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Characterizing omega-limit sets which are closed orbits

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Abstract

Let X be a vector field in a compact n-manifold M, $n \geqslant 2$. Given $\Sigma \subset M$ we say that $q \in M$ satisfies $(\mathrm{P})_\Sigma$ if the closure of the positive orbit of X through q does not intersect Σ, but, however, there is an open interval I with q as a boundary point such that every positive orbit through I intersects Σ. Among those q having saddle-type hyperbolic omega-limit set $\omega(q)$ the ones with $\omega(q)$ being a closed orbit satisfy $(\mathrm{P})_\Sigma$ for some *closed* subset Σ. The converse is true for $n = 2$ but not for $n \geqslant 4$. Here we prove the converse for $n = 3$. Moreover, we prove for $n = 3$ that if $\omega(q)$ is a *singular-hyperbolic set* [C. Morales, M. Pacifico, E. Pujals, On C^1 robust singular transitive sets for three-dimensional flows, C. R. Acad. Sci. Paris Sér. I 26 (1998) 81–86], [C. Morales, M. Pacifico, E. Pujals, Robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellers, Ann. of Math. (2) 160 (2) (2004) 375–432], then $\omega(q)$ is a closed orbit if and only if q satisfies $(\mathrm{P})_\Sigma$ for some Σ closed. This result improves [S. Bautista, Sobre conjuntos hiperbólicos-singulares (On singular-hyperbolic sets), thesis Uiversidade Federal do Rio de Janeiro, 2005 (in Portuguese)] and [C. Morales, M. Pacifico, Mixing attractors for 3-flows, Nonlinearity 14 (2001) 359–378].

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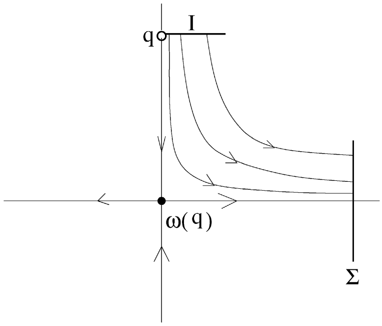

Fig. 1.

1. Introduction

This paper is motivated by two interesting properties related to the ordinary differential equation in Fig. 1. The first one is that the omega-limit set $\omega(q)$ of the point q in the figure is a hyperbolic singularity of saddle-type. The second one is that there is a *closed* subset Σ (the vertical segment in the figure) for which the following *Property* $(\mathrm{P})_\Sigma$ holds: The closure of the positive orbit of q does not intersect Σ, but, however, there is an open interval with q as a boundary point such that every positive orbit through I intersects Σ.

It is natural to ask how these properties are related *among those points q having saddle-type hyperbolic omega-limit set*. For example if $n \geqslant 2$ and $\omega(q)$ is a *closed orbit* (i.e. a singularity or a periodic orbit), then q satisfies $(\mathrm{P})_\Sigma$ for some *closed* subset Σ. The question is then whether the satisfaction of $(\mathrm{P})_\Sigma$ for some Σ closed implies that $\omega(q)$ is a closed orbit. Indeed, this is true for $n = 2$ (e.g. [11, pp. 145–146]) but false for $n \geqslant 4$ by the following counterexample:

Example 1. Let D^2 and S^1 be the two-dimensional closed unit disk and the unit circle, respectively. Consider the vector field X^0 in the solid torus $ST = D^2 \times S^1$ obtained from the suspension of the Smale Horseshoe in D^2 (see [11]). As is well known there is $x_0 \in ST$ whose omega-limit set H with respect to X^0 is a saddle-type hyperbolic set but not a closed orbit. Now define the vector field X in $ST \times [-1, 1]$ by $X(x, y) = (X^0(x), 2y)$, $\forall (x, y) \in ST \times [-1, 1]$. Fix $q = (x_0, 0)$. Then, $\omega(q) = H \times 0$ hence $\omega(q)$ is not a closed orbit but a saddle-type hyperbolic set. However, q satisfies $(\mathrm{P})_\Sigma$ for some closed subset Σ (e.g. take $\Sigma = ST \times 1$ with $I = \{(x_0, y) \in ST \times [-1, 1]\colon\ 0 < y \leqslant \frac{1}{2}\}$). Analogous counterexample can be constructed for $n \geqslant 4$.

Here we give positive answer for the question above when $n = 3$. Actually we do it *among those points q having singular-hyperbolic omega-limit set* [16,17]. More precisely, we prove that if $\omega(q)$ is a singular-hyperbolic set, then $\omega(q)$ is a closed orbit if and only if q satisfies $(\mathrm{P})_\Sigma$ for some closed subset Σ. This improves some previous results obtained in [2] and [14]. Let us state our result in a precise way.

Hereafter M will denote a compact 3-manifold and X will denote a C^1 vector field in M. Denote by X_t the flow generated by X, $t \in \mathbb{R}$. An orbit of X is a set of the form $O(p) = \{X_t(p)\colon\ t \in \mathbb{R}\}$ for some p. We denote by

$$O^+(p) = \{X_t(p)\colon t \geqslant 0\}$$

the positive orbit of p.

Now we state the precise definition of $(\mathrm{P})_\Sigma$. The closure and boundary operations will be denoted by $\mathrm{Cl}(\cdot)$ and $\partial(\cdot)$, respectively.

Definition 2. Given $\Sigma \subset M$ we say that $q \in M$ satisfies *property* $(\mathrm{P})_\Sigma$ if:

(1) $\mathrm{Cl}(O^+(q)) \cap \Sigma = \emptyset$;
(2) there is an open arc I in M with $q \in \partial I$ such that $O^+(x) \cap \Sigma \neq \emptyset$ for every $x \in I$.

Next we recall the definition of hyperbolic set. A compact invariant set H of X is *hyperbolic* if there are a continuous invariant tangent bundle decomposition $T_H M = E^s_H \oplus E^X_H \oplus E^u_H$ and positive constants K, λ such that

- E^s_H is *contracting*, i.e.,

$$\|DX_t(x)/E^s_x\| \leqslant Ke^{-\lambda t}, \quad \forall t > 0,\ \forall x \in H.$$

- E^u_H is *expanding*, i.e.,

$$\|DX_{-t}(x)/E^u_x\| \leqslant Ke^{-\lambda t}, \quad \forall t > 0,\ \forall x \in H.$$

- E^X_H is the subbundle generated by X in H.

If additionally $E^s_x \neq 0$ and $E^u_x \neq 0$ for all $x \in H$ then we say that H is *saddle-type*. A closed orbit is *hyperbolic* (of saddle-type) if it does as a compact invariant set.

Now we define dominated splitting and partially hyperbolic set. Given a linear operator L in a vector space V we define the minimum norm of L by

$$m(L) = \inf_{\|v\|=1} \|L(v)\|.$$

We denote by $\mathrm{Det}(L)$ the Jacobian of L. A continuous invariant tangent bundle splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ over a compact invariant set Λ is called *dominated* if $E^s_x \neq 0$ and $E^c_x \neq 0$ for all $x \in \Lambda$ and there are positive constants K, λ such that

$$\frac{\|DX_t(x)/E^s_x\|}{m(DX_t(x)/E^c_x)} \leqslant Ke^{-\lambda t}, \quad \forall t > 0,\ \forall x \in \Lambda.$$

We say that Λ is *partially hyperbolic* if it has a dominated splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ such that E^s_Λ is *uniformly contracting*, i.e.

$$\|DX_t(x)/E^s_x\| \leqslant Ke^{-\lambda t}, \quad \forall t > 0,\ \forall x \in \Lambda.$$

We say that the central direction E^c_Λ of a dominated splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ over Λ is *volume expanding* if $|\mathrm{Det}(DX_t(x)/E^c_x)| \geqslant K^{-1}e^{\lambda t}$, $\forall t > 0$, $\forall x \in \Lambda$.

Finally we define singular-hyperbolic set (e.g. [5,16,17]).

Definition 3. A compact invariant set of X is *singular-hyperbolic* if it is partially hyperbolic with volume expanding central direction and its singularities are hyperbolic.

Examples of singular-hyperbolic sets for three-dimensional vector fields are the hyperbolic sets of saddle-type (including hyperbolic closed orbits of saddle-type) and the geometric Lorenz attractor [1,8]. Define the *omega-limit set* of $q \in M$ by

$$\omega(q) = \Big\{x \in M\colon\ x = \lim_{n\to\infty} X_{t_n}(q) \text{ for some sequence } t_n \to \infty \text{ as } n \to \infty\Big\}.$$

With these definitions we can state our main result which improves ones in [2] and [14].

Theorem 4. *Let X be a C^1 vector field in a compact 3-manifold M. If $q \in M$ has singular-hyperbolic omega-limit set $\omega(q)$, then the following properties are equivalent*:

(1) *$\omega(q)$ is a closed orbit.*
(2) *q satisfies* $(\mathrm{P})_\Sigma$ *for some closed subset Σ.*

The idea of the proof is as follows. Fix X, M, q as in the statement. By the previous observations we only have to prove that (2) implies (1). Suppose that q satisfies $(\mathrm{P})_\Sigma$ for some closed subset Σ. In Theorem 8 we prove that $\omega(q)$ has what we call a *singular partition*: A finite disjoint collection of cross-sections $\mathcal{R}$, intersecting each non-singular orbit of $\omega(q)$, such that the boundary of every element of $\mathcal{R}$ does not intersect $\omega(q)$. In Theorem 11 we prove that if $\omega(q)$ is not a singularity, then there are $\delta > 0$, $S \in \mathcal{R}$, a sequence $\hat{q}_1, \hat{q}_2, \ldots \in S$ of points in the positive orbit of q and a sequence of intervals $\hat{J}_1, \hat{J}_2, \ldots \subset S$ in the positive orbit of the arc I in the definition of $(\mathrm{P})_\Sigma$ such that $\hat{q}_j$ is a boundary point of $\hat{J}_j$ and the length $\mathrm{Length}(\hat{J}_j)$ of $\hat{J}_j$ is at least δ $(\forall j)$. We obtain Theorem 4 from this property as follows. Let $\mathcal{F}^s(x, S)$ be the flow-projection onto S of the strong stable manifold through x. If $\hat{q}_i \notin \mathcal{F}^s(x, S)$ for infinitely many i's, then we get a contradiction by analyzing the relative position of the intervals $\hat{J}_j$ in S as in p. 371 of [14]. Therefore we can assume that $\hat{q}_i \in \mathcal{F}^s(x, S)$ for all i. In such a case $\omega(q)$ is a periodic orbit by Lemma 5.6, p. 369 in [14]. This ends the proof of Theorem 4.

The organization of the paper is as follows. In Section 2 we define singular partition and prove some topological properties of these partitions. In Section 3 we prove Theorem 8 dealing with the existence of singular partitions with small diameter for singular-hyperbolic omega-limit sets. In Section 4 we prove Theorem 11 which is the main property of the singular partition used here. In Section 5 we prove Theorem 4.

2. Singular partition

In this section we define singular partitions which is the main topological tool behind the proof of Theorem 4. Afterwards we give some properties of these partitions.

Hereafter we fix a compact 3-manifold M and a C^1 vector field X in M. A *cross-section* of X is a two-dimensional submanifold S transverse to X. We then denote by $\mathrm{Int}(S)$ and ∂S the interior and the boundary of S (as a submanifold). If $\mathcal{R} = \{S_1, \ldots, S_k\}$ is a collection of cross-sections of X we denote

$$\mathcal{R}' = \bigcup_{i=1}^{k} S_i, \qquad \partial\mathcal{R}' = \bigcup_{i=1}^{k} \partial S_i, \qquad \mathrm{Int}(\mathcal{R}') = \bigcup_{i=1}^{k} \mathrm{Int}(S_i).$$

The *diameter* of $\mathcal{R}$ will be the sum of the diameters of its elements. When the collection $\mathcal{R}$ is *disjoint* we can define a return map

$$\Pi_{\mathcal{R}} : \mathrm{Dom}(\Pi_{\mathcal{R}}) \subset \mathcal{R}' \to \mathcal{R}'$$

by

$$\Pi_{\mathcal{R}}(x) = X_{t(x)}(x),$$

where $t(x)$ is the *return time*, i.e. the first time $t > 0$ satisfying

$$X_t(x) \in \mathcal{R}'.$$

The definition below is a minor modification of Definition 6.2, p. 1586 in [15]. Denote by $\mathrm{Sing}(X)$ the set of singular points of a vector field X.

Definition 5. A *singular partition* of an invariant set H of X is a finite disjoint collection of cross-sections $\mathcal{R}$ of X such that $H \cap \partial\mathcal{R}' = \emptyset$ and

$$\mathrm{Sing}(X) \cap H = \bigl\{y \in H\colon\ X_t(y) \notin \mathcal{R}',\ \forall t \in \mathbb{R}\bigr\}.$$

Singular partition generalizes the concept of global cross-section [7] to include invariant sets with singularities. Actually a singular partition is equivalent to a global cross-section in the absence of singularities.

In the sequel we state some topological properties of the singular partitions. The first one is a direct consequence of the definition of singular partition (cf. [15]). For all compact invariant set Λ we define

$$W^s(\Lambda) = \bigl\{x \in M\colon\ \omega(x) \subset \Lambda\bigr\} \quad \text{and} \quad W^u(\Lambda) = \bigl\{x \in M\colon\ \alpha(x) \subset \Lambda\bigr\}. \tag{1}$$

If Λ reduces to a singularity σ then we write $W^s(\sigma)$ instead of $W^s(\{\sigma\})$ for simplicity. Analogously for $W^u(\sigma)$. For all $H \subset M$ we denote

$$W^s\bigl(\mathrm{Sing}(X) \cap H\bigr) = \bigcup_{\sigma \in \mathrm{Sing}(X)\cap H} W^s(\sigma).$$

Lemma 6. *If $\mathcal{R}$ is a singular partition of a compact invariant set H of X, then the following properties hold*:

(1) $(H \cap \mathcal{R}') \cap \mathrm{Dom}(\Pi_{\mathcal{R}}) \subset \mathrm{Int}(\mathrm{Dom}(\Pi_{\mathcal{R}}))$ *and $\Pi_{\mathcal{R}}$ is C^1 in a neighborhood of $H \cap \mathcal{R}'$ in $\mathcal{R}'$.*
(2) $(H \cap \mathcal{R}') \setminus \mathrm{Dom}(\Pi_{\mathcal{R}}) \subset W^s(\mathrm{Sing}(X) \cap H)$.

In the statement below we denote by $B_\delta(p)$ the δ-ball in $\mathcal{R}'$ centered at $p \in \mathcal{R}'$. Recall that $O^+(q) = \{X_t(q)\colon\ t \geqslant 0\}$ denotes the positive orbit of $q \in M$.

The lemma below plays the role of a claim in p. 370 of [14].

Lemma 7. *Let $q \in M$ be such that every singularity $\sigma \in \omega(q)$ is hyperbolic with one-dimensional unstable manifold $W^u(\sigma)$ (see Eq. (1)). If $\omega(q)$ is not a singularity and $\mathcal{R}$ is a singular partition of $\omega(q)$, then the following properties hold for $\Pi = \Pi_{\mathcal{R}}$:*

(1) *$O^+(q) \cap \mathcal{R}' = \{q_1, q_2, \ldots\}$ is an infinite sequence ordered in a way that $\Pi(q_n) = q_{n+1}$.*
(2) *There is $\delta > 0$ such that if $n \in \{1, 2, \ldots\}$ then either $B_\delta(q_n) \subset \mathrm{Dom}(\Pi)$ and $\Pi|_{B_\delta(q_n)}$ is C^1 or there is a curve $c_n \subset W^s(\mathrm{Sing}(X) \cap \omega(q)) \cap B_\delta(q_n)$ such that*

$$B_\delta^+(q_n) \subset \mathrm{Dom}(\Pi) \quad \textit{and} \quad \Pi|_{B_\delta^+(q_n)} \textit{ is } C^1,$$

where $B_\delta^+(q_n)$ denotes the connected component of $B_\delta(q_n) \setminus c_n$ containing q_n.

Proof. To prove item (1) notice that $\omega(q)$ contains regular orbits as it is not a singularity. Hence $\omega(q) \cap \mathcal{R}' \neq \emptyset$ because $\mathcal{R}$ is a singular partition of $\omega(q)$. Since each component of $\mathcal{R}$ is a cross-section of X we have that $O^+(q) \cap \mathcal{R}' = \{q_1, q_2, \ldots\}$ is a sequence whose accumulation points belong to $\omega(q) \cap \mathcal{R}'$. The sequence must be infinite for otherwise $\omega(q) \cap \mathcal{R}' = \emptyset$ a contradiction. Thus $q_n \in \mathrm{Dom}(\Pi)$ ($\forall n$) and clearly we can order the sequence in a way that $\Pi(q_n) = q_{n+1}$ ($\forall n$). This proves item (1) of the lemma.

To prove item (2) we proceed as in the proof of the aforementioned claim but now taking into account that $\omega(q) \cap \mathcal{R}' \not\subset \mathrm{Dom}(\Pi)$. To handle this problem we use Lemma 6 as follows.

To simplify the notation we write

$$H = \omega(q) \quad \text{and} \quad H^0 = H \cap \mathcal{R}'.$$

Then, $H^0 \neq \emptyset$. By Lemma 6 one has

(i) $H^0 \cap \mathrm{Dom}(\Pi) \subset \mathrm{Int}(\mathrm{Dom}(\Pi))$ and Π is C^1 in a neighborhood of H^0 in $\mathcal{R}'$;
(ii) $H^0 \setminus \mathrm{Dom}(\Pi) \subset W^s(\mathrm{Sing}(X) \cap H)$.

On the other hand, every singularity in $\omega(q)$ is hyperbolic with one-dimensional unstable manifold by hypothesis. It follows that the stable manifold of every $\sigma \in \mathrm{Sing}(X) \cap H$ is two-dimensional.

Now, we fix $x \in H^0 \setminus \mathrm{Dom}(\Pi)$ then $x \in \mathcal{R}' \cap W^s(\mathrm{Sing}(X) \cap H)$ by (ii). As $\mathcal{R}'$ and the stable manifolds of the singularities in $\mathrm{Sing}(X) \cap H$ are two-dimensional we have that x lies in a curve

$$c_x \subset \mathcal{R}' \cap W^s(\sigma_x)$$

for some $\sigma_x \in \mathrm{Sing}(X) \cap H$. By hypothesis we have that $W^u(\sigma_x)$ is one-dimensional, so $W^u(\sigma_x) \setminus \{\sigma_x\}$ consists of two connected components to be denote by W^+ and W^-. We have three possibilities for these components:

- $W^+ \subset H$ and $W^- \subset H$,
- $W^+ \subset H$ and $W^- \not\subset H$,
- $W^- \subset H$ and $W^+ \not\subset H$.

First suppose that $W^+ \subset H$ and $W^- \subset H$. It follows that $W^+ \cap \mathrm{Int}(\mathcal{R}') \neq \emptyset$ and $W^- \cap \mathrm{Int}(\mathcal{R}') \neq \emptyset$ since W^-, W^+ are regular orbits of H and $\mathcal{R}$ is a singular partition of $\omega(q) = H$. By using such non-empty intersections we can find $\delta_x > 0$ such that

(iii) $B_{\delta_x}(x) \setminus c_x \subset \mathrm{Dom}(\Pi)$ and $\Pi|_{B_{\delta_x}(x) \setminus c_x}$ is C^1.

Second suppose that $W^+ \subset H$ and $W^- \not\subset H$. As $W^+ \subset H$ and $\mathcal{R}'$ is a singular partition of H we have

(A) $W^+ \cap \mathrm{Int}(\mathcal{R}') \neq \emptyset$.

As $W^- \not\subset H$ we have

(B) $O^+(q)$ does not accumulate on W^-.

By using (A) and (B) we can find $\delta_x > 0$ such that the connected components

$$B^+_{\delta_x}(x) \quad \text{and} \quad B^-_{\delta_x}(x)$$

of $B_{\delta_x}(x) \setminus c_x$ are labeled in a way that

(iv) $B^+_{\delta_x}(x) \subset \mathrm{Dom}(\Pi)$, $\Pi|_{B^+_{\delta_x}(x)}$ is C^1 and $B^-_{\delta_x}(x) \cap O^+(q) = \emptyset$.

Third suppose that $W^- \subset H$ and $W^+ \not\subset H$. In this case we can proceed as in the second case to find $\delta_x > 0$ satisfying (iv).

Summarizing, for all $x \in H^0 \setminus \mathrm{Dom}(\Pi)$ we have found $\delta_x > 0$ satisfying either (iii) or (iv).

On the other hand, (i) implies that $H^0 \setminus \mathrm{Dom}(\Pi)$ is compact. Hence there are $x_1, \ldots, x_l \in H^0 \setminus \mathrm{Dom}(\Pi)$ such that

$$H^0 \setminus \mathrm{Dom}(\Pi) \subset \bigcup_{i=1}^{l} B_{\delta_{x_i}/2}(x_i). \tag{2}$$

Because the union in the right-hand side of (2) is open one has that

$$H^1 = H^0 \setminus \bigcup_{i=1}^{l} B_{\delta_{x_i}}(x_i)$$

is compact. By (2) one has

$$H^1 \subset H^0 \cap \mathrm{Dom}(\Pi).$$

By (i) we have that $\forall y \in H^1\ \exists \beta_y > 0$ such that

$$B_{\beta_y}(y) \subset \mathrm{Dom}(\Pi) \quad \text{and} \quad \Pi|_{B_{\beta_y}(y)} \text{ is } C^1. \tag{3}$$

It follows from the compactness of H^1 that $\exists y_1, \ldots, y_r$ (for some $r > 0$) such that

$$H^1 \subset \bigcup_{j=1}^{r} B_{\beta_{y_j}/2}(y_j). \tag{4}$$

Define

$$\delta = \min\{\delta_{x_i}/8, \beta_{y_j}/8\colon\ 1 \leqslant i \leqslant l,\ 1 \leqslant j \leqslant r\}.$$

Let us prove that this δ works.

By (2) and (4) we have that

$$\left\{B_{\delta_{x_i}}(x_i), B_{\beta_{y_j}}(y_j)\colon\ 1 \leqslant i \leqslant l,\ 1 \leqslant j \leqslant r\right\}$$

is an open covering of $H^0 = \omega(q) \cap \mathcal{R}'$. Then

$$q_n \in \left(\bigcup_{i=1}^{l} B_{\delta_{x_i}/2}(x_i)\right) \cup \left(\bigcup_{j=1}^{r} B_{\beta_{y_i}/2}(y_i)\right)$$

for n large enough. Hence for all n large we have either

$$q_n \in B_{\delta_{x_i}/2}(x_i) \quad \text{for some } 1 \leqslant i \leqslant l,$$

or

$$q_n \in B_{\beta_j/2}(y_j) \quad \text{for some } 1 \leqslant j \leqslant r.$$

Then, by the triangle inequality and the choice of δ we obtain

$$B_\delta(q_n) \subset B_{\delta_{x_i}}(x_i) \quad \text{or} \quad B_\delta(q_n) \subset B_{\beta_j}(y_j).$$

If $B_\delta(q_n) \subset B_{\beta_{y_j}}(y_j)$, then $B_\delta(q_n) \subset \mathrm{Dom}(\Pi)$ and $\Pi|_{B_\delta(q_n)}$ is C^1 by (3). In this case we are done.

If $B_\delta(q_n) \subset B_{\delta_{x_i}}(x_i)$ we define

$$c_n = c_{x_i} \cap B_\delta(q_n).$$

In this case we have two subcases, namely either (iii) or (iv) hold.

First assume that (iii) holds. Recalling that $B_\delta^+(q_n)$ is the connected component of $B_\delta(q_n) \setminus c_n$ containing q_n we have $B_\delta^+(q_n) \subset B_{\delta_{x_i}}(x_i) \setminus c_{x_i}$ therefore $B_\delta^+(q_n) \subset \mathrm{Dom}(\Pi)$ and $\Pi|_{B_\delta^+(q_n)}$ is C^1 by (iii).

Finally, if (iv) holds then $B_\delta^+(q_n) \subset B_{\delta_{x_i}}^+(x_i)$ since $q_n \in O^+(q)$ and $B_{\delta_{x_i}}^-(x_i) \cap O^+(q) = \emptyset$. Then the result follows from (iv). The lemma is proved. $\square$

3. Existence of singular partitions

In this section we shall prove the following existence result. Hereafter we fix a compact 3-manifold M and a C^1 vector field X in M.

Theorem 8. *Let $q \in M$ be a point satisfying $(\mathrm{P})_\Sigma$ for some closed subset Σ. If $\omega(q)$ is a singular-hyperbolic set, then for all $\delta > 0$ there is a singular partition of $\omega(q)$ with diameter less than δ.*

To prove this theorem we need some preliminary notations and results.

First of all, it follows from the Invariant Manifold Theory [10] that if Λ is a singular-hyperbolic set then the subbundle E^s_Λ of the singular-hyperbolic splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ over Λ can be extended to a continuous semi-invariant contracting subbundle E^s_U defined in a neighborhood U of Λ. We also have that E^s_U is integrable, i.e., tangent to a continuous contracting one-dimensional foliation W^{ss} in U. The leaf of W^{ss} at $x \in U$ will be denoted by $W^{ss}(x)$.

Hereafter we fix $q \in M$ satisfying $(\mathrm{P})_\Sigma$ for some closed subset Σ such that $\omega(q)$ is a singular-hyperbolic set of X.

With these notations we have the following lemma.

Lemma 9. *$W^{ss}(z) \cap \omega(q)$ has empty interior in $W^{ss}(z)$ for every $z \in \omega(q)$.*

Proof. The proof is by contradiction, that is, we assume that there is $z \in \omega(q)$ such that $W^{ss}(z) \cap \omega(q)$ has non-empty interior in $W^{ss}(z)$. Then, $\omega(q)$ contains an open interval in $W^{ss}(z)$. If we take x^* in this interval we obtain

$$W^{ss}_\epsilon\big(x^*\big) \subset \omega(q)$$

for some $\epsilon > 0$, where the operator $W^{ss}_\epsilon(\cdot)$ denotes the local strong stable manifold of diameter ϵ.

Next we proceed as in the proof of the main theorem in [13]:

Fix $0 < \epsilon^* < \epsilon$ and define

$$H = \Big\{ y = \lim_{n\to\infty} X_{t_n}(z_n) \text{ for some sequences } t_n \to -\infty \text{ and } z_n \in W^{ss}_{\epsilon^*}\big(x^*\big) \Big\}.$$

Clearly H is a compact invariant set. Moreover, $H \subset \omega(q)$ since $\omega(q)$ is compact invariant and $W^{ss}_{\epsilon^*}(x^*) \subset W^{ss}_\epsilon(x^*) \subset \omega(q)$.

We obtain the desired contradiction depending on whether H contains a singularity or not.

If H contains a singularity, then we get the contradiction exactly as in [13, p. 556] using $H \subset \omega(q)$ instead of the transitivity used there.

If H contains no singularities then it is a hyperbolic set. In addition, the continuity of $x \mapsto W^{ss}_{\epsilon^*}(x)$ implies

$$W^{ss}_{\epsilon^*}(y) \subset H, \quad \forall y \in \alpha\big(x^*\big).$$

As H is hyperbolic we have large unstable manifolds on $W^{ss}_{\epsilon^*}(y) \subset H$.

Then, the argument in [13] shows that $q \in H$ therefore $\omega(q) = H$. Thus $\omega(q)$ has no singularities, and so, by the Shadowing lemma for flows [9] applied to a pseudo-orbit derived from the positive orbit of q, we can find a periodic orbit O with large unstable manifold $W^u(O)$ nearby $W^{ss}_{\epsilon^*}(y)$.

As $W^u(O)$ is large we get in particular that it intersects $W^{ss}_{\epsilon^*}(y)$ transversally. Therefore $W^s(O) \subset H$ by the Inclination lemma [11] applied to the backward orbit of $W^{ss}_{\epsilon^*}(y)$. Then we get $\mathrm{Cl}(W^s(O)) \subset \omega(q)$ since $\omega(q)$ is compact invariant.

Therefore $\mathrm{Cl}(W^s(O))$ is a hyperbolic set contained in $\omega(q)$. But $W^s(O)$ is two-dimensional (as it is contained in a singular-hyperbolic set) so we can use $\mathrm{Cl}(W^s(O))$ to construct a hyperbolic repeller inside $\omega(q)$. From this we get that $\omega(q) = \mathrm{Cl}(W^s(O))$ is a hyperbolic repeller containing q.

Now, q satisfies $(\mathrm{P})_\Sigma$ hence $\omega(q)$ accumulates $W^s(O(q))$ by one-side only. Therefore $W^s(O(q))$ is what is called a *stable boundary leaf* of $\omega(q)$ (see [4]). As such leaves are formed by stable manifolds of periodic orbits (e.g. Lemme 1.6, p. 129 in [4] applied to $-X$) we conclude that q belongs to the stable manifold of a periodic orbit. It then follows that $\omega(q)$ is a periodic orbit, a contradiction since it contains the two-dimensional manifold $W^s(O)$. This proves the result. $\square$

A cross-section D of X is called *rectangle* if it is diffeomorphic to the square $[0,1] \times [0,1]$. In this case ∂D is a submanifold of M is formed by four curves $D^t_h, D^b_h, D^l_v, D^r_v$ (v for vertical, h for horizontal, l for left, r for right, t for top and b for bottom). One defines vertical and horizontal curves in D in the natural way. If D is a cross-section and $w \in H^* \cap D$ then we denote by $(H^* \cap D)_w$ the connected component of $H^* \cap D$ containing w.

If $x \in \Lambda$ belongs to a cross-section D of X we define $\mathcal{F}^s(x, D)$ as the connected component containing x of the projection of $W^{ss}(x)$ onto D along the flow of X.

Lemma 10. *For every $z \in \omega(q) \setminus \mathrm{Sing}(X)$ there is a rectangle R_z close to z with the following properties*:

(1) $z \in \mathrm{Int}(R_z)$.
(2) *If $x \in \omega(q) \cap R_z$ then $\mathcal{F}^s(x, R_z)$ is a horizontal curve in R_z.*
(3) $\omega(q) \cap \partial R_z = \emptyset$.

Proof. By Lemma 1, p. 184 in [12] for all $z \in \omega(q) \setminus \mathrm{Sing}(X)$ there is a rectangle R^0_z close to z satisfying (1) to (2) above. To conclude the proof we refine R^0_z to obtain R_z satisfying (1) to (3) via property $(\mathrm{P})_\Sigma$ and Lemma 9 in the following way:

First observe that the positive orbit of q intersects R^0_z into a sequence converging to z. If infinitely many elements of such a sequence belongs to $\mathcal{F}(z, R^0_z)$, then Lemma 5.6 in [14, p. 369] implies that $\omega(q)$ is a periodic orbit. In this case it is easy to find a rectangle R_z satisfying (1) to (3). Therefore we can assume that the positive orbit of q does not intersect $\mathcal{F}^s(z, R^0_z)$. Note that $\mathcal{F}(z, R^0_z)$ divides R^0_z in two subrectangles which will be refereed to as the top and the bottom ones. As the positive orbit of q does not intersect $\mathcal{F}^s(z, R^0_z)$ we conclude that it intersects either the top or the bottom subrectangle in a sequence converging to z. We shall assume that it does in the top subrectangle only. The proof for the remainder cases is similar (compare with cases I to IV in the proof of Lemma 1 in [12]).

As $\omega(q)$ is a singular-hyperbolic set but not a hyperbolic repeller, we have that $\mathcal{F}^s(z, R^0_z) \cap \omega(q)$ has empty interior in $\omega(q)$ by Lemma 9. Then, we can select two points a, b in opposite

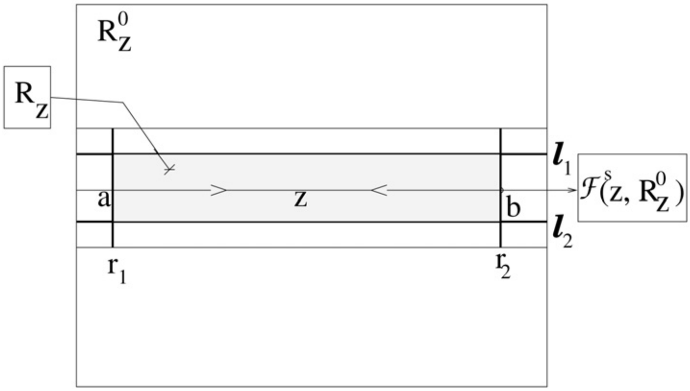

Fig. 2.

sides of z in $\mathcal{F}^s(z, R_z^0)$ which does not belong to $\omega(q) \cap R_z^0$. As $\omega(q)$ is closed we can also select two vertical curves r_1, r_2 centered at a and b, respectively, with the property that

$$\omega(q) \cap (r_1 \cup r_2) = \emptyset.$$

(See Fig. 2.) Now the positive orbit of q, which intersects the top subrectangle only, carries the positive orbit of I into R_z^0. Then, we can select a point in the positive orbit of I contained in the top subrectangle that is close to z. By taking the strong stable manifold through this point we get the horizontal curve l_1 in Fig. 2. The property of I in the definition of $(\mathrm{P})_\Sigma$ guarantees that $l_1 \cap \omega(q) = \emptyset$ (otherwise it would exist a point in I whose positive trajectory is asymptotic to one in $\omega(q)$ hence it does not intersect Σ). As the positive orbit of q does not intersect the bottom subrectangle by assumption we can also select a horizontal curve l_2 in the bottom subrectangle such that $\omega(q) \cap l_2 = \emptyset$. Then, we choose R_z as the subrectangle of R_z^0 fenced by l_1, l_2, r_1, r_2. $\square$

Proof of Theorem 8. Let $q \in M$ be a point satisfying $(\mathrm{P})_\Sigma$, for some closed subset Σ, such that $\omega(q)$ is a singular-hyperbolic set. We shall prove that for all δ there is a singular partition of $\omega(q)$ with diameter $\leqslant \delta$. For this we proceed as follows.

Since each $\sigma \in \mathrm{Sing}(X) \cap \omega(q)$ is hyperbolic we can shrink δ if necessary and apply the Grobman–Hartman theorem (say) to obtain

$$\mathrm{Sing}(X) \cap \omega(q) = \bigcap_{t \in \mathbb{R}} X_t \left(\bigcup_{\sigma \in \mathrm{Sing}(X) \cap \omega(q)} B_\delta(\sigma) \right). \tag{5}$$

Define

$$H = \omega(q) \setminus \left(\bigcup_{\sigma \in \omega(q) \cap \mathrm{Sing}(X)} B_\delta(\sigma) \right).$$

We can assume that $H \neq \emptyset$ for, otherwise, $\omega(q)$ would be a singularity by (5) and then the result follows. Clearly $H \subset \omega(q)$ and $H \cap \mathrm{Sing}(X) = \emptyset$. Then, for all $z \in H$ we can associate the rectangle R_z of diameter at most δ as in Lemma 10. For all $z \in H$ we define

$$V_z = \bigcup_{t \in (-1,1)} X_t(R_z).$$

Obviously $z \in V_z$ and by Lemma 10(1) we have that $\{V_z\colon z \in H\}$ is an open covering of H. But H is compact, so there is a finite subset $\{z_1, \dots, z_r\} \in H$ such that

$$H \subset \bigcup_{i=1} V_{z_i}.$$

By moving the rectangles $R_{z_1}, \dots, R_{z_r}$ along the flow of X as in [6, p. 189] (say) we can assume that the collection

$$\mathcal{R} = \{R_{z_1}, \dots, R_{z_r}\}$$

is pairwise disjoint.

We claim that $\mathcal{R}$ is a singular partition of $\omega(q)$. Indeed, we already know that the elements of $\mathcal{R}$ are pairwise disjoint. Now take $z \in \omega(q) \setminus \mathrm{Sing}(X)$. It follows from (5) that there is some $t \in \mathbb{R}$ such that

$$X_t(z) \notin \bigcup_{\sigma \in \mathrm{Sing}(X) \cap \omega(q)} B_\delta(\sigma).$$

But $X_t(z) \in \omega(q)$ since z does therefore $X_t(x) \in H$ by the definition of H. Hence $X_t(z) \in V_{z_i}$ for some i and then the orbit of z intersects R_{z_i} by the definition of V_{z_i}. The claim is proved. As $\mathcal{R}$ has diameter at most δ by construction we are done. This proves the result. □

4. Singular partitions and singular-hyperbolicity

In this section we relate singular partitions with singular-hyperbolicity. Hereafter we fix a compact 3-manifold M and a C^1 vector field X in M. The length of an arc J will be denoted by $\mathrm{Length}(J)$.

The following technical result is part of the proof of Theorem 4. We state it separately for the sake of clearness.

Theorem 11. *Let $q \in M$ be a point satisfying $(\mathrm{P})_\Sigma$ for some closed subset Σ such that $\omega(q)$ is a singular-hyperbolic set. Let $T_U M = \hat{E}^s_U \oplus \hat{E}^c_U$ be a continuous extension of the singular-hyperbolic splitting $T_{\omega(q)} M = E^s_{\omega(q)} \oplus E^c_{\omega(q)}$ of $\omega(q)$ to a neighborhood U of $\omega(q)$. Assume that $q \in U$ and that the interval I in the definition of $(\mathrm{P})_\Sigma$ is tangent to $\hat{E}^c_U$ and transverse to X. If $\omega(q)$ is not a singularity, then for every singular partition $\mathcal{R}$ of $\omega(q)$ there are $S \in \mathcal{R}$, $\delta > 0$, a sequence $\hat{q}_1, \hat{q}_2, \ldots \in S$ of points in the positive orbit of q and a sequence of intervals $\hat{J}_1, \hat{J}_2, \ldots \subset S$ in the positive orbit of I with $\hat{q}_j$ as a boundary point of $\hat{J}_j$ $(\forall j)$ such that*

$$\mathrm{Length}(\hat{J}_j) \geqslant \delta, \quad \forall j = 1, 2, 3, \ldots.$$

Proof. Assume that $\omega(q)$ is not a singularity and fix a singular partition $\mathcal{R}$ of $\omega(q)$. As $\omega(q)$ is not a singularity Theorem 4 in [3] implies that every singularity in $\omega(q)$ has one-dimensional unstable manifold. Then, Lemma 7 applied to $\mathcal{R}$ implies that the return map $\Pi = \Pi_{\mathcal{R}}$ associated to $\mathcal{R}$ satisfies the following properties:

(A) $O_X^+(q) \cap \mathcal{R}' = \{q_1, q_2, \ldots\}$ is an infinite sequence ordered in a way that $\Pi(q_i) = q_{i+1}$.
(B) There is $\delta > 0$ such that if $n \in \{1, 2, \ldots\}$ then either $B_\delta(q_n) \subset \mathrm{Dom}(\Pi)$ and $\Pi|_{B_\delta(q_n)}$ is C^1 or there is a curve $c_n \subset W_X^s(\mathrm{Sing}(X) \cap \omega(q)) \cap B_\delta(q_n)$ such that

$$B_\delta^+(q_n) \subset \mathrm{Dom}(\Pi) \quad \text{and} \quad \Pi|_{B_\delta^+(q_n)} \text{ is } C^1,$$

where $B_\delta^+(q_n)$ denotes the connected component of $B_\delta(q_n) \setminus c_n$ containing q_n.

We shall assume the second alternative in (B) since the first one is easier to handle.

We can assume that there is i_0 large such that $q_i \in \mathrm{Int}(\mathcal{R}')$ for all $i \geqslant i_0$. Otherwise $\omega(q) \cap \partial\mathcal{R}' \neq \emptyset$ and we get a contradiction because $\mathcal{R}$ is a singular partition of $\omega(q)$ (see Definition 5). We can assume $i_0 = 1$ without loss of generality. By (A) there is a sequence $n_1, n_2, \ldots \in \{1, \ldots, k\}$ such that

$$q_i \in S_{n_i}, \quad \forall i.$$

By using the positive orbit of I we can assume

$$I \subset S_{n_1} \cap \mathrm{Dom}(\Pi).$$

By shrinking I if necessary we can further assume that $I_1 \subset \mathrm{Int}(B_\delta^+(q_1))$, where δ comes from (B).

Define $I_1 = I$ and, inductively, the interval sequence $I_i = \Pi(I_{i-1}) = \Pi^i(I)$ as long as $I_{i-1} = \Pi^{i-1}(I) \subset B_\delta(q_{i-1})$.

Now we recall I is tangent to $\hat{E}_\Lambda^c$ and transverse to X by hypothesis. Then, the volume expansivity of E_Λ^c implies that Π is *expanding along* I (see [14, p. 370]).

Therefore the sequence $I_i = \Pi(I_{i-1})$ satisfies $\mathrm{Length}(I_i) \to \infty$ if $I_i \subset B_\delta^+(q_i)$ for all i. Since the elements of $\mathcal{R}$ have finite diameter we conclude that there is a first index i_1 such that

$$I_{i_1} \not\subset B_\delta^+(q_{i_1}).$$

On the other hand, the positive orbits starting in I_{i_1} meet Σ by $(\mathrm{P})_\Sigma$ while the ones in c_i do not for they go to $\mathrm{Sing}(X) \cap \omega(q)$ by (B). From this we conclude that

$$I_{i_1} \cap c_{i_1} = \emptyset.$$

Therefore, the connected component J_{i_1} of $I_{i_1} \cap B_\delta(q_{i_1})$ containing q_{i_1} joints q_{i_1} to some point in $\partial B_\delta(q_{i_1})$. This last assertion implies

$$\mathrm{Length}(J_{i_1}) \geqslant \delta.$$

In conclusion we have found an index i_1 and an interval $J_{i_1} \subset I_{i_1}$ (and so in the positive orbit of I) such that q_{i_1} is a boundary point of J_{i_1} and $\mathrm{Length}(J_{i_1}) \geqslant \delta$.

Repeating the argument we get a sequence $i_1, i_2, \ldots \in \{1, \ldots, k\}$, a sequence of points $q_{i_1}, q_{i_2}, \ldots$ with $q_{i_j} \in S_{i_j}$, and a sequence of intervals $J_{i_j} \subset S_{i_j}$ in the positive orbit of I such that q_{i_j} is a boundary point of J_{i_j} and $\mathrm{Length}(J_{i_j}) \geqslant \delta$.

As $\{1, \ldots, k\}$ is a finite set and contains i_j we can assume that $i_j = r$ for some fixed index $r \in \{1, \ldots, k\}$. Denoting $S = S_r$, $\hat{q}_j = q_{i_j}$ and $\hat{J}_j = J_{i_j}$ we get the result. $\square$

5. Proof of Theorem 4

Let X be a C^1 vector field in a compact 3-manifold M and $q \in M$. Suppose that $\omega(q)$ is a singular-hyperbolic set. We shall prove that $\omega(q)$ is a closed orbit if q satisfies $(\mathrm{P})_\Sigma$ for some closed subset Σ.

To start with we fix a neighborhood U of $\omega(q)$ where the singular-hyperbolic splitting $T_{\omega(q)}M = E^s_{\omega(q)} \oplus E^c_{\omega(q)}$ of $\omega(q)$ extends to a continuous splitting $T_U M = \hat{E}^s_U \oplus \hat{E}^c_U$. Let $W^{ss} = \{W^{ss}(x)\colon x \in U\}$ be the corresponding strong stable foliation (see the remark before Lemma 9). As U is a neighborhood of $\omega(q)$ we can assume that $q \in U$.

Let I be the interval in the definition of $(\mathrm{P})_\Sigma$. *We can assume that I is both tangent to $\hat{E}^c_U$ transverse to X*. Indeed, observe that there is $\epsilon > 0$ small such that the local strong stable manifold $W^{ss}_\epsilon(q)$ satisfies

$$I \cap \left(\bigcup_{-1 \leqslant t \leqslant 1} X_t\bigl(W^{ss}_\epsilon(q)\bigr) \right) = \emptyset.$$

(Otherwise it would exist $x \in I$ such that $O^+(x) \cap \Sigma = \emptyset$ as $O^+(x)$ is asymptotic to $O^+(q)$.) Then, we can use W^{ss} to project I onto an open interval $\hat{I}$, with q as a boundary point, such that $\hat{I}$ is tangent to $\hat{E}^c_U$ and transverse to X. As $\mathrm{Cl}(O^+(q))$ and Σ are disjoint we can enlarge Σ a bit using W^{ss} to obtain a closed subset $\hat{\Sigma}$ with $\mathrm{Cl}(O^+(q)) \cap \hat{\Sigma} = \emptyset$ such that $O^+(x) \cap \hat{\Sigma} \neq \emptyset$ for all $x \in \hat{I}$. Then, we can replace I by $\hat{I}$ and Σ by $\hat{\Sigma}$ if necessary in order to assume that I is tangent to $\hat{E}^c_U$ and transverse to X.

We have that $\omega(q)$ has a singular partition with arbitrarily small diameter $\mathcal{R} = \{S_1, \ldots, S_k\}$ by Theorem 8. We have $\mathcal{R}' \subset U$ (since $\mathcal{R}$ has small diameter) so the projection $\mathcal{F}^s(\cdot, S_i)$ of $\mathcal{F}^{ss}$ into S_i is well defined for every $i = 1, \ldots, k$.

As $\mathrm{Cl}(O^+(q))$ and Σ are disjoint there is a compact neighborhood $W \subset U$ of $\omega(q)$ such that

$$W \cap \Sigma = \emptyset.$$

Furthermore we can assume that

$$O^+(q) \subset W.$$

Because the diameter of the partition is small we can further assume that

$$\mathcal{R}' \subset \mathrm{Int}(W).$$

Now assume that $\omega(q)$ is not a singularity. As I is tangent to $\hat{E}^c_U$ and transverse to X we obtain $S, \hat{q}_i, \hat{J}_i$ and δ from Theorem 11. Let $x \in S$ be a limit point of $\hat{q}_i$.

If $\hat{q}_i \notin \mathcal{F}^s(x, S)$ for infinitely many i's we have a situation which is similar to that in Fig. 2 of [14, p. 371]: The splitting $T_{\omega(q)}M = E^s_{\omega(q)} \oplus E^c_{\omega(q)}$ is dominated and $\hat{J}_i$ is both tangent

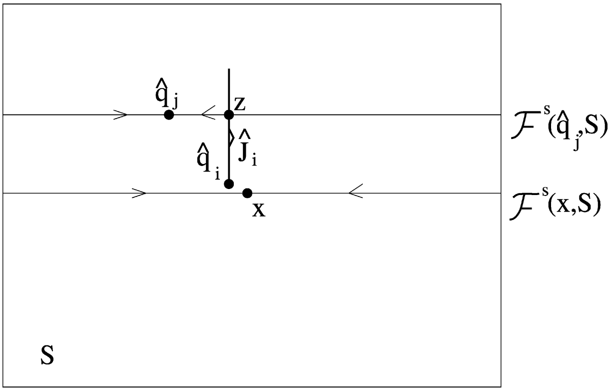

Fig. 3.

to $\hat{E}^c_U$ and transverse to X for all i. Therefore, the angle between the arcs $\hat{J}_i$ and the leaves $\{\mathcal{F}^s(y, S)\colon\ y \in S\}$ is bounded away from 0. As Length$(\hat{J}_i)$ is also bounded away from 0 and $\hat{q}_i \to x$ we eventually obtain an intersection point

$$z \in \hat{J}_i \cap \mathcal{F}^s(\hat{q}_j, S)$$

between $\hat{J}_i$ and $\mathcal{F}^s(\hat{q}_j, S)$ for some $i, j \in \mathbb{N}$ (see Fig. 3).

As $z \in \hat{J}_i$ we have that z is in the positive orbit of I so

$$O^+(z) \cap \Sigma \neq \emptyset.$$

But $z \in \mathcal{F}^s(\hat{q}_j, S)$ as well so $O^+(z)$ is asymptotic to $O^+(q)$ hence $O^+(z)$ cannot escape from W because $O^+(q) \subset W$. As $W \cap \Sigma = \emptyset$ we conclude that

$$O^+(z) \cap \Sigma = \emptyset$$

yielding a contradiction.

Therefore we can assume that $\hat{q}_i \in \mathcal{F}^s(x, S)$ for all i large. In this situation we can apply Lemma 5.6 in [14, p. 369] to obtain that $\omega(q)$ is a periodic orbit. The result follows.

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