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Connectifying some spaces

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Abstract

A Hausdorff space X is called (countably) connectifiable if there exists a connected Hausdorff space Y (with $|Y \setminus X| \leqslant \omega$ respectively) such that X embeds densely into Y. We prove that it is consistent with ZFC that there exists a regular dense in itself countable space which is not countably connectifiable giving thus a partial answer to Problem 3.9 of Watson and Wilson (1993). On the other hand we show that Martin's axiom implies that every countable dense in itself space X with $\pi w(X) < 2^\omega$ is countably connectifiable. We also establish that a separable metrizable space without open compact subsets can be densely embedded in a metric continuum.

Keywords: Connectifiable; Countably connectifiable; Connectifying family; Remainder; Extremally disconnected space

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0. Introduction

We study Hausdorff spaces which admit dense embeddings in connected Hausdorff spaces. Such spaces are called connectifiable and their connected extensions are referred to as connectifications. If Y is a connectification of X and $Y \setminus X$ is countable then Y is called an ω-connectification of X and if X has an ω-connectification, then it is called ω-connectifiable, or countably connectifiable.

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Although a characterization of Hausdorff connectifiable spaces is still unknown, significant progress in studying connectificability has been achieved in [1,2]. For example, the following classes of Hausdorff spaces are shown to be subclasses of the class of connectifiable spaces:

- paracompact, first countable spaces with a σ-locally finite π-base (in particular, the metric ones) and with no proper open compact subsets [2];
- countable spaces without isolated points [1];
- Tychonoff nowhere locally compact spaces with countable π-weight [1].

Also, various examples of nonconnectifiable spaces are given in those papers.

It was asked in [1] (Problem 3.9) whether every countable dense in itself Hausdorff space is countably connectifiable. We prove that there is a Tychonoff counterexample in every model of ZFC for which there exist P-points in $\beta\omega \setminus \omega$. On the other hand we show that if Martin's axiom holds, then every countable dense in itself Hausdorff space X with $\pi w(X) < 2^\omega$ is countably connectifiable.

Of course, countable connectifications of countable spaces have to be nonregular. However, we prove that every second countable regular space without open compact subspaces has a metrizable compact connectification. It was established in [2] that nowhere locally compact second countable regular spaces have a Tychonoff connectification.

1. Notations and terminology

All spaces under consideration are assumed to be Hausdorff. If X is a space then $T(X)$ is its topology and $T^*(X) = T(X) \setminus \{\emptyset\}$. If $A \subset X$ then

$$T(A, X) = \{U \in T(X):\ A \subset U\}$$

and $T(x, X) = T(\{x\}, X)$. An end of a proof of a statement or a substatement is marked by $\Box$. A clopen subset of a topological space is called proper, if neither it nor its complement are empty. If X is a space and $S \subset X$ is countable, then $S \to x$ says that the sequence S converges to x. We use the abbreviation BL for Booth's lemma. A space X is called Urysohn space if $x, y \in X$ & $x \neq y$ implies the existence of $U_x \in T(x, X)$ and $U_y \in T(y, X)$ with $\overline{U}_x \cap \overline{U}_y = \emptyset$. If τ is a cardinal, then $\exp\tau$ is 2^τ. All other notations are standard.

2. Connectifying countable spaces

A countable dense in itself space is connectifiable and it is ω-connectifiable if its π-weight is countable. These results were obtained in [1], where the following question was posed: [1, Problem 3.9]. Is every countable space without isolated points countably connectifiable? We show that a counterexample exists if $\beta\omega \setminus \omega$ has P-points.

Proposition 2.1. *Let X be a countable Tychonoff space without isolated points. Suppose that X is dense in a connected space Y. Let*

$$\mathcal{F}_Y = \left\{ \bigcap \{ \mathrm{cl}_{\beta X}(U \cap X)\colon\ U \in T(y,Y) \}\colon\ y \in Y \setminus X \right\}.$$

Then $\mathcal{F}_Y$ consists of nonempty compact subsets of $\beta X \setminus X$ and

(1) *for every proper clopen $O \subset \beta X$ we have $F \cap O \neq \emptyset \neq F \cap (\beta X \setminus O)$ for some $F \in \mathcal{F}_Y$;*

(2) *if X is extremally disconnected, then $\mathcal{F}_Y$ is a disjoint family.*

Proof. That every member of $\mathcal{F}_Y$ is compact and nonempty is evident. The space Y Hausdorff so that for any $x \in X$ and $y \in Y \setminus X$ there is a $U \in T(y,Y)$ with $\mathrm{cl}_Y(U) \not\ni x$ and hence $\mathrm{cl}_X(U \cap X) \not\ni x$. Therefore $\mathrm{cl}_{\beta X}(U \cap X) \not\ni x$ and the set

$$F_y = \bigcap \{ \mathrm{cl}_{\beta X}(U \cap X)\colon\ U \in T(y,Y) \}$$

does not contain x. This shows that all elements of $\mathcal{F}_Y$ lie outside X.

If O is a proper clopen subset of βX then so is $O \cap X$ (in X). Suppose that for all $y \in Y \setminus X$ we have $F_y \cap O = \emptyset$ or $F_y \cap (\beta X \setminus O) = \emptyset$. Then for every $y \in Y \setminus X$ there is a $U_y \in T(y,Y)$ with $(X \cap U_y) \cap O = \emptyset$ or $(X \cap U_y) \subset O$ the set F_y being an intersection of compact subsets of βX.

The space Y is connected, so there is a point $y \in \mathrm{cl}_Y(O \cap X) \cap \mathrm{cl}_Y(X \setminus O)$. Hence $U_y \cap (O \cap X) \neq \emptyset$ and $U_y \cap (X \setminus O) \neq \emptyset$ which is a contradiction, proving (1).

If X is extremally disconnected, then for distinct $y, z \in Y \setminus X$ take $U \in T(y,Y)$ and $V \in T(z,Y)$ with $U \cap V = \emptyset$. Then $\mathrm{cl}_{\beta X}(U \cap X) \cap \mathrm{cl}_{\beta X}(V \cap X) = \emptyset$ by the extremal disconnectedness of X. Therefore $F_y \cap F_z = \emptyset$ and (2) is proved. □

Corollary 2.2. *If X is a countable Tychonoff countably connectifiable space, then $\beta X \setminus X$ has a dense σ-compact subspace.*

Proof. Let Y be a countable connectification of X. Then the family $\mathcal{F}_Y$ defined in Proposition 2.1 is countable and

$$\bigcup \mathcal{F}_Y \subset \beta X \setminus X \subset \mathrm{cl}_{\beta X}\left(\bigcup \mathcal{F}_Y \right)$$

by (1). □

Corollary 2.3. *Let $\mathcal{M}$ be a model of ZFC in which there is a P-point in $\omega^* = \beta\omega \setminus \omega$. Then there is a countable dense in itself space $X \in \mathcal{M}$ which is not countably connectifiable.*

Proof. Let $X = \mathcal{G}_\omega$, where $\mathcal{G}_\omega$ is the space, constructed in [3] using P-points. Remark 1 of [3] states that $\beta X \setminus X$ has no dense σ-compact subset. Now Corollary 2.2 shows that X can not be countably connectifiable. □

We wish to thank J. Porter for bringing this example to our attention.

Our next step is to show that there are sufficiently many countable countably connectifiable spaces.

Proposition 2.4. *Let X be a Tychonoff countable dense in itself space. Suppose that X has a compactification bX for which there exist compact sets F_n, $n \in \omega$ with the following properties:*

(1) $F_n \cap F_m = \emptyset$ for different m and n;

(2) $F_n \subset bX \setminus X$ for all $n \in \omega$;

(3) for every pair (U, V) of nonempty disjoint open subsets of bX with $U \cup V$ dense in bX there is an $n \in \omega$ with $F_n \cap U \neq \emptyset \neq F_n \cap V$.

Then X is countably connectifiable (and we will call the family $\mathcal{F} = \{F_n: n \in \omega\}$ connectifying for X).

Proof. Let $Y = X \cup \{y_n: n \in \omega\}$, $y_n \notin X$ where X is open in Y and a base of open neighborhoods of a point y_n consists of sets $O_n(W) = \{y_n\} \cup (W \cap X)$, where $W \in T(F_n, bX)$.

We need only prove that Y is Hausdorff and connected (the density of X in Y being clear).

To separate y_n from an $x \in X$ take any $W \in T(F_n, bX)$ and $U \in T(x, bX)$ with $U \cap W = \emptyset$. Then $O_n(W) \cap (U \cap X) = \emptyset$. If $m \neq n$ then take disjoint W_1 and W_2 such that $W_1 \in T(F_n, bX)$, $W_2 \in T(F_m, bX)$. Then $O_n(W_1) \cap O_n(W_2) = \emptyset$ and so Y is Hausdorff.

To check that Y is connected, take any proper clopen $O \subset Y$. The set $O_1 = O \cap X$ is clopen in X, so that there are disjoint open sets U and V in bX such that $U \cap X = O_1$ and $V \cap X = X \setminus O_1$. It is clear that $\mathrm{cl}_{bX}(U \cup V) = bX$ so we can use (3) to find an $n \in \omega$ with $F_n \cap U \neq \emptyset \neq F_n \cap V$. It follows from $\mathrm{cl}_{bX}(O_1) = \mathrm{cl}_{bX}(U)$ and $\mathrm{cl}_{bX}(V) = \mathrm{cl}_{bX}(X \setminus O_1)$ that

$$F_n \cap \mathrm{cl}_{bX}(O_1) \neq \emptyset \neq F_n \cap \mathrm{cl}_{bX}(X \setminus O_1).$$

Hence $W \cap O_1 \neq \emptyset \neq W \cap (X \setminus O_1)$ for any $W \in T(F_n, bX)$. Thus $O_n(W) \cap O_1 \neq \emptyset$ and $O_n(W) \cap (X \setminus O_1) \neq \emptyset$ for all $W \in T(F_n, bX)$ so $y_n \in \mathrm{cl}_Y(O_1) \cap \mathrm{cl}_Y(X \setminus O_1)$ which is a contradiction. □

Corollary 2.5. *Let X be a countable Tychonoff space without isolated points. Suppose that X has a compactification bX for which there exists a sequence $\{(x_n, y_n): n \in \omega\}$ with the following properties:*

(1) $\{x_n, y_n\} \subset bX \setminus X$ for all $n \in \omega$;

(2) $\{x_n, y_n\} \cap \{x_m, y_m\} = \emptyset$ for different m and n;

(3) for every pair (U, V) of nonempty open subsets of bX there is an $n \in \omega$ with $x_n \in U$ and $y_n \in V$.

Then X is countably connectifiable (and we will call the sequence $\{(x_n, y_n): n \in \omega\}$ strongly connectifying for X).

Proof. Let $F_n = \{x_n, y_n\}$ and apply Proposition 2.4. □

Corollary 2.6. *Let X be a countable dense in itself extremally disconnected regular space. Then X is countably connectifiable iff there is a family $\{F_n\colon n \in \omega\}$ with the following properties:*

(1) F_n is a compact subset of $\beta X \setminus X$ for all $n \in \omega$;

(2) $F_n \cap F_m = \emptyset$ for different m and n;

(3) if U is a proper clopen subset of βX, then $F_n \cap U \neq \emptyset \neq F_n \cap (\beta X \setminus U)$ for some $n \in \omega$.

Proof. If X can be densely embedded into a countable connected Y, then the family $\mathcal{F}_Y$ constructed in Proposition 2.1 has properties (1)–(3) if we enumerate it with ω. This proves necessity. If $\{F_n\colon n \in \omega\}$ satisfies (1)–(3), then let $bX = \beta X$ and apply Proposition 2.4. □

The following result has been recently announced by Porter. We are not aware of the methods used in his proof, but we give one here to illustrate the usefulness of Corollary 2.5.

Corollary 2.7 (J.H. Porter). *Given an infinite ordinal $\beta \leqslant 2^\omega$, let M_α be a second countable regular space with $|M_\alpha| > 1$ for all $\alpha < \beta$. If X is a countable dense subset of $\prod\{M_\alpha\colon \alpha < \beta\}$, then it is ω-connectifiable.*

Proof. If β is countable, then X has a countable weight and we can apply the relevant results of [1].

If $\beta > \omega$, then we may assume that all M_α's are compact, metrizable and dense in themselves for if not we can replace M_α by a metrizable compactification of M_α and then consider the product of all disjoint countably infinite subproducts of

$$M = \prod\{M_\alpha\colon \alpha < \beta\}.$$

We are going construct a sequence $\{(x_n, y_n)\colon n \in \omega\}$ as in Corollary 2.5, where $bX = M$. Let $\pi_0\colon M \to M_0$ be the natural projection. Using the countability of $\pi_0(X)$ find for every $x_n \in X = \{x_i\colon i \in \omega\}$ a countable number of sequences

$$S^n_m = \{t^n_{mk}\colon k \in \omega\} \subset M_0 \setminus \pi_0(X)$$

such that

(i) $S^n_m \to \pi_0(x_n)$ for every $n, m \in \omega$;

(ii) $t^n_{mp} \neq t^n_{mq}$ if $p \neq q$ and $S^n_{m_1} \cap S^n_{m_2} = \emptyset$ if $m_1 \neq m_2$;

(iii) for all $m_1, m_2 \in \omega$ we have $S^{n_1}_{m_1} \cap S^{n_2}_{m_2} = \emptyset$ if $n_1 \neq n_2$.

Now for every pair $(m, n) \in (\omega \times \omega)$ define a sequence $T^n_m \subset M \setminus X$ as follows: $T^m_n = \{s^n_{mk}\colon k \in \omega\}$ where $s^n_{mk}(\alpha) = x_n(\alpha)$ for all $\alpha > 0$ and $s^n_{mk}(0) = t^n_{mk}$. It is clear that $T^n_m \to x_n$ for every $m \in \omega$. For every pair $(m, n) \in \omega$ such that $m \neq n$ let $F(m, n) = \{\{s^m_{nk}, s^n_{mk}\}\colon k \in \omega\}$. The set

$$F = \bigcup\{F(m, n)\colon m \neq n;\ m, n \in \omega\}$$

is countable, so $F = \{\{u_n, v_n\}\colon\ n \in \omega\}$. It is easy to check that the extension $bX = M$ and the set of pairs $\{\{u_n, v_n\}\colon\ n \in \omega\}$ satisfy all the conditions of the Corollary 2.5 for X. □

Corollary 2.8. *Any countable Tychonoff space can be embedded (maybe not densely) into a countable connected space.*

Proof. Indeed, any such space embeds into $D = \{0,1\}^{2^\omega}$. Add to it some countable dense subset of D and use Corollary 2.7. □

Theorem 2.9 [BL]. *Every countable dense in itself (Hausdorff) space X with $\pi w(X) < 2^\omega$ is ω-connectifiable.*

Proof. We first prove the theorem for regular spaces. To that end let us reduce it to the case when $w(X) < 2^\omega$. This will be achieved with the following

Lemma 2.10. *Let (X, t) be a countable regular space with a π-base γ. Then there exists a regular T_1-topology t^* on X such that $\gamma \subset t^* \subset t$ and $w(X, t^*) \leqslant |\gamma| \cdot \omega$.*

Proof. Being countable X is hereditarily Lindelöf. Hence for every open set V in X there exists a continuous function f_V from X to the unit segment $I = [0,1]$ such that $X \setminus V = f_V^{-1}(0)$. Let f be the diagonal product of functions f_V, $V \in \gamma \cup \mu$, where μ is a countable family of open sets in X separating points of X. Let $\delta = \gamma \cup \mu$. Then f is a continuous one-to-one mapping of X into I^δ, and $|\delta| \leqslant |\gamma| \cdot \omega$. Let $\mathcal{B}$ be a base of I^δ with $|\mathcal{B}| \leqslant |\delta|$. Then the topology t^* on X generated by the base $\{f^{-1}(O)\colon\ O \in \mathcal{B}\}$ has the required properties. □

Lemma 2.11 [BL]. *Suppose that X is a countable regular space without isolated points, and $w(X) < 2^\omega$. Then the remainder $\beta X \setminus X$ is separable (and dense in βX).*

Proof. By a theorem of Eda, Kamo and Nogura [4], under Booth's lemma every countable regular nonscattered space Y with $w(Y) < 2^\omega$ contains a copy of the rationals $\mathbb{Q}$. Let γ be a maximal disjoint family of subspaces of X homeomorphic to $\mathbb{Q}$. Then $Z = \bigcup\gamma$ is dense in X. We claim that for each $z \in Z$ one can find a sequence $S_z \subset \beta X \setminus X$ converging to z. Indeed, choose $C \in \gamma$ with $z \in C$. The point z has countable character in $K = \mathrm{cl}_{\beta X} C$, so that there exists a sequence $S_z \subset K \setminus X$ converging to z. Now put $D = \bigcup\{S_z\colon\ z \in Z\}$. Then D is countable and dense in βX. □

Lemma 2.12. *Let X be a countable regular space which is dense in a regular space Z and such that $Z \setminus X$ is separable and also dense in Z. Then X has a strongly connectifying sequence (and is therefore ω-connectifiable).*

Proof. Pick a countable D dense in $Z \setminus X$. Both D and X are dense in Z. For each $x \in X$ we have $\chi(x, Z) = \chi(x, X) \leqslant w(X) < 2^\omega$. Since D is dense and countable, BL implies that for each $x \in X$ there exists a sequence $S_x \subset D$, converging to x [5].

Let $\{x_n\colon n \in \omega\}$ be any enumeration of X. There exists an onto mapping $\varphi\colon \omega \setminus \{0\} \to \omega \times \omega$ such that $\varphi^{-1}(k,l)$ is infinite for each pair $(k,l) \in \omega \times \omega$. Let F_0 be any subset of $Z \setminus X$ with $|F_0| = 2$. Suppose that for some $n > 0$ we have already defined disjoint subsets F_m of $Z \setminus X$ for all $m < n$ such that $|F_m| = 2$. If $\varphi(n) = (k,l)$, choose points

$$y_n \in S_{x_k} \setminus \bigcup_{m<n} F_m \quad \text{and} \quad z_n \in S_{x_l} \setminus \bigcup_{m<n} F_m$$

with $y_n \neq z_n$ and put $F_n = \{y_n, z_n\}$.

The inductive construction being accomplished let us verify that the family $\mathcal{F} = \{F_n\colon n \in \omega\}$ is strongly connectifying for X. For $U, V \in T(\beta X)$ pick $x_k \in X \cap U$ and $x_l \in X \cap V$. By the choice of φ the set $M = \varphi^{-1}(k,l)$ is infinite and $F_n \cap S_{x_k} \neq \emptyset \neq F_n \cap S_{x_l}$ for each $n \in M$. Furthermore, both sets $S_{x_k} \setminus U$ and $S_{x_l} \setminus V$ are finite while the infinite family $\{F_n\colon n \in M\}$ is disjoint; hence there exists an $n \in M$ such that $F_n \cap U \neq \emptyset \neq F_n \cap V$. □

Now let us finish the Tychonoff case of Theorem 2.9. Denote by t the topology of X and choose a π-base γ for X with $|\gamma| < 2^\omega$. Apply Lemma 2.10 to find a regular T_1-topology t^* for X such that $\gamma \subset t^* \subset t$ and $|t^*| \leqslant |\gamma| < 2^\omega$. Let $Y = (X, t^*)$ and denote by id the identity mapping of X onto Y. Let $f\colon \beta X \to \beta Y$ be the continuous extension of id. By Lemma 2.12 there exists a strongly connectifying family $\mathcal{F}_Y$ for Y with $\bigcup \mathcal{F}_Y \subset \beta Y \setminus Y$. Now let $\mathcal{F}_X = \{f^{-1}(F)\colon F \in \mathcal{F}_Y\}$. We claim that $\mathcal{F}_X$ is connectifying for X.

Indeed, let U be a proper clopen subset of βX and $V = \beta X \setminus U$. There exist $U_1, V_1 \in \gamma$ such that $U_1 \subset U$ and $V_1 \subset V$. By the definition of the topology t^* on Y the sets $f(U_1)$ and $f(V_1)$ are open in Y, hence their closures in βY have nonempty interiors. Consequently, there is an $F \in \mathcal{F}_Y$ which intersects both these interiors. Therefore $f^{-1}(F) \cap U \neq \emptyset \neq f^{-1}(F) \cap V$ and so $\mathcal{F}_X$ is a connectifying family for X. □

Finally let us turn to the general case. Our main weapon will be the following lemma, which seems to be interesting in itself.

Lemma 2.13 [BL]. *Let X be a countable dense in itself space with $w(X) < 2^\omega$. Then X has a dense regular subspace.*

Proof. Let $\mathcal{B}$ be a base in X with $|\mathcal{B}| < 2^\omega$. The family $\mathcal{C}$ of boundaries of elements of $\mathcal{B}$ has the power less than continuum and for every finite subfamily γ of $\mathcal{C}$ we have $|B \setminus \bigcup \gamma| \geqslant \omega$ for every $B \in \mathcal{B}$. This enables us to use the Booth's lemma to find a subset $Y \subset X$ such that $Y \cap B$ is infinite for all $B \in \mathcal{B}$ and $Y \cap C$ is finite for all $C \in \mathcal{C}$. The first condition implies that Y is dense in X, while the second says that Y has a base, all elements of which have finite boundaries. This implies regularity of Y. Indeed, if $y \in Y$ and $U \in T(y, Y)$, then $y \in W \subset U$ for some W with finite boundary. Let $V \in T(y, Y)$ separate y from this boundary. Such a set V exists since Y is Hausdorff. It is clear that $\mathrm{cl}_Y(V \cap W) \subset U$ and we are done. □

Now if X is Hausdorff and $\pi w(X) = \tau < 2^\omega$, take some π-base γ in X with $|\gamma| = \tau$ and find some countable family $\mathcal{B}$ of open subsets of X separating all pairs of points of X. Generate a topology T_1 on X by the family $\gamma \cup \mathcal{B}$. Then T_1 is Hausdorff, $w(X, T_1) = \tau$ and $\gamma \subset T_1$. By Lemma 2.13 there is a dense regular subspace Y of (X, T_1).

For every point $x \in X \setminus Y$ let

$$\mathcal{F}_x = \{U \cap Y\colon\ U \in T(x, X)\}$$

and

$$F_x = \bigcap\{\mathrm{cl}_{\beta Y} A\colon\ A \in \mathcal{F}_x\}.$$

It is clear that the set $H = \bigcup\{F_x\colon\ x \in X \setminus Y\} \subset \beta Y \setminus Y$. The space $Z = \beta Y \setminus H$ is Čech complete and $Y \subset Z$.

We claim that $Z \setminus Y$ is separable. Indeed, if $P \subset Y$ is a copy of rationals, then $P_1 = \mathrm{cl}_Z P$ is Čech complete, so that $P_1 \setminus P$ is dense in P_1. Now apply the fact that every point of P has countable character in P_1 to conclude that $P_1 \setminus P$ is separable (see the proof of Lemma 2.11). Pick a countable disjoint family μ consisting of copies of rationals lying in Y with $\bigcup \mu$ dense in Y and for every $P \in \mu$ choose a countable $A_P \subset \mathrm{cl}_Z P \setminus P$ dense in $\mathrm{cl}_Z P$. Then the set $A = \bigcup\{A_P\colon\ P \in \mu\} \subset Z \setminus Y$ is dense in $Z \setminus Y$.

Now use Lemma 2.12 to find a countable strongly connectifying family for Y consisting of two-element subsets of $Z \setminus Y$. Let $\mathcal{U}_n$ be the family of corresponding open filters on Y. For each $U \in \mathcal{U}_n$ find $\widetilde{U} \in T(X)$ with $\widetilde{U} \cap Y = U$ and put $\mathcal{V}_n = \{\widetilde{U}\colon\ U \in \mathcal{U}_n\}$. Now let $Z = X \cup \{p_n\colon\ n \in \omega\}$ where a base at p_n is $\{\{p_n\} \cup U\colon\ U \in \mathcal{V}_n\}$.

To prove that Z is the required connectification, observe first that

$$U \cap V = \emptyset \iff \widetilde{U} \cap \widetilde{V} = \emptyset$$

which clearly implies that in $X \cup \{p_n\colon\ n \in \omega\}$ any two points $p_n \neq p_m$ can be separated. That any two points of X can be separated is evident and there can be no problem in separating points of $Z \setminus X$ and Y. Finally, take any $x \in X \setminus Y$ and $n \in \omega$.

Let P_n be the two-point set in $Z \setminus Y$, generating the filter $\mathcal{U}_n$. It follows from

$$P_n \cap F_x = \emptyset$$

that there is an open neighborhood U of the point x in X and $V \in T(P_n, \beta Y)$ with

$$(U \cap Y) \cap (V \cap Y) = \emptyset.$$

Let V_1 be a neighborhood of p_n with $V_1 \cap Y = V$. It is clear that U and V_1 are disjoint neighborhoods of x and p_n respectively and we established herewith the Hausdorffness of Z.

Finally, if U is a proper clopen subset of $X \cup \{p_n\colon\ n \in \omega\}$, then find $V, W \in \gamma$ such that $V \subset U$ and $W \subset X \setminus U$. The sets $V \cap Y$ and $W \cap Y$ are nonempty open disjoint subsets of Y. Hence there is an $n \in \omega$ with every element of $\mathcal{U}_n$ intersecting both V and

W. Therefore $p_n \in \mathrm{cl}_Z(U \cap X) \cap \mathrm{cl}_Z(X \setminus U)$ and we proved that Z is connected thus finishing Theorem 2.9. □

Remark 2.14. It is not possible to drop Booth's lemma from the hypothesis of Theorem 2.9. Indeed, there are models of ZFC with P-points in $\beta\omega \setminus \omega$ of character less than 2^ω [7] (we thank S. Watson for bringing this fact to our attention). Now let $X = \mathcal{G}_\omega$, where $\mathcal{G}_\omega$ is the space, constructed in [3] using P-points. It is immediate from the definition of $\mathcal{G}_\omega$, that the weight of X is less than continuum. Use Corollary 2.3 to see that X is not ω-connectifiable.

The following result extends what is known for countable spaces (see [1]) to some uncountable ones.

Proposition 2.15. *Let X be a dense in itself Hausdorff Urysohn space with $|X| < 2^\omega$. Then X is (not necessarily countably) connectifiable.*

Proof. Theorem 2.2 of [2] states that a space X is connectifiable in case no proper clopen subset of X is feebly compact ($\equiv$ there is no infinite locally finite families of nonempty open subsets of X) and the number of clopen subspaces of X is less than or equal to $\exp(\exp(\omega))$. Of course, the number of *all* subspaces of X does not exceed $\exp(\exp(\omega))$, so we have to prove only that no proper clopen subset of X is feebly compact. Indeed, if some proper clopen $U \subset X$ were feebly compact we could use the standard procedure to construct a Cantor tree of regular closed sets in U (i.e., take two closure-disjoint open subsets of U and the same inside each of them and so on). Feeble compactness clearly implies that every branch of this tree has nonempty intersection and so $|U| \geqslant 2^\omega$ which is a contradiction with $|X| < 2^\omega$. □

Corollary 2.16. *Let X be a dense in itself Tychonoff space with $|X| < 2^\omega$. Then X is connectifiable.*

Remark 2.17. Proposition 2.15 can not be proved for Hausdorff spaces (i.e., the word "Urysohn" can not be dropped from the hypothesis) because there even exist H-closed dense in themselves Hausdorff spaces X of power less than continuum. For any such X the space $X \oplus X$ is not connectifiable.

3. Trying to construct Tychonoff connectifications

All spaces considered in this section will be Tychonoff. In the paper of Porter and Woods [2, Theorem 5.7] it is proved that any nowhere locally compact separable metric space has a Tychonoff connectification. The following result strengthens this theorem.

Theorem 3.1. *Let X be a second countable Tychonoff space without nonempty open compact subsets. Then there is a metrizable connected compact Y with $X \subset Y = \overline{X}$.*

Proof. This will follow from several lemmas.

Lemma 3.2. *Let X be a second countable noncompact space. Then there is a metrizable compact Z such that $X \subset Z = \overline{X}$ and $Z \setminus X$ has no isolated points.*

Proof. The space $\beta X \setminus X$ has no isolated points for otherwise some closed neighborhood of a $y \in \beta X \setminus X$ would consist of a second countable subset plus $\{y\}$ which would imply existence of a countable network in this neighborhood and hence its metrizability. But then $x_n \to y$ for some sequence $S = \{x_n\colon n \in \omega\} \subset X$ while the closure of S in βX is homeomorphic to $\beta\omega$ the space X being normal. This gives a contradiction proving that there are no isolated points in $\beta X \setminus X$.

Let Z_0 be any metrizable compactification of X. Let A_0 be the set of isolated points of Z_0 (which of course has to be countable) and $\pi_0 : \beta X \to Z_0$ the natural map. For any $z \in A_0$ pick different $t_z, u_z \in \pi_0^{-1}(z)$ and a continuous map $f_z : \beta X \to [0,1]$ with $f_z(t_z) = 1,\ f_z(u_z) = 0$. Now let

$$\pi_1 = \pi_0 \Delta\big(\Delta\{f_z\colon z \in A_0\}\big) : \beta X \to Z_1,$$

where $Z_1 = \pi_1(\beta X)$ is a metrizable compact space. Clearly there exists a continuous onto map $\pi_0^1 : Z_1 \to Z_0$ with $\pi_0^1 \restriction X = id$.

Let $A_1 \subset Z_1 \setminus X$ be the (again countable) set of isolated points of $Z_1 \setminus X$. Use the same procedure to find a metrizable compactification Z_2 of X and a map $\pi_1^2 : Z_2 \to Z_1$. After having repeated this construction ω times we will have an inverse system

$$Z_0 \xleftarrow{\pi_0^1} Z_1 \xleftarrow{\pi_1^2} Z_2 \xleftarrow{\pi_2^3} \cdots \xleftarrow{\pi_{n-1}^n} Z_n \xleftarrow{\pi_n^{n+1}} \cdots$$

of metrizable compactifications of X such that $\pi_i^{i+1} \restriction X = id_X$. Now we have

$$\text{if } z \in A_i \ (\equiv \text{the set of isolated points of } Z_i \setminus X), \text{ then } |(\pi_i^{i+1})^{-1}(z)| \geqslant 2. \qquad (*)$$

We claim that the space $Z = \varprojlim Z_n$ is what required. Indeed, it is evident that Z is an extension of X. If $z \in Z \setminus X$ is isolated, then there is an $n \in \omega$ with $\varphi_n(z)$ isolated in $Z_n \setminus X$ (here $\varphi_n : Z \to Z_n$ is nth limit projection) because $\varphi_n(Z \setminus X) \subset Z_n \setminus X$ for all n. Now $\varphi_n(z)$ is not only isolated in $Z_n \setminus X$ but $|\varphi_n^{-1}(\varphi_n(z))| = 1$ which implies $|(\pi_n^{n+1})^{-1}(\varphi_n(z))| = 1$ while this contradicts $(*)$. □

Lemma 3.3. *Let Z be a second countable space with a totally bounded metric ρ. Let*

$$\gamma = \big\{U \subset Z\colon U \neq \emptyset \neq Z \setminus U \text{ and } U \text{ is clopen in } Z \text{ and } \rho(U, Z \setminus U) > 0\big\}.$$

Then for every $\varepsilon > 0$ the set $\{U \in \gamma\colon \rho(U, Z \setminus U) > \varepsilon\}$ is finite.

Proof. Indeed, if it were not so, then there would have been an infinite $\gamma' \subset \gamma$ with $\rho(U, Z \setminus U) > \varepsilon$ for all $U \in \gamma'$. Let $z_1, \ldots, z_n$ be an $\varepsilon/2$-net in Z with respect to the metric ρ; this exists by the total boundedness of ρ. Now for every $U \in \gamma'$ if $z_i \in U$ then

$$O_\varepsilon(z_i) = \big\{y \in Z\colon \rho(y, z_i) < \varepsilon\big\} \subset U \Longrightarrow U = \bigcup\big\{O_\varepsilon(z_i)\colon z_i \in U\big\}.$$

But there are only finitely many subsets of the finite family $\{O_\varepsilon(z_i)\colon\ i = 1,\dots,n\}$ so there will be two different $U, V \in \gamma'$ with

$$U = \bigcup\{O_\varepsilon(z_i)\colon\ i \in A\} = V$$

for some $A \subset \{1,\dots,n\}$, which is a contradiction. □

Lemma 3.4. *Suppose that we have a metrizable compact space Z with a metric ρ and a sequence $\{\{x_n, y_n\}\colon\ n \in \omega\}$ of disjoint two-element subsets of Z. If $\lim_{n\to\infty} \rho(x_n, y_n) = 0$ then $q(Z)$ will be a metrizable compact space, where q is a quotient map defined on Z by identifying the points x_n and y_n for all $n \in \omega$.*

Proof. We have a closed decomposition $\mathcal{F}$ of Z consisting of one- or two-point subsets of Z. To prove that the relevant quotient map gives a Hausdorff space (that's all we actually need) we must check that for any $F \in \mathcal{F}$ and for any open $U \supset F$ there is a $V \in T(F, Z)$ such that $G \cap V \neq \emptyset$ implies $G \subset U$ for any $G \in \mathcal{F}$ (see [6, p. 92]).

Choose an $\varepsilon > 0$ with $O_\varepsilon(F) \subset U$. Let $V = O_{\varepsilon/2}(F) \setminus A$, where $A = \{G \in \mathcal{F}\colon\ \mathrm{diam}(G) \geqslant \varepsilon/2\}$. The set V is an open neighborhood of F the set A being finite by $\lim_{n\to\infty} \rho(x_n, y_n) = 0$. It is straightforward that V is as required. □

Now let us take up to the proof of Theorem 3.1.

Using Lemma 3.2 find a metrizable (with a metric ρ) compact $Z \supset X$ with X dense in Z and $Z \setminus X$ perfect. The set γ of all proper clopen subsets of Z is countable and $\rho(U, Z \setminus U) > 0$ for any $U \in \gamma$. Let $\eta = \{U \cap (Z \setminus X)\colon\ U \in \gamma\}$. All elements of η are proper, because there are no open compact subsets of X. Clearly, for any $U \in \eta$ there is an $\varepsilon_U > 0$ such that $\rho(U, (Z \setminus X) \setminus U) > \varepsilon_U$. Hence by Lemma 3.3 we have $\lim_{U\in\eta} \rho(U, (Z \setminus X) \setminus U) = 0$.

Let $\eta = \{U_n\colon\ n \in \omega\}$ and let $x_n \in U_n \setminus X$, $y_n \in (Z \setminus X) \setminus U_n$ be such that $\rho(x_n, y_n) < 2\rho(U_n, (Z \setminus X) \setminus U_n)$ for all $n \in \omega$. Using perfectness of $Z \setminus X$ we can choose $\{x_n, y_n\} \subset Z \setminus X$ in such a way that $\{x_n, y_n\} \cap \{x_m, y_m\} = \emptyset$ if $m \neq n$.

Now identify x_n and y_n for all $n \in \omega$. The resulting space $Y = q(Z)$ will be a compact metrizable (by Lemma 3.4) extension of X.

We claim that Y is connected. Indeed, if U is a proper clopen subset of Y, then $q^{-1}(U)$ is a proper clopen subset of Z and therefore $q^{-1}(U) \cap (Z \setminus X) = U_n$ for some $n \in \omega$ which is impossible, because $q^{-1}(U)$ is saturated with respect to $\{\{x_n, y_n\}\colon\ n \in \omega\}$. This contradiction proves Theorem 3.1. □

Corollary 3.5. *Let X be a locally separable metric space without open compact subspaces. Then X has a Tychonoff connectification.*

Proof. It is well known, that every such space is a discrete union of separable metrizable spaces. Evidently, none of these clopen separable metric summands has proper compact open subspaces. Hence we can use Theorem 3.1 to densely embed X into a discrete union of connected metrizable compact spaces. Now every summand of this union has a point which is not in X. Pick it and identify all the points thus chosen. The resulting space will be a connected Tychonoff extension of X. □

4. Formulating unsolved problems

In this section we pose twenty natural questions we did not succeed in solving while working at this paper. Of course we could ask many more, but the following are representative, illustrate the wide scope of the field and appear to be interesting.

Question 4.1. Is it consistent with ZFC that every countable dense in itself Tychonoff space is countably connectifiable?

Question 4.2. Is it consistent with ZFC that every countable dense in itself Hausdorff Urysohn space is countably connectifiable?

Question 4.3. Is it consistent with ZFC that every countable dense in itself Hausdorff space is countably connectifiable?

Question 4.4. Let X be a countable dense in itself Tychonoff space such that $\beta X \setminus X$ has a dense σ-compact subspace. Is then X countably connectifiable?

Question 4.5. Let X be a countable dense in itself Tychonoff ω-connectifiable space. Does X have a strongly connectifying family (see Corollary 2.5)?

Question 4.6. Let X be a countable dense in itself Tychonoff sequential space. Is then X countably connectifiable?

Question 4.7. Let X be a countable dense in itself Tychonoff Frechet–Urysohn space. Is then X countably connectifiable?

Question 4.8. Let X be a countable dense in itself Tychonoff space with $\beta X \setminus X$ separable. Is then X countably connectifiable?

Question 4.9. Let X be a countable dense in itself Tychonoff space. Does X have a Tychonoff (then clearly uncountable!) connectification?

Question 4.10. Let X be a Tychonoff space with a countable network and without open compact subspaces. Does X have a Tychonoff connectification?

Question 4.11. Let X be metric space without open compact subspaces. Does X have a Tychonoff connectification?

Question 4.12. Let X be metric space without open compact subspaces. Does X have a metrizable connectification?

Question 4.13. Let X be a countable dense in itself Hausdorff space. Does X have a dense Tychonoff subspace?

Question 4.14. Let X be a countable dense in itself Hausdorff space. Does X have a dense Urysohn subspace?

Question 4.15. Let X be a countable dense in itself Hausdorff Urysohn space. Does X have a dense Tychonoff subspace?

Question 4.16. Let X be a countable dense in itself Hausdorff space which has a dense Tychonoff subspace. Is then X countably connectifiable?

Question 4.17. Let X be a countable dense in itself Hausdorff space which has an ω-connectifiable dense subspace. Is then X countably connectifiable?

Question 4.18. Let X be a countable dense in itself Hausdorff Urysohn space which has an ω-connectifiable dense subspace. Is then X countably connectifiable?

Question 4.19. Let X be a countable dense in itself Tychonoff space which has an ω-connectifiable dense subspace. Is then X countably connectifiable?

Question 4.20. Let G be a countable nondiscrete Hausdorff topological group. Is G countably connectifiable?

References

[1] S. Watson and R.G. Wilson, Embeddings in connected spaces, Houston J. Math. 19 (1993) 469–481.
[2] J.R. Porter and G.R. Woods, Subspaces of connected spaces, to appear.
[3] A. Dow, A.V. Gubbi and A. Szymański, Rigid Stone spaces within ZFC, Proc. Amer. Math. Soc. 102 (1988) 745–748.
[4] K. Eda, I. Kamo and T. Nogura, Spaces which contain a copy of the rationals, J. Math. Soc. Japan (1990) 103–112.
[5] M.E. Rudin, Martin's axiom, in: J. Barwise, ed., Handbook of Mathematical Logic (North-Holland, Amsterdam, 1977) 409–501.
[6] A.V. Arhangel'skii and V.I. Ponomarev, General Topology in Problems and Exercises (Nauka, Moscow, 1974) (in Russian).
[7] M. Bell and K. Kunen, On the PI character of ultrafilters, C. R. Math. Rep. Acad. Sci. Canada 3 (1981) 351–356.