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Weakly hyperbolic systems with Hölder continuous coefficients

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Abstract

We study the Cauchy Problem for a hyperbolic system with multiple characteristics and non-smooth coefficients depending on time. We prove in particular that, if the leading coefficients are α-Hölder continuous, and the system has size $m \leqslant 3$, then the Problem is well posed in each Gevrey class of exponent $s < 1 + \alpha/m$.

Keywords: Hyperbolic systems; Hölder coefficients; Gevrey well-posedness

1. Introduction

We consider the Cauchy problem, on $[0, T] \times \mathbf{R}_x$, for the system

$$\begin{cases} \partial_t U = A(t)\partial_x U + B(t)U, \\ U(0, x) = U_0(x), \end{cases} \tag{1}$$

where $U \in \mathbf{C}^m$, $A(t)$ is an $m \times m$ matrix with *real eigenvalues* $\{\lambda_1(t), \dots, \lambda_m(t)\}$. We say that (1) is well posed in a class $\mathscr{X}$ of functions on $\mathbf{R}_x$, when, for all $U_0 \in \mathscr{X}^m$, it admits a unique solution $U \in C^1([0, T], \mathscr{X}^m)$.

If the entries of $A(t)$ are sufficiently smooth functions of t (e.g., of class C^2), we know by Bronshtein [1] and Kajitani [9] (see also [5]) that (1) is well posed in the

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Gevrey class $\gamma^s = \gamma^s(\mathbf{R}_x)$ provided

$$1 < s < 1 + \frac{1}{m-1}.$$

When the leading coefficients are only Hölder continuous, i.e., $A(t) \in C^{0,\alpha}$ for some $\alpha \leqslant 1$, we expect a similar conclusion with $1 < s < \bar{s}$, for some smaller bound $\bar{s} = \bar{s}(m, \alpha)$. The first result in this direction, due to Colombini et al. [4], was concerned with the scalar equation

$$\partial_t^2 u = a(t)\partial_x^2 u + b(t)\partial_x u, \quad a(t) \geqslant 0, \quad a(t) \in C^{0,\alpha},$$

for which the γ^s well-posedness for $s < 1 + \alpha/2$ was proved. This upper bound is sharp.

Subsequently, such a result was extended by Nishitani [11] to the second-order equations with coefficients also depending on x, and, finally, by Ohya and Tarama [12] to any *scalar equation of order m*. In the last case, the range of s for γ^s well-posedness is

$$1 < s < 1 + \frac{\alpha}{m}.$$

The purpose of this paper is to investigate the *vector case*, and prove that the same range of well-posedness holds for any $m \times m$ system (1), at least for $m \leqslant 3$:

Theorem 1. *Let $m = 2, 3$. Assume that $A(t)$ is hyperbolic, i.e., has real eigenvalues $\lambda_j(t)$, and $A(t) \in C^{0,\alpha}([0, T])$, $B(t) \in C^0([0, T])$. Therefore, (1) is well posed in γ^s for all $s < 1 + \alpha/m$, more precisely for*

$$1 < s < 1 + \frac{\alpha}{r} \quad (r = 2, 3),$$

where r is the maximum multiplicity of the $\lambda_j(t)$.

If $r = 1$, i.e., in the strictly hyperbolic case, we have γ^s well-posedness for

$$1 < s < \frac{1}{1-\alpha}.$$

It should be mentioned that case $r = 1$ was already proved by Jannelli [6] in full generality, i.e., for a differential system with arbitrary size and x-depending coefficients, and then extended by Cicognani [2] to pseudodifferential systems. We also recall that Kajitani [10] (cf. Yuzawa [13]) proved the γ^s well-posedness for any size m, but with a smaller range of s than in Theorem 1:

$$1 < s < 1 + \min\{\alpha/(r+1), (2-\alpha)/(2r-1)\}.$$

In this paper we also prove a result of well-posedness for a special class of systems with arbitrary size m: systems (1) where the *square* of the matrix $A(t)$ is Hermitian.

Note that, if $A(t)$ is Hermitian, then (1) is a *symmetric system*, hence the Cauchy Problem is well posed in C^∞ no matter how regular the coefficients are. However, A^2 may be Hermitian even if A is not; for instance, A^2 is Hermitian for any 2×2 hyperbolic matrix A *with trace zero*.

Theorem 2. *If $A(t)$ is hyperbolic, $A(t) \in C^{0,\alpha}([0,T])$, $B(t) \in C^0([0,T])$, and*

$$A(t)^2 \text{ is Hermitian,} \tag{2}$$

then (1) *is well posed in γ^s for*

$$1 < s < 1 + \frac{\alpha}{2}.$$

If, in addition, $\lambda_1(t)^2 + \cdots + \lambda_m(t)^2 \neq 0$ for all t, then (1) *is well posed for*

$$1 < s < \frac{1}{1-\alpha}.$$

Remark 1. By (2), the condition $\sum \lambda_j(t)^2 \neq 0$ is equivalent to $A(t)^2 \neq 0$.

Remark 2. Case $m = 2$ of Theorem 1 can be easily derived from Theorem 2: indeed, it is not restrictive to assume that the 2×2 matrix $A(t)$ has trace zero (see Section 2), which implies that $A(t)^2$ is Hermitian. Case $m = 2$ of Theorem 1 is also a special case of case $m = 3$; indeed, any 2×2 system can be viewed as a 3×3 system with maximum multiplicity $r \leqslant 2$. However, we prefer to give here a direct proof of Theorem 1 even for $m = 2$.

Remark 3. The conclusions of Theorems 1 and 2 can easily be extended to spatial dimension $n \geqslant 2$. Here, for the sake of simplicity, we shall consider only the one-dimensional case.

Our proof of Theorem 1 is rather elementary, relying on an appropriate choice of the energy function. To define such an energy, we suitably approximate the characteristic invariants of $A(t)$ and apply the Hamilton–Cayley equation. Due to its simplicity, case $m = 2$ will be treated in a direct way (see Section 3), while case $m = 3$ (see Section 5) can be better understood in the framework of *quasi-symmetrizers* introduced in [5] (see also [7,8]).

2. Preliminaries

In order to prove Theorem 1, we can assume that the matrix $A(t)$ satisfies

$$\operatorname{tr}(A(t)) = 0, \quad \forall t \in [0,T]. \tag{3}$$

Indeed, if we put $U(t,x) = \widetilde{U}(t, x + \int_0^t \mathrm{tr}(A(\tau))\,d\tau/m)$, we can reduce (1) to

$$\begin{cases} \partial_t \widetilde{U} = \widetilde{A}(t)\partial_x \widetilde{U} + B(t)\widetilde{U}, \\ \widetilde{U}(0,x) = U_0(x), \end{cases}$$

where the matrix $\widetilde{A}(t) \equiv A(t) - \{\mathrm{tr}(A(t))/m\}I$ is traceless. Note that, if $\widetilde{U}$ belongs to $C^1([0,T],[\gamma^s]^m)$, then also $U \in C^1([0,T],[\gamma^s]^m)$.

By a standard argument based on Holmgren uniqueness theorem and on Paley–Wiener theorem (see for instance [4] or [3]), the γ^s well-posedness of (1) follows from the a priori estimate in $\widehat{\gamma^s}$ of $\widehat{U}(t,\xi)$, the Fourier transform w.r. to x of a smooth solution $U(t,x)$ with compact support in $\mathbf{R}_x$ for each t.

Now, by Fourier transform (1) yields

$$\begin{cases} V' = i\xi A(t)V + B(t)V, \\ V(0,\xi) = V_0(\xi), \end{cases} \tag{4}$$

where $V = \hat{U}(t,\xi)$, and a compactly supported function $f(x)$ belongs to $\gamma^s(\mathbf{R})$ if and only if, for some $C, \delta > 0$, one has

$$|\widehat{f}(\xi)| \leqslant Ce^{-\delta|\xi|^{1/s}} \quad \text{for } |\xi| \geqslant 1.$$

Thus, to conclude that $U(t,x) \in C^1([0,T],(\gamma^s)^m)$ for all $s < \sigma$, it will be sufficient to prove that there are some v and C for which

$$|V(t,\xi)| \leqslant |\xi|^v |V_0(\xi)| e^{C|\xi|^{1/\sigma}} \quad \text{for } |\xi| \geqslant 1. \tag{5}$$

Given a non-negative function $\varphi \in C_0^\infty(\mathbf{R})$ with $\int_{-\infty}^{\infty} \varphi(\tau)\,d\tau = 1$, and $0 < \varepsilon \leqslant 1$, we extend $A(t)$ as a Hölder function on all of $\mathbf{R}$, constant outside of $]0,T[$, and define the mollified matrix

$$A_\varepsilon(t) = \int_{-\infty}^{\infty} A(t - \varepsilon\tau)\varphi(\tau)\,d\tau. \tag{6}$$

Since $A(t) \in C^{0,\alpha}$, we can find a constant M for which

$$||A_\varepsilon(t)|| \leqslant M, \quad ||A_\varepsilon{}'(t)|| \leqslant M\varepsilon^{\alpha-1}, \quad ||A_\varepsilon(t) - A(t)|| \leqslant M\varepsilon^\alpha, \tag{7}$$

for all $t \in [0,T]$, where $||\cdot||$ denotes the matrix norm.

3. Proof of Theorem 1 in case $m = 2$

For the sake of brevity, we shall limit ourselves to assuming $B(t) \equiv 0$, the general case requiring only minor changes. We put

$$h_A(t) = -\det(A(t)), \quad h_{A_\varepsilon}(t) = -\det(A_\varepsilon(t)), \quad h_\varepsilon(t) = \Re h_{A_\varepsilon}(t).$$

Note that $h_A(t) \geqslant 0$, by (3), whereas $h_{A_\varepsilon}(t)$ is only complex valued. The characteristic equation and the Hamilton–Cayley equality have, respectively, the forms:

$$\lambda^2 - h_A(t) = 0, \quad A(t)^2 - h_A(t)I = 0.$$

Since $\mathrm{tr}(A_\varepsilon(t)) = \mathrm{tr}(A(t)) = 0$, we also get

$$A_\varepsilon(t)^2 - h_{A_\varepsilon}(t)I = 0. \tag{8}$$

From (7) we obtain, for possibly a larger constant M,

$$|h_{A_\varepsilon}{}'(t)| \leqslant M\varepsilon^{\alpha-1}, \quad |h_{A_\varepsilon}(t) - h_A(t)| \leqslant M\varepsilon^\alpha,$$

hence

$$|h_\varepsilon{}'(t)| \leqslant M\varepsilon^{\alpha-1}, \quad |h_\varepsilon(t) - h_A(t)| \leqslant M\varepsilon^\alpha, \quad |\Im h_{A_\varepsilon}(t)| \leqslant M\varepsilon^\alpha. \tag{9}$$

Now, having fixed a constant M which fulfills (7) and (9), we define, for any solution $V(t,\xi)$ of (4) and for any ε, the energy

$$E(t,\xi) = |A_\varepsilon(t)V|^2 + \{h_\varepsilon(t) + 2M\varepsilon^\alpha\}|V|^2. \tag{10}$$

From (9) we have, observing that $h_A(t) \geqslant c > 0$ in the strictly hyperbolic case,

$$h_\varepsilon(t) + 2M\varepsilon^\alpha \geqslant h_A(t) + M\varepsilon^\alpha \geqslant \begin{cases} c & \text{if } r = 1, \\ M\varepsilon^\alpha & \text{if } r = 2, \end{cases}$$

hence

$$C(M)|V|^2 \geqslant E(t,\xi) \geqslant \begin{cases} |A_\varepsilon(t)V|^2 + c|V|^2 & \text{if } r = 1, \\ |A_\varepsilon(t)V|^2 + M\varepsilon^\alpha|V|^2 & \text{if } r = 2. \end{cases} \tag{11}$$

Differentiating the energy w.r.t. time, and using (4), we find the equality

$$\begin{aligned} E'(t,\xi) &= 2\Re(A_\varepsilon V', A_\varepsilon V) + 2\Re(A_\varepsilon{}'V, A_\varepsilon V) + h_\varepsilon{}'|V|^2 + 2\{h_\varepsilon + 2M\varepsilon^\alpha\}\Re(V', V) \\ &= -2\xi\Im(A_\varepsilon^2 V, A_\varepsilon V) - 2\xi\Im(A_\varepsilon\{A - A_\varepsilon\}V, A_\varepsilon V) + 2\Re(A_\varepsilon{}'V, A_\varepsilon V) + h_\varepsilon{}'|V|^2 \\ &\quad - 2\{h_\varepsilon + 2M\varepsilon^\alpha\}\xi\Im(A_\varepsilon V, V) - 2\{h_\varepsilon + 2M\varepsilon^\alpha\}\xi\Im(\{A - A_\varepsilon\}V, V) \\ &\equiv I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned}$$

Recalling that $\Re h_{A_\varepsilon} = h_\varepsilon$ we see, by (8), that

$$\Im(A_\varepsilon^2 V, A_\varepsilon V) = h_\varepsilon \Im(V, A_\varepsilon V) + \Im h_{A_\varepsilon} \Re(V, A_\varepsilon V),$$

hence, by (7) and (10), we find

$$I_1 + I_5 = -2\xi\Im h_{A_\varepsilon}\Re(V, A_\varepsilon V) - 4M\varepsilon^\alpha\xi\Im(A_\varepsilon V, V) \leqslant 6M\varepsilon^\alpha|\xi||V||A_\varepsilon V|,$$

$$I_2 \leqslant 2|\xi|\,||A_\varepsilon||\,||A - A_\varepsilon||\,|V|\,|A_\varepsilon V| \leqslant 2M^2\varepsilon^\alpha|\xi||V||A_\varepsilon V|,$$

$$I_3 \leqslant 2||A_\varepsilon'||\,|V|\,|A_\varepsilon V| \leqslant 2M\varepsilon^{\alpha-1}|V|\,|A_\varepsilon V|,$$

$$I_4 \leqslant |h_\varepsilon'|\,|V|^2 \leqslant M\varepsilon^{\alpha-1}|V|^2,$$

$$I_6 \leqslant 2|\xi|\,||A - A_\varepsilon||\{h_\varepsilon + 2M\varepsilon^\alpha\}|V|^2 \leqslant 2M\varepsilon^\alpha|\xi|E(t,\xi).$$

Thus, choosing

$$\varepsilon = \begin{cases} |\xi|^{-1} & \text{if } r = 1, \\ |\xi|^{-1/(1+\alpha/2)} & \text{if } r = 2, \end{cases}$$

and recalling (11), we find a constant $C = C(M)$ such that, for all $|\xi| \geqslant 1$,

$$E'(t,\xi) \leqslant \begin{cases} CE(t,\xi)\{\varepsilon^\alpha|\xi| + \varepsilon^{\alpha-1}\} \leqslant 2CE(t,\xi)|\xi|^{1-\alpha} & \text{if } r = 1, \\ CE(t,\xi)\{\varepsilon^{\alpha/2}|\xi| + \varepsilon^{-1}\} \leqslant 2CE(t,\xi)|\xi|^{1/(1+\alpha/2)} & \text{if } r = 2. \end{cases}$$

Gronwall's inequality and (11) yield estimate (5) with $\sigma = 1/(1-\alpha)$ or $\sigma = 1 + \alpha/2$, respectively. This concludes the proof of Theorem 1 for $m = 2$.

4. Proof of Theorem 2

Theorem 2 can be proved in a similar way to Theorem 1 for $m = 2$, but we do not need to suppose (3). We still assume $B \equiv 0$.

Let us first observe that $||A_\varepsilon^2 - A^2|| \leqslant (||A_\varepsilon|| + ||A||)||A_\varepsilon - A||$, thus recalling that $A^2 = (A^2)^*$, we can choose a constant M large enough to satisfy, besides (7),

$$||A_\varepsilon(t)^2 - A(t)^2|| \leqslant M\varepsilon^\alpha, \quad ||A_\varepsilon(t)^2 - (A_\varepsilon(t)^2)^*|| \leqslant M\varepsilon^\alpha. \tag{12}$$

Then we define, instead of (10), the following energy:

$$E(t,\xi) = |A_\varepsilon(t)V|^2 + \Re(\{A_\varepsilon(t)^2 + 2M\varepsilon^\alpha\}V, V).$$

By the first inequality in (12) we derive

$$\Re(\{A_\varepsilon(t)^2 + 2M\varepsilon^\alpha\}V, V) \geqslant (A(t)^2V, V) + M\varepsilon^\alpha|V|^2.$$

But the Hermitian matrix A^2 has eigenvalues $\lambda_j^2 \geqslant 0$, hence we see that $(A^2V, V) \geqslant 0$, while $(A^2V, V)|V|^{-2} \geqslant c > 0$ when $\lambda_l^2 + \cdots + \lambda_m^2 \neq 0$. Thus, we obtain the estimates

$$C(M)|V|^2 \geqslant E(t,\xi) \geqslant \begin{cases} |A_\varepsilon(t)V|^2 + c|V|^2 & \text{if } \lambda_1^2 + \cdots + \lambda_m^2 \neq 0, \\ |A_\varepsilon(t)V|^2 + M\varepsilon^\alpha |V|^2 & \text{if } \lambda_1^2 + \cdots + \lambda_m^2 \geqslant 0. \end{cases} \tag{13}$$

We differentiate the energy and use (2) and (4) to get the equality

$$\begin{aligned} E'(t,\xi) &= 2\Re(A_\varepsilon V', A_\varepsilon V) + 2\Re(A_\varepsilon{}' V, A_\varepsilon V) + \Re(\{A_\varepsilon^2\}' V, V) \\ &\quad + \Re(\{A_\varepsilon^2 + A_\varepsilon^{2^*} + 4M\varepsilon^\alpha\} V', V) \\ &= -2\xi\Im(A_\varepsilon^2 V, A_\varepsilon V) \\ &\quad - 2\xi\Im(A_\varepsilon\{A - A_\varepsilon\}V, A_\varepsilon V) + 2\Re(A_\varepsilon{}' V, A_\varepsilon V) + \Re(\{A_\varepsilon^2\}' V, V) \\ &\quad - \xi\Im(\{A_\varepsilon^2 + A_\varepsilon^{2^*} + 4M\varepsilon^\alpha\}A_\varepsilon V, V) - \xi\Im(\{A_\varepsilon^2 + A_\varepsilon^{2^*} + 4M\varepsilon^\alpha\}(A - A_\varepsilon)V, V) \\ &\equiv I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned}$$

Using (7) and the second inequality in (12), we find a constant $C = C(M)$ for which

$$\begin{aligned} I_1 + I_5 &= -\xi\Im[2(A_\varepsilon^2 V, A_\varepsilon V) + (\{A_\varepsilon^2 + A_\varepsilon^{2^*}\}A_\varepsilon V, V)] - 4M\varepsilon^\alpha\xi\Im(A_\varepsilon V, V) \\ &= -\xi\Im[(\{A_\varepsilon^2 - A_\varepsilon^{2^*}\}V, A_\varepsilon V)] - 4M\varepsilon^\alpha\xi\Im(A_\varepsilon V, V) \leqslant C\varepsilon^\alpha|\xi||V||A_\varepsilon V|, \end{aligned}$$

$$I_2 \leqslant C\varepsilon^\alpha|\xi||V||A_\varepsilon V|, \quad I_3 \leqslant C\varepsilon^{\alpha-1}|V||A_\varepsilon V|, \qquad I_4 \leqslant C\varepsilon^{\alpha-1}|V|^2,$$

$$I_6 \leqslant |\xi|\,||A_\varepsilon^2 + A_\varepsilon^{2^*} + 4M\varepsilon^\alpha||^{1/2}||A - A_\varepsilon||\,|V|\sqrt{2E(t)} \leqslant C\varepsilon^\alpha|\xi|\,|V|\sqrt{E(t)}.$$

Note that, to estimate I_6, we have applied the Schwarz's inequality for the scalar product (TV, V) where $T \equiv T^* = A_\varepsilon^2 + A_\varepsilon^{2^*} + 4M\varepsilon^\alpha \geqslant 0$, to get

$$|(TSV, V)| \leqslant (TSV, SV)^{1/2}(TV, V)^{1/2} \leqslant ||T||^{1/2}||S|||V|(TV, V)^{1/2},$$

where $S = A - A_\varepsilon$. Also note that $E(t) = |A_\varepsilon V|^2 + (TV, V)/2$.

In conclusion, recalling (13) and choosing

$$\varepsilon = \begin{cases} |\xi|^{-1} & \text{if } \lambda_1^2 + \cdots + \lambda_m^2 \neq 0, \\ |\xi|^{-1/(1+\alpha/2)} & \text{if } \lambda_1^2 + \cdots + \lambda_m^2 \geqslant 0, \end{cases}$$

we obtain the following estimate for $|\xi| \geqslant 1$:

$$E'(t,\xi) \leqslant \begin{cases} CE(t,\xi)[\varepsilon^{\alpha}|\xi| + \varepsilon^{\alpha-1}] \leqslant 2CE(t,\xi)|\xi|^{1-\alpha} & \text{if } \lambda_1^2 + \cdots + \lambda_m^2 \neq 0, \\ CE(t,\xi)[\varepsilon^{\alpha/2}|\xi| + \varepsilon^{-1}] \leqslant 2CE(t,\xi)|\xi|^{1/(1+\alpha/2)} & \text{if } \lambda_1^2 + \cdots + \lambda_m^2 \geqslant 0. \end{cases}$$

This yields (5) with $\sigma = 1/(1-\alpha)$, or $\sigma = 1 + \alpha/2$, respectively. Hence, the conclusion of Theorem 2 follows.

5. Proof of Theorem 1 in case $m = 3$

We now define

$$h_A(t) = \det(A(t)) = \lambda_1(t)\lambda_2(t)\lambda_3(t),$$

$$k_A(t) = \sum_{1 \leqslant i,j \leqslant 3} \{a_{ij}(t)a_{ji}(t) - a_{ii}(t)a_{jj}(t)\} = \frac{1}{2}\sum_{j=1}^{3} \lambda_j(t)^2,$$

thus, by (3), the characteristic equation and the Hamilton–Cayley equality are

$$\lambda^3 - k_A(t)\lambda - h_A(t) = 0, \quad A(t)^3 - k_A(t)A(t) - h_A(t)I = 0.$$

By the assumption of hyperbolicity, we see that $k_A(t)$ is a non-negative function, and, in particular, $k_A(t) \geqslant c > 0$ when $r \leqslant 2$. Moreover we have

$$\triangle_A(t) \equiv \prod_{1 \leqslant i < j \leqslant 3} (\lambda_i(t) - \lambda_j(t))^2 = 4k_A(t)^3 - 27h_A(t)^2 \geqslant 0.$$

Since $\mathrm{tr}(A_\varepsilon(t)) = \mathrm{tr}(A(t)) = 0$, the regularized matrix (6) satisfies the equality

$$A_\varepsilon(t)^3 - k_{A_\varepsilon}(t)A_\varepsilon(t) - h_{A_\varepsilon}(t)I = 0. \tag{14}$$

However, the eigenvalues of $A_\varepsilon(t)$ may be non-real, thus $k_{A_\varepsilon}(t)$ and $h_{A_\varepsilon}(t)$ are complex valued. To overcome this difficulty, we introduce the real functions

$$h_\varepsilon(t) = \Re h_{A_\varepsilon}(t), \quad k_\varepsilon(t) = \{\{\Re k_{A_\varepsilon}(t) + M\varepsilon^{\alpha}\}^{3/2} + 12M^{3/2}\varepsilon^{\alpha}\}^{2/3}. \tag{15}$$

Here M is a constant $\geqslant 1$, which is chosen large enough to satisfy, besides (7), the following inequalities on $[0, T]$:

$$\begin{cases} |h_\varepsilon(t) - h_A(t)| \leqslant M\varepsilon^{\alpha}, \quad |\Im h_{A_\varepsilon}(t)| \leqslant M\varepsilon^{\alpha}, \quad |h_\varepsilon{}'(t)| \leqslant M\varepsilon^{\alpha-1}, \\ |k_{A_\varepsilon}(t)| \leqslant M, \quad |k_{A_\varepsilon}(t) - k_A(t)| \leqslant M\varepsilon^{\alpha}, \quad |k_{A_\varepsilon}{}'(t)| \leqslant M\varepsilon^{\alpha-1}, \end{cases} \tag{16}$$

which imply, in particular,

$$|\Re k_{A_\varepsilon}{}'(t)| \leqslant M\varepsilon^{\alpha-1}, \quad |\Re k_{A_\varepsilon}(t) - k_A(t)| \leqslant M\varepsilon^{\alpha}, \quad |\Im k_{A_\varepsilon}(t)| \leqslant M\varepsilon^{\alpha}. \tag{17}$$

We also define

$$\triangle_\varepsilon(t) = 4k_\varepsilon(t)^3 - 27h_\varepsilon(t)^2. \tag{18}$$

Next we show that $\triangle_\varepsilon(t) \geqslant 0$, thus $z^3 - k_\varepsilon(t)z + h_\varepsilon(t)$ is a *hyperbolic polynomial*, and we also prove some crucial estimates on $k_\varepsilon(t)$:

Lemma 1. *There exist constants $C = C(M)$ and $c > 0$, such that*

$$k_\varepsilon(t) \geqslant \begin{cases} c & \text{if } r = 1, 2, \\ M\varepsilon^{2\alpha/3} & \text{if } r = 3, \end{cases} \tag{19}$$

$$|k_\varepsilon{}'(t)| \leqslant C\varepsilon^{\alpha-1}, \quad |k_\varepsilon(t) - k_{A_\varepsilon}(t)| \leqslant C\varepsilon^\alpha k_\varepsilon(t)^{-1/2}, \tag{20}$$

$$\triangle_\varepsilon(t) \geqslant \begin{cases} c & \text{if } r = 1, \\ M^{3/2}\varepsilon^\alpha k_\varepsilon(t)^{3/2} & \text{if } r = 2, 3. \end{cases} \tag{21}$$

Moreover, we have

$$|h_\varepsilon(t)| \leqslant \sqrt{\frac{4}{27}}\, k_\varepsilon(t)^{3/2}. \tag{22}$$

Proof. We write for brevity (15) in the form

$$k_\varepsilon(t) = \{\widetilde{k}_\varepsilon(t)^{3/2} + 12M^{3/2}\varepsilon^\alpha\}^{2/3} \quad \text{where} \quad \widetilde{k}_\varepsilon(t) = \Re k_{A_\varepsilon}(t) + M\varepsilon^\alpha,$$

and observe that, by (17), we have

$$\widetilde{k}_\varepsilon(t) = \{\Re k_{A_\varepsilon}(t) - k_A(t)\} + k_A(t) + M\varepsilon^\alpha \geqslant k_A(t) \geqslant \begin{cases} c & \text{if } r = 1, 2, \\ 0 & \text{if } r = 3. \end{cases}$$

This yields (19). Let us now prove (20). From (15) and (17) it follows that

$$|k_\varepsilon{}'| = |\widetilde{k}_\varepsilon{}'|\widetilde{k}_\varepsilon^{1/2}\{\widetilde{k}_\varepsilon^{3/2} + 12M^{3/2}\varepsilon^\alpha\}^{-1/3} \leqslant |\widetilde{k}_\varepsilon{}'| = |\Re k_{A_\varepsilon}{}'| \leqslant M\varepsilon^{\alpha-1}.$$

Moreover we get, since $k_\varepsilon(t) \geqslant \widetilde{k}_\varepsilon(t)$,

$$|k_\varepsilon - \widetilde{k}_\varepsilon| = \frac{\{k_\varepsilon^{3/2} - \widetilde{k}_\varepsilon{}^{3/2}\}\{k_\varepsilon^{3/2} + \widetilde{k}_\varepsilon{}^{3/2}\}}{k_\varepsilon^2 + k_\varepsilon\widetilde{k}_\varepsilon + \widetilde{k}_\varepsilon^2} \leqslant \frac{12M^{3/2}\varepsilon^\alpha \cdot 2k_\varepsilon^{3/2}}{k_\varepsilon^2} = 24M^{3/2}\varepsilon^\alpha k_\varepsilon^{-1/2},$$

and hence, using again (17),

$$|k_\varepsilon - k_{A_\varepsilon}| \leqslant |k_\varepsilon(t) - \widetilde{k}_\varepsilon(t)| + |\widetilde{k}_\varepsilon(t) - \Re k_{A_\varepsilon}(t)| + |\Im k_{A_\varepsilon}(t)| \leqslant C\varepsilon^\alpha k_\varepsilon^{-1/2}.$$

This completes the proof of (20).

To prove (21), we first derive the following estimate by (16) and (17), recalling that $\widetilde{k}_\varepsilon(t) \geqslant k_A(t)$, $\varepsilon \leqslant 1$,

$$\begin{aligned} |\widetilde{k}_\varepsilon^{3/2} - k_A^{3/2}| &= |\widetilde{k}_\varepsilon - k_A| \frac{\widetilde{k}_\varepsilon + \widetilde{k}_\varepsilon^{1/2} k_A^{1/2} + k_A}{\widetilde{k}_\varepsilon^{1/2} + k_A^{1/2}} \\ &\leqslant \{|\Re k_{A_\varepsilon} - k_A| + M\varepsilon^\alpha\} \frac{3\widetilde{k}_\varepsilon}{\widetilde{k}_\varepsilon^{1/2}} \\ &\leqslant 2M\varepsilon^\alpha 3\widetilde{k}_\varepsilon^{1/2} \leqslant 2M\varepsilon^\alpha 3(|\Re k_{A_\varepsilon}| + M\varepsilon^\alpha)^{1/2} \\ &\leqslant 6\sqrt{2} M^{3/2} \varepsilon^\alpha, \end{aligned} \tag{23}$$

Then, we write

$$\triangle_\varepsilon = 4\{2k_\varepsilon^{3/2} + \sqrt{27} h_\varepsilon\}\{2k_\varepsilon^{3/2} - \sqrt{27} h_\varepsilon\}. \tag{24}$$

We know that

$$\{2k_A^{3/2} + \sqrt{27} h_A\}\{2k_A^{3/2} - \sqrt{27} h_A\} = \triangle_A(t) \geqslant 0 \quad \text{and} \quad k_A(t) \geqslant 0,$$

thus

$$\{2k_A(t)^{3/2} \pm \sqrt{27} h_A(t)\} \geqslant 0. \tag{25}$$

For each fixed $t \in [0, T]$, we have either $h_\varepsilon(t) \geqslant 0$ or $h_\varepsilon(t) \leqslant 0$. In the first case, we have $\{2k_\varepsilon(t)^{3/2} + \sqrt{27} h_\varepsilon(t)\} \geqslant k_\varepsilon(t)^{3/2}$, while, by (16), (23) and (25), we obtain

$$\begin{aligned} &\{2k_\varepsilon(t)^{3/2} - \sqrt{2}7 h_\varepsilon(t)\} \\ &\quad = 24M^{3/2}\varepsilon^\alpha + \{2\widetilde{k}_\varepsilon^{3/2} - \sqrt{27} h_\varepsilon\} \\ &\quad = 24M^{3/2}\varepsilon^\alpha + 2\{\widetilde{k}_\varepsilon^{3/2} - k_A^{3/2}\} + \{2k_A^{3/2} - \sqrt{27} h_A\} + \sqrt{27}(h_A - h_\varepsilon) \\ &\quad \geqslant 24M^{3/2}\varepsilon^\alpha - 2|\widetilde{k}_\varepsilon{}^{3/2} - k_A^{3/2}| + \{2k_A^{3/2} - \sqrt{27} h_A\} - \sqrt{27}|h_A - h_\varepsilon| \\ &\quad \geqslant [24 - 12\sqrt{2} - \sqrt{27}] M^{3/2}\varepsilon^\alpha + \{2k_A^{3/2} - \sqrt{27} h_A\} \\ &\quad \geqslant M^{3/2}\varepsilon^\alpha. \end{aligned}$$

In the same way, when $h_\varepsilon(t) \leqslant 0$ we obtain

$$\{2k_\varepsilon^{3/2} - \sqrt{27} h_\varepsilon(t)\} \geqslant k_\varepsilon(t)^{3/2}, \quad \{2k_\varepsilon(t)^{3/2} + \sqrt{27} h_\varepsilon(t)\} \geqslant M^{3/2}\varepsilon^\alpha.$$

Thus, in both the cases we get by (24)

$$\triangle_\varepsilon(t) \geqslant 4M^{3/2}\varepsilon^\alpha k_\varepsilon(t)^{3/2}.$$

In the special case when $r = 1$, the discriminant $\triangle_A(t)$ is strictly positive, hence both the inequalities in (25) are strict, and we conclude that $\triangle_\varepsilon(t) \geqslant c > 0$.

Finally, (22) follows directly from (21) and definition (18) of $\triangle_\varepsilon(t)$. □

In the following lemma, we exhibit an exact (but possibly non-coercive) symmetrizer $Q_\varepsilon(t)$ for the 3×3 Sylvester matrix whose characteristic polynomial is the polynomial $z^3 - k_\varepsilon(t)z + h_\varepsilon(t)$. We also give a lower estimate for such a symmetrizer $Q_\varepsilon(t)$, which will be decisive in our proof.

Lemma 2. *Let us define*

$$A_\varepsilon^\sharp(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ h_\varepsilon(t) & k_\varepsilon(t) & 0 \end{pmatrix}, \quad Q_\varepsilon(t) = \begin{pmatrix} k_\varepsilon(t)^2 & 3h_\varepsilon(t) & -k_\varepsilon(t) \\ 3h_\varepsilon(t) & 2k_\varepsilon(t) & 0 \\ -k_\varepsilon(t) & 0 & 3 \end{pmatrix}. \tag{26}$$

Then, the matrix $Q_\varepsilon(t)$ is Hermitian and satisfies

$$Q_\varepsilon(t)A_\varepsilon^\sharp(t) = A_\varepsilon^\sharp(t)^* Q_\varepsilon(t). \tag{27}$$

$$(Q_\varepsilon(t)W, W) \geqslant c|L_\varepsilon(t)W|^2 \quad \textit{for all} \;\; W \in \mathbf{C}^3, \quad c > 0, \tag{28}$$

where

$$L_\varepsilon(t) = \triangle_\varepsilon(t)^{1/2} \begin{pmatrix} k_\varepsilon(t)^{-1/2} & 0 & 0 \\ 0 & k_\varepsilon(t)^{-1} & 0 \\ 0 & 0 & k_\varepsilon(t)^{-3/2} \end{pmatrix}.$$

Proof. Eq. (27) follows from definitions (26). Let us prove (28). Since

$$L_\varepsilon^{-1} = (L_\varepsilon^{-1})^* = \triangle_\varepsilon^{-1/2} \begin{pmatrix} k_\varepsilon^{1/2} & 0 & 0 \\ 0 & k_\varepsilon & 0 \\ 0 & 0 & k_\varepsilon^{3/2} \end{pmatrix},$$

we have

$$(L_\varepsilon^{-1})^* Q_\varepsilon L_\varepsilon^{-1} = \frac{k_\varepsilon^3}{\triangle_\varepsilon} \widetilde{Q}_\varepsilon, \tag{29}$$

where

$$\widetilde{Q}_\varepsilon(t) \equiv [\widetilde{q}_{ij}(t)]_{1\leqslant i,j\leqslant 3} = \begin{pmatrix} 1 & 3h_\varepsilon k_\varepsilon^{-3/2} & -1 \\ 3h_\varepsilon k_\varepsilon^{-3/2} & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}.$$

Now, by (22) we see that $||\widetilde{Q}_\varepsilon(t)||\leqslant C$ on $[0,T]$. Moreover, by (19) and (20), the determinant and the minor determinants of $\widetilde{Q}_\varepsilon(t)$ satisfy

$$\det \widetilde{Q}_\varepsilon(t) = 4 - \frac{27h_\varepsilon^2}{k_\varepsilon^3} = \frac{\triangle_\varepsilon}{k_\varepsilon^3} > 0,$$

$$\widetilde{q}_{11}(t)\widetilde{q}_{22}(t) - \widetilde{q}_{12}(t)\widetilde{q}_{21}(t) = 2 - \frac{9h_\varepsilon^2}{k_\varepsilon^3} = \frac{2}{3} + \frac{\triangle_\varepsilon}{3k_\varepsilon^3} > 0, \quad \widetilde{q}_{11}(t) = 1 > 0.$$

This implies that the eigenvalues $\mu_1(t), \mu_2(t), \mu_3(t)$ of $\widetilde{Q}_\varepsilon(t)$ are non-negative, and thus we have, for $\{i,j,k\} = \{1,2,3\}$,

$$\mu_i(t) = \frac{\mu_i(t)\mu_j(t)\mu_k(t)}{\mu_j(t)\mu_k(t)} \geqslant \frac{\det(\widetilde{Q}_\varepsilon(t))}{||\widetilde{Q}_\varepsilon(t)||^2} \geqslant c\frac{\triangle_\varepsilon(t)}{k_\varepsilon(t)^3}, \quad c>0.$$

Hence we get

$$(\widetilde{Q}_\varepsilon(t)\widetilde{W}, \widetilde{W}) \geqslant c\frac{\triangle_\varepsilon(t)}{k_\varepsilon(t)^3}|\widetilde{W}|^2 \quad \text{for all } \widetilde{W}\in\mathbf{C}^3,$$

and consequently, taking $\widetilde{W} = L_\varepsilon(t)W$ and recalling (29),

$$(Q_\varepsilon(t)W, W) = \frac{k_\varepsilon(t)^3}{\triangle_\varepsilon(t)}(\widetilde{Q}_\varepsilon(t)\widetilde{W}, \widetilde{W}) \geqslant c|\widetilde{W}|^2 = c|L_\varepsilon(t)W|^2. \qquad \square$$

Lemma 2 also is applicable to 9×9 block-matrices whose blocks are 3×3 matrices of scalar type. Indeed, denoting by I the 3×3 identity matrix, we have:

Lemma 3. *Let us define the* 9×9 *matrices*

$$\mathscr{A}_\varepsilon(t) = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ h_\varepsilon(t)I & k_\varepsilon(t)I & 0 \end{pmatrix}, \quad \mathscr{Q}_\varepsilon(t) = \begin{pmatrix} k_\varepsilon(t)^2 I & 3h_\varepsilon(t)I & -k_\varepsilon(t)I \\ 3h_\varepsilon(t)I & 2k_\varepsilon(t)I & 0 \\ -k_\varepsilon(t)I & 0 & 3I \end{pmatrix}. \tag{30}$$

Therefore, $\mathcal{Q}_\varepsilon(t)$ *is Hermitian and satisfies*

$$\mathcal{Q}_\varepsilon(t)\mathcal{A}_\varepsilon(t) = \mathcal{A}_\varepsilon(t)^*\mathcal{Q}_\varepsilon(t), \tag{31}$$

$$(\mathcal{Q}_\varepsilon(t)\mathcal{W}, \mathcal{W}) \geqslant c|\mathcal{L}_\varepsilon(t)\mathcal{W}|^2, \quad \forall\, \mathcal{W} \in \mathbf{C}^9, \quad c>0, \tag{32}$$

where

$$\mathcal{L}_\varepsilon(t) = \triangle_\varepsilon(t)^{1/2}\begin{pmatrix} k_\varepsilon(t)^{-1/2}I & 0 & 0 \\ 0 & k_\varepsilon(t)^{-1}I & 0 \\ 0 & 0 & k_\varepsilon(t)^{-3/2}I \end{pmatrix}. \tag{33}$$

Proof. Since the 3×3 submatrices in $\mathcal{A}_\varepsilon(t)$, $\mathcal{Q}_\varepsilon(t)$ and $\mathcal{L}_\varepsilon(t)$ consist of the 3×3 identity matrix I, (31) and (32) can be easily derived from (27) and (28), respectively. □

Now, we transform the 3×3 system (4) into a 9×9 system whose principal part is the block Sylvester matrix $\mathcal{A}_\varepsilon(t)$ of Lemma 3. We deduce from (4) that

(i) $V' = i\xi AV + BV = i\xi A_\varepsilon V + i\xi(A - A_\varepsilon)V + BV,$
(ii) $(A_\varepsilon V)' = i\xi A_\varepsilon^2 V + i\xi A_\varepsilon(A - A_\varepsilon)V + A_\varepsilon' V + A_\varepsilon BV,$
(iii)
$$\begin{aligned}(A_\varepsilon^2 V)' &= i\xi A_\varepsilon^3 V + i\xi A_\varepsilon^2(A - A_\varepsilon)V + (A_\varepsilon^2)'V + A_\varepsilon^2 BV \\ &= [i\xi h_\varepsilon V + i\xi k_\varepsilon A_\varepsilon V] - \xi\Im h_{A_\varepsilon}V + i\xi(k_{A_\varepsilon} - k_\varepsilon)A_\varepsilon V \\ &\quad + i\xi A_\varepsilon^2(A - A_\varepsilon)V + (A_\varepsilon^2)'V + A_\varepsilon^2 BV,\end{aligned}$$

where, in the last equality, we have used the Hamilton–Cayley equality (14).

Putting

$$\mathcal{V} \equiv \mathcal{V}(t,\xi) = \begin{pmatrix} V \\ A_\varepsilon V \\ A_\varepsilon^2 V \end{pmatrix} \in \mathbf{C}^9,$$

we combine together (i), (ii) and (iii) to get the 9×9 system:

$$\mathcal{V}' = i\xi\mathcal{A}_\varepsilon(t)\mathcal{V} + i\xi\mathcal{R}_\varepsilon(t)\mathcal{V} - \xi\mathcal{P}_\varepsilon(t)\mathcal{V} + \mathcal{D}_\varepsilon(t)\mathcal{V} + \mathcal{B}_\varepsilon(t)\mathcal{V}, \tag{34}$$

where $\mathscr{A}_\varepsilon(t)$ is defined in (30), and

$$\mathscr{R}_\varepsilon(t) = \begin{pmatrix} A - A_\varepsilon & 0 & 0 \\ A_\varepsilon(A - A_\varepsilon) & 0 & 0 \\ A_\varepsilon^2(A - A_\varepsilon) & 0 & 0 \end{pmatrix}, \quad \mathscr{P}_\varepsilon(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \Im h_{A_\varepsilon} I & -i(k_{A_\varepsilon} - k_\varepsilon)I & 0 \end{pmatrix},$$

$$\mathscr{D}_\varepsilon(t) = \begin{pmatrix} 0 & 0 & 0 \\ A_\varepsilon{}' & 0 & 0 \\ (A_\varepsilon^2)' & 0 & 0 \end{pmatrix}, \quad \mathscr{B}_\varepsilon(t) = \begin{pmatrix} B & 0 & 0 \\ A_\varepsilon B & 0 & 0 \\ A_\varepsilon^2 B & 0 & 0 \end{pmatrix}.$$

Then, recalling (30), we define the energy

$$E(t, \xi) = (\mathscr{Q}_\varepsilon(t)\mathscr{V}, \mathscr{V}).$$

By definition (33) of $\mathscr{L}_\varepsilon(t)$, using (19) and (21), we see that

$$|\mathscr{L}_\varepsilon(t)\mathscr{W}|^2 \geqslant c_1 \triangle_\varepsilon(t) k_\varepsilon(t)^{-1} |\mathscr{W}|^2 \geqslant c_2 \varepsilon^{4\alpha/3} |\mathscr{W}|^2, \tag{35}$$

hence, remarking that $||\mathscr{Q}_\varepsilon(t)|| \leqslant C$, and $|V|^2 \leqslant |\mathscr{V}|^2 \leqslant C|V|^2$, we deduce from (32) and (35):

$$c\varepsilon^{4\alpha/3}|V|^2 \leqslant E(t, \xi) \leqslant C|V|^2. \tag{36}$$

By (31) and (34), considering that $\mathscr{Q}_\varepsilon$ is Hermitian, we get the equality

$$\begin{aligned} E'(t, \xi) &= (\mathscr{Q}_\varepsilon'\mathscr{V}, \mathscr{V}) + (\mathscr{Q}_\varepsilon\mathscr{V}', \mathscr{V}) + (\mathscr{Q}_\varepsilon\mathscr{V}, \mathscr{V}') \\ &= (\mathscr{Q}_\varepsilon'\mathscr{V}, \mathscr{V}) + i\xi(\{\mathscr{Q}_\varepsilon\mathscr{A}_\varepsilon - \mathscr{A}_\varepsilon^*\mathscr{Q}_\varepsilon^*\}\mathscr{V}, \mathscr{V}) \\ &\quad + (\mathscr{Q}_\varepsilon\{i\xi\mathscr{R}_\varepsilon - \xi\mathscr{P}_\varepsilon + \mathscr{D}_\varepsilon + \mathscr{B}_\varepsilon\}\mathscr{V}, \mathscr{V}) + \overline{(\mathscr{Q}_\varepsilon\{i\xi\mathscr{R}_\varepsilon - \xi\mathscr{P}_\varepsilon + \mathscr{D}_\varepsilon + \mathscr{B}_\varepsilon\}\mathscr{V}, \mathscr{V})} \\ &= (\mathscr{Q}_\varepsilon'\mathscr{V}, \mathscr{V}) - 2\xi\Im(\mathscr{Q}_\varepsilon\mathscr{R}_\varepsilon\mathscr{V}, \mathscr{V}) - 2\xi\Re(\mathscr{Q}_\varepsilon\mathscr{P}_\varepsilon\mathscr{V}, \mathscr{V}) + 2\Re(\mathscr{Q}_\varepsilon\mathscr{D}_\varepsilon\mathscr{V}, \mathscr{V}) \\ &\quad + 2\Re(\mathscr{Q}_\varepsilon\mathscr{B}_\varepsilon\mathscr{V}, \mathscr{V}). \end{aligned}$$

In order to prove the energy estimate, we use the following:

Lemma 4. *If $\mathscr{S}$ be a 9×9 matrix, then we have, for all $\mathscr{W} \in \mathbf{C}^9$,*

$$(\mathscr{S}\mathscr{W}, \mathscr{W}) \leqslant C||\mathscr{L}_\varepsilon^{-1}\mathscr{S}\mathscr{L}_\varepsilon^{-1}||(\mathscr{Q}_\varepsilon\mathscr{W}, \mathscr{W}), \tag{37}$$

$$(\mathscr{Q}_\varepsilon\mathscr{S}\mathscr{W}, \mathscr{W}) \leqslant C||\mathscr{L}_\varepsilon^{-1}(\mathscr{S}^*\mathscr{Q}_\varepsilon\mathscr{S})\mathscr{L}_\varepsilon^{-1}||^{1/2}(\mathscr{Q}_\varepsilon\mathscr{W}, \mathscr{W}). \tag{38}$$

Proof. Eq. (37) follows directly from (32); indeed, noting that $\mathscr{L}_\varepsilon^* = \mathscr{L}_\varepsilon$, we find

$$(\mathscr{S}\mathscr{W}, \mathscr{W}) = (\mathscr{L}_\varepsilon^{-1}\mathscr{S}\mathscr{L}_\varepsilon^{-1}\mathscr{L}_\varepsilon\mathscr{W}, \mathscr{L}_\varepsilon^*\mathscr{W}) \leqslant ||\mathscr{L}_\varepsilon^{-1}\mathscr{S}\mathscr{L}_\varepsilon^{-1}||\,|\mathscr{L}_\varepsilon(t)\mathscr{W}|^2$$
$$\leqslant \frac{1}{c}||\mathscr{L}_\varepsilon^{-1}\mathscr{S}\mathscr{L}_\varepsilon^{-1}||(\mathscr{Q}_\varepsilon\mathscr{W}, \mathscr{W}).$$

To prove (38), we use the Schwarz's inequality for the scalar product $\langle \mathscr{Y}, \mathscr{W} \rangle \equiv (\mathscr{Q}_\varepsilon\mathscr{Y}, \mathscr{W})$, and (37) with $\mathscr{S}^*\mathscr{Q}_\varepsilon\mathscr{S}$ in place of $\mathscr{S}$. Thus we obtain

$$(\mathscr{Q}_\varepsilon\mathscr{S}\mathscr{W}, \mathscr{W}) \leqslant (\mathscr{Q}_\varepsilon\mathscr{S}\mathscr{W}, \mathscr{S}\mathscr{W})^{1/2}(\mathscr{Q}_\varepsilon\mathscr{W}, \mathscr{W})^{1/2}$$
$$\leqslant C||\mathscr{L}_\varepsilon^{-1}(\mathscr{S}^*\mathscr{Q}_\varepsilon\mathscr{S})\mathscr{L}_\varepsilon^{-1}||^{1/2}(\mathscr{Q}_\varepsilon\mathscr{W}, \mathscr{W}). \qquad \square$$

By (37) and (38), it follows that

$$E'(t,\xi) \leqslant CE(t,\xi)\{||\mathscr{L}_\varepsilon^{-1}\mathscr{Q}_\varepsilon{}'\mathscr{L}_\varepsilon^{-1}|| + |\xi|\,||\mathscr{L}_\varepsilon^{-1}(\mathscr{R}_\varepsilon^*\mathscr{Q}_\varepsilon\mathscr{R}_\varepsilon)\mathscr{L}_\varepsilon^{-1}||^{1/2}$$
$$+ |\xi|\,||\mathscr{L}_\varepsilon^{-1}(\mathscr{P}_\varepsilon^*\mathscr{Q}_\varepsilon\mathscr{P}_\varepsilon)\mathscr{L}_\varepsilon^{-1}||^{1/2} + ||\mathscr{L}_\varepsilon^{-1}(\mathscr{D}_\varepsilon^*\mathscr{Q}_\varepsilon\mathscr{D}_\varepsilon)\mathscr{L}_\varepsilon^{-1}||^{1/2}$$
$$+ ||\mathscr{L}_\varepsilon^{-1}(\mathscr{B}_\varepsilon^*\mathscr{Q}_\varepsilon\mathscr{B}_\varepsilon)\mathscr{L}_\varepsilon^{-1}||^{1/2}\}.$$

Now we estimate the five summands on the right-hand side. To this end, let us firstly observe that, for any 9×9 block matrix $\mathscr{S} = [S_{ij}]_{1 \leqslant i,j \leqslant 3}$, one has

$$\mathscr{L}_\varepsilon^{-1}\mathscr{S}\mathscr{L}_\varepsilon^{-1} = \frac{1}{\triangle_\varepsilon}[k_\varepsilon^{(i+j)/2}S_{ij}]_{1 \leqslant i,j \leqslant 3}. \tag{39}$$

i) *Estimate of* $||\mathscr{L}_\varepsilon^{-1}\mathscr{Q}_\varepsilon{}'\mathscr{L}_\varepsilon^{-1}||$: By using (39), we see that

$$\mathscr{L}_\varepsilon^{-1}\mathscr{Q}_\varepsilon{}'\mathscr{L}_\varepsilon^{-1} = \frac{k_\varepsilon^{3/2}}{\triangle_\varepsilon}\begin{pmatrix} 2k_\varepsilon^{1/2}k_\varepsilon{}'I & 3h_\varepsilon{}'I & -k_\varepsilon^{1/2}k_\varepsilon{}'I \\ 3h_\varepsilon{}'I & 2k_\varepsilon^{1/2}k_\varepsilon{}'I & 0 \\ -k_\varepsilon^{1/2}k_\varepsilon{}'I & 0 & 0 \end{pmatrix},$$

thus, by (16) and (20), we get

$$||\mathscr{L}_\varepsilon^{-1}\mathscr{Q}_\varepsilon{}'\mathscr{L}_\varepsilon^{-1}|| \leqslant \frac{k_\varepsilon^{3/2}}{\triangle_\varepsilon}C\{k_\varepsilon^{1/2}|k_\varepsilon{}'| + |h_\varepsilon{}'|\} \leqslant \frac{k_\varepsilon^{3/2}}{\triangle_\varepsilon}C_1\varepsilon^{\alpha-1}. \tag{40}$$

ii) *Estimate of* $||\mathscr{L}_\varepsilon^{-1}(\mathscr{P}_\varepsilon^*\mathscr{Q}_\varepsilon\mathscr{P}_\varepsilon)\mathscr{L}_\varepsilon^{-1}||$: By the equality

$$\begin{pmatrix} 0 & 0 & Y_1^* \\ 0 & 0 & Y_2^* \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} k^2I & 3hI & -I \\ 3hI & 2kI & 0 \\ -kI & 0 & 3I \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ Y_1 & Y_2 & 0 \end{pmatrix} = 3\begin{pmatrix} Y_1^*Y_1 & Y_1^*Y_2 & 0 \\ Y_2^*Y_1 & Y_2^*Y_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and by (39), we find

$$\mathscr{L}_\varepsilon^{-1}(\mathscr{P}_\varepsilon^* \mathscr{Q}_\varepsilon \mathscr{P}_\varepsilon)\mathscr{L}_\varepsilon^{-1} = \frac{3k_\varepsilon}{\triangle_\varepsilon}\begin{pmatrix} (\Im h_{A_\varepsilon})^2 I & -ik_\varepsilon^{1/2}(k_{A_\varepsilon}-k_\varepsilon)\Im h_{A_\varepsilon} I & 0 \\ ik_\varepsilon^{1/2}\overline{(k_{A_\varepsilon}-k_\varepsilon)}\Im h_{A_\varepsilon} I & k_\varepsilon |k_{A_\varepsilon}-k_\varepsilon|^2 I & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, by (16) and (20),

$$||\mathscr{L}_\varepsilon^{-1}(\mathscr{P}_\varepsilon^* \mathscr{Q}_\varepsilon \mathscr{P}_\varepsilon)\mathscr{L}_\varepsilon^{-1}|| \leqslant \frac{k_\varepsilon}{\triangle_\varepsilon} C\Big\{\varepsilon^{2\alpha} + k_\varepsilon^{1/2}|k_{A_\varepsilon}-k_\varepsilon|\varepsilon^\alpha + k_\varepsilon |k_{A_\varepsilon}-k_\varepsilon|^2\Big\} \leqslant \frac{k_\varepsilon}{\triangle_\varepsilon} C_2 \varepsilon^{2\alpha}. \quad (41)$$

To compute the products $\mathscr{X}^* \mathscr{Q}_\varepsilon \mathscr{X}$ with $\mathscr{X} = \mathscr{R}_\varepsilon, \mathscr{D}_\varepsilon, \mathscr{B}_\varepsilon$, we note that

$$\begin{pmatrix} X_1^* & X_2^* & X_3^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} k_\varepsilon^2 I & 3h_\varepsilon I & -k_\varepsilon I \\ 3h_\varepsilon I & 2k_\varepsilon I & 0 \\ -k_\varepsilon I & 0 & 3I \end{pmatrix}\begin{pmatrix} X_1 & 0 & 0 \\ X_2 & 0 & 0 \\ X_3 & 0 & 0 \end{pmatrix} = Z_\varepsilon \mathscr{J}, \quad (42)$$

where

$$Z_\varepsilon = k_\varepsilon^2 X_1^* X_1 + 3h_\varepsilon(X_1^* X_2 + X_2^* X_1) - k_\varepsilon(X_1^* X_3 + X_3^* X_1 - 2X_2^* X_2) + 3X_3^* X_3$$

and

$$\mathscr{J} = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

iii) *Estimate of* $||\mathscr{L}_\varepsilon^{-1}(\mathscr{R}_\varepsilon^* \mathscr{Q}_\varepsilon \mathscr{R}_\varepsilon)\mathscr{L}_\varepsilon^{-1}||$: From (42) with $X_j = A_\varepsilon^{j-1}(A - A_\varepsilon), j = 1, 2, 3,$ recalling (39), we see that

$$\mathscr{L}_\varepsilon^{-1}(\mathscr{R}_\varepsilon^* \mathscr{Q}_\varepsilon \mathscr{R}_\varepsilon)\mathscr{L}_\varepsilon^{-1} = \frac{k_\varepsilon}{\triangle_\varepsilon} F_\varepsilon \mathscr{J},$$

where

$$F_\varepsilon = (A - A_\varepsilon)^* \{k_\varepsilon^2 I + 3h_\varepsilon(A_\varepsilon + A_\varepsilon^*) - k_\varepsilon(A_\varepsilon - A_\varepsilon^*)^2 + 3A_\varepsilon^{*^2} A_\varepsilon^2\}(A - A_\varepsilon).$$

Hence, by using (7), we get

$$||\mathscr{L}_\varepsilon^{-1}(\mathscr{R}_\varepsilon^* \mathscr{Q}_\varepsilon \mathscr{R}_\varepsilon)\mathscr{L}_\varepsilon^{-1}|| \leqslant \frac{k_\varepsilon}{\triangle_\varepsilon} C||A - A_\varepsilon||^2 \leqslant \frac{k_\varepsilon}{\triangle_\varepsilon} C_3 \varepsilon^{2\alpha}. \quad (43)$$

iv) *Estimate of* $||\mathscr{L}_\varepsilon^{-1}(\mathscr{D}_\varepsilon^*\mathscr{Q}_\varepsilon\mathscr{D}_\varepsilon)\mathscr{L}_\varepsilon^{-1}||$: From (42) with $X_1 = 0, X_2 = A_\varepsilon{}'$ and $X_3 = (A_\varepsilon^2)'$, by (39) we see that

$$\mathscr{L}_\varepsilon^{-1}(\mathscr{D}_\varepsilon^*\mathscr{Q}_\varepsilon\mathscr{D}_\varepsilon)\mathscr{L}_\varepsilon^{-1} = \frac{k_\varepsilon}{\triangle_\varepsilon}G_\varepsilon\mathscr{J},$$

where $G_\varepsilon = 2k_\varepsilon A_\varepsilon{}'^* A_\varepsilon{}' + 3(A_\varepsilon^2)'^*(A_\varepsilon^2)'$. Hence we get, by using (7),

$$||\mathscr{L}_\varepsilon^{-1}(\mathscr{D}_\varepsilon^*\mathscr{Q}_\varepsilon\mathscr{D}_\varepsilon)\mathscr{L}_\varepsilon^{-1}|| \leqslant \frac{k_\varepsilon}{\triangle_\varepsilon}C||A_\varepsilon{}'||^2 \leqslant \frac{k_\varepsilon}{\triangle_\varepsilon}C_4\varepsilon^{2(\alpha-1)}. \tag{44}$$

v) Estimate of $||\mathscr{L}_\varepsilon^{-1}(\mathscr{B}_\varepsilon^*\mathscr{Q}_\varepsilon\mathscr{B}_\varepsilon)\mathscr{L}_\varepsilon^{-1}||$: From (42) with $X_1 = B, X_2 = A_\varepsilon B$, $X_3 = A_\varepsilon^2 B$, and by using (39), we see that

$$\mathscr{L}_\varepsilon^{-1}(\mathscr{B}_\varepsilon^*\mathscr{Q}_\varepsilon\mathscr{B}_\varepsilon)\mathscr{L}_\varepsilon^{-1} = \frac{k_\varepsilon}{\triangle_\varepsilon}H_\varepsilon\mathscr{J},$$

where

$$H_\varepsilon = B^*\{k_\varepsilon^2 + 3h_\varepsilon(A_\varepsilon + A_\varepsilon^*) - k_\varepsilon(A_\varepsilon - A_\varepsilon^*)^2 + 3A_\varepsilon^{*^2}A_\varepsilon^2\}B.$$

Hence

$$||\mathscr{L}_\varepsilon^{-1}(\mathscr{B}_\varepsilon^*\mathscr{Q}_\varepsilon\mathscr{B}_\varepsilon)\mathscr{L}_\varepsilon^{-1}|| \leqslant \frac{k_\varepsilon}{\triangle_\varepsilon}||H_\varepsilon|| \leqslant C_5\frac{k_\varepsilon}{\triangle_\varepsilon}||B(t)||^2. \tag{45}$$

From (40), (41), (43)–(45) and (19), (21), recalling that $||B(t)|| \leqslant C$ and $\varepsilon \leqslant 1$, and choosing

$$\varepsilon = \begin{cases} |\xi|^{-1} & \text{if } r = 1, \\ |\xi|^{-1/(1+\alpha/2)} & \text{if } r = 2, \\ |\xi|^{-1/(1+\alpha/3)} & \text{if } r = 3, \end{cases}$$

we obtain the following estimate, for $|\xi| \geqslant 1$,

$$\begin{aligned} E'(t,\xi) &\leqslant C_6E(t,\xi)\left[\varepsilon^{\alpha-1}\frac{k_\varepsilon^{3/2}}{\triangle_\varepsilon} + \varepsilon^\alpha\frac{k_\varepsilon^{1/2}}{\triangle_\varepsilon^{1/2}}|\xi| + \varepsilon^{\alpha-1}\frac{k_\varepsilon^{1/2}}{\triangle_\varepsilon^{1/2}}\right] \\ &\leqslant \begin{cases} C_7E[\varepsilon^{\alpha-1}k_\varepsilon^{3/2} + \varepsilon^\alpha k_\varepsilon^{1/2}|\xi| + \varepsilon^{\alpha-1}k_\varepsilon^{1/2}] & \text{if } r = 1, \\ C_7E[\varepsilon^{-1} + \varepsilon^{\alpha/2}k_\varepsilon^{-1/4}|\xi| + \varepsilon^{\alpha/2-1}k_\varepsilon^{-1/4}] & \text{if } r = 2, 3, \end{cases} \\ &\leqslant \begin{cases} CE[\varepsilon^\alpha|\xi| + \varepsilon^{\alpha-1}] \leqslant 2CE|\xi|^{1-\alpha} & \text{if } r = 1, \\ CE[\varepsilon^{\alpha/2}|\xi| + \varepsilon^{-1}] \leqslant 2CE|\xi|^{1/(1+\alpha/2)} & \text{if } r = 2, \\ CE[\varepsilon^{\alpha/3}|\xi| + \varepsilon^{-1}] \leqslant 2CE|\xi|^{1/(1+\alpha/3)} & \text{if } r = 3, \end{cases} \end{aligned}$$

which gives, by (36), the required a priori estimate (5) with σ equal, respectively, to $1/(1-\alpha), 1+\alpha/2$, or $1+\alpha/3$. This concludes the proof of Theorem 1 for $m = 3$.

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