On multivariate projection operators

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Received 10 April 2008; accepted 24 November 2008
Available online 3 December 2008

Communicated by C.K. Chui and H.N. Mhaskar
Dedicated to the memory of G.G. Lorentz

Abstract

This paper deals with multivariate Fourier series considering triangular type partial sums. Among others we give the exact order of the corresponding operator norm. Moreover, a generalization of the so-called Faber–Marcinkiewicz–Berman theorem has been proved.

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Keywords: Rectangular partial sums; Triangular partial sums; Operator norm of the multivariate Fourier series; Projection operators

1. Introduction

1.1

Multivariate Fourier series has been the object of an intensive study. We may refer the classical works Zygmund [1, Ch. XVII] and Stein, Weiss [2, Ch. VII]. This paper is another contribution to this subject. To step further we need some notations.

Let $\mathbb{R}^d$ (direct product) be the Euclidean $d$-dimensional space ($d \geq 1$, fixed) and let $\mathbb{T}^d = \mathbb{R}^d \ (\text{mod} \ 2\pi \mathbb{Z}^d)$ denote the $d$-dimensional torus, where $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots \}$.

Further, let $C(\mathbb{T}^d)$ denote the space of (complex valued) continuous functions on $\mathbb{T}^d$. By definition they are $2\pi$-periodic in each variable.

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For \( g \in C(\mathbb{T}^d) \) we define its Fourier series by

\[
g(\vartheta) \sim \sum_{k} \hat{g}(k)e^{ik \cdot \vartheta}, \quad \hat{g}(k) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} g(t)e^{-ik \cdot t} dt, \tag{1.1}\]

where in the above vector notation \( \vartheta = (\vartheta_1, \vartheta_2, \ldots, \vartheta_d) \in \mathbb{T}^d \), \( k = (k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d \) and \( k \cdot \vartheta = \sum_{l=1}^{d} k_l \vartheta_l \) (scalar product).

The rectangular \( n \)th partial sum of the Fourier series is defined by

\[
S_{nd}^{[r]}(g, \vartheta) := \sum_{|k| \leq n} \hat{g}(k)e^{ik \cdot \vartheta} \quad (n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}); \tag{1.2}
\]

the triangular one is

\[
S_{nd}^{[t]}(g, \vartheta) := \sum_{|k|_1 \leq n} \hat{g}(k)e^{ik \cdot \vartheta} \quad (n \in \mathbb{N}_0). \tag{1.3}
\]

Above, \( |k|_{\infty} = \max_{1 \leq l \leq d} |k_l| \) and \( |k|_1 = \sum_{k=1}^{d} |k_l| \) (they are the \( l_p \) norms of the multiindex \( k \)). The names “rectangular” and “triangular” refer to the shape of the corresponding indices of terms when \( d = 2 \) and \( 0 \leq k_1, k_2, |k|_{\infty} \leq n, |k|_1 \leq n \) respectively.

In a way the investigation of the \( S_{nd}^{[t]} \) is apparent: in many cases in essence it is a one variable problem (see the above works [1,2]).

However there are only relatively few works dealing with the triangular (or \( l_1 \)) summability: see Herriot [3].

In a recent paper [4] Berens and Xu prove the analogue of the famous Fejér paper on the (\( C, 1 \)) mean of \( S_n(f) \) (here and later \( S_n \) stands for \( S_{n1} \equiv S_{n1}^{[r]} \equiv S_{n1}^{[t]} \): actually it turns out that \( (C, 2d - 1) \) in a way corresponds to \( (C, 1) \) for any \( d \geq 1 \). Moreover the explicit formula for the corresponding Dirichlet kernel \( D_{nd} \) is crucial: it gives the possibility to prove many theorem for \( S_{nd} \equiv S_{nd}^{[r]} \) analogous to statements on \( S_n \) (this formula which actually gives \( D_{nd} \) as a divided difference of univariable function was developed earlier by Xu [5]).

Exploiting this relation we prove some theorems on \( S_{nd}^{[r]} \); as it turns out they can be used to investigate \( S_{nd}^{[t]} \), too.

2. New results

2.1

Introducing the notations

\[
D_{nd}(\vartheta) = \sum_{|k|_1 \leq n} e^{ik \cdot \vartheta} \quad (n \geq 1), \tag{2.1}
\]

where \( k \in \mathbb{Z}^d \), one can see that

\[
S_{nd}(g, \vartheta) = (g \ast D_{nd})(\vartheta) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} g(\vartheta - t)D_{nd}(t)dt = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} g(\vartheta + t)D_{nd}(t)dt, \tag{2.2}
\]

where as before, \( g \in C(\mathbb{T}^d), \vartheta, t \in \mathbb{T}^d \) (cf. (1.3) or [2, Chs I, VII], [4]).
Let \( \|g\| := \max_{\vartheta \in \mathbb{T}^d} |g(\vartheta)| \),

\[
\|S_{nd}\| := \max_{g \in C(\mathbb{T}^d)} \|S_{nd}(g, \vartheta)\| \quad (n \geq 1)
\]

and

\[
\|g\|_p := \left( \int_{\mathbb{T}^d} |g(\vartheta)|^p d\vartheta \right)^{1/p}
\]

if \( g \in L^p := \{ \text{the set of all measurable } 2\pi \text{ periodic (in each variable) functions on } \mathbb{T}^d \}, \)

\( 1 \leq p < \infty \).

We state

**Theorem 2.1.** We have, for any fixed \( d \geq 1 \),

\[
\|D_{nd}\|_1 = \|S_{nd}\| \sim (\log n)^d \quad (n \geq 2). \tag{2.3}
\]

**Remark.** The general case of the upper estimation (when \( d \geq 4 \)) is due to Professor Gábor Halász [6]. We shall give another argument applying a formula of Xu.

The lower estimation also belongs to Gábor Halász [6]. The proof is based on his original argument; we thank him for communicating the ideas.

2.2

One of the most characteristic properties of the Fourier series in one dimension is the so-called Faber–Marcinkiewicz–Berman theorem, namely that the operator \( S_n \) has the smallest norm among all projection operators (cf. [7, p. 281] for other details). This part extends the above statement for \( S_{nd}, d \geq 1 \).

Let \( T_{nd} \) be the space of trigonometric polynomials of form

\[
\sum_{|k|_1 \leq n} \left( a_k \cos(k \cdot \vartheta) + b_k \sin(k \cdot \vartheta) \right),
\]

where \( k = (k_1, k_2, \ldots, k_d) \) and \( k_1, \ldots, k_d \geq 0 \), arbitrary real numbers. Moreover, let \( T_{nd} \) be a linear trigonometric projection operator on \( C(\mathbb{T}^d) \), i.e. \( T_{nd}(g, \vartheta) = g(\vartheta) \) for \( g \in T_{nd} \) and \( T_{nd}(g, \vartheta) \in T_{nd} \) for other \( g \in C(\mathbb{T}^d) \).

**Theorem 2.2.** For any linear trigonometric projection operator \( T_{nd} \), one has

\[
\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} T_{nd}(g_t, \vartheta - t) dt = S_{nd}(g, \vartheta) \quad (g \in C(\mathbb{T}^d)), \tag{2.4}
\]

\[
\|T_{nd}\| \geq \|S_{nd}\|, \tag{2.5}
\]

where \( g_t(\vartheta) = g(\vartheta + t) \) is the \( t \)-translation operator.

\[\text{1} \text{ Here and later } a_n \sim b_n \text{ means that } 0 < c_1 \leq a_n b_n^{-1} \leq c_2 \text{ where } c, c_1, c_2, \ldots \text{ are positive constants, not depending on } n; \text{ they may denote different values in different formulae.}\]
Here we formulate an analogue of Theorem 2.2 which can be applied for the norm estimation of projection onto the space \( \Pi_{nd} \) (the algebraic analogue of \( T_{nd} \); cf. [7, p. 284]). Let \( T_{nd} \) be a linear trigonometric projection operator defined on the even functions in \( C(T^{d}) \). We extend it to all of \( C(T^{d}) \) by defining \( T_{nd}^{*}(g) := T_{nd}(G) \), where \( 2G(\vartheta) = g(\vartheta) + g(-\vartheta) \). Obviously, \( \| T_{nd}^{*} \| = \| T_{nd} \| \).

**Theorem 2.3.** We have for \( g \in C(T^{d}) \)

\[
\frac{1}{(2\pi)^{d}} \int_{T^{d}} T_{nd}^{*}(g + g_{-t}, \vartheta - t)dt = S_{nd}^{*}(g, \vartheta) = S_{nd}(G, \vartheta),
\]

\( \| T_{nd} \| = \| T_{nd}^{*} \| \geq \frac{1}{2} \| S_{nd}^{*} \| = \frac{1}{2} \| S_{nd} \|. \) (2.6) (2.7)

A simple consequence is

**Theorem 2.4.** If \( P_{nd} \) is a projection of \( C(I^{d}) \) onto \( \Pi_{nd} \) then

\[
\| P_{nd} \| \geq \frac{1}{2} \| S_{nd} \|. \) (2.8)

Above, \( P_{nd} \) is a projection of \( C(I^{d}) \) (:= the set of continuous functions of \( d \)-variables on \( I^{d} = [-1, 1]^{d} \)) onto \( \Pi_{nd} \) iff it is linear, \( P_{nd}(p) = p \) if \( p \in \Pi_{nd} \) and \( P_{nd}(f) \in \Pi_{nd} \) for any \( f \in C(I^{d}) \).

As an application of Theorem 2.4, see the papers [8–10] dealing with the two-dimensional Lagrange interpolation defined on special node systems.

**Remark.** Using our methods we intend to settle the “critical cases” in the paper [11, p. 290.] in our forthcoming work.

3. **Proofs**

3.1. **Proof of Theorem 2.1**

3.1.1

The relation

\[
\| D_{nd} \|_{1} = \| S_{nd} \|
\]

(3.1)

is a simple consequence of the Riesz representation theorem (see [12, IV. 6.3] or [13]).

3.1.2

Next, we prove the relation

\[
\| D_{nd} \|_{1} \sim (\log n)^{d} \quad (n \geq 2).
\]
Our basic tool is the relation proved by Xu (cf. [5, Lemma 1])

\[ D_{nd}(\vartheta) = (-1)^{\frac{d+1}{2}} \sum_{l=1}^{d} \frac{2 \cos \frac{\vartheta_l}{2} (\sin \frac{\vartheta_l}{2})^{d-2} \text{soc} \frac{2n+1}{2} \vartheta_l}{\prod_{j=1 \atop j \neq l}^{d} (\cos \vartheta_l - \cos \vartheta_j)}, \] (3.2)

where the function \( \text{soc}(\sin \text{ or } \cos) \) is defined by

\[ \text{soc}\vartheta := \begin{cases} 
\sin \vartheta & \text{if } d \text{ is odd}, \\
\cos \vartheta & \text{if } d \text{ is even}. 
\end{cases} \] (3.3)

3.1.3

First we prove that \( \|D_{nd}\|_1 \leq c (\log n)^d \). Indeed, let

\[ e_d := \left\{ \vartheta : |\vartheta_k| \leq n^{-d}, |\vartheta_k - \vartheta_l| \leq n^{-d}, \ 1 \leq k \neq l \leq d \right\}. \] (3.4)

Obviously \( |e_d| \leq A(d)n^{-d} \), where \( A(d) > 0 \), fixed. If \( E_d := \mathbb{T}^d \setminus e_d \) we get

\[ \int_{\mathbb{T}^d} |D_{nd}| = \int_{e_d} |D_{nd}| + \int_{E_d} |D_{nd}|. \] (3.5)

By (2.1), \( |D_{nd}(\vartheta)| \leq 2^d \left( \frac{2n+1}{d} \right) \leq B(d)n^d \), whence

\[ \int_{e_d} |D_{nd}| \leq A(d)B(d). \]

At the second integral of (3.5) we estimate only one term on the right-hand side of (3.2) on \( F_d := E_d \cap [0, \frac{\pi}{2}]^d \) and then we use symmetry. By

\[ \sin \vartheta_1 \leq 2 \sin \frac{\vartheta_1 + \vartheta_j}{2} \quad (2 \leq j \leq d, \ \vartheta \in F_d) \] (3.6)

(consider the cases \( \vartheta_1 \leq \vartheta_j \) or \( \vartheta_1 > \vartheta_j \)) and

\[ \cos \vartheta_1 - \cos \vartheta_j = 2 \sin \frac{\vartheta_1 + \vartheta_j}{2} \sin \frac{\vartheta_1 - \vartheta_j}{2}, \] (3.7)

we get an estimate for \( s_1 \) (the first term of (3.2)) as follows:

\[ |s_1| \leq c \cdot \frac{\prod_{j=2}^{d} \sin \frac{\vartheta_1+\vartheta_j}{2}}{\sin \vartheta_1 \prod_{j=2}^{d} \sin \frac{\vartheta_1+\vartheta_j}{2} \sin \frac{\vartheta_1-\vartheta_j}{2}} \left| \prod_{j=2}^{d} \frac{\sin \frac{\vartheta_1-\vartheta_j}{2}}{\sin \frac{\vartheta_1-\vartheta_j}{2}} \right| \]
\[ \leq \frac{c_1}{\sin \vartheta_1 \prod_{j=2}^{d} \left| \sin \frac{\vartheta_1-\vartheta_j}{2} \right|} (\vartheta \in F_d). \]
Then, using the substitution $u_1 = \vartheta_1, u_j = \vartheta_j - \vartheta_1$, whence $\vartheta_1 = u_1, \vartheta_j = u_1 + u_j (2 \leq j \leq d)$, we get

$$\int_{F_d} |s_1| d\vartheta \leq c_1 \int_{F_d} \frac{1}{\sin \vartheta \prod_{j=2}^{d} \left| \frac{\vartheta_1 - \vartheta_j}{2} \right|} d\vartheta$$

$$\leq c_2 V_d \int_{\frac{\pi}{n^d}} \int_{\frac{\pi}{n^d}} \ldots \int_{\frac{\pi}{n^d}} \frac{1}{\prod_{j=1}^{d} u_j} du_1 du_2 \ldots du_d$$

$$= c_2 \left( \log \frac{\pi n^d}{2} \right)^d \sim (\log n)^d,$$

where if

$$U_d := \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 1 & \ldots & 0 \\ \ldots & & & & \ldots \\ 1 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

is the matrix of the transformation, then $V_d = |\det U_d| = 1$.

3.1.4

Here we prove that

$$\|D_{nd}\|_1 \geq c (\log n)^d \quad (n \in \mathbb{N}) \quad (3.8)$$

with a constant $c > 0$ independent of $n$.

Our proof is based on Fejér’s classical example (see [14, Vol. 2, Ch. 2/1]). We suggest the interested reader to examine it carefully.

For every $n \in \mathbb{N}$ we shall construct a trigonometric polynomial $f_n(t) = \sum c_j e^{i j t}$ with

$$\|f_n\| \leq 1 \quad \text{and} \quad |S_{nd}(f_n, 0)| \geq c (\log n)^d \quad (n \in \mathbb{N}), \quad (3.9)$$

where $c > 0$ is independent of $n$.

With these polynomials we have

$$\|D_{nd}\|_1 = \|S_{nd}\| = \max_{g \in C(T^d)} \|S_{nd}(g, \cdot)\| \geq |S_{nd}(f_n, 0)| \|g\| \leq 1$$

which proves (3.8).

For the construction of $f_n$ first we choose real numbers $\alpha_j, \beta_j (j = 1, 2, \ldots, d)$ for which

$$1 > \alpha_1 > \beta_1 > \alpha_2 > \beta_2 > \cdots > \alpha_d > \beta_d > 0. \quad (3.10)$$
Starting from Fejér’s classical example $F_m$ (see (3.12)) let us consider the trigonometric polynomials

\[ F_j(t) := \sum_{|k| = [n^{\beta_j}]} \frac{1}{k} e^{ikt} \quad (t \in [0, 2\pi), 1 \leq j \leq d). \tag{3.11} \]

As we know for the trigonometric polynomials

\[ F_m(t) := \sum_{0 < |l| \leq m} \frac{1}{l} e^{ilt} \quad (t \in \mathbb{R}, m \in \mathbb{N}) \tag{3.12} \]

we have

\[ |F_m(t)| = 2 \left| \sum_{l=1}^{m} \frac{\sin lt}{l} \right| \leq 4 \sqrt{\pi} \quad (t \in \mathbb{R}, m \in \mathbb{N}) \]

(see [15,16], [14, Vol. I, (118)]). Therefore we get

\[ |F_j(t)| = |F_{[n^{\alpha_j}]}(t) - F_{[n^{\beta_j}]_{-1}}(t)| \leq |F_{[n^{\alpha_j}]}(t)| + |F_{[n^{\beta_j}]_{-1}}(t)| \leq 8 \sqrt{\pi} =: M. \tag{3.13} \]

Using $e_j := (0, \ldots, 0, 1, 0, \ldots, 0)(1 \leq j \leq d)$ (the canonical unit vectors of $\mathbb{R}^d$) we define the polynomial $g_n(t) := M^d f_n(t)$ as follows

\[ g_n(t) := e^{intd} \cdot F_d(-e_d \cdot t) \cdot \prod_{j=1}^{d-1} F_j ((e_j - e_d) \cdot t) \]

\[ = e^{intd} \sum_{|k| = [n^{\alpha_d}]} \frac{e^{-ikdt_d}}{k_d} \prod_{j=1}^{d-1} \left( \sum_{|k| = [n^{\beta_j}]} \frac{1}{k} e^{i(k_jt_j-k_jt_d)} \right), \tag{3.14} \]

where $\prod_{j=1}^{d} \ldots := 1$ and $(e_j - e_d) \cdot t = t_j - t_d$.

Using (3.13) we obtain that $|f_n(t)| \leq 1(t \in \mathbb{T}^d)$, i.e. the first requirement of (3.9) holds.

The polynomial $g_n(t)$ can be written as

\[ g_n(t) = \sum_{k_1, \ldots, k_d} \frac{1}{k_1k_2 \cdots k_d} e^{ikt}, \]

where $k := (k_1, k_2, \ldots, k_{d-1}, n - k_1 - k_2 - \cdots - k_d)$ and the notation $\sum_{k_1, \ldots, k_d}^*$ means that we take the summation for indices $[n^{\beta_j}] \leq |k_j| \leq [n^{\alpha_j}](1 \leq j \leq d)$.

The triangular partial sum of the Fourier series of $g_n$ at the point $0$ is given by

\[ S_{nd}(g_n, 0) = \sum_{k_1, \ldots, k_d}^* \frac{1}{k_1k_2 \cdots k_d}. \tag{3.15} \]

Now we prove that in this sum appear only the positive indices $k_1, \ldots, k_d$. 
First we remark that $n - k_1 - k_2 - \cdots - k_d \geq 0$. Then, by $|k_j| - k_j \geq 0 (1 \leq j \leq d)$, we get
\[
|k|_1 = |k_1| + |k_2| + \cdots + |k_{d-1}| + |n - k_1 - k_2 - \cdots - k_d| \\
= \sum_{j=1}^{d-1} (|k_j| - k_j) + n - k_d \geq n - k_d,
\]
whence we obtain that $k_d \geq 0$. Applying (3.10) we have $k_d > 0$.

Let us suppose that for a fixed index $j^*(1 \leq j^* \leq d-1)$ we have $k_{j^*} < 0$. Since $-k_{j^*} - k_d > 0$ for $n \geq n_0$ (see (3.10)), we would get
\[
|k|_1 = \sum_{j=1}^{d-1} (|k_j| - k_j) + n - k_d \geq -2k_{j^*} + n - k_d > n - k_{j^*} > n
\]
— a contradiction. This means that in (3.15) appear only positive indices $k_1, k_2, \ldots, k_{d-1}$, indeed. Therefore we get (see (3.14))
\[
S_{nd}(f_n, 0) = \frac{1}{M^d} \prod_{j=1}^{d} \left( \sum_{k=[n^{\beta_j}]}^{[n_j]} \frac{1}{k_j} \right) \geq c(\log n)^d
\]
which proves the second requirement of (3.9).

Thus the proof of (3.8) is complete. □

3.2. Proof of Theorem 2.2.

It is analogous to the classical argument (cf. [7, p. 282]).

3.2.1

First we verify that the integrand in (2.4) is a continuous function of $(\vartheta, t) \in \mathbb{T}^d \times \mathbb{T}^d$. Indeed, if $\vartheta_0, t_0$ are fixed then
\[
|T_{nd}(g_t, \vartheta) - T_{nd}(g_{t_0}, \vartheta_0)| \leq |T_{nd}(g_t, \vartheta) - T_{nd}(g_{t_0}, \vartheta_0)|
\]
\[
+ |T_{nd}(g_{t_0}, \vartheta) - T_{nd}(g_{t_0}, \vartheta_0)| =: \delta_1 + \delta_2.
\]
Above, $\delta_1 \leq ||T_{nd}|| \|g_t - g_{t_0}\| < \varepsilon$ if $\|t - t_0\|$ is small; on the other hand if $\|\vartheta - \vartheta_0\|$ is small then $\delta_2 < \varepsilon$, too.

So the integral (2.4) is well defined. Denoting it by $A(g, \vartheta)$ we prove that $A(g, \vartheta) = S_{nd}(g, \vartheta)$ for any $g \in C(\mathbb{T}^d)$.

First, we verify that $A(g)$ is linear trigonometric projection operator. It is enough to prove the boundedness of $A(g)$. But it follows by
\[
|A(g, \vartheta)| \leq \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} |T_{nd}(g_t, \vartheta - t)|dt
\]
\[
\leq \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \|T_{nd}\| \|g_t\|dt = \|T_{nd}\| \|g_t\|,
\]
whence $\|A\| \leq \|T_{nd}\|$. 

3.2.2
Moreover, we verify that
(a) for $p \in \mathcal{T}_{nd}$, $A(p) = S_{nd}(p)$. Indeed, by $p_t(\vartheta) \in \mathcal{T}_{nd}$ we have that $T_{nd}(p_t, \vartheta - t) = p_t(\vartheta - t) = p_t(\vartheta)$, whence by (2.4), $A(p) = p$, which by $S_{nd}(p) = p$ (cf. [7, p.209]), gives $A(p) = S_{nd}(p)$ if $p \in \mathcal{T}_{nd}$.
(b) Now let $p(\vartheta) = \cos(m \cdot \vartheta), |m|_1 > n$. We prove that $A(p, \vartheta) = 0$. Indeed, by
$$p_t(\vartheta) = \cos(m \cdot \vartheta) \cos(m \cdot t) - \sin(m \cdot \vartheta) \sin(m \cdot t)$$
(to verify, see the exponential form, say) we get
$$\int_{\mathbb{T}^d} T_{nd}(p, \vartheta - t) \cos(m \cdot t) dt$$
$$= - \int_{\mathbb{T}^d} T_{nd}(q, \vartheta - t) \sin(m \cdot t) dt =: I_1 + I_2,$$
where $q(\vartheta) = \sin(m \cdot \vartheta)$. Consider the first integral, say. By definition, $T_{nd}(p, \vartheta - t)$ is a linear combination of terms having the form $\cos(k \cdot t)|k|_1 \leq n$, whence in $I_1$ we have integrals of type
$$\int_{\mathbb{T}^d} \cos(k \cdot t) \cos(m \cdot t) dt.$$ 
(3.16)
But they are equal to zero. Indeed, as it is well known (and easy to see) that the functions
$$\{e^{i(r \cdot t)}; r \in \mathbb{Z}_0^d, t \in \mathbb{T}^d\}$$
are mutually orthogonal, i.e.
$$\int_{\mathbb{T}^d} e^{i(r \cdot t)} e^{-is \cdot t} dt \neq 0 \quad \text{iff } r = s;$$
whence by
$$2 \cos(r \cdot \vartheta) = e^{i(r \cdot \vartheta)} + e^{-i(r \cdot \vartheta)},$$
$$2i \sin(r \cdot \vartheta) = e^{i(r \cdot \vartheta)} - e^{-i(r \cdot \vartheta)}$$
(one may verify them by induction for $d$), the functions
$$\{e, \cos(r \cdot t), \sin(s \cdot t)\};$$
$r, s \in \mathbb{N}^d, e = \{1\}^d$ are mutually orthogonal, too. That means, in (3.16) using $|k|_1 < |m|_1$ we get $k \neq m$, i.e. the integral is zero, indeed.

The other cases are similar. Using that $S_{nd}(p) = S_{nd}(q) = 0(|m|_1 > n)$, we get $A(p) = A(q) = S_{nd}(p) = S_{nd}(q) = 0$ at case (b).

3.2.3
Summarizing, we obtained that $S_{nd}(g) = A(g)$ for all trigonometric polynomials, whatever their degree. Using that they form a dense set in $C(\mathbb{T}^d)$ (cf. [2, Theorem 1.7, p. 248]), the operators $A$ and $S_{nd}$ coincide. \hfill $\Box$

3.2.4
Now (2.5) is immediate: By (2.4) we write
$$|S_{nd}(g, \vartheta)| \leq \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} |T_{nd}(g, \vartheta - t)| dt \leq \|T_{nd}\| \|g\|. \hfill \Box$$
3.3. Proof of Theorem 2.3.

It is very similar to the previous one. For the sake of variety we use the system \{e^{\imath r t}\}.

Let \(g(\vartheta) = e^{\imath k \cdot \vartheta}, |k|_1 \leq n\). Then by \(g_{\pm t}(\vartheta) = e^{\imath k(\vartheta \pm t)}\), whence

\[
T_{nd}^* (g_t + g_{-t}, \vartheta - t) = 2 \cos (k \cdot (\vartheta - t)) \cos (k \cdot t).
\]

So the integrand in (2.6) is equal to \(\cos(k \cdot \vartheta)\) (simple calculation) which is also the right-hand side of (2.6).

(b) If \(g(\vartheta) = e^{\imath m \cdot \vartheta}, |m|_1 > n\), then the integrand is \(T(\vartheta - t) \cos (m \cdot t)\), where \(T \in T_{nd}\), i.e. the left-hand side is zero, which holds for the right-hand side, too.

Using the above facts, we can get (2.6) as before (cf. 3.2.1–3.2.3).

Finally, using the obvious fact \(\|T_{nd}\| = \|T_{nd}^*\|\), (2.7) easily comes from (2.6) (see 3.2.4). □

3.4. Proof of Theorem 2.4.

The map

\[
V : f(x) \to f(\cos \vartheta) \equiv g(\vartheta)
\]

is an isometric operator with \(\|V\| = 1\). If we define the projection in Theorem 2.3 by

\[
T_{nd} = VP_{nd}V^{-1},
\]

then by (2.7)

\[
\frac{1}{2} \|S_{nd}\| \leq \|T_{nd}^*\| = \|T_{nd}\| = \|VP_{nd}V^{-1}\|
\]

as it was stated. □

Acknowledgments

The authors thank the referees for their comments and suggestions. At the same time we express again our sincere thanks to Professor Gábor Halász. The work was supported by the Hungarian National Scientific Research Foundation (OTKA) under Grant Nos. T37299 and T047132.

References

[16] L. Fejér, A függvény szakadásának meghatározása Fourier-féle sorából (Hungarian), Mat. és Term Értesítő 31 (1913) 385–415.