frac{\partial \mathbf{u}}{\partial \nu} \right)^k = a_{ij}^{k\ell} \frac{\partial u^\ell}{\partial x_j} N_i \tag{1.1}$$

denote the conormal derivatives of $\mathbf{u}$ on $\partial\Omega$. We are interested in the L^p Neumann type boundary value problem

$$\begin{cases} \mathcal{L}(\mathbf{u}) = \mathbf{0} & \text{in } \Omega, \\ \dfrac{\partial \mathbf{u}}{\partial \nu} = \mathbf{f} \in L^p(\partial\Omega) & \text{on } \partial\Omega, \\ (\nabla \mathbf{u})^* \in L^p(\partial\Omega), \end{cases} \tag{1.2}$$

where $(\nabla \mathbf{u})^*$ denotes the nontangential maximal function of $\nabla \mathbf{u}$, and the boundary data $\mathbf{f}$ is taken in the sense of nontangential convergence. We will assume that $a_{ij}^{k\ell}$, $1 \leqslant i, j \leqslant n$, $1 \leqslant k$, $\ell \leqslant m$, are real constants and satisfy the symmetry condition $a_{ij}^{k\ell} = a_{ji}^{\ell k}$ and the strong ellipticity condition

$$\mu_0 |\xi|^2 \leqslant a_{ij}^{k\ell} \xi_i^k \xi_j^\ell \leqslant \frac{1}{\mu_0} |\xi|^2, \tag{1.3}$$

for some $\mu_0 > 0$ and any $\xi = (\xi_i^k) \in \mathbb{R}^{nm}$. Let $\| \cdot \|_p$ denote the norm in $L^p(\partial\Omega)$ with respect to the surface measure $d\sigma$ on $\partial\Omega$. The following is one of main results of the paper.

Theorem 1.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geqslant 4$, with a connected boundary. Then there exists $\varepsilon > 0$ depending only on n, m, μ_0 and Ω such that, given any $\mathbf{f} \in L^p(\partial\Omega)$ with $\int_{\partial\Omega} \mathbf{f} d\sigma = \mathbf{0}$ and*

$$\frac{2(n-1)}{n+1} - \varepsilon < p < 2, \tag{1.4}$$

the Neumann type problem (1.2) *has a unique* (*up to constants*) *solution $\mathbf{u}$. Furthermore, the solution $\mathbf{u}$ satisfies the estimate $\|(\nabla\mathbf{u})^*\|_p \leqslant C\|\mathbf{f}\|_p$ and may be represented by a single layer potential with a density in $L^p(\partial\Omega)$.*

Theorem 1.1 will be proved by the method of layer potentials. Let $\Gamma(x) = (\Gamma^{k\ell}(x))_{m \times m}$ denote the matrix of fundamental solutions for operator $\mathcal{L}$ on $\mathbb{R}^n$. For $\mathbf{g} \in L^p(\partial\Omega)$, let $\mathcal{S}(\mathbf{g})$ and $\mathcal{D}(\mathbf{g})$ denote the single and double layer potentials respectively with density $\mathbf{g}$, defined by

$$\left(\mathcal{S}(\mathbf{g})\right)^k(x) = \int_{\partial\Omega} \Gamma^{k\ell}(y-x) g^\ell(y) \, d\sigma(y), \tag{1.5}$$

$$\left(\mathcal{D}(\mathbf{g})\right)^k(x) = \int_{\partial\Omega} \left\{ \frac{\partial}{\partial\nu(y)} \Gamma_k(y-x) \right\}^\ell g^\ell(y) \, d\sigma(y), \tag{1.6}$$

where $\Gamma_k(x) = (\Gamma^{k1}(x), \dots, \Gamma^{km}(x))$ is the kth row of $\Gamma(x)$. Let $\mathbf{u} = \mathcal{S}(\mathbf{g})$ and $\mathbf{v} = \mathcal{D}(\mathbf{g})$, then $\mathcal{L}(\mathbf{u}) = \mathcal{L}(\mathbf{v}) = \mathbf{0}$ in $\mathbb{R}^n \setminus \partial\Omega$. Moreover,

$$\frac{\partial\mathbf{u}_+}{\partial\nu} = \left(\frac{1}{2}I + \mathcal{K}\right)\mathbf{g}, \qquad \frac{\partial\mathbf{u}_-}{\partial\nu} = \left(-\frac{1}{2}I + \mathcal{K}\right)\mathbf{g}, \tag{1.7}$$

$$\mathbf{v}_+ = \left(-\frac{1}{2}I + \mathcal{K}^*\right)\mathbf{g}, \qquad \mathbf{v}_- = \left(\frac{1}{2}I + \mathcal{K}^*\right)\mathbf{g}, \tag{1.8}$$

on $\partial\Omega$, where I denotes the identity operator, and $\pm$ indicate the nontangential limits taken from $\Omega_+ = \Omega$ and $\Omega_- = \mathbb{R}^n \setminus \overline{\Omega}$ respectively. We remark that in (1.7)–(1.8), $\mathcal{K}$ is a singular integral operator on $\partial\Omega$ and $\mathcal{K}^*$ is the adjoint of $\mathcal{K}$. By [4], $\mathcal{K}$ and $\mathcal{K}^*$ are bounded on $L^p(\partial\Omega)$, and $\|(\nabla\mathbf{u})^*\|_p + \|(\mathbf{v})^*\|_p \leqslant C\|\mathbf{g}\|_p$ for any $1 < p < \infty$. In view of the trace formulas (1.7), the L^p Neumann type problem (1.2) is reduced to that of the invertibility of the operator $(1/2)I + \mathcal{K}$ on $L^p(\partial\Omega)$ (modulo a finite dimensional subspace). Similarly, because of (1.8), one may solve the L^p Dirichlet problem

$$\begin{cases} \mathcal{L}(\mathbf{u}) = \mathbf{0} & \text{in } \Omega, \\ \mathbf{u} = \mathbf{f} \in L^p(\partial\Omega) & \text{on } \partial\Omega, \\ (\mathbf{u})^* \in L^p(\partial\Omega), \end{cases} \tag{1.9}$$

by showing that $-(1/2)I + \mathcal{K}^*$ is invertible on $L^p(\partial\Omega)$. This is the so-called method of layer potentials for solving boundary value problems.

For $n \geqslant 2$, the invertibility of $\pm(1/2)I + \mathcal{K}$ on $L^p(\partial\Omega)$ was indeed established in [9,12] (also see [11,16,17]) for $2 - \varepsilon < p < 2 + \varepsilon$, where $\varepsilon > 0$ depends on the Lipschitz character of Ω. To do this, the main step is to show that for suitable solutions of $\mathcal{L}(\mathbf{u}) = \mathbf{0}$ in $\mathbb{R}^n \setminus \partial\Omega$, one has

$$\left\| \frac{\partial \mathbf{u}_+}{\partial \nu} \right\|_2 \sim \|\nabla_t \mathbf{u}_+\|_2 \quad \text{and} \quad \left\| \frac{\partial \mathbf{u}_-}{\partial \nu} \right\|_2 + \|\mathbf{u}\|_2 \sim \|\nabla_t \mathbf{u}_-\|_2 + \|\mathbf{u}\|_2, \tag{1.10}$$

where $\nabla_t \mathbf{u}$ denotes the tangential derivatives of $\mathbf{u}$ on $\partial\Omega$. As in the case of Laplace's equation [30], the proof of (1.10) relies on the Rellich type identities.

If we let $\mathbf{u} = \mathcal{S}(\mathbf{g})$ in (1.10), since $\nabla_t \mathbf{u}_+ = \nabla_t \mathbf{u}_-$ a.e. on $\partial\Omega$, we obtain

$$\left\| \frac{\partial \mathbf{u}_+}{\partial \nu} \right\|_2 + \|\mathbf{u}\|_2 \sim \left\| \frac{\partial \mathbf{u}_-}{\partial \nu} \right\|_2 + \|\mathbf{u}\|_2. \tag{1.11}$$

It follows that

$$\|\mathbf{g}\|_2 \leqslant \left\| \frac{\partial \mathbf{u}_+}{\partial \nu} \right\|_2 + \left\| \frac{\partial \mathbf{u}_-}{\partial \nu} \right\|_2 \leqslant C \left\| \left(\pm \frac{1}{2} I + \mathcal{K} \right) \mathbf{g} \right\|_2 + C \|\mathcal{S}(\mathbf{g})\|_2. \tag{1.12}$$

This is essentially enough to deduce the invertibility of $\pm(1/2)I + \mathcal{K}$ and hence of $\pm(1/2)I + \mathcal{K}^*$ on $L^2(\partial\Omega)$, modulo some finite dimensional subspaces. By a perturbation argument of A.P. Calderón, the invertibility can be extended to $L^p(\partial\Omega)$ for p close to 2. As a consequence, the L^p Dirichlet and Neumann type problems are solved for $2 - \varepsilon < p < 2 + \varepsilon$.

For Laplace's equation on Lipschitz domains, the invertibility of the corresponding operators $\pm(1/2)I + \mathcal{K}$ on $L^p(\partial\Omega)$ was established for the sharp ranges of p's in [6] (the case $p = 2$ is in [30]). The method used in [6] relies on the classical Hölder estimates for solutions of second order elliptic equations of divergence form with bounded measurable coefficients. Because of this, the extension of the results in [6] to elliptic systems has only been successful in the lower dimensional case ($n = 2$ or 3) [7]. As we mentioned in the beginning of this section, we recently introduced a new approach to the L^p Dirichlet problem for $p > 2$ in [25,26]. Roughly speaking, this approach reduces the solvability of the L^p Dirichlet problem to a weak reverse Hölder inequality on $I(P, r)$ with exponent p for L^2 solutions whose Dirichlet data vanish on $I(P, 3r)$. Here $I(P, r) = B(P, r) \cap \partial\Omega$, where $P \in \partial\Omega$ and $0 < r < r_0$, is a surface ball on $\partial\Omega$. Combined with the $W^{1,2}$ regularity estimate $\|(\nabla\mathbf{u})^*\|_2 \leqslant C\|\nabla_t \mathbf{u}\|_2$, this allows us to establish the solvability of the L^p Dirichlet problem (1.9) for $n \geqslant 4$ and

$$2 < p < \frac{2(n-1)}{n-3} + \varepsilon_1. \tag{1.13}$$

In this paper we will show that if $\mathbf{v} = \mathcal{D}(\mathbf{g})$ is a double layer potential, then

$$\left\| (\mathbf{v})^* \right\|_p \sim \|\mathbf{v}_\pm\|_p, \tag{1.14}$$

for any p satisfying (1.13), where the nontangential maximal function $(\mathbf{v})^*$ is defined using nontangential approach regions from both sides of $\partial\Omega$. Since $g = \mathbf{v}_- - \mathbf{v}_+$ by (1.8), estimate (1.14) implies that $\pm(1/2)I + \mathcal{K}^*$ are invertible on $L^p(\partial\Omega)$. By duality, $\pm(1/2)I + \mathcal{K}$ are invertible on $L^p(\partial\Omega)$ for p in the dual range (1.4).

By a refinement of the approach used in [25,26], we may reduce the proof of (1.14) to the weak reverse Hölder inequality

$$\left\{\frac{1}{r^{n-1}}\int\limits_{I(P,r)}\left|(\mathbf{v})^*\right|^p d\sigma\right\}^{1/p}\leqslant C\left\{\frac{1}{r^{n-1}}\int\limits_{I(P,2r)}\left|(\mathbf{v})^*\right|^2 d\sigma\right\}^{1/2}, \tag{1.15}$$

where $\mathbf{v}=\mathcal{D}(\mathbf{g})$, and either $\mathbf{v}_+=\mathbf{0}$ or $\mathbf{v}_-=\mathbf{0}$ on $I(P,3r)$. The proof of (1.15) relies on applications of localized L^2 estimates (or Rellich identities) on the domains $B(P,r)\cap\Omega_\pm$. It also depends on the fact that

$$\frac{\partial\mathbf{v}_+}{\partial\nu}=\frac{\partial\mathbf{v}_-}{\partial\nu}\quad\text{on }\partial\Omega \tag{1.16}$$

for any double layer potential $\mathbf{v}$. This crucial fact allows us to estimate the L^2 norm of $\nabla\mathbf{v}_\pm$ on $I(P,r)$ by the L^2 norm of $\nabla_t\mathbf{v}_\mp$ on $I(P,2r)$ respectively, plus some lower order terms. See Lemma 2.4. We mention that the upper bound of p in (1.13) is dictated by the use of Sobolev inequality on $I(P,r)$. Whether this upper bound is necessary for the invertibility of $\pm(1/2)I+\mathcal{K}^*$ on $L^p(\partial\Omega)$ for second order elliptic systems remains open.

In this paper we also study the traction boundary value problem for the system of elastostatics

$$\begin{cases}\mu\Delta\mathbf{u}+(\lambda+\mu)\nabla(\operatorname{div}\mathbf{u})=\mathbf{0}&\text{in }\Omega,\\\lambda(\operatorname{div}\mathbf{u})N+\mu\big(\nabla\mathbf{u}+(\nabla\mathbf{u})^T\big)N=\mathbf{f}\in L^p(\partial\Omega),\\(\nabla\mathbf{u})^*\in L^p(\partial\Omega),\end{cases} \tag{1.17}$$

where $\mu>0$, $\lambda>-2\mu/n$ are Lamé constants, and T indicates the transpose of a matrix. One may put (1.17) in the general form of (1.2) with

$$a_{ij}^{k\ell}=\mu\delta_{ij}\delta_{k\ell}+\lambda\delta_{ik}\delta_{j\ell}+\mu\delta_{i\ell}\delta_{jk} \tag{1.18}$$

for $i,j,k,\ell=1,2,\ldots,n$. It is easy to verify that the coefficients satisfy the Legendre–Hadamard ellipticity condition

$$a_{ij}^{k\ell}\xi_i\xi_j\eta^k\eta^\ell\geqslant\mu|\xi|^2|\eta|^2\quad\text{for any }\xi,\eta\in\mathbb{R}^n. \tag{1.19}$$

However they do not satisfy the strong elliptic condition (1.3). Thus Rellich type identities alone are not strong enough to give estimate (1.10). Nevertheless, this difficulty was overcome in [9] by establishing a Korn type inequality on $\partial\Omega$. Consequently, the L^p traction problem (1.17) was solved in [9] for $|p-2|<\varepsilon$. In the case $n=2$ or 3, the problem was solved in [7] for the optimal range $1<p<2+\varepsilon$. Here we will show that with a few modifications, the proof of Theorem 1.1 may be used to solve the L^p traction problem for p in the same range given in (1.4). More specifically, let Ψ denote the space of vector valued functions $\mathbf{g}=(g^1,\ldots,g^n)$ on $\mathbb{R}^n$ satisfying $D_ig^j+D_jg^i=0$ for $1\leqslant i,j\leqslant n$. It is easy to show that $\mathbf{g}\in\Psi$ if and only if $\mathbf{g}(x)=Ax+\mathbf{b}$, where $\mathbf{b}\in\mathbb{R}^n$ and A is a real skew-symmetric matrix, $A^T=-A$. Let

$$L_\Psi^p(\partial\Omega)=\left\{\mathbf{f}\in L^p(\partial\Omega):\int\limits_{\partial\Omega}\mathbf{f}\cdot\mathbf{g}\,d\sigma=0\text{ for all }\mathbf{g}\in\Psi\right\}. \tag{1.20}$$

Theorem 1.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geqslant 4$, with a connected boundary. Then there exists $\varepsilon > 0$ depending only on λ, μ, n and Ω such that for any $\mathbf{f} \in L_\psi^p(\partial\Omega)$ with p satisfying* (1.4), *the traction problem* (1.17) *has a solution $\mathbf{u}$, unique up to elements of Ψ. Furthermore, the solution $\mathbf{u}$ satisfies the estimate $\|(\nabla\mathbf{u})^*\|_p \leqslant C\|\mathbf{f}\|_p$ and may be represented by a single layer potential with a density in $L^p(\partial\Omega)$.*

The general program we outlined above for the second order systems should apply to higher order elliptic equations and systems, once the L^2 invertibility of the layer potentials is established. In the second part of this paper, we study the biharmonic Neumann problem

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ \rho\Delta u + (1-\rho)\dfrac{\partial^2 u}{\partial N^2} = f \in L^p(\partial\Omega) & \text{on } \partial\Omega, \\ \dfrac{\partial}{\partial N}\Delta u + \dfrac{1}{2}(1-\rho)\dfrac{\partial}{\partial T_{ij}}\left(\dfrac{\partial^2 u}{\partial N \partial T_{ij}}\right) = \Lambda \in W_0^{-1,p}(\partial\Omega) & \text{on } \partial\Omega, \end{cases} \quad (1.21)$$

where $\frac{\partial}{\partial T_{ij}} = N_i D_j - N_j D_i$, and $W_0^{-1,p}(\partial\Omega)$ denotes the space of bounded linear functionals Λ on $W^{1,p'}(\partial\Omega)$ such that $\Lambda(1) = 0$. The L^p Neumann problem (1.21) was recently formulated and studied by G. Verchota in [32], where the solvability was established for $p \in (2 - \varepsilon, 2 + \varepsilon)$ by the method of layer potentials. The following is the second main result of the paper.

Theorem 1.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geqslant 4$, with a connected boundary. Let $(1/(1-n)) < \rho < 1$. Then there exists $\varepsilon > 0$ such that given any $f \in L^p(\partial\Omega)$ and $\Lambda \in W_0^{-1,p}(\partial\Omega)$ with $2(n-1)/(n+1) - \varepsilon < p < 2$, there exists a biharmonic function u, unique up to linear functions, satisfying* (1.21) *and $(\nabla\nabla u)^* \in L^p(\partial\Omega)$. Moreover, there exists a constant C depending only on n, p, ρ and Ω so that*

$$\left\|(\nabla\nabla u)^*\right\|_p \leqslant C\left\{\|\Lambda\|_{W^{-1,p}(\partial\Omega)} + \|f\|_p\right\}, \quad (1.22)$$

and the solution u may be represented by a single layer potential. If $n = 2$ or 3, above results hold for $1 < p < 2$.

We refer the reader to Remark 7.3 for the ranges of p's for which the L^p Dirichlet problem for the biharmonic equation is uniquely solvable. In particular the sharp ranges are known in the case $2 \leqslant n \leqslant 7$.

In the last part of this paper we apply the method used above for systems and the biharmonic equation to the classical layer potentials for Laplace's equation. This allows us to recover the sharp L^p results in [6], without the use of the Hardy spaces. In fact we are able to establish the following stronger result.

Theorem 1.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geqslant 3$, with a connected boundary. Then there exists $\delta > 0$ depending only on n and Ω, such that*

$$\frac{1}{2}I + \mathcal{K} : L_0^2\left(\partial\Omega, \frac{d\sigma}{\omega}\right) \to L_0^2\left(\partial\Omega, \frac{d\sigma}{\omega}\right),$$

$$-\frac{1}{2}I + \mathcal{K}^* : L^2(\partial\Omega, \omega\, d\sigma) \to L^2(\Omega, \omega\, d\sigma) \tag{1.23}$$

are isomorphisms for any $A_{1+\delta}$ weight ω on $\partial\Omega$.

We remark that the sharp L^p invertibility of $(1/2)I + \mathcal{K}$ and $-(1/2)I + \mathcal{K}^*$ follows from Theorem 1.4 by an extrapolation theorem, due to Rubio de Francia [22]. Theorem 1.4 allows us to solve the Neumann problem for Laplace's equation with boundary data in $L^2(\partial\Omega, \frac{d\sigma}{\omega})$. This, combined with the weighted regularity estimate in [24], shows that

$$\left\| \frac{\partial u}{\partial N} \right\|_{L^2(\partial\Omega, \frac{d\sigma}{\omega})} \sim \|\nabla_t u\|_{L^2(\partial\Omega, \frac{d\sigma}{\omega})}, \tag{1.24}$$

if $\Delta u = 0$ in Ω and $(\nabla u)^* \in L^2(\partial\Omega, \frac{d\sigma}{\omega})$ with $\omega \in A_{1+\delta}(\partial\Omega)$.

The paper is organized as follows. Throughout Sections 2–4, we will assume that the coefficients $a_{ij}^{k\ell}$ of $\mathcal{L}$ satisfy the symmetry condition $a_{ij}^{k\ell} = a_{ji}^{\ell k}$ and the strong ellipticity condition (1.3). In Section 2 we prove the reverse Hölder inequality (1.15). See Theorem 2.6. This is used in Section 3 to establish the invertibility of $\pm(1/2)I + \mathcal{K}^*$ on L^p. The proof of Theorem 1.1 is given in Section 4, while the proof of Theorem 1.2 can be found in Section 5. Sections 6 and 7 deal with the biharmonic equation. The corresponding reverse Hölder inequality for biharmonic functions is proved in Section 6. The proof of Theorem 1.3 is given in Section 7. Finally the classical layer potentials are studied in Section 8, where the proof of Theorem 1.4 can be found. We point out that the usual conventions on repeated indices and on constants are used throughout the paper.

2. Reverse Hölder inequalities

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Denote $\Omega_+ = \Omega$ and $\Omega_- = \mathbb{R}^n \setminus \overline{\Omega}$. For continuous function u in $\Omega_\pm$, the nontangential maximal function $(u)^*_\pm$ on $\partial\Omega$ is defined by

$$(\mathbf{u})^*_\pm(P) = \sup\{|\mathbf{u}(x)| : x \in \Omega_\pm \text{ and } x \in \gamma(P)\}, \tag{2.1}$$

where $\gamma(P) = \{x \in \mathbb{R}^n \setminus \partial\Omega : |x - P| < 2\,\mathrm{dist}(x, \partial\Omega)\}$.

Assume $0 \in \partial\Omega$ and

$$\Omega \cap B(0, r_0) = \{(x', x_n) \in \mathbb{R}^n : x_n > \psi(x')\} \cap B(0, r_0), \tag{2.2}$$

where $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function, and $\psi(0) = 0$. For $r > 0$, we let

$$I_r = \{(x', \psi(x')) \in \mathbb{R}^{n-1} : |x_1| < r, \dots, |x_{n-1}| < r\}, \tag{2.3}$$

and

$$\begin{aligned}
D_r^+ &= \{(x', x_n) : |x_1| < r, \dots, |x_{n-1}| < r, \ \psi(x') < x_n < \psi(x') + r\}, \\
D_r^- &= \{(x', x_n) : |x_1| < r, \dots, |x_{n-1}| < r, \ \psi(x') - r < x_n < \psi(x')\}.
\end{aligned} \tag{2.4}$$

Note that if $0 < r < cr_0$, then $I_r \subset \partial\Omega$ and $D_r^\pm \subset \Omega_\pm$.

We begin with a boundary Cacciopoli's inequality.

Lemma 2.1. *Suppose that* $\mathcal{L}(\mathbf{u}) = \mathbf{0}$ *in* $\Omega_\pm$ *and* $(\nabla \mathbf{u})^*_\pm \in L^2(I_{2r})$ *for some* $0 < 2r < cr_0$. *Then*

$$\int_{D_r^\pm} |\nabla \mathbf{u}|^2 \, dx \leqslant \frac{C}{r^2} \int_{D_{2r}^\pm} |\mathbf{u}|^2 \, dx + C \int_{I_{2r}} \left| \frac{\partial \mathbf{u}_\pm}{\partial \nu} \right| |\mathbf{u}_\pm| \, d\sigma. \tag{2.5}$$

Proof. The proof is rather standard. We first choose a nonnegative function $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi = 1$ in D_r^+, $\varphi = 0$ in $\Omega \setminus D_{2r}^+$ and $|\nabla \varphi| \leqslant C/r$. Let $a(\xi, \eta) = a_{ij}^{k\ell} \xi_i^k \eta_j^\ell$ for $\xi = (\xi_i^k)$, $\eta = (\eta_j^\ell) \in \mathbb{R}^{mn}$. It follows from the integration by parts that

$$\int_\Omega a(\xi, \xi) \varphi^2 \, dx = -2 \int_\Omega a(\xi, \eta) \varphi \, dx + \int_{\partial \Omega} \frac{\partial \mathbf{u}_+}{\partial \nu} \cdot \mathbf{u}_+ \varphi^2 \, d\sigma, \tag{2.6}$$

where $\xi = (\xi_i^k) = (\frac{\partial u^k}{\partial x_i})$ and $\eta = (\eta_j^\ell) = (u^\ell \frac{\partial \varphi}{\partial x_j})$. Since $a(\xi, \xi) \geqslant 0$ for any $\xi \in \mathbb{R}^{mn}$, by the Cauchy inequality, we have

$$\left| a(\xi, \eta) \right| \leqslant a(\xi, \xi)^{1/2} a(\eta, \eta)^{1/2} \leqslant \frac{1}{4} a(\xi, \xi) + a(\eta, \eta). \tag{2.7}$$

This, together with (2.6), gives

$$\int_\Omega a(\xi, \xi) \varphi^2 \, dx \leqslant 4 \int_\Omega a(\eta, \eta) \varphi \, dx + \int_{\partial \Omega} \frac{\partial \mathbf{u}_+}{\partial \nu} \cdot \mathbf{u}_+ \varphi^2 \, d\sigma. \tag{2.8}$$

Since $a(\xi, \xi) \geqslant \mu_0 |\nabla \mathbf{u}|^2$, estimate (2.5) for the case D_r^+ follows easily from (2.8). It is clear that the argument above also applies to the case D_r^-. □

Lemma 2.2. *Suppose that* $\mathcal{L}(\mathbf{u}) = \mathbf{0}$ *in* $\Omega_\pm$ *and* $(\nabla \mathbf{u})^*_\pm \in L^2(I_{2r})$ *for some* $0 < 2r < cr_0$. *Then*

$$\int_{I_r} |\nabla \mathbf{u}_\pm|^2 \, d\sigma \leqslant C \int_{I_{2r}} \left| \frac{\partial \mathbf{u}_\pm}{\partial \nu} \right|^2 \, d\sigma + \frac{C}{r} \int_{D_{2r}^\pm} |\nabla \mathbf{u}|^2 \, dx, \tag{2.9}$$

$$\int_{I_r} |\nabla \mathbf{u}_\pm|^2 \, d\sigma \leqslant C \int_{I_{2r}} |\nabla_t \mathbf{u}_\pm|^2 \, d\sigma + \frac{C}{r} \int_{D_{2r}^\pm} |\nabla \mathbf{u}|^2 \, dx, \tag{2.10}$$

where $\nabla_t \mathbf{u}$ *denotes the tangential derivatives of* $\mathbf{u}$ *on* $\partial \Omega$.

Proof. To show (2.9), we observe that the L^2 Neumann problem is solvable, uniquely up to constants, on $D_{sr}^\pm$ for any $1 < s < 3/2$. This yields

$$\int_{I_r} |\nabla \mathbf{u}_\pm|^2 \, d\sigma \leqslant \int_{\partial D_{sr}^\pm} |\nabla \mathbf{u}|^2 \, d\sigma \leqslant C \int_{\partial D_{sr}^\pm} \left| \frac{\partial \mathbf{u}}{\partial \nu} \right|^2 \, d\sigma$$

$$\leqslant C \int_{I_{2r}} \left| \frac{\partial \mathbf{u}_{\pm}}{\partial \nu} \right|^2 d\sigma + C \int_{\Omega_{\pm} \cap \partial D_{sr}^{\pm}} |\nabla \mathbf{u}|^2 d\sigma. \tag{2.11}$$

Estimate (2.9) now follows by integrating both sides of (2.11) with respect to s over interval $(1, 3/2)$. Similarly, estimate (2.10) follows by applying the regularity estimate

$$\int_{\partial D_{sr}^{\pm}} |\nabla \mathbf{u}|^2 d\sigma \leqslant C \int_{\partial D_{sr}^{\pm}} |\nabla_t \mathbf{u}|^2 d\sigma \tag{2.12}$$

for the Dirichlet problem on $D_{sr}^{\pm}$. We remark that the regularity estimate (2.12) and hence (2.10) in fact hold for elliptic systems satisfying the Legendre–Hadamard ellipticity condition (1.19) [11,13,16]. This will be used in the proof of Theorem 1.2. $\quad\square$

In order to handle the solid integrals like those in (2.9)–(2.10), we introduce a localized nontangential maximal function,

$$(\mathbf{u})_{\pm}^{*,r}(P) = \sup\{|\mathbf{u}(x)|: x \in \Omega_{\pm}, \ |x - P| < cr \text{ and } |x - P| < 2\operatorname{dist}(x, \partial\Omega)\} \tag{2.13}$$

where $c > 0$, depending on $\|\nabla\psi\|_{\infty}$ and n, is sufficiently small.

Lemma 2.3. *Let $\mathbf{u}$ be a continuous function on $D_{2r}^{\pm}$. Then*

$$\left\{ \frac{1}{r^n} \int_{\substack{x \in D_r^{\pm} \\ \delta(x) \leqslant cr}} |\mathbf{u}|^p \, dx \right\}^{1/p} \leqslant C \left\{ \frac{1}{r^{n-1}} \int_{I_{2r}} |(\mathbf{u})_{\pm}^{*,r}|^q \, d\sigma \right\}^{1/q} \tag{2.14}$$

where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ and $1 < q < p < nq/(n-1)$.

Proof. We only consider the case D_r^+. Note that if $x = (x', x_n) \in D_r^+$ and $\delta(x) \leqslant cr$, then $|\mathbf{u}(x)| \leqslant (\mathbf{u})_+^{*,r}(y', \psi(y'))$ for $|y' - x'| \leqslant c\delta(x)$. Hence, if $0 < \alpha < n - 1$,

$$\begin{aligned}
\left|\mathbf{u}(x)\right|\delta^{\alpha}(x) &\leqslant C \int_{|Q-P|<c\delta(x)} \frac{(\mathbf{u})_+^{*,r}(Q)}{|P-Q|^{n-1-\alpha}} \, d\sigma(Q) \\
&\leqslant C \int_{|Q-P|<cr} \frac{(\mathbf{u})_+^{*,r}(Q)}{|P-Q|^{n-1-\alpha}} \, d\sigma(Q),
\end{aligned} \tag{2.15}$$

where $P = (x', \psi(x'))$. It follows that if $\alpha p < 1$,

$$\int_{\substack{x \in D_r^+ \\ \delta(x) \leqslant cr}} |\mathbf{u}(x)|^p \, dx \leqslant C r^{1-\alpha p} \int_{I_r} d\sigma(P) \left\{ \int_{|Q-P|<cr} \frac{(\mathbf{u})_+^{*,r}(Q)}{|P-Q|^{n-1-\alpha}} \, d\sigma(Q) \right\}^p. \tag{2.16}$$

This leads to the desired estimate (2.14) by the $L^q - L^p$ bounds of the fractional integrals on $\partial \Omega$ [28], where $1 < q < p$ and $(1/q) - (1/p) = \alpha/(n-1)$. Finally we observe that the condition $\alpha p < 1$ is equivalent to $p < qn/(n-1)$. $\quad \square$

Lemma 2.4. *Suppose that* $\mathcal{L}(\mathbf{u}) = \mathbf{0}$ *in* $\mathbb{R}^n \setminus \partial \Omega$. *Assume that* $\mathbf{u}_+ = \mathbf{0}$ *on* I_{32r} *and* $(\nabla \mathbf{u})_+^* + (\nabla \mathbf{u})_-^* \in L^2(I_{32r})$ *for some* $0 < 32r < cr_0$. *Then*

$$
\int\limits_{I_r} |\nabla \mathbf{u}_-|^2 \, d\sigma \leqslant \frac{C}{r^2} \int\limits_{I_{4r}} |\mathbf{u}_-|^2 \, d\sigma + \frac{C}{r^3} \int\limits_{D_{32r}^+ \cup D_{32r}^-} |\mathbf{u}|^2 \, dx
$$

$$
+ C \int\limits_{I_{4r}} \left| \frac{\partial \mathbf{u}_+}{\partial \nu} - \frac{\partial \mathbf{u}_-}{\partial \nu} \right|^2 \, d\sigma. \tag{2.17}
$$

Similarly, if $\mathbf{u}_- = \mathbf{0}$ *on* I_{32r}, *we have*

$$
\int\limits_{I_r} |\nabla \mathbf{u}_+|^2 \, d\sigma \leqslant \frac{C}{r^2} \int\limits_{I_{4r}} |\mathbf{u}_+|^2 \, d\sigma + \frac{C}{r^3} \int\limits_{D_{32r}^+ \cup D_{32r}^-} |\mathbf{u}|^2 \, dx
$$

$$
+ C \int\limits_{I_{4r}} \left| \frac{\partial \mathbf{u}_+}{\partial \nu} - \frac{\partial \mathbf{u}_-}{\partial \nu} \right|^2 \, d\sigma. \tag{2.18}
$$

Proof. Assume $\mathbf{u}_+ = \mathbf{0}$ on I_{32r}. By using (2.9) and (2.5) as well as the Cauchy inequality, we have

$$
\int\limits_{I_r} |\nabla \mathbf{u}_-|^2 \, d\sigma \leqslant C \int\limits_{I_{4r}} \left| \frac{\partial \mathbf{u}_-}{\partial \nu} \right|^2 \, d\sigma + \frac{C}{r^2} \int\limits_{I_{8r}} |\mathbf{u}_-|^2 \, d\sigma
$$

$$
+ \frac{C}{r^3} \int\limits_{D_{4r}^-} |\mathbf{u}|^2 \, dx. \tag{2.19}
$$

Similarly, by (2.10) and (2.5), we obtain

$$
\int\limits_{I_{4r}} \left| \frac{\partial \mathbf{u}_+}{\partial \nu} \right|^2 \, d\sigma \leqslant \frac{C}{r^3} \int\limits_{D_{32r}^+} |\mathbf{u}|^2 \, dx, \tag{2.20}
$$

where we have used the assumption $\mathbf{u}_+ = \mathbf{0}$ and hence $\nabla_t \mathbf{u}_+ = \mathbf{0}$ on I_{32r}. Using

$$
\left| \frac{\partial \mathbf{u}_-}{\partial \nu} \right| \leqslant \left| \frac{\partial \mathbf{u}_+}{\partial \nu} \right| + \left| \frac{\partial \mathbf{u}_+}{\partial \nu} - \frac{\partial \mathbf{u}_-}{\partial \nu} \right|,
$$

it is not hard to see that (2.17) follows from (2.19) and (2.20). The proof of (2.18) is exactly the same. $\quad \square$

Observe that estimates (2.17) and (2.18), together with the Sobolev inequality

$$\left\{ \frac{1}{|I_r|} \int_{I_r} |\mathbf{u}|^{p_n}\, d\sigma \right\}^{1/p_n} \leqslant Cr \left\{ \frac{1}{|I_r|} \int_{I_r} |\nabla_t \mathbf{u}|^2\, d\sigma \right\}^{1/2} + C \left\{ \frac{1}{|I_r|} \int_{I_r} |\mathbf{u}|^2\, d\sigma \right\}^{1/2}, \quad (2.21)$$

where $p_n = 2(n-1)/(n-3)$ for $n \geqslant 4$, and p_3 may be any exponent in $(2, \infty)$, allow us to control the L^{p_n} average of $\mathbf{u}$ over I_r by its L^2 average over I_{4r}, provided we can handle the last two terms in the right sides of (2.17) and (2.18). Since we will apply (2.17)–(2.18) to solutions given by the double layer potentials plus possible corrections, the term involving $\frac{\partial \mathbf{u}_+}{\partial \nu} - \frac{\partial \mathbf{u}_-}{\partial \nu}$ is negligible in view of (1.16). In order to manage the remaining solid integrals, it will be convenient to work with the nontangential maximal function of $\mathbf{u}$.

If $\mathbf{u}$ is a function on $\mathbb{R}^n \setminus \partial\Omega$, we let $(\mathbf{u})^*(P) = \max\{(\mathbf{u})^*_+(P), (\mathbf{u})^*_-(P)\}$ and

$$(\mathbf{u})^{*,r}(P) = \sup\{|\mathbf{u}(x)|\colon x \in \gamma(P) \text{ and } |x - P| < cr\}, \quad (2.22)$$

for $P \in \partial\Omega$, where $c > 0$ is sufficiently small. By a simple geometric observation, we have

$$\left\{ \frac{1}{|I_r|} \int_{I_r} |(\mathbf{u})^*|^p\, d\sigma \right\}^{1/p}$$

$$\leqslant \left\{ \frac{1}{|I_r|} \int_{I_r} |(\mathbf{u})^{*,r}|^p\, d\sigma \right\}^{1/p} + \frac{C}{|I_{2r}|} \int_{I_{2r}} |(\mathbf{u})^*|\, d\sigma \quad (2.23)$$

for any $p > 1$.

Lemma 2.5. *Let $\bar{p} > 2$. Suppose that the $L^{\bar{p}}$ Dirichlet problem for operator $\mathcal{L}$ is uniquely solvable for any bounded Lipschitz domain in $\mathbb{R}^n$. Then for any $2(n-1)/n < p \leqslant 2$,*

$$\left\{ \frac{1}{|I_r|} \int_{I_r} |(\mathbf{u})^*|^{\bar{p}}\, d\sigma \right\}^{1/\bar{p}}$$

$$\leqslant C \left\{ \frac{1}{|I_{4r}|} \int_{I_{4r}} (|\mathbf{u}_+| + |\mathbf{u}_-|)^{\bar{p}}\, d\sigma \right\}^{1/\bar{p}} + C \left\{ \frac{1}{|I_{4r}|} \int_{I_{4r}} |(\mathbf{u})^*|^p\, d\sigma \right\}^{1/p}, \quad (2.24)$$

where $\mathcal{L}(\mathbf{u}) = \mathbf{0}$ in $\mathbb{R}^n \setminus \partial\Omega$ and $(\mathbf{u})^* \in L^{\bar{p}}(I_{4r})$.

Proof. Since the $L^{\bar{p}}$ Dirichlet problem is solvable on the Lipschitz domain $D^{\pm}_{sr}$, we have

$$\int_{I_r} |(\mathbf{u})^{*,r}|^{\bar{p}}\, d\sigma \leqslant C \int_{\partial D^+_{sr}} |\mathbf{u}|^{\bar{p}}\, d\sigma + \int_{\partial D^-_{sr}} |\mathbf{u}|^{\bar{p}}\, d\sigma \quad (2.25)$$

for $s \in (3/2, 2)$. It follows by an integration in s over $(3/2, 2)$ that

$$\int_{I_r} |(\mathbf{u})^{*,r}|^{\bar{p}} \, d\sigma \leqslant C \int_{I_{2r}} (|\mathbf{u}_+| + |\mathbf{u}_-|)^{\bar{p}} \, d\sigma + \frac{C}{r} \int_{D_{2r}^+ \cup D_{2r}^-} |\mathbf{u}|^{\bar{p}} \, dx. \qquad (2.26)$$

This, together with estimates (2.23) and (2.14), yields that

$$\left\{ \frac{1}{|I_r|} \int_{I_r} |(\mathbf{u})^*|^{\bar{p}} \, d\sigma \right\}^{1/\bar{p}}$$

$$\leqslant C \left\{ \frac{1}{|I_{3r}|} \int_{I_{3r}} (|\mathbf{u}_+| + |\mathbf{u}_-|)^{\bar{p}} \, d\sigma \right\}^{1/\bar{p}} + C \left\{ \frac{1}{|I_{3r}|} \int_{I_{3r}} |(\mathbf{u})^*|^q \, d\sigma \right\}^{1/q} \qquad (2.27)$$

for any $q > (n-1)\bar{p}/n$. Since the L^q Dirichlet problem for $\mathcal{L}$ is also uniquely solvable for any $2 \leqslant q < p$, it is not hard to see that one may deduce estimate (2.24) for $p = 2$ from (2.27) by using above argument repeatedly to decrease the exponent q in (2.27) to 2. From here another application of the argument reduces the exponent from 2 to any q in $(2(n-1)/n, 2)$. $\quad \square$

Finally we are ready to state and prove the desired reverse Hölder inequality for elliptic systems.

Theorem 2.6. *Suppose that* $\mathcal{L}(\mathbf{u}) = 0$ *in* $\mathbb{R}^n \setminus \partial\Omega$ *and* $n \geqslant 4$. *Assume that either* $\mathbf{u}_+ = 0$ *or* $\mathbf{u}_- = 0$ *on* I_{64r}. *Then, if* $(\nabla\mathbf{u})^* \in L^2(I_{64r})$ *and* $(\mathbf{u})^* \in L^{p_n}(I_{64r})$, *we have*

$$\left\{ \frac{1}{|I_r|} \int_{I_r} |(\mathbf{u})^*|^{p_n} \, d\sigma \right\}^{1/p_n}$$

$$\leqslant C \left\{ \frac{1}{|I_{64r}|} \int_{I_{64r}} |(\mathbf{u})^*|^2 \, d\sigma \right\}^{1/2} + Cr \left\{ \frac{1}{r^{n-1}} \int_{I_{32r}} \left| \frac{\partial\mathbf{u}_+}{\partial\nu} - \frac{\partial\mathbf{u}_-}{\partial\nu} \right|^2 \, d\sigma \right\}^{1/2}, \qquad (2.28)$$

where $p_n = 2(n-1)/(n-3)$. *If* $n = 3$, *estimate* (2.28) *holds for any* $p_3 > 2$.

Proof. It is proved in [25] that if $2 < p < 2(n-1)/(n-3) + \varepsilon$, the L^p Dirichlet problem is uniquely solvable for any bounded Lipschitz domain in $\mathbb{R}^n$. Thus estimate (2.24) holds for $\bar{p} = p_n$. This, combined with the Sobolev inequality (2.21), gives

$$\left\{ \frac{1}{|I_r|} \int_{I_r} |(\mathbf{u})^*|^{p_n} \, d\sigma \right\}^{1/p_n}$$

$$\leqslant Cr \left\{ \frac{1}{|I_{4r}|} \int_{I_{4r}} (|\nabla_t \mathbf{u}_+| + |\nabla_t \mathbf{u}_-|)^2 \, d\sigma \right\}^{1/2} + \left\{ \frac{1}{|I_{4r}|} \int_{I_{4r}} |(\mathbf{u})^*|^2 \, d\sigma \right\}^{1/2}. \qquad (2.29)$$

We now use (2.17)–(2.18) to estimate the term in (2.29) with the tangential derivatives. Note that the solid integrals in (2.17)–(2.18) are easily bounded by the maximal function $(\mathbf{u})^*$. Estimate (2.28) then follows. $\quad\square$

3. Invertibility of double layer potentials in L^p

Given $\mathbf{g} \in L^p(\partial\Omega)$ for some $1 < p < \infty$, let $\mathbf{u} = \mathcal{D}(\mathbf{g})$ be the double layer potential defined in (1.6). Then $\mathbf{u}_+ = (-(1/2)I + \mathcal{K}^*)\mathbf{g}$ and $\mathbf{u}_- = ((1/2)I + \mathcal{K}^*)\mathbf{g}$ on $\partial\Omega$. Moreover, we have $(\nabla\mathbf{u})^* \in L^p(\partial\Omega)$ and $\frac{\partial \mathbf{u}_+}{\partial \nu} = \frac{\partial \mathbf{u}_-}{\partial \nu}$ on $\partial\Omega$, if $\nabla_t \mathbf{g} \in L^p(\partial\Omega)$.

Since Ω_- is connected, the kernel of operator $(1/2)I + \mathcal{K}$ on $L^2(\partial\Omega)$ is of dimension m. Suppose $\{\mathbf{f}_\ell, \ \ell = 1, \ldots, m\}$ spans the kernel. Then $\int_{\partial\Omega} \mathbf{f}_\ell \, d\sigma \neq \mathbf{0}$, and $\mathcal{S}(\mathbf{f}_\ell)$ is a nonzero constant vector in $\overline{\Omega}$. Let

$$\mathcal{X}^p(\partial\Omega) = \left\{ \mathbf{f} \in L^p(\partial\Omega) \colon \int\limits_{\partial\Omega} \mathbf{f} \cdot \mathbf{f}_\ell \, d\sigma = 0, \ \text{for all } \ell = 1, \ldots, m \right\} \tag{3.1}$$

for $p \geqslant 2$. Since $\mathcal{S} \colon L^p(\partial\Omega) \to W^{1,p}(\partial\Omega)$ is invertible for some $p > 2$ [13], $\mathbf{f}_\ell \in L^p(\partial\Omega)$ for some $p > 2$. Thus the space $\mathcal{X}^p$ is also well defined for $p > 2 - \varepsilon$. It was proved in [9] that

$$\frac{1}{2}I + \mathcal{K}^* \colon \mathcal{X}^p(\partial\Omega) \to \mathcal{X}^p(\partial\Omega),$$

$$-\frac{1}{2}I + \mathcal{K}^* \colon L^p(\partial\Omega) \to L^p(\partial\Omega) \tag{3.2}$$

are isomorphisms if $n \geqslant 3$ and $|p - 2| < \varepsilon$. In the case $n = 3$, the operators in (3.2) are isomorphisms for $2 - \varepsilon < p < \infty$ [7]. The goal of this section is to establish the invertibility of $\pm(1/2)I + \mathcal{K}^*$ for $n \geqslant 4$ and $2 < p < (2(n-1)/(n-3)) + \varepsilon$.

Theorem 3.1. *There exists $\varepsilon > 0$, depending on n, m, μ_0 and the Lipschitz character of Ω, such that the operators $\pm(1/2)I + \mathcal{K}^*$ in (3.2) are isomorphisms for $n \geqslant 4$ and $2 < p < 2(n-1)/(n-3) + \varepsilon$.*

The proof of Theorem 3.1 is based on a real variable argument, inspired by a paper of Caffarelli and Peral [3] (see also [33]). In [25,26], the argument was used to solve the L^p Dirichlet problem for elliptic systems and higher order elliptic equations. This real variable argument may be considered as a dual and refined version of the celebrated Calderón–Zygmund Lemma. We should mention that a similar argument with a different motivation was also used in [2] (see also [1]).

The real variable argument may be formulated as follows.

Theorem 3.2. *Let Q_0 be a cube in $\mathbb{R}^n$ and $F \in L^1(2Q_0)$. Let $p > 1$ and $f \in L^q(2Q_0)$ for some $1 < q < p$. Suppose that for each dyadic subcube Q of Q_0 with $|Q| \leqslant \beta|Q_0|$, there exist two integrable functions F_Q and R_Q on $2Q$ such that $|F| \leqslant |F_Q| + |R_Q|$ on $2Q$, and*

$$\left\{ \frac{1}{|2Q|} \int\limits_{2Q} |R_Q|^p \, dx \right\}^{1/p} \leqslant C_1 \left\{ \frac{1}{|\alpha Q|} \int\limits_{\alpha Q} |F| \, dx + \sup_{Q' \supset Q} \frac{1}{|Q'|} \int\limits_{Q'} |f| \, dx \right\}, \tag{3.3}$$

$$\frac{1}{|2Q|}\int\limits_{2Q}|F_Q|\,dx \leqslant C_2\sup_{Q'\supset Q}\frac{1}{|Q'|}\int\limits_{Q'}|f|\,dx, \tag{3.4}$$

where $C_1, C_2 > 0$ and $0 < \beta < 1 < \alpha$. Then

$$\left\{\frac{1}{|Q_0|}\int\limits_{Q_0}|F|^q\,dx\right\}^{1/q} \leqslant \frac{C}{|2Q_0|}\int\limits_{2Q_0}|F|\,dx + C\left\{\frac{1}{|2Q_0|}\int\limits_{2Q_0}|f|^q\,dx\right\}^{1/q}, \tag{3.5}$$

where $C > 0$ is a constant depending only on p, q, C_1, C_2, α, β and n.

We postpone the proof of Theorem 3.2 to the end of this section.

Remark 3.3. Because of the local nature of Theorem 3.2, it may be extended easily to each coordinate patch of $\partial\Omega$. Indeed, assume that $0 \in \partial\Omega$ and $\Omega \cap B(0, r_0)$ is given by (2.2). Consider the map $\Phi : \partial D = \{(x', \psi(x')) : x' \in \mathbb{R}^{n-1}\} \to \mathbb{R}^{n-1}$, defined by $\Phi(x', \psi(x')) = x'$. We say $Q \subset \partial D$ is a surface cube of ∂D if $\Phi(Q)$ is a cube of $\mathbb{R}^{n-1}$. Moreover, a dilation of Q may be defined by $\alpha Q = \Phi^{-1}(\alpha\Phi(Q))$. With these notations, one may state the extension of Theorem 3.2 to ∂D in exactly the same manner as for the case of $\mathbb{R}^{n-1}$. Of course in the case of ∂D, the constant C in (3.5) also depends on $\|\nabla\psi\|_\infty$.

Proof of Theorem 3.1. We will give the proof for the invertibility of $(1/2)I + \mathcal{K}^*$ on $\mathcal{X}^p(\partial\Omega)$. The case of $-(1/2)I + \mathcal{K}^*$ on $L^p(\partial\Omega)$ is similar and slightly easier.

Let $\mathbf{f} \in \mathcal{X}^p(\partial\Omega) \cap W^{1,2}(\partial\Omega)$ for some $p > 2$. Since $(1/2)I + \mathcal{K}^*$ is invertible on $\mathcal{X}^2(\partial\Omega)$ and on $W^{1,2}(\partial\Omega)/\text{span}\{\mathbf{f}_1, \dots, \mathbf{f}_m\}$, there exists $\mathbf{g} \in \mathcal{X}^2(\partial\Omega) \cap W^{1,2}(\partial\Omega)$ such that $((1/2)I + \mathcal{K}^*)\mathbf{g} = \mathbf{f}$ and $\|\mathbf{g}\|_2 \leqslant C\|\mathbf{f}\|_2$. Let $\mathbf{u} = \mathcal{D}(\mathbf{g})$ in $\mathbb{R}^n \setminus \partial\Omega$. We will show that there exists $\varepsilon > 0$, depending only on n, m, μ_0 and Ω, such that if $2 < p < p_n + \varepsilon$,

$$\left\{\frac{1}{s^{n-1}}\int\limits_{B(P,s)\cap\partial\Omega}|(\mathbf{u})^*|^p\,d\sigma\right\}^{1/p}$$

$$\leqslant C\left\{\frac{1}{s^{n-1}}\int\limits_{B(P,Cs)\cap\partial\Omega}|(\mathbf{u})^*|^2\,d\sigma\right\}^{1/2} + C\left\{\frac{1}{s^{n-1}}\int\limits_{B(P,Cs)\cap\partial\Omega}|\mathbf{f}|^p\,d\sigma\right\}^{1/p}, \tag{3.6}$$

for any $P \in \partial\Omega$ and $s > 0$ small. Since $|\mathbf{g}| = |\mathbf{u}_+ - \mathbf{u}_-| \leqslant 2(\mathbf{u})^*$, by covering $\partial\Omega$ with a finite number of small balls, estimate (3.6) implies that

$$\|\mathbf{g}\|_p \leqslant C\|\mathbf{g}\|_2 + C\|\mathbf{f}\|_p \leqslant C\|\mathbf{f}\|_p. \tag{3.7}$$

This shows that $(1/2)I + \mathcal{K}^* : \mathcal{X}^p(\partial\Omega) \to \mathcal{X}^p(\partial\Omega)$ is invertible, since $\mathcal{X}^p(\partial\Omega) \cap W^{1,2}(\partial\Omega)$ is dense in $\mathcal{X}^p(\partial\Omega)$.

To prove (3.6), we use Theorems 3.2 and 2.6. By translation and rotation, we may assume that $P = 0$ and $B(0, r_0) \cap \Omega$ is given by (2.2). We consider the surface cube $Q_0 = I_s$, defined in (2.3) for $0 < s < cr_0$. Let Q be a small subcube of Q_0. Choose $\varphi \in C_0^1(\mathbb{R}^n)$ such that $\varphi = 1$

on $200Q$, $\varphi = 0$ in $\partial\Omega \setminus 300Q$ and $|\nabla\varphi| \leqslant C/r$, where r is the diameter of Q. Since $L^2(\partial\Omega) = \mathcal{X}^2(\partial\Omega) \oplus \mathbb{R}^m$, there exist $\mathbf{g}_Q \in \mathcal{X}^2(\partial\Omega) \cap W^{1,2}(\partial\Omega)$ and $\mathbf{b} \in \mathbb{R}^m$ such that

$$\mathbf{f}\varphi = \left(\frac{1}{2}I + \mathcal{K}^*\right)\mathbf{g}_Q + \mathbf{b} \quad \text{on } \partial\Omega, \tag{3.8}$$

and $\|\mathbf{f}\varphi\|_2 \sim \|\mathbf{g}_Q\|_2 + |\mathbf{b}|$. Let $\mathbf{v} = \mathcal{D}(\mathbf{g}_Q) + \mathbf{b}$ in $\mathbb{R}^n \setminus \partial\Omega$ and $\mathbf{w} = \mathbf{u} - \mathbf{v}$.

We will apply Theorem 3.2 with $F = |(\mathbf{u})^*|^2$, $f = |\mathbf{f}|^2$ and

$$F_Q = 2|(\mathbf{v})^*|^2 \quad \text{and} \quad R_Q = 2|(\mathbf{w})^*|^2. \tag{3.9}$$

Note that by the L^2 estimates,

$$\frac{1}{|2Q|} \int_{2Q} |F_Q|\,d\sigma \leqslant \frac{C}{|Q|} \int_{\partial\Omega} |(\mathbf{v})^*|^2 \,d\sigma \leqslant \frac{C}{|Q|}\{\|\mathbf{g}_Q\|_2^2 + |\mathbf{b}|^2\}$$

$$\leqslant \frac{C}{|200Q|} \int_{200Q} |\mathbf{f}|^2 \,d\sigma. \tag{3.10}$$

This gives condition (3.4). To verify (3.3), we observe that $\mathbf{w}_- = \mathbf{u}_- - \mathbf{v}_- = \mathbf{f}(1 - \varphi)$ on $\partial\Omega$. Hence $\mathbf{w}_- = \mathbf{0}$ on $200Q$. Also note that $(\nabla\mathbf{w})^* \in L^2(\partial\Omega)$ since $\mathbf{g}$, $\mathbf{g}_Q \in W^{1,2}(\partial\Omega)$. It follows that $(\mathbf{w})^* \in L^{p_n}(\partial\Omega)$ (see e.g. [23, p. 1094]). Since $\mathbf{w} = \mathcal{D}(\mathbf{g}) - \mathcal{D}(\mathbf{g}_Q) - \mathbf{b}$, we have $\frac{\partial\mathbf{w}_+}{\partial\nu} = \frac{\partial\mathbf{w}_-}{\partial\nu}$ on $\partial\Omega$. Thus we may apply Theorem 2.6 to obtain

$$\left\{\frac{1}{|Q'|} \int_{Q'} |(\mathbf{w})^*|^{p_n} \,d\sigma\right\}^{1/p_n} \leqslant C\left\{\frac{1}{|64Q'|} \int_{64Q'} |(\mathbf{w})^*|^2 \,d\sigma\right\}^{1/2}, \tag{3.11}$$

where Q' is any subcube of Q. It is well known that the reverse Hölder inequalities like (3.11) have the self-improving property (see e.g. [14]). This implies that there exists $\varepsilon > 0$, depending only on n, $\|\nabla\psi\|_\infty$ and the constant C in (3.11), such that

$$\left\{\frac{1}{|Q|} \int_{Q} |(\mathbf{w})^*|^{\bar{p}} \,d\sigma\right\}^{1/\bar{p}} \leqslant C\left\{\frac{1}{|2Q|} \int_{2Q} |(\mathbf{w})^*|^2 \,d\sigma\right\}^{1/2} \tag{3.12}$$

where $\bar{p} = p_n + \varepsilon$. The right side of (3.12) may be estimated using $(\mathbf{w})^* \leqslant (\mathbf{u})^* + (\mathbf{v})^*$ and then (3.10). Thus condition (3.3) in Theorem 3.2 holds for $p = p_n + \varepsilon$. Consequently, estimate (3.6) holds for $2 < p < p_n + \varepsilon$. The proof is complete. $\quad\square$

We now give the proof of Theorem 3.2. The argument is essentially the same as that in the proof of Lemma 2.18 in [25]. We shall need a localized Hardy–Littlewood maximal function

$$M_Q(g)(x) = \sup_{\substack{Q' \ni x \\ Q' \subset Q}} \frac{1}{|Q'|} \int_{Q'} |g|\,dx \tag{3.13}$$

for $x \in Q$, where Q' is a subcube of Q.

Proof of Theorem 3.2. For $\lambda > 0$, let

$$E(\lambda) = \big\{x \in Q_0 \colon M_{2Q_0}(F)(x) > \lambda\big\}. \tag{3.14}$$

We claim that for any $1 < q < p$, it is possible to choose three constants $0 < \delta < 1$, $\gamma > 0$ and $C_0 > 0$ depending only on n, C_1, C_2, α, β in (3.3)–(3.4) and p, q such that

$$\big|E(A\lambda)\big| \leqslant \delta\big|E(\lambda)\big| + \big|\big\{x \in Q_0 \colon M_{2Q_0}(f)(x) > \gamma\lambda\big\}\big| \tag{3.15}$$

for all $\lambda > \lambda_0$, where $A = (2\delta)^{-1/q}$ and

$$\lambda_0 = \frac{C_0}{|2Q_0|} \int\limits_{2Q_0} |F| \, dx. \tag{3.16}$$

Multiplying both sides of (3.15) by λ^{q-1} and then integrating the resulting inequality in $\lambda \in (\lambda_0, \Lambda)$, we obtain

$$\int\limits_{\lambda_0}^{\Lambda} \lambda^{q-1}\big|E(A\lambda)\big| \, d\lambda \leqslant \delta \int\limits_{\lambda_0}^{\Lambda} \lambda^{q-1}\big|E(\lambda)\big| \, d\lambda + C_\gamma \int\limits_{2Q_0} |f|^q \, dx, \tag{3.17}$$

where we have used the fact that M_{2Q_0} is bounded on L^q. By a change of variables in the left side of (3.17), we may deduce that

$$A^{-q}\big(1 - \delta A^q\big) \int\limits_0^{\Lambda} \lambda^{q-1}\big|E(\lambda)\big| \, d\lambda \leqslant C|Q_0|\lambda_0^q + C_\gamma \int\limits_{2Q_0} |f|^q \, dx. \tag{3.18}$$

Note that $\delta A^q = 1/2 < 1$. Let $\Lambda \to \infty$ in (3.18). This gives

$$\int\limits_{Q_0} |F|^q \, dx \leqslant C|Q_0|\lambda_0^q + C \int\limits_{2Q_0} |f|^q \, dx, \tag{3.19}$$

which is (3.5) in view of (3.16).

To prove (3.15), we first note that $|E(\lambda)| \leqslant C_n|Q_0|/C_0$ for any $\lambda > \lambda_0$. This follows from the weak $(1,1)$ estimate for M_{2Q_0}. Thus we may choose $C_0 = 2C_n/\delta$ so that $|E(\lambda)| < \delta|Q_0|$ for any $\lambda > \lambda_0$. We now fix $\lambda > \lambda_0$. Since $E(\lambda)$ is open relative to Q_0, we may write $E(\lambda) = \bigcup_k Q_k$, where Q_k are maximal dyadic subcubes of Q_0 contained in $E(\lambda)$. By choosing δ sufficiently small, we may certainly assume that $|Q_k| < \beta|Q_0|$ and $(\alpha + 64)Q_k \subset 2Q_0$.

We will show that it is possible to choose $\delta > 0$ and $\gamma > 0$ so that

$$\big|E(A\lambda) \cap Q_k\big| \leqslant \delta|Q_k|, \tag{3.20}$$

whenever $\{x \in Q_k \colon M_{2Q_0}(f)(x) \leqslant \gamma\lambda\} \neq \emptyset$. Clearly, estimate (3.15) follows from (3.20) by summation.

Let Q_k be such a maximal dyadic subcube. Observe that

$$M_{2Q_0}(F)(x) \leqslant \max\{M_{2Q_k}(F)(x), C_n\lambda\}, \tag{3.21}$$

for any $x \in Q_k$. This is because Q_k is maximal and so

$$\frac{1}{|Q'|}\int_{Q'}|F|\,dx \leqslant C_n\lambda \tag{3.22}$$

for any $Q' \cap Q_k \neq \emptyset$ and $|Q'| \geqslant c_n|Q_k|$. We may assume that $A > C_n$. Then

$$\begin{aligned}
|E(A\lambda) \cap Q_k| &\leqslant \left|\left\{x \in Q_k\colon M_{2Q_k}(F) > A\lambda\right\}\right| \\
&\leqslant \left|\left\{x \in Q_k\colon M_{2Q_k}(F_{Q_k})(x) > \frac{A\lambda}{2}\right\}\right| \\
&\quad + \left|\left\{x \in Q_k\colon M_{2Q_k}(R_{Q_k})(x) > \frac{A\lambda}{2}\right\}\right| \\
&\leqslant \frac{C_n}{A\lambda}\int_{2Q_k}|F_{Q_k}|\,dx + \frac{C_{n,p}}{(A\lambda)^p}\int_{2Q_k}|R_{Q_k}|^p\,dx, \tag{3.23}
\end{aligned}$$

where we have used $|F| \leqslant |F_{Q_k}| + |R_{Q_k}|$ on $2Q_k$ as well as weak $(1,1)$, weak (p,p) bounds of M_{2Q_k}.

By assumption (3.4), we have

$$\begin{aligned}
\int_{2Q_k}|F_{Q_k}|\,dx &\leqslant C_2|2Q_k|\sup_{2Q_0 \supset Q' \supset Q_k}\frac{1}{|Q'|}\int_{Q'}|f|\,dx \\
&\leqslant C_2|2Q_k| \cdot \gamma\lambda, \tag{3.24}
\end{aligned}$$

where the last inequality follows from the fact $\{x \in Q_k\colon M_{2Q_0}(f) \leqslant \gamma\lambda\} \neq \emptyset$. Similarly, we may use (3.3) and (3.22) to obtain

$$\begin{aligned}
\int_{2Q_k}|R_{Q_k}|^p\,dx &\leqslant C_1^p \cdot |2Q_k|\left\{\frac{1}{|\alpha Q_k|}\int_{\alpha Q_k}|F|\,dx + \gamma\lambda\right\}^p \\
&\leqslant C_{n,\alpha}C_1^p|Q_k|\{\lambda + \gamma\lambda\}^p. \tag{3.25}
\end{aligned}$$

We now use (3.24) and (3.25) to estimate the right side of (2.23). This yields

$$\begin{aligned}
|E(A\lambda) \cap Q_k| &\leqslant |Q_k|\left\{\frac{C_nC_2\gamma}{A} + \frac{C_{n,\alpha,p}C_1^p}{A^p}\right\} \\
&= \delta|Q_k|\left\{C_nC_2\gamma\delta^{-\frac{1}{q}-1} + C_{n,p,\alpha}C_1^p\delta^{\frac{p}{q}-1}\right\}. \tag{3.26}
\end{aligned}$$

Finally we observe that since $q < p$, it is possible to choose $\delta > 0$ so small that

$$C_{n,p,\alpha} C_1^p \delta^{\frac{p}{q}-1} < (1/4).$$

After δ is chosen, we then choose $\gamma > 0$ so small that $C_n C_2 \gamma \delta^{-\frac{1}{q}-1} < 1/4$. This finishes the proof of (3.20) and thus the theorem. $\quad\square$

The following weighted version of Theorem 3.2 will be used in Section 8.

Theorem 3.4. *Under the same assumption as in Theorem* 3.2, *we have*

$$\left\{\frac{1}{\omega(Q_0)} \int\limits_{Q_0} |F|^q \omega \, dx\right\}^{1/q} \leqslant \frac{C}{|2Q_0|} \int\limits_{2Q_0} |F| \, dx + C \left\{\frac{1}{\omega(2Q_0)} \int\limits_{2Q_0} |f|^q \omega \, dx\right\}^{1/q}, \quad (3.27)$$

where ω is an A_q weight on $2Q_0$ with the property that for some $\eta > q/p$,

$$\frac{\omega(E)}{\omega(Q)} \leqslant C \left(\frac{|E|}{|Q|}\right)^\eta, \quad (3.28)$$

for any $E \subset Q \subset Q_0$.

Proof. Fix $1 < q < p$. Since $\eta > q/p$, we may choose $q_1 \in (q, p)$ so that $\eta > q/q_1$. Let $A = (2\delta)^{-1/q_1}$ in the proof of Theorem 3.2. Note that if $|E(A\lambda) \cap Q_k| \leqslant \delta |Q_k|$, then $\omega(E(A\lambda) \cap Q_k) \leqslant C \delta^\eta \omega(Q_k)$. This follows from (3.28). Thus

$$\omega\big(E(A\lambda)\big) \leqslant C\delta^\eta \omega\big(E(\lambda)\big) + \omega\big\{x \in Q_0 \colon M_{2Q_0}(f) > \gamma\lambda\big\}, \quad (3.29)$$

for any $\lambda \geqslant \lambda_0$. We now multiply both sides of (3.29) by λ^{q-1} and integrate the resulting inequality in λ from λ_0 to Λ. By a change of variable, we obtain

$$\big(A^{-q} - C\delta^\eta\big) \int\limits_0^\Lambda \lambda^{q-1} \omega\big(E(\lambda)\big) \, d\lambda \leqslant C\lambda_0^q \omega(Q_0) + C_\delta \int\limits_{Q_0} \big|M_{2Q_0}(f)\big|^q \omega \, dx$$

$$\leqslant C\lambda_0^q \omega(Q_0) + C_\delta \int\limits_{2Q_0} |f|^q \omega \, dx, \quad (3.30)$$

where the second inequality follows from the well-known property of M_{2Q_0} on $L^q(2Q_0, \omega \, dx)$ with A_q weight ω (see e.g. [29]). Finally we note that since $\delta > q/q_1$, we have $A^{-q} - C\delta^\eta = (2\delta)^{q/q_1} - C\delta^\eta > 0$ if $\delta > 0$ is sufficiently small. Estimate (3.27) follows from (3.30) by letting $\Lambda \to \infty$. $\quad\square$

Remark 3.5. If condition (3.3) holds for any $1 < p < \infty$ (constant C_1 may depend on p), then estimate (3.27) in Theorem 3.4 holds for any $\omega \in A_q$. This is because $w \in A_q$ implies condition (3.28) for some $\eta = \eta(\omega) > 0$.

4. The L^p boundary value problems for elliptic systems

In this section we give the proof of Theorem 1.1 stated in the Introduction. Let

$$L_0^p(\partial\Omega) = \left\{ \mathbf{f} \in L^p(\partial\Omega) : \int_{\partial\Omega} \mathbf{f}\, d\sigma = \mathbf{0} \right\}. \tag{4.1}$$

Theorem 4.1. *There exists $\varepsilon_1 > 0$, depending on n, m, μ_0, and the Lipschitz character of Ω, such that operators $(1/2)I + \mathcal{K} : L_0^p(\partial\Omega) \to L_0^p(\partial\Omega)$ and $-(1/2)I + \mathcal{K} : L^p(\partial\Omega) \to L^p(\partial\Omega)$ are invertible for $2(n-1)/(n+1) - \varepsilon_1 < p < 2$.*

Proof. Let $p_0 = 2(n-1)/(n-3) + \varepsilon$, where $\varepsilon > 0$ is given in Theorem 3.1. Note that $p_0' < 2(n-1)/(n+1)$. Since $-(1/2)I + \mathcal{K}^* : L^p(\partial\Omega) \to L^p(\partial\Omega)$ is invertible for $2 < p < p_0$, by duality, we see that $-(1/2)I + \mathcal{K} : L^p(\partial\Omega) \to L^p(\partial\Omega)$ is invertible for $p_0' < p < 2$.

Let $\mathbf{f} \in L_0^p(\partial\Omega)$ for some $p_0' < p < 2$. Given any $\mathbf{g} \in L^{p'}(\partial\Omega)$, since $L^{p'}(\partial\Omega) = \mathcal{X}^{p'}(\partial\Omega) \oplus \mathbb{R}^m$ and $(1/2)I + \mathcal{K}^*$ is invertible on $\mathcal{X}^{p'}(\partial\Omega)$ by Theorem 3.1, there exist $\mathbf{h} \in \mathcal{X}^{p'}(\partial\Omega)$ and $\mathbf{b} \in \mathbb{R}^m$ such that $\mathbf{g} = ((1/2)I + \mathcal{K}^*)\mathbf{h} + \mathbf{b}$ and $\|\mathbf{g}\|_{p'} \sim \|\mathbf{h}\|_{p'} + |\mathbf{b}|$. Thus

$$\left| \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{g}\, d\sigma \right| = \left| \int_{\partial\Omega} \left(\frac{1}{2}I + \mathcal{K}\right)\mathbf{f} \cdot \mathbf{h}\, d\sigma \right|$$

$$\leqslant \left\| \left(\frac{1}{2}I + \mathcal{K}\right)\mathbf{f} \right\|_p \|\mathbf{h}\|_{p'} \leqslant C \left\| \left(\frac{1}{2}I + \mathcal{K}\right)\mathbf{f} \right\|_p \|\mathbf{g}\|_{p'}. \tag{4.2}$$

It follows by duality that $\|\mathbf{f}\|_p \leqslant C \|((1/2)I + \mathcal{K})\mathbf{f}\|_p$ for any $\mathbf{f} \in L_0^p(\partial\Omega)$. This shows that $(1/2)I + \mathcal{K} : L_0^p(\partial\Omega) \to L_0^p(\partial\Omega)$ is one-to-one and the range is closed. Note that the range is also dense in $L_0^p(\partial\Omega)$. This is because the operator is known to be invertible on $L_0^2(\partial\Omega)$. Thus we have proved that $(1/2)I + \mathcal{K}$ is invertible on $L_0^p(\partial\Omega)$ for any $p_0' < p < 2$. $\quad\square$

Proof of Theorem 1.1. The existence follows directly from the invertibility of $(1/2)I + \mathcal{K}$ on $L_0^p(\partial\Omega)$ for $2(n-1)/(n+1) - \varepsilon_1 < p < 2$.

In order to prove the uniqueness, we construct a matrix of the Neumann functions

$$G_\nu^x(y) = \Gamma(x-y) - W^x(y), \tag{4.3}$$

where for each $x \in \Omega$, W^x is a matrix solution of the L^2 Neumann problem (1.2) with boundary data

$$\frac{\partial}{\partial\nu(y)}\{\Gamma(x-y)\} + \frac{1}{|\partial\Omega|}I_{m\times m}. \tag{4.4}$$

In (4.4), $I_{m\times m}$ denotes the $m \times m$ identity matrix. By the $L^{2+\varepsilon}$ estimates for the Neumann problem, we have $(\nabla W^x)^* \in L^p(\partial\Omega)$ for some $p > 2$. Consequently, $(W^x)^* \in L^{p_1}(\partial\Omega)$ for some $p_1 > 2(n-1)/(n-3)$ (see [23, p. 1094]).

Suppose now that $\mathcal{L}(\mathbf{u}) = \mathbf{0}$ in Ω, $(\nabla\mathbf{u})^* \in L^p(\partial\Omega)$ and $\frac{\partial\mathbf{u}}{\partial\nu} = \mathbf{0}$ on $\partial\Omega$. Note that if $p > \max(p_0', p_1')$, then $(\nabla\mathbf{u})^*(W^x)^* \in L^1(\partial\Omega)$. Similarly, one may show that $(\mathbf{u})^*(\nabla W^x)^* \in$

$L^1(\partial\Omega)$. Thus one can use the integration by parts, justified by the Lebesgue dominated convergence theorem, to obtain the representation formula

$$\mathbf{u}(x) = \int_{\partial\Omega} G_\nu^x(y)\frac{\partial\mathbf{u}}{\partial\nu}\,d\sigma(y) - \int_{\partial\Omega}\frac{\partial G_\nu^x}{\partial\nu}\mathbf{u}(y)\,d\sigma(y) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\mathbf{u}\,d\sigma. \tag{4.5}$$

Hence $\mathbf{u}$ is constant in Ω. The proof is finished. $\quad\square$

Remark 4.2. Theorem 1.1 also holds in the exterior domain $\Omega_- = \mathbb{R}^n \setminus \overline{\Omega}$ if one imposes additional condition $|\mathbf{u}(x)| = O(|x|^{n-2})$ as $|x| \to \infty$. In this case the mean zero condition on $\mathbf{f}$ is not needed. The proof is similar.

Remark 4.3. Since $-(1/2)I + \mathcal{K}^*$ is invertible on $L^p(\partial\Omega)$ for $2 < p < 2(n-1)/(n-3)+\varepsilon$, the unique solution of the L^p Dirichlet problem (1.9), which was solved in [25], may be represented by the double layer potential

$$\mathbf{u}(x) = \mathcal{D}\left(\left(-\frac{1}{2}I + \mathcal{K}^*\right)^{-1}(\mathbf{f})\right)(x). \tag{4.6}$$

Since $L^p(\partial\Omega) = \mathcal{X}^p(\partial\Omega) \oplus \mathbb{R}^m$, in the case of Ω_-, the solution may be represented as $\mathbf{u} = \mathcal{D}(\mathbf{g}) + \mathcal{S}(\mathbf{h})$, where $\mathbf{g} \in \mathcal{X}^p(\partial\Omega)$, $\mathbf{h} \in \mathrm{Ker}((1/2)I + \mathcal{K})$, and $\|\mathbf{u}\|_p \sim \|\mathbf{g}\|_p + \|\mathbf{h}\|_p$.

Remark 4.4. The Dirichlet problem with boundary data in $W^{1,p}(\partial\Omega)$ for the elliptic systems satisfying the Legendre–Hadamard condition (1.19) was solved in [25] for $n \geqslant 4$ and $2(n-1)/(n+1) - \varepsilon < p < 2$. This, combined with Theorem 1.1, gives

$$\left\|\frac{\partial\mathbf{u}}{\partial\nu}\right\|_p \sim \|\nabla_t\mathbf{u}\|_p$$

for any solution of (1.2) with p in the range (1.4).

5. The traction boundary value problem

Throughout this section we assume that

$$\mathcal{L}(\mathbf{u}) = -\mu\Delta\mathbf{u} - (\lambda + \mu)\nabla(\mathrm{div}\,\mathbf{u}) \quad \text{in } \Omega, \tag{5.1}$$

$$\frac{\partial\mathbf{u}}{\partial\nu} = \lambda(\mathrm{div}\,\mathbf{u})N + \mu\big(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\big)N \quad \text{on } \partial\Omega. \tag{5.2}$$

If we write $(\mathcal{L}(\mathbf{u}))^k = -a_{ij}^{k\ell}D_iD_ju^\ell$, the conormal derivatives (5.2) correspond to the choice of coefficients given by (1.18). Note that $a_{ij}^{k\ell}$ do not satisfy the strong ellipticity condition (1.3). However one has

$$a_{ij}^{k\ell}\frac{\partial u^k}{\partial x_i}\frac{\partial u^\ell}{\partial x_j} = \lambda|\mathrm{div}\,\mathbf{u}|^2 + \frac{\mu}{2}\big|\nabla\mathbf{u} + (\nabla\mathbf{u})^T\big|^2 \sim \big|\nabla\mathbf{u} + (\nabla\mathbf{u})^T\big|^2. \tag{5.3}$$

Using this observation, by establishing a Korn type inequality on the boundary, Dahlberg, Kenig and Verchota were able to strength the Rellich type inequalities. This allows them to show that

$$\frac{1}{2}I + \mathcal{K} : L^p_\psi(\partial\Omega) \to L^p_\psi(\partial\Omega),$$

$$-\frac{1}{2}I + \mathcal{K} : L^p(\partial\Omega) \to L^p(\partial\Omega), \tag{5.4}$$

are invertible for $|p - 2| < \varepsilon$ and $n \geqslant 2$ [9], where $L^p_\psi(\partial\Omega)$ is defined in (1.20). In the case $n = 2$ or 3, it was proved in [7] that the operators in (5.4) are invertible for the optimal range $1 < p < 2 + \varepsilon$. The goal of this section is to prove the following.

Theorem 5.1. *There exists $\varepsilon > 0$, depending on n, λ, μ and the Lipschitz character of Ω, such that the operators in (5.4) are invertible if $n \geqslant 4$ and $2(n-1)/(n+1) - \varepsilon < p < 2$.*

Let $\mathrm{Ker}((1/2)I + \mathcal{K})$ denote the kernel of operator $(1/2)I + \mathcal{K}$ on $L^2(\partial\Omega)$. If $\mathbf{u} = \mathcal{S}(\mathbf{g})$ for some $\mathbf{g} \in \mathrm{Ker}((1/2)I + \mathcal{K})$, then $\frac{\partial \mathbf{u}_+}{\partial \nu} = \mathbf{0}$ on $\partial\Omega$. It follows from (5.3) and the integration by parts that $\nabla\mathbf{u} + (\nabla\mathbf{u})^T = 0$ in Ω. Thus $\mathcal{S}(\mathbf{g})|_\Omega \in \Psi$. It is not hard to show that the map $\mathbf{g} \to \mathcal{S}(\mathbf{g})|_\Omega$ from $\mathrm{Ker}((1/2)I + \mathcal{K})$ to Ψ is bijective. Suppose $\{\mathbf{g}_k \colon k = 1, 2, \ldots, n(n+1)/2\}$ spans $\mathrm{Ker}((1/2)I + \mathcal{K})$. Since $\mathcal{S} : L^p(\partial\Omega) \to W^{1,p}(\partial\Omega)$ is invertible for p close to 2 [13], $\mathbf{g}_k \in L^{q_0}(\partial\Omega)$ for some $q_0 > 2$. Define

$$\mathbf{T}^p(\partial\Omega) = \left\{ \mathbf{f} \in L^p(\partial\Omega) \colon \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{g}_k \, d\sigma = 0 \text{ for } k = 1, 2, \ldots, n(n+1)/2 \right\} \tag{5.5}$$

for $p \geqslant q_0'$.

Theorem 5.2. *There exists $\varepsilon > 0$ such that operators*

$$\frac{1}{2}I + \mathcal{K}^* : \mathbf{T}^p(\partial\Omega) \to \mathbf{T}^p(\partial\Omega),$$

$$-\frac{1}{2}I + \mathcal{K}^* : L^p(\partial\Omega) \to L^p(\partial\Omega), \tag{5.6}$$

are invertible for $n \geqslant 4$ and $2 < p < 2(n-1)/(n-3) + \varepsilon$.

Theorem 5.1 follows from Theorem 5.2 by duality. The case for $-(1/2)I + \mathcal{K}$ is obvious. To see that $(1/2)I + \mathcal{K}$ is invertible on $L^p_\psi(\partial\Omega)$, we apply the same duality argument as in the proof of Theorem 4.1. To do this, we only need to show that $L^{p'}(\partial\Omega) = \mathbf{T}^{p'}(\partial\Omega) \oplus \Psi$. By a dimensional consideration, it suffices to prove that $\mathbf{T}^{p'}(\partial\Omega) \cap \Psi = \{0\}$. To this end, let $\mathbf{g} \in \mathbf{T}^{p'}(\partial\Omega) \cap \Psi$. Then $\mathbf{g} = \mathcal{S}(\mathbf{h})$ on $\partial\Omega$ for some $\mathbf{h} \in \mathrm{Ker}((1/2)I + \mathcal{K})$. Let $\mathbf{u} = \mathcal{S}(\mathbf{h})$ in $\mathbb{R}^n$. Since $\mathbf{h} = \frac{\partial \mathbf{u}_+}{\partial \nu} - \frac{\partial \mathbf{u}_-}{\partial \nu} = -\frac{\partial \mathbf{u}_-}{\partial \nu}$, we obtain

$$\int_{\Omega_-} a^{k\ell}_{ij} \frac{\partial u^k}{\partial x_i} \frac{\partial u^\ell}{\partial x_j} \, dx = -\int_{\partial\Omega} \frac{\partial \mathbf{u}_-}{\partial \nu} \cdot \mathbf{u} \, d\sigma = \int_{\partial\Omega} \mathbf{h} \cdot \mathbf{g} \, d\sigma = 0, \tag{5.7}$$

where the last equality follows from the fact that **g** is in the range of $(1/2)I + \mathcal{K}^*$ on $L^2(\partial\Omega)$. One may deduce from (5.7) that $\mathbf{u}|_{\Omega_-} \in \Psi$. This implies that $\frac{\partial\mathbf{u}_-}{\partial v} = \mathbf{0}$ and thus $\mathbf{h} = \mathbf{0}$.

Since the proof of Theorem 5.2 uses the same line of argument as in the proof of Theorem 3.1, we will only point out the necessary modification needed here.

First, because of (5.3), estimate (2.5) is replaced by

$$\int_{D_r^\pm} \left|\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right|^2 dx \leqslant \frac{C}{r^2}\int_{D_{2r}^\pm} |\mathbf{u}|^2\,dx + C\int_{I_{2r}} \left|\frac{\partial\mathbf{u}_\pm}{\partial v}\right| |\mathbf{u}_\pm|\,d\sigma. \tag{5.8}$$

The proof is exactly the same.

Next, estimate (2.9) needs to be modified, as we used

$$\|\nabla\mathbf{u}\|_{L^2(\partial D_{sr}^\pm)} \leqslant C\left\|\frac{\partial\mathbf{u}}{\partial v}\right\|_{L^2(\partial D_{sr}^\pm)} \tag{5.9}$$

for any L^2 solutions. In the case of (5.1), we know that estimate (5.9) is true for one of such solutions, **v**, given by a single layer potential with density $((1/2)I + \mathcal{K})^{-1}(\frac{\partial\mathbf{u}}{\partial v})$. If **u** is another solution with the same traction boundary data on $\partial D_{sr}^\pm$, then $\mathbf{w} = \mathbf{u} - \mathbf{v} = Ax + \mathbf{b} \in \Psi$. It follows that

$$\int_{\partial D_{sr}^\pm} |\nabla\mathbf{u}|^2\,d\sigma \leqslant C\int_{\partial D_{sr}^\pm} |\nabla\mathbf{v}|^2\,d\sigma + Cr^{n-1}|A|^2$$

$$\leqslant C\int_{\partial D_{sr}^\pm} \left|\frac{\partial\mathbf{u}}{\partial v}\right|^2 d\sigma + Cr^{n-1}|A|^2. \tag{5.10}$$

Since **w** is a linear function and thus harmonic, we have

$$\int_{D_{sr}^\pm} |\nabla\mathbf{w}|^2\,dx \leqslant \int_{\partial D_{sr}^\pm} |\mathbf{w}|\,|\nabla\mathbf{w}|\,d\sigma. \tag{5.11}$$

It follows that

$$|A| \leqslant \frac{C}{r^n}\int_{\partial D_{sr}^\pm} |\mathbf{w}|\,d\sigma \leqslant \frac{C}{r^n}\int_{\partial D_{sr}^\pm} (|\mathbf{u}| + |\mathbf{v}|)\,d\sigma$$

$$\leqslant \frac{C}{r}\left\{\frac{1}{r^{n-1}}\int_{\partial D_{sr}^\pm} |\mathbf{u}|^2\,d\sigma\right\}^{1/2} + C\left\{\frac{1}{r^{n-1}}\int_{\partial D_{sr}^\pm} \left|\frac{\partial\mathbf{u}}{\partial v}\right|^2 d\sigma\right\}^{1/2}. \tag{5.12}$$

This, together with (5.10), gives

$$\int_{\partial D_{sr}^\pm} |\nabla\mathbf{u}|^2\,d\sigma \leqslant C\int_{\partial D_{sr}^\pm} \left|\frac{\partial\mathbf{u}}{\partial v}\right|^2 d\sigma + \frac{C}{r^2}\int_{\partial D_{sr}^\pm} |\mathbf{u}|^2\,d\sigma. \tag{5.13}$$

By integrating both sides of (5.13) in $s \in (1, 3/2)$, we obtain

$$\int_{I_r} |\nabla \mathbf{u}_\pm|^2 \, d\sigma \leqslant C \int_{I_{2r}} \left| \frac{\partial \mathbf{u}_\pm}{\partial \nu} \right|^2 d\sigma + \frac{C}{r} \int_{D_{2r}^\pm} \left| \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right|^2 dx$$

$$+ \frac{C}{r^3} \int_{D_{2r}^\pm} |\mathbf{u}|^2 \, dx. \tag{5.14}$$

This replaces estimate (2.9). The extra term in (5.14) is harmless.

Finally in the proof of Lemma 2.4, we used estimate (2.5) to estimate the solid integral of $|\nabla \mathbf{u}|^2$ on $D_{sr}^\pm$. In the case of (5.1), we consider $\mathbf{v} = \mathbf{u} - Ax$, where

$$A = \frac{1}{2|D_{sr}^\pm|} \int_{D_{sr}^\pm} \left(\nabla \mathbf{u} - (\nabla \mathbf{u})^T \right) dx. \tag{5.15}$$

Then by Korn's inequality (see [9, Lemma 1.18]), we have

$$\int_{D_{sr}^\pm} |\nabla \mathbf{v}|^2 \, dx \leqslant C \int_{D_{sr}^\pm} \left| \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right|^2 dx. \tag{5.16}$$

Note that the integration by parts gives

$$|A| \leqslant \frac{C}{r^n} \int_{\partial D_{sr}^\pm} |\mathbf{u}| \, d\sigma. \tag{5.17}$$

It follows that

$$\int_{D_{sr}^\pm} |\nabla \mathbf{u}|^2 \, dx \leqslant C \int_{D_{sr}^\pm} \left| \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right|^2 dx + C r^n |A|^2$$

$$\leqslant C \int_{D_{sr}^\pm} \left| \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right|^2 dx + \frac{C}{r} \int_{\partial D_{sr}^\pm} |\mathbf{u}|^2 \, d\sigma. \tag{5.18}$$

We now integrate both sides of (5.18) in $s \in (1, 3/2)$. This yields

$$\int_{D_r^\pm} |\nabla \mathbf{u}|^2 \, dx \leqslant C \int_{D_{2r}^\pm} \left| \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right|^2 dx + \frac{C}{r} \int_{I_{2r}} |\mathbf{u}_\pm|^2 \, d\sigma$$

$$+ \frac{C}{r^2} \int_{D_{2r}^\pm} |\mathbf{u}|^2 \, dx. \tag{5.19}$$

Estimate (5.19), combined with (5.8), allows us to bound the solid integral of $|\nabla \mathbf{u}|^2$ in the same manner as in the strong elliptic case. Because of this, Lemma 2.4 and therefore Theorem 2.6 hold for the system of elastostatics. Consequently, Theorem 5.2 is proved using the same line of argument as in the proof of Theorem 3.1. We should point out that since $a_{ij}^{k\ell}$ satisfy the Legendre–Hadamard ellipticity condition, the L^p Dirichlet problem is solved for $2 < p < 2(n-1)/(n-3) + \varepsilon$ and $n \geqslant 4$ in [25]. This is used in the proof of Theorem 5.2. We omit the details.

We end this section with

Proof of Theorem 1.2. The existence follows from the invertibility of $(1/2)I + \mathcal{K}$ on $L_\psi^p(\partial\Omega)$ for p in the range given in (1.4). As in the case of Theorem 1.1, to prove the uniqueness, one constructs a matrix Neumann function $G_\nu^x(y) = \Gamma(x - y) - W^x(y)$, where W^x is a matrix whose ith row is an L^2 solution of (1.17) with the traction boundary data

$$\frac{\partial}{\partial \nu(y)}\{\Gamma_i(y - x)\} - \sum_{k=1}^{\frac{n(n+1)}{2}} C_{i,k}^x \{A_k y + \mathbf{b}_k\}. \tag{5.20}$$

Here $\{A_k y + \mathbf{b}_k, \ k = 1, 2, \ldots, n(n+1)/2\}$ is an orthonormal basis of Ψ with respect to the $L^2(\partial\Omega)$ norm, and

$$C_{i,k}^x = \int_{\partial\Omega} \frac{\partial}{\partial \nu(y)}\{\Gamma_i(y - x)\} \cdot (A_k y + \mathbf{b}_k)\, d\sigma(y) = -(A_k x + b_k)^i \tag{5.21}$$

so that the functions in (5.20) belong to $L_\psi^2(\partial\Omega)$. The same argument as in the proof of Theorem 1.1 shows that if $\mathcal{L}(\mathbf{u}) = \mathbf{0}$ in Ω, $(\nabla \mathbf{u})^* \in L^p(\partial\Omega)$ for some $p > 2(n-1)/(n+1) - \varepsilon$, and $\frac{\partial \mathbf{u}}{\partial \nu} = \mathbf{0}$ on $\partial\Omega$, then

$$\mathbf{u}(x) = -\int_{\partial\Omega} \frac{\partial G_\nu^x}{\partial \nu} \mathbf{u}\, d\sigma = (A_k x + \mathbf{b}_k) \int_{\partial\Omega} \{A_k y + \mathbf{b}_k\} \cdot \mathbf{u}(y)\, d\sigma(y). \tag{5.22}$$

Thus $\mathbf{u} \in \Psi$. This finishes the proof. □

6. Reverse Hölder inequalities for biharmonic functions

For simplicity, we will assume that $1/(1 - n) < \rho < 1$. Some modifications are needed in the case $\rho = 1/(1 - n)$. Following [32], we let

$$M_\rho(u) = \rho \Delta u + (1 - \rho)\frac{\partial^2 u}{\partial N^2} = \rho \Delta u + (1 - \rho)N_i N_j D_i D_j u,$$

$$K_\rho(u) = \frac{\partial \Delta u}{\partial N} + \frac{1}{2}(1 - \rho)\frac{\partial}{\partial T_{ij}}\left(\frac{\partial^2 u}{\partial N \partial T_{ij}}\right)$$

$$= \frac{\partial \Delta u}{\partial N} + \frac{1}{2}(1 - \rho)(N_i D_j - N_j D_i)(N_k(N_i D_j - N_j D_i)D_k u), \tag{6.1}$$

where $\frac{\partial}{\partial T_{ij}} = N_i D_j - N_j D_i$. Observe that $N_i N_j \frac{\partial u}{\partial T_{ij}} = 0$.

Assume $0 \in \partial\Omega$ and $\Omega \cap B(0, r_0)$ is given by (2.2). Let $W^{1,2}(I_r)$ denote the space of functions f on I_r such that $|\nabla_t f| \in L^2(I_r)$, where I_r is defined in (2.3). We will use the scale-invariant norm

$$\|f\|_{W^{1,2}(I_r)} = \left\{ \int\limits_{I_r} |\nabla_t f|^2 \, d\sigma + \frac{1}{r^2} \int\limits_{I_r} |f|^2 \, d\sigma \right\}^{1/2} \tag{6.2}$$

for $W^{1,2}(I_r)$, whose dual space is denoted by $W^{-1,2}(I_r)$.

The following is a boundary Cacciopoli inequality.

Lemma 6.1. *Suppose $\Delta^2 u = 0$ in $\Omega_\pm$ and $(\nabla\nabla u)^*_\pm \in L^2(I_{3r})$. Then*

$$\int\limits_{D_r^\pm} |\nabla\nabla u|^2 \, dx \leqslant C \|u\varphi\|_{W^{1,2}(I_{2r})} \|\varphi K_\rho(u)\|_{W^{-1,2}(I_{2r})}$$

$$+ C \left\| \frac{\partial(u\varphi^2)}{\partial N} \right\|_2 \|M_\rho(u)\|_{L^2(I_{2r})}$$

$$+ \frac{C}{r^2} \int\limits_{D_{2r}^\pm} |\nabla u|^2 \, dx + \frac{C}{r^2} \int\limits_{I_{2r}} |u||\nabla u| \, d\sigma, \tag{6.3}$$

where φ is a function in $C_0^\infty(B(0, (3/2)r))$ such that $\varphi = 1$ in $B(0, r)$, $0 \leqslant \varphi \leqslant 1$ and $|\nabla\varphi| \leqslant C/r$.

Proof. Let $v = u\varphi^2$. It follows from the integration by parts and $\Delta^2 u = 0$ in $\Omega_\pm$ that

$$\int\limits_{\partial\Omega} \left\{ v K_\rho(u) - \frac{\partial v}{\partial N} M_\rho(u) \right\} d\sigma$$

$$= \mp \int\limits_{\Omega_\pm} \left\{ (1 - \rho) D_i D_j v \cdot D_i D_j u + \rho \Delta v \cdot \Delta u \right\} dx. \tag{6.4}$$

Note that

$$D_i D_j v \cdot D_i D_j u = \varphi^2 |\nabla\nabla u|^2 + 4\varphi D_i u D_j \varphi \cdot D_i D_j u + u D_i D_j \varphi^2 \cdot D_i D_j u \tag{6.5}$$

and $\Delta v \cdot \Delta u = \varphi^2 |\Delta u|^2 + 4\varphi D_i u D_i \varphi \cdot \Delta u + u \Delta\varphi^2 \cdot \Delta u$. The second term in the right side of (6.5) can be absorbed by the first term using the Cauchy inequality with an ε. To handle the last term in the right side of (6.5), one uses the integration by parts again. This produces the last integral in (6.3). Finally, to finish the proof, we observe that

$$(1 - \rho)|\nabla\nabla u|^2 + \rho|\Delta u|^2 \geqslant c_\rho |\nabla\nabla u|^2, \tag{6.6}$$

if $1/(1 - n) < \rho < 1$ (see [32]). $\square$

Remark 6.2. If, in addition, in Lemma 6.1 we assume that $u_\pm = |\nabla u_\pm| = 0$ on I_{2r}, then

$$\int_{D_r^\pm} |\nabla\nabla u|^2 \, dx \leqslant \frac{C}{r^2} \int_{D_{2r}^\pm} |\nabla u|^2 \, dx. \tag{6.7}$$

This is the usual boundary Cacciopoli's inequality for the biharmonic equation.

Remark 6.3. It follows from (6.3) and the Cauchy inequality with an ε that

$$\int_{D_r^\pm} |\nabla\nabla u|^2 \, dx \leqslant \varepsilon r \big\| \varphi K_\rho(u) \big\|_{W^{-1,2}(\partial\Omega)}^2 + \varepsilon r \big\| M_\rho(u) \big\|_{L^2(I_{2r})}^2$$

$$+ \frac{C_\varepsilon}{r} \int_{I_{2r}} |\nabla u|^2 \, d\sigma + \frac{C}{r^2} \int_{D_{2r}^\pm} |\nabla u|^2 \, dx. \tag{6.8}$$

We remark that the integrals in (6.3) which involve $|u|^2$ on I_{2r} may be handled by replacing $|u|^2$ with $|u - c|^2$ and using the Poincaré inequality.

Our next lemma relies on the following Rellich type identity discovered by G. Verchota [32, pp. 232–233] for the biharmonic equation,

$$\frac{1}{2} \int_{\partial\Omega} \langle N, \alpha \rangle \big\{ (1 - \rho)|\nabla\nabla u|^2 + \rho|\Delta u|^2 \big\} \, d\sigma$$

$$= \int_{\partial\Omega} \frac{\partial}{\partial N} (\alpha \cdot \nabla u) M_\rho(u) \, d\sigma - \int_{\partial\Omega} (\alpha \cdot \nabla u) K_\rho(u) \, d\sigma$$

$$\pm (1 - \rho) \int_{\Omega_\pm} E_{ij}(\alpha, u) L_{ij}(u) \, dx, \tag{6.9}$$

where $L_{ij} = D_i D_j + \theta \delta_{ij} \Delta$ and

$$E_{ij}(\alpha, u) = \frac{1}{2} \operatorname{div}(\alpha) L_{ij}(u) - L_{ij}(\alpha) \cdot \nabla u - 2 D_i \alpha \cdot \nabla D_j u - 2\theta \delta_{ij} D_k \alpha \cdot \nabla D_k u.$$

In (6.9), $\alpha \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$ is a vector field and u is a suitable biharmonic function in $\Omega_\pm$. Also θ is related to ρ by $\rho = (n\theta + n\theta^2)/(1 + 2\theta + n\theta^2)$. With identity (6.9), Verchota was able to extend the method of layer potentials from second order equations and systems to the fourth order biharmonic equation. This identity will also play a crucial role in our study of the L^p biharmonic Neumann problem.

Lemma 6.4. *Under the same assumption as in Lemma* 6.1, *we have*

$$\int_{I_r} |\nabla\nabla u|^2 \, d\sigma \leqslant C \|\varphi K_\rho(u)\|^2_{W^{-1,2}(I_{2r})} + C \|M_\rho(u)\|^2_{L^2(I_{2r})}$$

$$+ \frac{C}{r^2} \int_{I_{2r}} |\nabla u|^2 \, d\sigma + \frac{C}{r} \int_{D^{\pm}_{2r}} |\nabla\nabla u|^2 \, dx + \frac{C}{r^3} \int_{D^{\pm}_{2r}} |\nabla u|^2 \, dx, \qquad (6.10)$$

where $\varphi \in C^\infty(B(0, (3/2)r))$ is the same function as in Lemma 6.1.

Proof. Let $\alpha = -\mathbf{e}_n \varphi^2$ where $\mathbf{e}_n = (0, \ldots, 0, 1)$. We apply the Rellich identity (6.9) on the Lipschitz domain $D^{\pm}_{sr}$, where $s \in (3/2, 2)$. Since $\langle N, -\mathbf{e}_n \rangle \geqslant c > 0$ on I_{2r}, this gives

$$c \int_{I_{sr}} |\varphi \nabla\nabla u|^2 \, d\sigma$$

$$\leqslant C \int_{\Omega_\pm \cap \partial D^{\pm}_{sr}} |\nabla\nabla u|^2 \, d\sigma + C \|\varphi \nabla u\|_{W^{1,2}(I_{2r})} \|\varphi K_\rho(u)\|_{W^{-1,2}(I_{2r})}$$

$$+ C \|\nabla(\alpha \cdot \nabla u)\|_{L^2(I_{2r})} \|M_\rho(u)\|_{L^2(I_{2r})}$$

$$+ \frac{C}{r} \int_{D^{\pm}_{2r}} |\nabla\nabla u|^2 \, dx + \frac{C}{r^3} \int_{D^{\pm}_{2r}} |\nabla u|^2 \, dx. \qquad (6.11)$$

Using the Cauchy inequality with an ε, it is not hard to see that the higher order terms in $\|\varphi \nabla u\|_{W^{1,2}(I_{2r})}$ and $\|\nabla(\alpha \cdot \nabla u)\|_{L^2(I_{2r})}$ may be absorbed by the left side of (6.11). Finally a familiar integration in s over $(3/2, 2)$ enables us to handle the first term in the right side of (6.10), as in Section 2. $\quad\square$

Remark 6.5. Suppose $\Delta^2 u = 0$ in $\Omega_\pm$ and $(\nabla\nabla u)^*_\pm \in L^2(I_{3r})$. If $u_\pm = |\nabla u_\pm| = 0$ on I_{2r}, then

$$\int_{I_r} |\nabla\nabla u|^2 \, d\sigma \leqslant \frac{C}{r^3} \int_{D^{\pm}_{2r}} |\nabla u|^2 \, dx. \qquad (6.12)$$

This follows from the regularity estimate [31]

$$\int_{\partial D^{\pm}_{sr}} |\nabla\nabla u|^2 \, d\sigma \leqslant C \int_{\partial D^{\pm}_{sr}} |\nabla_t \nabla u|^2 \, d\sigma, \qquad (6.13)$$

together with estimate (6.7), by an integration in $s \in (3/2, 2)$.

Recall that $(\nabla\nabla u)^* = \max\{(\nabla\nabla u)^*_+, (\nabla\nabla u)^*_-\}$ for functions u defined in $\mathbb{R}^n \setminus \partial\Omega$.

Lemma 6.6. *Suppose $\Delta^2 u = 0$ in $\mathbb{R}^n \setminus \partial\Omega$ and $(\nabla\nabla u)^* \in L^2(I_{32r})$. Assume that either $u_+ = |\nabla u_+| = 0$ or $u_- = |\nabla u_-| = 0$ on I_{32r}. Then*

$$\int_{I_r} |\nabla\nabla u_\pm|^2 \, d\sigma \leqslant \frac{C}{r^2} \int_{I_{8r}} \left\{ |\nabla u_+|^2 + |\nabla u_-|^2 \right\} d\sigma + \frac{C}{r^3} \int_{D_{16r}^+ \cup D_{16r}^-} |\nabla u|^2 \, dx$$

$$+ C \left\| \varphi_1 \left[K_\rho(u_+) - K_\rho(u_-) \right] \right\|_{W^{-1,2}(I_{4r})}^2$$

$$+ C \left\| \varphi_2 \left[K_\rho(u_+) - K_\rho(u_-) \right] \right\|_{W^{-1,2}(I_{4r})}^2$$

$$+ C \left\| M_\rho(u_+) - M_\rho(u_-) \right\|_{L^2(I_{4r})}^2, \tag{6.14}$$

where φ_1, φ_2 *are two functions in* $C_0^\infty(B(0, 4r))$ *with the properties that* $0 \leqslant \varphi_i \leqslant 1$ *and* $|\nabla\varphi_i| \leqslant C/r$ *for* $i = 1, 2$.

Proof. Assume that $u_+ = |\nabla u_+| = 0$ on I_{32r}. By (6.10) and (6.8), we obtain

$$\int_{I_r} |\nabla\nabla u_-|^2 \, d\sigma \leqslant C \left\| \varphi_1 K_\rho(u_-) \right\|_{W^{-1,2}(I_{4r})}^2 + C \left\| \varphi_2 K_\rho(u_-) \right\|_{W^{-1,2}(I_{4r})}^2$$

$$+ C \left\| M_\rho(u_-) \right\|_{L^2(I_{4r})}^2$$

$$+ \frac{C}{r^2} \int_{I_{4r}} |\nabla u_-|^2 \, d\sigma + \frac{C}{r^3} \int_{D_{4r}^-} |\nabla u|^2 \, dx \tag{6.15}$$

where $\varphi_1 \in C_0^\infty(B(0, (3/2)r))$ and $\varphi_2 \in C_0^\infty(B(0, 3r))$. In view of (6.14) and (6.15), we need to estimate $\|\varphi_i K_\rho(u_+)\|_{W^{-1,2}(I_{4r})}^2$, $i = 1, 2$, and $\|M_\rho(u_+)\|_{L^2(I_{4r})}^2$. Clearly, by Remark 6.5,

$$\left\| M_\rho(u_+) \right\|_{L^2(I_{4r})}^2 \leqslant C \int_{I_{4r}} |\nabla\nabla u_+|^2 \, d\sigma \leqslant \frac{C}{r^3} \int_{D_{8r}^+} |\nabla u|^2 \, dx. \tag{6.16}$$

Finally, since $\operatorname{supp}\varphi_i \subset B(0, 3r)$, the term $\|\varphi_i K_\rho(u_+)\|_{W^{-1,2}(I_{4r})}^2$ is bounded by

$$C \left\| \varphi_i \frac{\partial}{\partial N}(\Delta u_+) \right\|_{W^{-1,2}(I_{4r})} + C \|\nabla\nabla u_+\|_{L^2(I_{4r})}^2$$

$$\leqslant C \left\| \varphi_i \frac{\partial}{\partial N}(\Delta u_+) \right\|_{W^{-1,2}(\partial D_{sr})}^2 + C \|\nabla\nabla u_+\|_{L^2(I_{4r})}^2$$

$$\leqslant C \|\Delta u_+\|_{L^2(\partial D_{sr})}^2 + C \|\nabla\nabla u_+\|_{L^2(I_{4r})}^2$$

$$\leqslant C \int_{I_{5r}} |\nabla\nabla u_+|^2 \, d\sigma + C \int_{\Omega \cap \partial D_{sr}^+} |\nabla\nabla u|^2 \, d\sigma, \tag{6.17}$$

for any $s \in (4, 5)$, where we have used the L^2 regularity estimate in D_{sr}^+ for Laplace's equation in the second inequality. With (6.12) and (6.7) at our disposal, the desired estimate for $\|\varphi_i K_\rho(u_+)\|_{W^{-1,2}(I_{4r})}^2$ now follows from (6.17) by an integration in $s \in (4, 5)$. The case $u_- = |\nabla u_-| = 0$ on I_{32r} is exactly the same. This completes the proof. $\square$

As in Section 2, estimate (6.14) leads to a reverse Hölder inequality.

Theorem 6.7. *Under the same assumption as in Lemma* 6.6*, we have*

$$\left\{ \frac{1}{|I_r|} \int_{I_r} \left| (\nabla u)^* \right|^{p_n} d\sigma \right\}^{1/p_n} \leqslant C \left\{ \frac{1}{|I_{32r}|} \int_{I_{32r}} \left| (\nabla u)^* \right|^2 d\sigma \right\}^{1/2}$$

$$+ C \left\| \varphi_1 \left[K_\rho(u_+) - K_\rho(u_-) \right] \right\|^2_{W^{-1,2}(I_{4r})}$$

$$+ C \left\| \varphi_2 \left[K_\rho(u_+) - K_\rho(u_-) \right] \right\|^2_{W^{-1,2}(I_{4r})}$$

$$+ C \left\| M_\rho(u_+) - M_\rho(u_-) \right\|^2_{L^2(I_{4r})}, \tag{6.18}$$

where $p_n = 2(n-1)/(n-3)$ for $n \geqslant 4$. If $n = 2$ or 3, estimate (6.18) holds for any $2 < p_n < \infty$.

Proof. The proof is similar to that of Theorem 2.6 with ∇u in the place of **u**. We leave the details to the reader. However we should remark that the proof uses the solvability of the L^{p_n} Dirichlet problem for the biharmonic equation on any bounded Lipschitz domains. But this has been established in [18] for $n = 2$ or 3, and in [25] for $n \geqslant 4$. □

7. The L^p biharmonic Neumann problem

This section is devoted to the proof of Theorem 1.3. We begin with the definition of the biharmonic layer potentials introduced by Verchota in [32]. Fix $x \in \mathbb{R}^n$, let $B^x = B^x(y)$ denote the fundamental solution for operator Δ^2 with pole at x, given by

$$B^x(y) = \begin{cases} \dfrac{1}{2(n-2)(n-4)\omega_n} \cdot \dfrac{1}{|x-y|^{n-4}}, & n = 3 \text{ or } n \geqslant 5, \\[2mm] -\dfrac{1}{4\omega_4} \log |x-y|, & n = 4, \\[2mm] -\dfrac{1}{8\pi} |x-y|^2 \big(1 - \log |x-y|\big), & n = 2. \end{cases} \tag{7.1}$$

Given $(F, g) \in W^{1,p}(\partial\Omega) \times L^p(\partial\Omega)$ for $1 < p < \infty$, the double layer potential for the biharmonic equation is defined by

$$w(x) = \mathcal{D}_\rho(F, g)(x) = \int_{\partial\Omega} \left\{ K_\rho(B^x)(y) F(y) + M_\rho(B^x)(y) g(y) \right\} d\sigma(y), \tag{7.2}$$

for $x \in \mathbb{R}^n \setminus \partial\Omega$. Clearly $\Delta^2 w = 0$ in $\mathbb{R}^n \setminus \partial\Omega$. By computing $K_\rho(B^x)$ and $M_\rho(B^x)$ in (7.2), one may show that

$$w(x) = \int_{\partial\Omega} \left\{ \frac{\partial \Gamma^x}{\partial N} F + \Gamma^x g + (1-\rho) \frac{\partial}{\partial T_{jk}} D_k B^x \cdot \left(N_i \frac{\partial F}{\partial T_{ij}} - N_j g \right) \right\} d\sigma, \tag{7.3}$$

where $\Gamma^x = \Delta B^x$ is the fundamental solution for Δ with pole at x. Also

$$D_\ell w(x) = -\int_{\partial\Omega} \left\{ D_i \Gamma^x \cdot \frac{\partial F}{\partial T_{\ell i}} + D_\ell \Gamma^x \cdot g \right\} d\sigma$$

$$- (1-\rho) \int_{\partial\Omega} \left\{ \frac{\partial}{\partial T_{jk}} D_k D_\ell B^x \cdot \left(N_i \frac{\partial F}{\partial T_{ij}} - N_j g \right) \right\} d\sigma. \tag{7.4}$$

It follows by [4] that

$$\left\| (\nabla w)^* \right\|_p \leqslant C \left\{ \|\nabla_t F\|_p + \|g\|_p \right\}. \tag{7.5}$$

To compute the nontangential limits of w and ∇w, one uses

$$\lim_{\substack{x \to P \in \partial\Omega \\ x \in \Omega_\pm \cap \gamma(P)}} \int_{\partial\Omega} D_i D_j D_k B^x \cdot f \, d\sigma$$

$$= \pm \frac{1}{2} N_i N_j N_k f(P) + \text{p.v.} \int_{\partial\Omega} D_i D_j D_k B^P \cdot f \, d\sigma. \tag{7.6}$$

This, together with (7.3)–(7.4), gives

$$\left(w_\pm, -\frac{\partial w_\pm}{\partial N} \right) = \left(\pm \frac{1}{2} + \mathcal{K}_\rho^* \right)(F, g), \tag{7.7}$$

where $\mathcal{K}_\rho^*$ is a bounded operator on $W^{1,p}(\partial\Omega) \times L^p(\partial\Omega)$.

For $(\Lambda, f) \in W^{-1,p}(\partial\Omega) \times L^p(\partial\Omega)$ with $1 < p < \infty$, the single layer potential is defined by

$$v(x) = \mathcal{S}(\Lambda, f)(x) = \Lambda \left(B^x(\cdot) \right) - \int_{\partial\Omega} \frac{\partial B^x}{\partial N} f \, d\sigma. \tag{7.8}$$

Clearly $\Delta^2 v = 0$ in $\mathbb{R}^n \setminus \partial\Omega$. By writing $\Lambda = \frac{\partial h_{ij}}{\partial T_{ij}} + h_0$ with $h_{ij}, h_0 \in L^p(\partial\Omega)$ so that

$$\Lambda(B^x) = \int_{\partial\Omega} \left\{ -\frac{\partial B^x}{\partial T_{ij}} h_{ij} + B^x h_0 \right\} d\sigma, \tag{7.9}$$

one sees that

$$\left\| (\nabla\nabla v)^* \right\|_p \leqslant C \left\{ \|\Lambda\|_{W^{-1,p}(\partial\Omega)} + \|f\|_p \right\} \tag{7.10}$$

for $1 < p < \infty$ by [4]. Also

$$\left(K_\rho(v)_\pm, M_\rho(v)_\pm \right) = \left(\mp \frac{1}{2} I + \mathcal{K}_\rho \right)(\Lambda, f) \tag{7.11}$$

where operator $\mathcal{K}_\rho$, whose adjoint is $\mathcal{K}_\rho^*$ in (7.4), is bounded on $W^{-1,p}(\partial\Omega) \times L^p(\partial\Omega)$. We point out that the trace of $K_\rho(v)_\pm$ in (7.11) is taken in the sense of distribution, i.e.,

$$K_\rho(v)_\pm(\phi) = \lim_{k\to\infty} \int_{\partial\Omega_k^\pm} K_\rho(v)\phi\,d\sigma, \qquad (7.12)$$

for $\phi \in C_0^1(\mathbb{R}^n)$, where $\Omega_k^\pm$ is a sequence of smooth domains which approximate $\Omega_\pm$ from inside, respectively [30]. Because of (7.6), to prove (7.11), we only need to take care of the term $\frac{\partial}{\partial N}\Delta v$. To do this, we note that

$$\Delta v = -\int_{\partial\Omega} \left\{ \frac{\partial\Gamma^x}{\partial T_{ij}} h_{ij} + \frac{\partial\Gamma^x}{\partial N} f \right\} d\sigma + \int_{\partial\Omega} \Gamma^x h_0 \, d\sigma$$

$$= D_j \int_{\partial\Omega} \Gamma^x \{ N_i h_{ij} - N_i h_{ji} + N_j f \} d\sigma + \int_{\partial\Omega} \Gamma^x h_0 \, d\sigma. \qquad (7.13)$$

This allows us to express $\frac{\partial}{\partial N}\Delta v$ on $\partial\Omega_k$ in terms of tangential derivatives plus a higher order term,

$$\frac{\partial\Delta v}{\partial N} = \frac{\partial}{\partial T_{\ell j}} D_\ell \int_{\partial\Omega} \Gamma^x \{ N_i h_{ij} - N_i h_{ji} + N_j f \} d\sigma + \frac{\partial}{\partial N} \int_{\partial\Omega} \Gamma^x h_0 \, d\sigma. \qquad (7.14)$$

We remark that the computation of the trace operators in [32] used the harmonic extension of functions in $W^{1,p'}(\partial\Omega)$ to Ω. On general Lipschitz domains, this would require $p > 2 - \varepsilon$.

Let $\mathbf{X}^p(\partial\Omega)$ denote the subspace of $W^{-1,p}(\partial\Omega) \times L^p(\partial\Omega)$ whose elements (Λ, f) satisfy

$$\Lambda(1) = 0 \quad \text{and} \quad \Lambda(x_j) = \int_{\partial\Omega} f N_j \, d\sigma \quad \text{for } j = 1, \dots, n. \qquad (7.15)$$

One of the main results in [32] is that

$$\frac{1}{2} I + \mathcal{K}_\rho : W^{-1,p}(\partial\Omega) \times L^p(\partial\Omega) \to W^{-1,p}(\partial\Omega) \times L^p(\partial\Omega),$$

$$-\frac{1}{2} I + \mathcal{K}_\rho : \mathbf{X}^p(\partial\Omega) \to \mathbf{X}^p(\partial\Omega) \qquad (7.16)$$

are isomorphisms for $p \in (2 - \varepsilon, 2 + \varepsilon)$. Let $\{(\Lambda_j^*, f_j^*): \ j = 0, 1, \dots, n\}$ be the set of the affine equilibrium distributions (see [32, p. 261]). This set spans the kernel of $-(1/2)I + \mathcal{K}_\rho$ on $W^{-1,2}(\partial\Omega) \times L^2(\partial\Omega)$. It follows from (7.16) and duality that for p close to 2,

$$\frac{1}{2} I + \mathcal{K}_\rho^* : W^{1,p}(\partial\Omega) \times L^p(\partial\Omega) \to W^{1,p}(\partial\Omega) \times L^p(\partial\Omega),$$

$$-\frac{1}{2} I + \mathcal{K}_\rho^* : \mathbf{Z}^p(\partial\Omega) \to \mathbf{Z}^p(\partial\Omega), \qquad (7.17)$$

are isomorphisms, where $\mathbf{Z}^p(\partial\Omega)$ is a subspace of $W^{1,p}(\partial\Omega) \times L^p(\partial\Omega)$ whose elements (F, g) satisfy

$$\Lambda_j^*(F) + \int_{\partial\Omega} f_j^* g \, d\sigma = 0 \quad \text{for } j = 0, 1, \dots, n. \tag{7.18}$$

Note that $\mathbf{Z}^p(\partial\Omega)$ is well defined for $p > 2 - \varepsilon$.

Theorem 7.1. *There exists $\varepsilon > 0$ such that the operators in (7.17) are isomorphisms for $2 < p < 2(n-1)/(n-3) + \varepsilon$ and $n \geqslant 4$. If $n = 2$ or 3, the operators in (7.17) are isomorphisms for any $2 < p < \infty$.*

Theorem 7.1 follows from Theorem 6.7 by the same line of argument that we used to prove Theorem 3.1. To carry out the proof, we need to compute the Neumann trace of the double layer potential. Let $WA_2^p(\partial\Omega)$ denote the space of Whitney arrays $\dot{f} = \{f_0, f_1, \dots, f_n\} \subset W^{1,p}(\partial\Omega)$ which satisfy the compatibility conditions $\frac{\partial f_0}{\partial T_{ij}} = N_i f_j - N_j f_i$ for $1 \leqslant i < j \leqslant n$ [31].

Lemma 7.2. *Let $\dot{f} = \{f_0, f_1, \dots, f_n\} \in WA_2^p(\partial\Omega)$. Let $w(x) = \mathcal{D}_\rho(F, g)$ with $F = f_0$ and $g = -N_i f_i$. Then $(\nabla\nabla w)^* \in L^p(\partial\Omega)$ and*

$$\big(K_\rho(w)_+, M_\rho(w)_+\big) = \big(K_\rho(w)_-, M_\rho(w)_-\big), \tag{7.19}$$

on $\partial\Omega$.

Proof. Using (7.4) and the compatibility conditions, we have

$$D_\ell w(x) = -\int_{\partial\Omega} \left\{ D_i \Gamma^x \cdot \frac{\partial F}{\partial T_{\ell i}} + D_\ell \Gamma^x \cdot g + (1-\rho)\frac{\partial}{\partial T_{ik}} D_k D_\ell B^x \cdot f_i \right\} d\sigma$$

$$= \int_{\partial\Omega} \frac{\partial \Gamma^x}{\partial N} f_\ell \, d\sigma + \int_{\partial\Omega} \left\{ \Gamma^x \cdot \frac{\partial f_i}{\partial T_{\ell i}} + (1-\rho) D_k D_\ell B^x \cdot \frac{\partial f_i}{\partial T_{ik}} \right\} d\sigma. \tag{7.20}$$

It follows that

$$D_j D_\ell w(x) = \int_{\partial\Omega} \left\{ D_i \Gamma^x \cdot \frac{\partial f_\ell}{\partial T_{ij}} + D_j \Gamma^x \cdot \frac{\partial f_i}{\partial T_{i\ell}} + (1-\rho) D_j D_k D_\ell B^x \cdot \frac{\partial f_i}{\partial T_{ki}} \right\} d\sigma. \tag{7.21}$$

By [4], this implies $\|(\nabla\nabla w)^*\|_p \leqslant C \sum_i \|\nabla_t f_i\|_p < \infty$. Also it follows from (7.6) that

$$D_j D_\ell w_+ - D_j D_\ell w_- = N_i \frac{\partial f_\ell}{\partial T_{ij}} + N_j \frac{\partial f_i}{\partial T_{i\ell}} + (1-\rho) N_j N_k N_\ell \frac{\partial f_i}{\partial T_{ki}}. \tag{7.22}$$

This yields that $M_\rho(w)_+ = M_\rho(w)_-$ on $\partial\Omega$ by a simple computation. To find $K_\rho(w)_\pm = \frac{\partial}{\partial N}\Delta w_\pm + (1-\rho)\frac{\partial}{\partial T_{ij}}(N_\ell N_i D_j D_\ell w)_\pm$ on $\partial\Omega$, we note that by (7.21),

$$\Delta w(x) = (1 - \rho) \int_{\partial \Omega} D_j \Gamma^x \cdot \frac{\partial f_i}{\partial T_{ji}} \, d\sigma. \tag{7.23}$$

Thus we may write

$$\frac{\partial \Delta w}{\partial N} = (1 - \rho) \frac{\partial}{\partial T_{\ell j}} \int_{\partial \Omega} D_\ell \Gamma^x \cdot \frac{\partial f_i}{\partial T_{ji}} \, d\sigma. \tag{7.24}$$

It then follows from (7.24), (7.21) and (7.6) that

$$
\begin{aligned}
& \left[K_\rho(w)_+ - K_\rho(w)_- \right](\phi) \\
&= (1 - \rho) \int_{\partial \Omega} N_\ell \frac{\partial f_i}{\partial T_{ji}} \cdot \frac{\partial \phi}{\partial T_{j\ell}} \, d\sigma \\
&\quad + (1 - \rho) \int_{\partial \Omega} N_\ell N_i \left\{ N_m \frac{\partial f_\ell}{\partial T_{mj}} + N_j \frac{\partial f_m}{\partial T_{m\ell}} + (1 - \rho) N_j N_k N_\ell \frac{\partial f_m}{\partial T_{km}} \right\} \frac{\partial \phi}{\partial T_{ji}} \, d\sigma \\
&= (1 - \rho) \int_{\partial \Omega} \left\{ N_\ell \frac{\partial f_i}{\partial T_{ji}} \cdot \frac{\partial \phi}{\partial T_{j\ell}} + N_\ell N_i N_m \frac{\partial f_\ell}{\partial T_{mj}} \cdot \frac{\partial \phi}{\partial T_{ji}} \right\} d\sigma \\
&= \int_{\partial \Omega} \left\{ N_i N_j \frac{\partial f_\ell}{\partial T_{j\ell}} - \frac{\partial f_\ell}{\partial T_{i\ell}} - N_\ell N_m \frac{\partial f_\ell}{\partial T_{mi}} \right\} D_i \phi \, d\sigma \\
&= 0,
\end{aligned}
$$

where we have used the compatibility condition

$$N_i \frac{\partial f_\ell}{\partial T_{jk}} = N_k \frac{\partial f_\ell}{\partial T_{ji}} - N_j \frac{\partial f_\ell}{\partial T_{ki}}$$

for $k = \ell$ in the last step. This finishes the proof. $\quad\square$

Proof of Theorem 7.1. We will give the proof of the invertibility of $-(1/2)I + \mathcal{K}_\rho^*$ on $\mathbf{Z}^p(\partial \Omega)$. The case for $(1/2)I + \mathcal{K}_\rho^*$ on $W^{1,p}(\partial \Omega) \times L^p(\partial \Omega)$ is similar.

Let $(G, h) \in \mathbf{Z}^p(\partial \Omega)$ for some $2 < p < \infty$. Since $-(1/2)I + \mathcal{K}_\rho^*$ is invertible on $\mathbf{Z}^2(\partial \Omega)$, there exists $(F, g) \in \mathbf{Z}^2(\partial \Omega)$ so that $(-(1/2)I + \mathcal{K}_\rho^*)(F, g) = (G, h)$. Let $u(x) = \mathcal{D}_\rho(F, g)$ be the double layer potential. We will show that if $n \geqslant 4$ and $2 < p < p_n + \varepsilon$, or if $n = 2, 3$ and $2 < p < \infty$,

$$
\left\{ \frac{1}{s^{n-1}} \int_{B(P,s) \cap \partial \Omega} \left| (\nabla u)^* \right|^p \, d\sigma \right\}^{1/p} \leqslant C \left\{ \frac{1}{s^{n-1}} \int_{B(P,Cs) \cap \partial \Omega} \left| (\nabla u)^* \right|^2 \, d\sigma \right\}^{1/2}
$$

$$
+ C \left\{ \frac{1}{s^{n-1}} \int_{B(P,Cs) \cap \partial \Omega} \left(|\nabla_t G| + |h| \right)^p \, d\sigma \right\}^{1/p}, \tag{7.25}
$$

for any $P \in \partial\Omega$ and $s > 0$ small. Since $(F, g) = (u_+ - u_-, -\frac{\partial u_+}{\partial N} + \frac{\partial u_-}{\partial N})$, by covering $\partial\Omega$ with a finite number of small balls, we obtain

$$
\begin{aligned}
\|\nabla_t F\|_p + \|g\|_p &\leqslant C \|(\nabla u)^*\|_p \leqslant C \{ \|(\nabla u)^*\|_2 + \|\nabla_t G\|_p + \|h\|_p \} \\
&\leqslant C \{ \|\nabla_t F\|_2 + \|g\|_2 + \|\nabla_t G\|_p + \|h\|_p \} \\
&\leqslant C \{ \|\nabla_t G\|_p + \|h\|_p \}.
\end{aligned}
\tag{7.26}
$$

This shows that $-(1/2)I + \mathcal{K}_\rho^*$ is invertible on $\mathbf{Z}^p(\partial\Omega)$. Note that by a density argument, we may assume that $(G, h) = (f_0, -f_i N_i)$ for some $\{f_0, f_1, \ldots, f_n\} \in WA_2^2(\partial\Omega)$. This would imply that $(F, g) = (\tilde{f}_0, -\tilde{f}_i N_i)$ for some $\{\tilde{f}_0, \tilde{f}_1, \ldots, \tilde{f}_n\} \in WA_2^2(\partial\Omega)$ by [32, p. 265]. Consequently $(\nabla\nabla u)^* \in L^2(\partial\Omega)$ by Lemma 7.2.

To establish estimate (7.25), we may assume that $P = 0$ and $B(0, r_0) \cap \Omega$ is given by (2.2). Let $Q_0 = I_s$ be a surface cube defined in (2.3). For any subcube Q of Q_0, we choose a function $\varphi = \varphi_Q \in C_0^2(\mathbb{R}^n)$ such that $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ in $100Q$, $\varphi = 0$ outside of $200Q$, and $|\nabla\varphi| \leqslant C/r$, $|\nabla\nabla\varphi| \leqslant C/r^2$ where r is the diameter of Q. Let

$$
\beta = \frac{1}{|200Q|} \int\limits_{200Q} G \, d\sigma.
\tag{7.27}
$$

Since

$$
W^{1,2}(\partial\Omega) \times L^2(\partial\Omega) = \mathbf{Z}^2(\partial\Omega) \oplus \operatorname{span}\{(1, 0), (x_j, -N_j), \ j = 1, \ldots, n\},
\tag{7.28}
$$

there exist $(F_Q, g_Q) \in \mathbf{Z}^2(\partial\Omega)$ and $(\alpha_0, \alpha_1, \ldots, \alpha_n) \in \mathbb{R}^{n+1}$ such that

$$
\begin{aligned}
&\left((G - \beta)\varphi, -h\varphi - (G - \beta)\frac{\partial\varphi}{\partial N} \right) \\
&\quad = \left(-\frac{1}{2}I + \mathcal{K}_\rho^* \right)(F_Q, g_Q) + \alpha_0(1, 0) + \alpha_j(x_j, -N_j), \\
&\left\| (G - \beta)\varphi \right\|_{W^{1,2}(\partial\Omega)} + \left\| h\varphi + (G - \beta)\frac{\partial\varphi}{\partial N} \right\|_2 \\
&\quad \sim \|F_Q\|_{W^{1,2}(\partial\Omega)} + \|g_Q\|_2 + \sum_{j=0}^{n} |\alpha_j|.
\end{aligned}
\tag{7.29}
$$

Let $v(x) = \mathcal{D}_\rho(F_Q, g_Q) + \alpha_0 + \alpha_j x_j$ and $w = u - v - \beta = \mathcal{D}_\rho(F - F_Q, g - g_Q) - \beta$. Note that

$$
\left(w_-, -\frac{\partial w_-}{\partial N} \right) = \left((G - \beta)(1 - \varphi), -h(1 - \varphi) + (G - \beta)\frac{\partial\varphi}{\partial N} \right).
\tag{7.30}
$$

Thus $w_- = |\nabla w_-| = 0$ on $100Q$. Since $(-(1/2)I + \mathcal{K}_\rho^*)(F_Q, g_Q)$ is given by an array in $WA_2^2(\partial\Omega)$, we may deduce that (F_Q, g_Q) is also given by an array in $WA_2^2(\partial\Omega)$. It fol-

lows from Lemma 7.2 that $(\nabla\nabla w)^* \in L^2(\partial\Omega)$ and $(M_\rho(w)_+, K_\rho(w)_+) = (M_\rho(w)_-, K_\rho(w)_-)$ on $\partial\Omega$. This allows us to apply Theorem 6.7. We obtain

$$\left\{ \frac{1}{|Q'|} \int_{Q'} |(\nabla w)^*|^{p_n} \, d\sigma \right\}^{1/p_n} \leqslant C \left\{ \frac{1}{|32Q'|} \int_{32Q'} |(\nabla w)^*|^2 \, d\sigma \right\}^{1/2} \tag{7.31}$$

for any subcube Q' of Q. Since the reverse Hölder inequality (7.31) is self-improving [14], in the case $n \geqslant 4$, this means that there exists $\varepsilon > 0$ depending only on $\|\psi\|_\infty$, n and the constant C in (7.31) so that

$$\left\{ \frac{1}{|Q|} \int_Q |(\nabla w)^*|^{\bar{p}} \, d\sigma \right\}^{1/\bar{p}} \leqslant C \left\{ \frac{1}{|64Q|} \int_{64Q} |(\nabla w)^*|^2 \, d\sigma \right\}^{1/2}$$

$$\leqslant C \left\{ \frac{1}{|64Q|} \int_{64Q} |(\nabla u)^*|^2 \, d\sigma \right\}^{1/2}$$

$$+ C \left\{ \frac{1}{|64Q|} \int_{64Q} |(\nabla v)^*|^2 \, d\sigma \right\}^{1/2}, \tag{7.32}$$

where $\bar{p} = p_n + \varepsilon$.

Finally we note that by (7.29)

$$\left\{ \int_{\partial\Omega} |(\nabla v)^*|^2 \, d\sigma \right\}^{1/2} \leqslant C \left\{ \int_{\partial\Omega} \left(|\nabla_t F_Q| + |g_Q| \right)^2 \, d\sigma \right\}^{1/2} + \sum_{j=1}^n |\alpha_j|$$

$$\leqslant C \left\{ \int_{200Q} \left(|\nabla_t G| + |h| \right)^2 \, d\sigma \right\}^{1/2}, \tag{7.33}$$

where we also used the Poincaré inequality. With (7.33) and (7.32), estimate (7.25) follows by Theorem 3.2. This completes the proof of Theorem 7.1. □

Remark 7.3. The L^p Dirichlet problem for the biharmonic equation

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u = F \in W^{1,p}(\partial\Omega), \quad \dfrac{\partial u}{\partial N} = g \in L^p(\partial\Omega) & \text{on } \partial\Omega, \\ (\nabla u)^* \in L^p(\partial\Omega), \end{cases} \tag{7.34}$$

is uniquely solvable if

$$
\begin{aligned}
&n = 2, 3, && 2 - \varepsilon < p \leqslant \infty, \\
&n = 4, && 2 - \varepsilon < p < 6 + \varepsilon, \\
&n = 5, 6, 7, && 2 - \varepsilon < p < 4 + \varepsilon, \\
&n \geqslant 8, && 2 - \varepsilon < p < 2 + \frac{4}{n - \lambda_n} + \varepsilon,
\end{aligned}
\tag{7.35}
$$

where $\lambda_n = (n + 10 + 2\sqrt{2(n^2 - n + 2)})/7$. See [8,18,25,27]. The ranges of p's in (7.35) are known to be sharp in the case $2 \leqslant n \leqslant 7$ [18]. This implies that the ranges of p's in Theorem 7.1 are sharp for $n = 2, 3, 4, 5$.

Corollary 7.4. *Let $2 < p < p_n + \varepsilon$ for $n \geqslant 4$ and $2 < p < \infty$ for $n = 2$ or 3. The unique solution to the Dirichlet problem* (7.34) *for the biharmonic equation with boundary data* (F, g) *is given by*

$$
u(x) = \mathcal{D}_\rho \left(\left(\frac{1}{2} I + \mathcal{K}_\rho^* \right)^{-1} (F, g) \right).
\tag{7.36}
$$

By duality and an argument similar to that in the proof of Theorem 5.1, we may deduce the following from Theorem 7.1.

Theorem 7.5. *There exists $\varepsilon > 0$ such that the operators $\pm(1/2)I + \mathcal{K}_\rho$ in* (7.16) *are isomorphisms for $n \geqslant 4$ and $2(n-1)/(n+1) - \varepsilon < p < 2$. If $n = 2$ or 3, the operators are isomorphisms for $1 < p < 2$.*

Proof of Theorem 1.3. The existence follows from the invertibility of $-(1/2)I + \mathcal{K}_\rho$ on $\mathbf{X}^p(\partial\Omega)$, while the uniqueness was proved in [32, p. 273] by constructing a Neumann function. $\quad\square$

8. The classical layer potentials on weighted spaces

In this section we consider the classical layer potentials for Laplace's equation $\Delta u = 0$ in Ω. In order to be consistent with our notation for elliptic systems, we shall use the fundamental solution for $\mathcal{L} = -\Delta$ in the definitions of single and double layer potentials. It is well known that the operators $(1/2)I + \mathcal{K} : L_0^p(\partial\Omega) \to L_0^p(\partial\Omega)$ and $-(1/2)I + \mathcal{K} : L^p(\partial\Omega) \to L^p(\partial\Omega)$ are isomorphisms for $n \geqslant 2$ and $1 < p < 2 + \varepsilon$. The case $p = 2$ was proved in [30], using Rellich identities as we indicated in Section 1. The sharp range $1 < p < 2 + \varepsilon$ was obtained in [6]. This was done by establishing L^1 estimates for solutions of the Neumann and regularity problems with boundary data in the atomic Hardy spaces. It follows by duality that $(1/2)I + \mathcal{K}^*$ and $-(1/2)I + \mathcal{K}^*$ are isomorphisms on $L^p(\partial\Omega)/\{h_0\}$ and $L^p(\partial\Omega)$ respectively, where $2 - \varepsilon_1 < p < \infty$ and h_0 is a function which spans the kernel of $(1/2)I + \mathcal{K}$ on $L^2(\partial\Omega)$.

With the method from previous sections, it is possible to recover the sharp L^p invertibility in [6] without the use of the Hardy spaces. To do this, we will prove directly that $(1/2)I + \mathcal{K}^* : L^p(\partial\Omega)/\{h_0\} \to L^p(\partial\Omega)/\{h_0\}$ and $-(1/2)I + \mathcal{K}^* : L^p(\partial\Omega) \to L^p(\partial\Omega)$ are invertible for $2 - \varepsilon_1 < p < \infty$. In fact we shall prove a stronger result. Let $\mathcal{X}^2(\partial\Omega, \omega \, d\sigma)$ denote the space of functions f in $L^2(\partial\Omega, \omega \, d\sigma)$ such that $\int_{\partial\Omega} f h_0 \, d\sigma = 0$.

Theorem 8.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geqslant 3$, with a connected boundary. Then there exists $\delta \in (0, 1]$ depending only on n and the Lipschitz character of Ω such that the operators*

$$(1/2)I + \mathcal{K}^* : \mathcal{X}^2(\partial\Omega, \omega\,d\sigma) \to \mathcal{X}^2(\partial\Omega, \omega\,d\sigma),$$

$$-(1/2)I + \mathcal{K}^* : L^2(\partial\Omega, \omega\,d\sigma) \to L^2(\partial\Omega, \omega\,d\sigma), \tag{8.1}$$

are isomorphisms for any $A_{1+\delta}$ weight ω on $\partial\Omega$.

We refer the reader to [29] for the theory of A_p weights. In particular the boundedness of operator $\mathcal{K}^*$ on $L^2(\partial\Omega, \omega\,d\sigma)$ with $\omega \in A_2(\partial\Omega)$ follows from [4] and the standard weighted inequalities for Calderón–Zygmund operators. Also, by the Hölder inequality, $L^2(\partial\Omega, \omega\,d\sigma) \subset L^p(\partial\Omega)$ if $\omega \in A_{1+\delta}(\partial\Omega)$ and $p = 2/(1+\delta)$. Since $h_0 \in L^q(\partial\Omega)$ for some $q > 2$, this implies that the space $\mathcal{X}^2(\partial\Omega, \omega\,d\sigma)$ is well defined if $\omega \in A_{1+\delta}$ and $\delta > 0$ is sufficiently small.

Note that by an extrapolation theorem of Rubio de Francia (see e.g. [10]), Theorem 8.1 yields the L^p invertibility of $\pm(1/2)I + \mathcal{K}^*$ for the sharp range $2 - \varepsilon < p < \infty$. Furthermore, by duality, we obtain the following.

Theorem 8.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geqslant 3$, with a connected boundary. Then there exists $\delta \in (0, 1]$ depending only on n and the Lipschitz character of Ω such that the operators $(1/2)I + \mathcal{K}$ and $-(1/2)I + \mathcal{K}$ are isomorphisms on $L_0^2(\partial\Omega, \frac{d\sigma}{\omega})$ and $L^2(\partial\Omega, \frac{d\sigma}{\omega})$ respectively, for any $A_{1+\delta}$ weight ω on $\partial\Omega$.*

Here $L_0^2(\partial\Omega, \frac{d\sigma}{\omega})$ denotes the space of functions f in $L^2(\partial\Omega, \frac{d\sigma}{\omega})$ such that $\int_{\partial\Omega} f\,d\sigma = 0$. To prove Theorem 8.2, one uses the fact that $L^2(\partial\Omega, \omega\,d\sigma) = \mathcal{X}^2(\partial\Omega, \omega\,d\sigma) \oplus \mathbb{R}$ and precedes as in the proof of Theorem 4.1.

As in the L^p case, the invertibility of $(1/2)I + \mathcal{K}$ on $L^2(\partial\Omega, \frac{d\sigma}{\omega})$ gives us the existence for the Neumann problem with boundary data in the weighted L^2 space. Since $L^2(\partial\Omega, \frac{d\sigma}{\omega}) \subset L^p(\partial\Omega)$ for some $p > 1$, the uniqueness follows from the uniqueness for the L^p Neumann problem [6].

Corollary 8.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geqslant 3$, with a connected boundary. Then there exists $\delta \in (0, 1]$ depending only on n and the Lipschitz character of Ω such that given any $g \in L_0^2(\partial\Omega, \frac{d\sigma}{\omega})$ with $\omega \in A_{1+\delta}(\partial\Omega)$, there exists a harmonic function u on Ω, unique up to constants, such that $\frac{\partial u}{\partial N} = g$ and $(\nabla u)^* \in L^2(\partial\Omega, \frac{d\sigma}{\omega})$. Moreover, the solution u satisfies*

$$\left\| (\nabla u)^* \right\|_{L^2(\partial\Omega, \frac{d\sigma}{\omega})} \leqslant C \|g\|_{L^2(\partial\Omega, \frac{d\sigma}{\omega})}, \tag{8.2}$$

and is given by the single layer potential with density $((1/2)I + \mathcal{K})^{-1}(g)$.

Remark 8.4. The condition $\omega \in A_{1+\delta}$ in Theorems 8.1 and 8.2 (and in Corollary 8.3) is sharp in the context of A_p weights. This is because they imply the sharp ranges of p's for the L^p invertibility. However in the case $n \geqslant 4$, there are weights ω which are not in the sharp A_p class and for which $\pm(1/2)I + \mathcal{K}$ are invertible on $L^2(\partial\Omega, \frac{d\sigma}{\omega})$. Indeed, consider the power weight $\omega_\alpha = |Q - Q_0|^\alpha$, where $Q_0 \in \partial\Omega$ and $\alpha > 1 - n$. It is shown in [24] that $(1/2)I + \mathcal{K}$ and $-(1/2)I + \mathcal{K}$ are invertible on $L_0^2(\partial\Omega, \frac{d\sigma}{\omega_\alpha})$ and $L^2(\partial\Omega, \frac{d\sigma}{\omega_\alpha})$ respectively, if $1 - n < \alpha < n - 3 + \varepsilon$. However we observe that $\omega_\alpha \in A_{1+\delta}$ if and only if $1 - n < \alpha < (n-1)\delta$.

It remains to prove Theorem 8.1. To do this, we need to establish a reverse Hölder inequality similar to (2.28), but with p_n replaced by any exponent $p > 2$. Since $|\nabla u|$ on the boundary is only L^q integrable for some $q > 2$, the Sobolev inequality is not useful in higher dimensions. Instead we use the following Morrey space estimate (see e.g. [14, Ch. 3]),

$$\sup_{I(P_0,R)} |u| \leqslant \frac{C}{R^{n-1}} \int\limits_{I(P_0,2R)} |u| \, d\sigma + C_\lambda R^{\frac{\lambda-n+3}{2}} \sup_{\substack{0<r<R \\ P \in I(P_0,R)}} \left\{ r^{-\lambda} \int\limits_{I(P,r)} |\nabla_t u|^2 \, d\sigma \right\}^{1/2} \quad (8.3)$$

where $\lambda > n - 3$ and $I(P,r) = B(P,r) \cap \partial\Omega$ for $P \in \partial\Omega$ and $0 < r < r_0$.

Assume $0 \in \partial\Omega$ and $\Omega \cap B(0, r_0)$ is given by (2.2).

Lemma 8.5. *Suppose $\Delta u = 0$ in $\Omega_\pm$. Assume that $(\nabla u)_\pm^* \in L^2(I_{4R})$ and $u_\pm = 0$ on I_{4R} for some $0 < 4R < cr_0$. Then there exists $\lambda > n - 3$ depending only on n and Ω such that*

$$\sup_{0<r<R} r^{-\lambda} \int\limits_{I_r} |\nabla u_\pm|^2 \, d\sigma \leqslant \frac{C}{R^{\lambda+3}} \int\limits_{D_{4R}^\pm} |u|^2 \, dx. \quad (8.4)$$

Proof. Since $u_\pm = 0$ in I_{4R}, we may use (2.10) and (2.5) to obtain

$$\int\limits_{I_r} |\nabla u_\pm|^2 \, d\sigma \leqslant \frac{C}{r^3} \int\limits_{D_{4r}^\pm} |u|^2 \, dx. \quad (8.5)$$

By the boundary Hölder estimates, we have

$$|u(x)|^2 \leqslant C \left(\frac{r}{R} \right)^\delta \frac{1}{R^n} \int\limits_{D_{4R}^\pm} |u|^2 \, dx, \quad (8.6)$$

for any $x \in D_{4r}^\pm$, where $\delta > 0$ depends only on n and Ω. Estimate (8.4) with $\lambda = n - 3 + \delta$ now follows easily from (8.5) and (8.6).   □

Lemma 8.6. *Suppose that $\Delta u = 0$ in $\Omega_\pm$ and $(\nabla u)_\pm^* \in L^2(I_{4R})$ for some $0 < 4R < cr_0$. Then there exists $\lambda > n - 3$ depending only on n and Ω such that*

$$\sup_{0<r<R} r^{-\lambda} \int\limits_{I_r} |\nabla u_\pm|^2 \, d\sigma$$

$$\leqslant C \sup_{0<r<2R} r^{-\lambda} \int\limits_{I_r} \left| \frac{\partial u_\pm}{\partial N} \right|^2 \, d\sigma + \frac{C}{R^{\lambda+3}} \int\limits_{D_{4R}^\pm} |u|^2 \, dx + \frac{C}{R^{\lambda+1}} \int\limits_{I_{4R}} |u_\pm| \left| \frac{\partial u_\pm}{\partial N} \right| \, d\sigma. \quad (8.7)$$

Proof. We use the following estimate established in [24, Lemma 4.18, p. 2855],

$$\int\limits_{I_r} |\nabla u|^2 \, d\sigma \leqslant Cr^{\lambda_0} \int\limits_{\partial D_{sR}^\pm} \frac{|\frac{\partial u}{\partial N}|^2}{\{|P| + r\}^{\lambda_0}} \, d\sigma(P), \quad (8.8)$$

where $n - 3 < \lambda_0 < n - 3 + \varepsilon$. It follows that if $n - 3 < \lambda < \lambda_0$,

$$\sup_{0 < r < R} r^{-\lambda} \int_{I_r} |\nabla u|^2 \, d\sigma \leqslant C \sup_{0 < r < 2R} r^{-\lambda} \int_{I_r} \left| \frac{\partial u}{\partial N} \right|^2 \, d\sigma + \frac{C}{R^\lambda} \int_{\Omega_\pm \cap \partial D_{sR}^\pm} |\nabla u|^2 \, d\sigma, \quad (8.9)$$

for $1 < s < 2$. Estimate (8.7) now follows by an integration in s over $(1, 2)$ and using (2.5). $\quad \square$

Lemma 8.7. *Suppose that* $\Delta u = 0$ *in* $\mathbb{R}^n \setminus \partial \Omega$ *and* $(\nabla u)_+^* + (\nabla u)_-^* \in L^2(I_{16R})$ *for some* $0 < 16R < cr_0$. *Assume that either* $u_+ = 0$ *or* $u_- = 0$ *on* I_{16R}. *Then there exists* $\lambda > n - 3$ *and* $p_0 < 2$ *depending only on n and* Ω *such that*

$$\sup_{0 < r < R} r^{-\lambda} \int_{I_r} |\nabla u_\pm|^2 \, d\sigma \leqslant C \sup_{0 < r < 2R} r^{-\lambda} \int_{I_r} \left| \frac{\partial u_+}{\partial N} - \frac{\partial u_-}{\partial N} \right|^2 \, d\sigma$$

$$+ \frac{C}{R^{\lambda+1}} \int_{I_{8R}} \left(|u_+| + |u_-| \right) \left| \frac{\partial u_+}{\partial N} - \frac{\partial u_-}{\partial N} \right| \, d\sigma$$

$$+ \frac{C}{R^{\lambda+3}} \int_{D_{16R}^+ \cup D_{16R}^-} |u|^2 \, dx$$

$$+ C R^{n-\lambda-3} \left\{ \frac{1}{R^{n-1}} \int_{I_{8R}} \left(|u_+| + |u_-| \right)^{p_0} \, d\sigma \right\}^{2/p_0}. \quad (8.10)$$

Proof. We only consider the case $u_+ = 0$ on I_{16R}. The case for u_- is exactly the same.

The estimate for $r^{-\lambda} \int_{I_r} |\nabla u_+|^2 \, d\sigma$ is contained in (8.4). To estimate $r^{-\lambda} \int_{I_r} |\nabla u_-|^2 \, d\sigma$, in view of (8.7) and (8.4), we only need to take care of the term

$$\frac{1}{R^{\lambda+1}} \int_{I_{4R}} |u_-| \left| \frac{\partial u_-}{\partial N} \right| \, d\sigma. \quad (8.11)$$

To this end, first we replace $|\frac{\partial u_-}{\partial N}|$ in (8.11) by $|\frac{\partial u_+}{\partial N}|$, since the difference is bounded by the second term in the right side of (8.10). Next we use the Hölder inequality. This reduces the problem to the estimation of

$$R^{n-\lambda-1} \left\{ \frac{1}{R^{n-1}} \int_{I_{4R}} \left| \frac{\partial u_+}{\partial N} \right|^{p_0'} \, d\sigma \right\}^{2/p_0'}. \quad (8.12)$$

Finally we use the $L^{p_0'}$ estimate for the regularity problem on D_{sR}^+ for $s \in (4, 5)$ and then a familiar integration in s to bound the term in (8.12) by

$$C R^{n-\lambda-1} \left\{ \frac{1}{R^n} \int_{D_{5R}^+} |\nabla u|^{p_0'} \, dx \right\}^{2/p_0'} \leqslant \frac{C}{R^{\lambda+1}} \int_{D_{6R}^+} |\nabla u|^2 \, dx \leqslant \frac{C}{R^{\lambda+3}} \int_{D_{16R}^+} |u|^2 \, dx, \quad (8.13)$$

where we have used a higher integrability estimate in the first inequality (see e.g. [14]). We remark that $L^{p_0'}$ regularity estimate holds if p_0 is close to 2 [6]. This completes the proof of (8.10). □

We now are ready to prove the desired reverse Hölder inequality.

Theorem 8.8. *Suppose that $\Delta u = 0$ in $\mathbb{R}^n \setminus \partial\Omega$ and $(\nabla u)_+^* + (\nabla u)_-^* \in L^2(I_{300R})$ for some $0 < 300R < cr_0$. Also assume that $\frac{\partial u_+}{\partial N} = \frac{\partial u_-}{\partial N}$ on I_{300R} and that either $u_+ = 0$ or $u_- = 0$ on I_{300R}. Then for any $2 < q < \infty$,*

$$\left\{ \frac{1}{R^{n-1}} \int_{I_R} |(u)^*|^q \, d\sigma \right\}^{1/q} \leqslant C_q \left\{ \frac{1}{R^{n-1}} \int_{I_{300R}} |(u)^*|^{p_0} \, d\sigma \right\}^{1/p_0}, \tag{8.14}$$

where $p_0 < 2$ depends only on n and Ω.

Proof. It follows from (8.3) and (8.10) that

$$\left\{ \frac{1}{R^{n-1}} \int_{I_R} (|u_+| + |u_-|)^q \, d\sigma \right\}^{1/q}$$

$$\leqslant C \left\{ \frac{1}{R^n} \int_{D_{32R}^+ \cup D_{32R}^-} |u|^2 \, dx \right\}^{1/2} + C \left\{ \frac{1}{R^{n-1}} \int_{I_{16R}} (|u_+| + |u_-|)^{p_0} \, d\sigma \right\}^{1/p_0}$$

$$\leqslant C \left\{ \frac{1}{R^{n-1}} \int_{I_{64R}} |(u)^*|^{p_0} \, d\sigma \right\}^{1/p_0}, \tag{8.15}$$

where we also used (2.14) for the second inequality. Since the L^p Dirichlet problem for Laplace's equation is solvable for any $p \geqslant 2$ (this follows from the L^2 solvability and the maximum principle), estimate (8.14) follows from (8.15) and (2.24). □

Proof of Theorem 8.1. We only give the proof for the invertibility of $(1/2)I + \mathcal{K}^*$ on $\mathcal{X}^2(\partial\Omega, \omega \, d\sigma)$. The case of $-(1/2)I + \mathcal{K}^*$ is similar.

Let $f \in \mathcal{X}^2(\partial\Omega, \omega \, d\sigma) \cap W^{1,2}(\partial\Omega)$. Since $(1/2)I + \mathcal{K}^*$ is invertible on $W^{1,2}(\partial\Omega)/\{h_0\}$ and $L^2(\partial\Omega)/\{h_0\}$ [30], there exists $g \in W^{1,2}(\partial\Omega)$ such that $((1/2)I + \mathcal{K}^*)g = f$ and $\|g\|_2 \leqslant C\|f\|_2$. We need to show that

$$\int_{\partial\Omega} |g|^2 \omega \, d\sigma \leqslant C \int_{\partial\Omega} |f|^2 \omega \, d\sigma. \tag{8.16}$$

To this end, we fix $P_0 \in \partial\Omega$ and $s > 0$ sufficiently small. Let $u = \mathcal{D}(g)$. We will show that there exists $p_0 < 2$ such that

$$\left\{ \int\limits_{I(P_0,s)} \left|(u)^*\right|^2 \omega \, d\sigma \right\}^{1/2} \leqslant C \left\{ \omega\big(I(P_0, Cs)\big) \right\}^{1/2} \left\{ \frac{1}{s^{n-1}} \int\limits_{I(P_0,Cs)} \left|(u)^*\right|^{p_0} \, d\sigma \right\}^{1/p_0}$$

$$+ C \left\{ \int\limits_{I(P_0,Cs)} |f|^2 \omega \, d\sigma \right\}^{1/2}, \tag{8.17}$$

for all $\omega \in A_{2/p_0}(\partial\Omega)$. Note that $\|g\|_{p_0} \leqslant C \|f\|_{p_0}$ if p_0 is close to 2. Thus the first term in the right side of (8.17) is bounded by

$$C_s \left\{ \omega(\partial\Omega) \right\}^{1/2} \|g\|_{p_0} \leqslant C_s \left\{ \omega(\partial\Omega) \right\}^{1/2} \|f\|_{p_0} \leqslant C_s \|f\|_{L^2(\partial\Omega, \omega \, d\sigma)}. \tag{8.18}$$

Since $|g| \leqslant 2(u)^*$, estimate (8.16) follows from (8.17) and (8.18) by covering $\partial\Omega$ with a finite number of small surface balls.

We will use Theorem 3.4 to prove (8.17). We may assume that $P_0 = 0$ and $B(0, r_0) \cap \Omega$ is given by (2.2). Let Q be a small subcube of I_s. We proceed as in the proof of Theorem 3.1 to choose function $\varphi = \varphi_Q \in C_0^1(\mathbb{R}^n)$ and then g_Q so that $f\varphi = ((1/2)I + \mathcal{K}^*)(g_Q) + b$ and $\|f\varphi\|_{p_0} \sim \|g_Q\|_{p_0} + |b|$. Let

$$F = \left|(u)^*\right|^{p_0}, \qquad R_Q = 2^{p_0-1}\left|(w)^*\right|^{p_0}, \quad \text{and} \quad F_Q = 2^{p_0-1}\left|(v)^*\right|^{p_0}, \tag{8.19}$$

where $p_0 < 2$ is given in Theorem 8.8, $v = \mathcal{D}(g_Q) + b$ and $w = u - v$. Since $w_- = f(1 - \varphi)$ and $\frac{\partial w_+}{\partial N} = \frac{\partial w_-}{\partial N}$, by Theorem 8.8, we have

$$\left\{ \frac{1}{|2Q|} \int\limits_{2Q} |R_Q|^p \, d\sigma \right\}^{1/p} \leqslant \frac{C}{|Q|} \int\limits_{600Q} |R_Q| \, d\sigma \tag{8.20}$$

for any $p > (2/p_0)$. Also note that

$$\|F_Q\|_1 = \left\|(v)^*\right\|_{p_0}^{p_0} \leqslant C \left\{ \|g_Q\|_{p_0} + |b| \right\}^{p_0} \leqslant C \|f\varphi\|_{p_0}^{p_0}. \tag{8.21}$$

This shows that conditions (3.3) and (3.4) in Theorem 3.2 hold for any $1 < p < \infty$. It then follows from Theorem 3.4 and Remark 3.5 with $q = (2/p_0)$ that estimate (8.17) holds for any $w \in A_{2/p_0}(\partial\Omega)$. This completes the proof. $\quad\square$

Remark 8.9. If $\omega \in A_{1+\delta}(\partial\Omega)$, the Dirichlet problem for Laplace's equation with boundary data in $L^2(\partial\Omega, \omega \, d\sigma)$ is uniquely solvable. This follows easily from [5]. In [24], we solved the regularity problem with data in $W^{1,2}(\partial\Omega, \frac{d\sigma}{\omega})$ for $\omega \in A_{1+\delta}(\partial\Omega)$, and established the sharp estimate

$$\left\|(\nabla u)^*\right\|_{L^2(\partial\Omega, \frac{d\sigma}{\omega})} \leqslant C \|\nabla_t u\|_{L^2(\partial\Omega, \frac{d\sigma}{\omega})}. \tag{8.22}$$

This, together with (8.2), gives the Rellich estimate (1.24) in the weighted L^2 space.

References

[1] P. Auscher, On necessary and sufficient conditions for L^p estimates of Riesz transform associated to elliptic operators on $\mathbb{R}^n$ and related estimates, Mem. Amer. Math. Soc. 186 (2007).

[2] P. Auscher, T. Coulhon, X.T. Duong, S. Hofmann, Riesz transforms on manifolds and heat kernel regularity, Ann. Sci. École Norm. Sup. Paris 37 (2004) 911–957.

[3] L.A. Caffarelli, I. Peral, On $W^{1,p}$ estimates for elliptic equations in divergence form, Comm. Pure Appl. Math. 51 (1998) 1–21.

[4] R. Coifman, A. McIntosh, Y. Meyer, L'intégrale de Cauchy définit un opérateur borńe sur L_2 pour les courbes lipschitziennes, Ann. of Math. 116 (1982) 361–387.

[5] B. Dahlberg, On the Poisson integral for Lipschitz and C^1 domains, Studia Math. 66 (1979) 13–24.

[6] B. Dahlberg, C. Kenig, Hardy spaces and the Neumann problem in L^p for Laplace's equation in Lipschitz domains, Ann. of Math. 125 (1987) 437–465.

[7] B. Dahlberg, C. Kenig, L^p estimates for the three-dimensional systems of elastostatics on Lipschitz domains, in: Lecture Notes Pure Appl. Math., vol. 122, 1990, pp. 621–634.

[8] B. Dahlberg, C. Kenig, G. Verchota, The Dirichlet problem for the biharmonic equation in a Lipschitz domain, Ann. Inst. Fourier (Grenoble) 36 (1986) 109–135.

[9] B. Dahlberg, C. Kenig, G. Verchota, Boundary value problems for the systems of elastostatics in Lipschitz domains, Duke Math. J. 57 (1988) 795–818.

[10] J. Duoandikoetxea, Fourier Analysis, Grad. Stud. in Math., vol. 29, Amer. Math. Soc., 2000.

[11] E. Fabes, Layer potential methods for boundary value problems on Lipschitz domains, in: Lecture Notes in Math., vol. 1344, 1988, pp. 55–80.

[12] E. Fabes, C. Kenig, G. Verchota, Boundary value problems for the Stokes system on Lipschitz domains, Duke Math. J. 57 (1988) 769–793.

[13] W. Gao, Boundary value problems on Lipschitz domains for general elliptic systems, J. Funct. Anal. (1991) 377–399.

[14] M. Giaquinta, Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems, Ann. of Math. Stud., vol. 105, Princeton Univ. Press, 1983.

[15] D. Jerison, C. Kenig, The Neumann problem on Lipschitz domains, Bull. Amer. Math. Soc. 4 (1981) 203–207.

[16] C. Kenig, Elliptic boundary value problems on Lipschitz domains, in: Beijing Lectures in Harmonic Analysis, in: Ann. of Math. Stud., vol. 112, 1986, pp. 131–183.

[17] C. Kenig, Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems, CBMS Reg. Conf. Ser. Math., vol. 83, Amer. Math. Soc., Providence, RI, 1994.

[18] J. Pipher, G. Verchota, The Dirichlet problem in L^p for the biharmonic equation on Lipschitz domains, Amer. J. Math. 114 (1992) 923–972.

[19] J. Pipher, G. Verchota, A maximum principle for biharmonic functions in Lipschitz and C^1 domains, Comment. Math. Helv. 68 (1993) 385–414.

[20] J. Pipher, G. Verchota, Dilation invariant estimates and the boundary Garding inequality for higher order elliptic operators, Ann. of Math. 142 (1995) 1–38.

[21] J. Pipher, G. Verchota, Maximum principle for the polyharmonic equation on Lipschitz domains, Potential Anal. 4 (1995) 615–636.

[22] J.L. Rubio de Francia, Factorization theory and the A_p weights, Amer. J. Math. 106 (1984) 533–547.

[23] Z. Shen, Boundary value problems in Morrey spaces for elliptic systems on Lipschitz domains, Amer. J. Math. 125 (2003) 1079–1115.

[24] Z. Shen, Weighted estimates in L^2 for Laplace's equation on Lipschitz domains, Trans. Amer. Math. Soc. 357 (2004) 2843–2870.

[25] Z. Shen, The L^p Dirichlet problem for elliptic systems on Lipschitz domains, Math. Res. Lett. 13 (2006) 143–159.

[26] Z. Shen, Necessary and sufficient conditions for the solvability of the L^p Dirichlet problem on Lipschitz domains, Math. Ann. 336 (2006) 697–725.

[27] Z. Shen, On estimates of biharmonic functions on Lipschitz domains, J. Geometric Analysis 16 (2006) 721–734.

[28] E. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, 1970.

[29] E. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton Univ. Press, Princeton, NJ, 1993.

[30] G. Verchota, Layer potentials and regularity for the Dirichlet problem for Laplace's equation, J. Funct. Anal. 59 (1984) 572–611.

[31] G. Verchota, The Dirichlet problem for the polyharmonic equation in Lipschitz domains, Indiana Univ. Math. J. 39 (1990) 671–702.

[32] G. Verchota, The biharmonic Neumann problem in Lipschitz domains, Acta Math. 194 (2005) 217–279.

[33] L. Wang, A geometric approach to the Calderón–Zygmund estimates, Acta Math. Sinica (Engl. Ser.) 19 (2003) 381–396.