Complexity Estimates for the Schmüdgen Positivstellensatz

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Let $K$ be a closed basic set in $\mathbb{R}^n$ given by the polynomial inequalities $\phi_1 \geq 0, \ldots, \phi_m \geq 0$ and let $\Sigma$ be the semiring generated by the $\phi_k$ and the squares in $R[x_1, \ldots, x_n]$. Schmüdgen has shown that if $K$ is compact then any polynomial function strictly positive on $K$ belongs to $\Sigma$. Easy consequences are (1) $f \geq 0$ on $K$ if and only if $f \in R^+ + \Sigma$ (Positivstellensatz) and (2) if $f \geq 0$ on $K$ but $f \notin \Sigma$ then as $d$ tends to $0^+$, in any representation of $f + d$ as an element of $\Sigma$ in terms of the $\phi_k$, the squares and semiring operations, the integer $N(d)$ which is the minimum over all representations of the maximum degree of the summands must become arbitrarily large. A one-dimensional example is analyzed to obtain asymptotic lower and upper bounds of the form $cd^{1/2} \leq N(d) \leq Cd^{-1/2} \log (1/d)$. © 1996 Academic Press, Inc.

1. Introduction

Following Hilbert’s illustrious contribution, the international mathematical community accepts the German word “nullstellensatz” for algebraic conditions characterizing the vanishing of one function on the common zero set of some collection of other functions. In the same spirit we use “positivstellensatz” in real algebraic and semialgebraic geometry for analogous conditions characterizing definiteness of a function on the set of solutions of a system of real equations and inequalities (Lam [3] has even made the witty suggestion of using “stellensätze” as a comprehensive term for all such theorems.) In 1990 Schmüdgen [4], in the course of characterizing moment sequences of positive Borel measures on subsets of $\mathbb{R}^n$, obtained a new result of this kind. He discovered an algebraic characterization of strict positivity for polynomial functions on compact basic semialgebraic sets, that is, sets which are the common nonnegativity set of a finite collec-
tion of polynomials $\Phi = \{\phi_1, \phi_2, \ldots, \phi_p\}$. For its statement we introduce the semiring $\Sigma(\Phi)$ in $R[x_1, x_2, \ldots, x_n]$ generated by the squares and the elements of $\Phi$. We also use the notation $\Phi \geq 0$ to mean the system of inequalities obtained by requiring each element of $\Phi$ to be nonnegative.

**Theorem 1 (Schmüdgen).** If the set $K = \{\Phi \geq 0\}$ is compact and $g$ is a polynomial function strictly positive on $K$ then $g \in \Sigma(\Phi)$.

Unlike the algebraic stellensätze which give necessary and sufficient conditions this expresses a necessary condition for strict positivity. However, using existence of greatest lower bounds in the reals, it is trivial to restate it in terms of a necessary and sufficient condition.

**Theorem 2 (Positivstellensatz).** If the set $K = \{\Phi \geq 0\}$ is compact then $g$ is strictly positive on $K$ if and only if $g \in R^+ + \Sigma(\Phi)$.

**Proof.** $f > 0$ implies $f \geq c > 0$. Hence $f = c/2 + f - c/2 \in R^+ + \Sigma(\Phi)$.

Theorem 1 is perhaps surprising since for noncompact $K$ or nonstrictly positive $f$ the body of theory arising from Hilbert’s seventeenth problem [1] shows that in general the semiring $\Sigma(\Phi)$, while it obviously consists of functions nonnegative on $K$, is very far from the collection of all such functions. An important aspect of the various algebraic stellensätze is their validity over more general ground fields. In this case, however, although Schmüdgen’s proof depends partially on methods and results of real algebra and semialgebraic geometry, the theorem is, in fact, a result of analysis depending on properties of the real numbers. This is signalized by the appeal to the topological property of compactness (although this is nicely described algebraically by including an inequality $\|x^n\| - r^2 \leq 0$ in the definition of $K$). If it were purely a result of algebra one might guess that it would be true over a real closed field containing infinitesimals for which the definiteness of $f$ takes the form $f \geq \tau$ where $\tau$ is infinitely small. Extracting ordinary parts would then give the conclusion over $R$ with the condition of strict positivity relaxed to nonnegativity. But this is not possible as the following example shows.

**Example.** Let $K = [-1, 1] = \{x \mid (1 - x^2)^3 \geq 0\}$ and $f = 1 - x^2$. Recalling that in the single variable case sums of squares and nonnegative polynomial functions coincide, the semiring $\Sigma(\Phi)$ then consists of polynomials of the form $P(1 - x^2)^3 + Q$ where $P$ and $Q$ are globally nonnegative. But although $f$ is nonnegative on $K$ it cannot be that $1 - x^2 = P(1 - x^2)^3 + Q$ since the implied vanishing of $Q$ at $x = 1$ would necessarily be of even order which in turn would imply the contradiction that $1 - x^2$ vanishes there to at least order two. However, if $1 - x^2$ is replaced by $1 - x^2 + d$ where $d$ is positive then we conclude that there exist nonnegative $P$ and $Q$ for which

$$P(1 - x^2)^3 + Q.$$
This relation, perhaps the simplest nontrivial consequence of Theorem 1, is an object of study in this paper. We first obtain a kind of negative consequence of (1), Theorem 4, showing an adverse dependence of (1) on \( d \) in the form of a necessary lower bound for the degree of \( P \) which becomes large as \( d \) becomes small. We then give a positive result, Theorem 5, which implies an asymptotic upper bound for the minimum degrees of \( P \) and \( Q \) sufficient to represent a general \( f \) in case the interval \([-1, 1]\) is determined by a general system of inequalities. While we can hardly hope that these special results on representations in a single variable on a single interval are typical of the general case, we believe they do shed some light on the nature of Schmüdgen's result and the methods appropriate to its study.

### 2. A LOWER COMPLEXITY BOUND

As a preliminary we first reason rigorously to show that the degrees of \( P \) and \( Q \) in (1) must become large as \( d \) tends to 0. Otherwise we could select a sequence \( d_n \) in (1) tending to 0 with corresponding \( P_n \) and \( Q_n \) remaining bounded in degree. The positivity of \( P_n \) and \( Q_n \) together with (1) would then imply that \( P \) and \( Q \) lie in a finite dimensional ball defined by the supremum norm on the interval \([-1/2, 1/2]\]. Extracting a convergent subsequence and using the equivalence in finite dimensions of convergence in any norm and pointwise convergence we would then obtain in the limit \( Q \) and consequently also \( P \), satisfying a contradictory relation of the form (1) with \( d = 0 \). Thus the degrees of \( P \) and \( Q \) must grow as \( d \) becomes small. With simple elaborations (which we omit) this argument also proves the following theorem. For its statement we require precise notions of representation and degree. By representing \( f \) we mean expressing \( f \) in terms of the elements of \( \Phi \), the squares and semiring operations. By the degree of a representation we mean the highest degree of any summand. For example, if \( \Phi = \{(1 - x^2)^3\} \) and \( f = 5/3 - x^2 \) then \( f = 1/3 + 4(1 - x^2)^3/3 + x^2(2x^2 - 3)^2/3 \) is a representation of degree 6. The significance of a representation from an algorithmic or constructivist point of view is that, without further reasoning or calculation, it bears immediate witness to the definiteness of \( f \) on \( K \).

**Theorem 3.** Suppose \( \{\phi \geq 0\} \) is compact. \( \{\Phi > 0\} \) is nonempty, \( \Phi > 0 \) implies \( f \equiv 0 \), and \( f \) does not belong to \( \Sigma(\Phi) \). For \( d > 0 \) let \( N = N(f, d) \) be the least integer for which \( f + d \) has a representation of degree \( N \) as an element of \( \Sigma(\Phi) \). Then \( N \) grows without bound as \( d \) tends to 0.

Our first hard result is an asymptotic lower bound for the degree of \( P \)
in any representation (1). Regarding this degree as a simple natural measure of the complexity of relation (1) we quantify its dependence on $d$ with the following lower estimate.

**THEOREM 4.** Let $N(d)$ be the least degree of any $P$ for which the relations
\[ 1 - x^2 + d = P(1 - x^2)^3 + Q, \quad P, \ Q \geq 0 \]
hold. Then there is a constant $C$ such that $N(d) \geq Cd^{1/2}$.

**Proof.** We use some tools of analysis, particularly approximation theory. A classical extremal property of the Chebyshev polynomials $T_n(x)$ ensures that if degree $(P) = n$ then for $a \geq 1$,
\[ \max_{|x| \leq a} |P(x)| \leq |T_n(a)| \max_{|x| \leq 1} |P(x)|. \]  

We require a consequence of this which relates bounds on intervals larger and smaller than $[-1, 1]$. For $0 < r < 1$ we have
\[ \max_{x^2 \leq (1 - r)^2} |P(x)| = \max_{x^2 \leq (1 - r)^2} |P(x(1 - r)^{1/2})| \]
\[ \leq T_n(1/(1 - r)) \max_{x^2 \leq 1} |P(x(1 - r)^{1/2})| \]
\[ = T_n(1/(1 - r)) \max_{x^2 \leq 1} |P(x)|. \]  

Next (1) and the positivity of $Q$ imply that
\[ \max_{x^2 \leq 1 - r} P(x) \leq dr^3 + r^2. \]  

Let $u = x^2 - 1$. Then for $1 \leq x^2 \leq 1/(1 - r)$ or equivalently for
\[ 0 \leq u \leq r/(1 - r) \]
(3) and (4) imply that
\[ d \geq u - u^2(dr^3 + r^2)T_n(1/(1 - r)). \]  

Since
\[ T_n(x) = \frac{1}{2} \left[ (x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n \right] \]
some algebra shows that estimate (6) can be weakened to
Here any choice of $r$ and $u$ satisfying (5) gives a valid inequality. The choice $r = n^{-2}$ and $u = r/2$ gives

$$n^2 d \geq \frac{1}{2} - \frac{1}{8} (n^2 d + 1)\{e + o(1)\}$$

or

$$\left\{1 - \frac{e}{8} + o(1)\right\}n^2 d \geq \frac{1}{2} - \frac{1}{8} \{e + o(1)\}$$

which for large $n$ implies $n^2 d \geq c > 0$, completing the proof.

3. AN UPPER BOUND

We continue to consider only the special case $K = [-1, 1]$ but now consider a general polynomial function $f$ nonnegative there. We also make no special assumptions about the system of polynomial inequalities determining this set. We give an asymptotic upper degree bound within which it is sure that a representation can be found. This depends explicitly on attributes of $f$. In contrast, the system of inequalities, which we regard as fixed, enters only through the constant in the asymptotic estimate.

**Theorem 5.** Given a system of polynomial inequalities $\Phi \geq 0$ with solution set $[-1, 1]$ there is a constant $C = C(\Phi)$ such that any polynomial function of degree $m$ satisfying $0 < d \leq f \leq M$ on $[-1, 1]$ has a representation as an element of $\Sigma(\Phi)$ of degree at most

$$C(Mm^2/d)^{1/2} \log(Mm^2/d).$$

**Proof.** We first show that there is a single element $\sigma$ of $\Sigma(\Phi)$ such that $[-1, 1] = \{\sigma \geq 0\}$. The following algebraic argument gives a first step in this direction. Of course $1 - x^2 \geq 0$ determines the interval but $1 - x^2$ is not in general an element of $\Sigma(\Phi)$. However, since by hypothesis $\Phi \geq 0$ implies $1 - x^2 \geq 0$ we know by the polynomial positivstellensatz [1, 5] that for some integer $k$ and for some elements $\sigma_1$ and $\sigma_2$ of $\Sigma(\Phi)$ we have
\[(1 - x^3)(1 - x^3)^{2k} + \sigma_1) = \sigma_2.\]

Hence
\[\sigma_3 = ([1 - x^3]^{2k} + \sigma_1)\sigma_2 = ([1 - x^3]^{2k} + \sigma_1)^2(1 - x^2)\]
is an element of \(\Sigma(\Phi)\) which is strictly positive on \((-1, 1)\), changes sign at \(\pm 1\), and is nonpositive outside \([-1, 1]\). Thus \(\{\sigma_3 \geq 0\}\) consists of this interval together with at most finitely many exterior points.

We next aim at excluding these exterior points. This requires recourse to approximation theory. Choose an interval \([x_0, x_1]\) with \(x_0 > 1\) containing the moduli of these points. Then at each point of \([-x_1, -x_0] \cup [x_0, x_1]\) there is an element of \(\Phi\) negative there. Hence by compactness and a standard partition of unity argument there are elements \(f_1, \ldots, f_q\) of \(\Phi\) and globally nonegative continuous functions \(a_1, \ldots, a_q\) such that
\[\sum a_j f_j \geq 1\] on \([-x_1, x_1]\), and are globally positive. For example, replacing the \(P_j\) by \(Q_j = P_j + \varepsilon T_{2N}(x/x_1)\) for sufficiently small \(\varepsilon\) and sufficiently large \(N\) we can accomplish this. Then for possibly smaller \(\varepsilon\) and larger \(N\) we have that \(\sigma_4 = \sum Q_j \phi_j\) is an element of \(\Sigma(\Phi)\) satisfying \(\sigma_4 \leq -1\) on \([-x_1, -x_0] \cup [x_0, x_1]\). We next consider \(\sigma_5 = \sigma_3 + \delta \sigma_3 (1 - x^2)^{2q}\). If \(2p\) exceeds the orders of the zeroes of \(\sigma_3\) at \(\pm 1\) then the sign changes of \(\sigma_3\) are not affected. Then, for sufficiently small \(\delta, \sigma_5\) inherits the strict positivity of \(\sigma_3\) on \((-1, 1)\). Thus \(\{\sigma_5 \geq 0\}\) consists of \([-1, 1]\) together with possibly a set exterior to \([-x_1, x_1]\). Since on this exterior set \(\sigma_5\) is strictly negative, for sufficiently large \(L\) and \(r, \sigma = \sigma_5 + L(1 + x^2)\sigma_3\) will be an element of \(\Sigma(\Phi)\) having exactly \([-1, 1]\) as it nonnegativity set.

The balance of the argument will be to produce and analyze a representation of \(f\) in \(\Sigma(\{\sigma\})\). We introduce the following normalizations. Replacing \(f\) by \(f/M\) and \(d\) by \(d/M\) we can suppose that \(M = 1\). Also since \(\sigma\) is bounded above we can suppose that it is bounded by 1. We will show for sufficiently large \(n\) that \(f\) has a representation of the form
\[f = \frac{d}{2} \sigma(1 + T_{2n}) + Q.\]

Globally \(T_{2n} \geq -1\) so this will be a representation provided that \(Q\) is nonnegative. We establish this by considering the sign of \(Q\) in three subdomains.

Subdomain 1: \([-1, 1]\). The conditions \(f \geq d, \sigma \leq 1,\) and \(0 \leq (1 + T_{2n})/2 \leq 1\) imply that
\[ Q = f - \frac{d}{2} \sigma(1 + T_{2n}) \] (7)

is nonnegative.

Subdomain 2: \( \{1 \leq \{|x| \leq 1 + d/4m^2\} \). By continuity there is a larger interval \([-1, -a, 1 + a]\) on which \( f \geq d/2 \). We quantify the size of this interval using the estimate of Bernstein [2] according to which polynomials of degree \( m \) satisfy

\[
\max_{-1 \leq x \leq 1} |f'| \leq m^2 \max_{-1 \leq x \leq 1} |f|.
\]

Combining this with the previous Chebyshev estimate and the upper bound 1 for \( f \) we have

\[
\max_{-1 \leq x \leq 1+a} |f'| \leq m^2 T_{m-1}(1 + a).
\]

Hence on subdomain 2

\[ f(x) \geq f(1) - m^2|x - 1|T_{m-1}(1 + a) \geq d - am^2 T_{m-1}(1 + a). \]

This will imply that \( f \geq d/2 \) provided that

\[ am^2 T_{m-1}(1 + a) \leq d/2. \] (8)

Using the upper bound \((x + \sqrt{x^2 - 1})^n\) for \( T_n(x)\), (8) will be satisfied if

\[ am^2(1 + a + \sqrt{2a + a^2})^{m-1} \leq d/2. \]

Letting \( a = b/m^2 \) this becomes

\[ be^{\sqrt{2b+o(1)}} \leq d/2 \]

where here the order symbol refers to asymptotic dependence on small \( d \). For small \( d \) this is satisfied by \( b \leq d/4 \). Hence \( a \leq d/4m^2 \) ensures that \( f \geq d/2 \) on subdomain 2. Since \( \sigma \) is negative on this domain, Eq. (7) also gives \( d/2 \) as a lower bound for \( Q \).

Subdomain 3: \(|x| \geq 1 + d/4m^2\). Here \( \sigma \) is strictly negative and its modulus at \( x \) is bounded below by a constant multiple of some power of the distance from \( x \) to its zero set \([-1, 1]\). So it has a negative upper bound of the form \(-A(d/4m^2)^k\). Using this upper bound, the Chebyshev estimate for \( f(x) \) outside \([-1, 1]\) in terms of the bound 1 on \([-1, 1]\), and the bounds
\[(|x| + \sqrt{x^2 - 1})^n/2 \leq |T_n(x)| \leq (|x| + \sqrt{x^2 - 1})^n.\]

Eq. (7) yields

\[Q \geq - (|x| + \sqrt{x^2 - 1})^m + \frac{Ad}{2} \left( \frac{d}{4m^2} \right)^k (|x| + \sqrt{x^2 - 1})^{2n}.\]

Thus for \(Q\) to be nonnegative it suffices that

\[\frac{Ad}{2} \left( \frac{d}{4m^2} \right)^k (|x| + \sqrt{x^2 - 1})^{2n-m} \geq 1\]

on subdomain 3. Sufficient in turn is

\[\frac{Ad}{2} \left( \frac{d}{4m^2} \right)^{k+1} \left( 1 + \sqrt{\frac{d}{2m^2}} \right)^{2n-m} \geq 1.\]

Solving this inequality for \(n\) and making elementary estimations we find that this is so if

\[n \geq C \left( \frac{m^3}{d^2} \right)^{1/2} \log \left( \frac{m^3}{d^2} \right)\]

where \(C\) depends on \(A\) and \(k\) which depend only on \(\sigma\). In terms of the normalized estimates this is the conclusion of the theorem.

As a consequence, for the special relation (1) we obtain the rather close lower and upper estimates for \(N(d)\) of \(O(d^{-1/2})\) and \(O(d^{-1/2} \log(1/d))\) respectively.

REFERENCES