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Monotone Power Method in Indefinite Metric and Inertia Theorem for Matrices

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ABSTRACT

The usual power method for matrices is generalized for contractions in indefinite metric spaces. This generalization unifies the power method and the inertia theorem in a natural way.

1. INTRODUCTION

Let us begin by recalling the *power method*. This method is used to compute the magnitude of the eigenvalues of a matrix (see for example [B, F, or LT]) and is based on the following considerations. Let A be an $r \times r$ matrix, and denote by $\lambda_1, \dots, \lambda_r$ the eigenvalues of A counting multiplicities. We denote the magnitudes of the eigenvalues of A by

$$\mu_j = |\lambda_j| \qquad (j = 1, \dots, r) \tag{1.1}$$

and assume that the eigenvalues are ordered so that

$$\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_r. \tag{1.2}$$

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For each $j = 1, \dots, r$ we denote by P_j the Riesz projection

$$P_j = \frac{1}{2\pi i} \int_{\Gamma_j} (\lambda I - A)^{-1} \, d\lambda, \tag{1.3}$$

where Γ_j is a smooth Jordan curve in $\mathbb{C}$ containing $\{\lambda_1, \dots, \lambda_j\}$ in its interior and $\{\lambda_{j+1}, \dots, \lambda_r\} \setminus \{\lambda_1, \dots, \lambda_j\}$ in its exterior. We also set $P_0 = 0$, the zero operator in $\mathbb{C}^r$, and put $\mu_0 = 0$.

The spaces Im P_j are nested, namely,

$$\{0\} = \operatorname{Im} P_o \subset \operatorname{Im} P_1 \subset \cdots \subset \operatorname{Im} P_r = \mathbb{C}^r. \tag{1.4}$$

Now let $x_0 \in \mathbb{C}^r$ be an arbitrary nonzero vector, and define a sequence $(x_n)_{n=0}^{\infty}$ of vectors in $\mathbb{C}^r$ via the recursion

$$x_{n+1} = Ax_n \qquad (n = 0, 1, \dots), \tag{1.5}$$

with the initial data x_0. Then the limit

$$\lim_{n \to \infty} \|x_n\|^{1/n} = \mu_j \tag{1.6}$$

holds, where $j \in \{1, \dots, r\}$ is an index that is uniquely determined by the condition

$$x_0 \in \operatorname{Im} P_j \setminus \operatorname{Im} P_{j-1}. \tag{1.7}$$

Thus, for almost all vectors x_0, namely, for all vectors x_0 in $\mathbb{C}^r \setminus \operatorname{Im} P_{i-1}$, where $i = \min\{j \in \{1, \dots, r\} : \mu_j = \mu_r\}$, the limit of $\|x_n\|^{1/n}$ is μ_r.

Let us also remark that although the sequence $(\|x_n\|)_{n=0}^{\infty}$ need not be monotone, for each number

$$\mu > \|A\|$$

the sequence $(\mu^{-n}\|x_n\|)_{n=0}^{\infty}$ is monotone decreasing to zero. In fact,

$$\mu^{-(n+1)}\|x_{n+1}\| = \mu^{-(n+1)}\|Ax_n\| \leqslant \left(\frac{\|A\|}{\mu}\right)\mu^{-n}\|x_n\| \qquad (n = 0, 1, \dots).$$

We now introduce a new inner product on $\mathbb{C}^r$, given in terms of a self-adjoint matrix G of order r. We consider three cases of G of increasing generality.

We begin by considering the case in which G is positive definite. In this case one can introduce a new norm on $\mathbb{C}^r$ via

$$\|x\|_G = \sqrt{\langle Gx, x\rangle} \qquad (x \in \mathbb{C}^r),$$

where $\langle\cdot,\cdot\rangle$ is the ordinary inner product in $\mathbb{C}^r$. This new norm is equivalent to the original norm in $\mathbb{C}^r$ because

$$\|G^{-1/2}\|^{-1}\,\|x\| \leqslant \|x\|_G \leqslant \|G^{1/2}\|\,\|x\| \qquad (x \in \mathbb{C}^r).$$

Therefore the limit (1.6) implies that for each nonzero vector $x_0 \in \mathbb{C}^r$ the limit

$$\lim_{n\to\infty} \|x_n\|_G^{1/n} = \mu_j$$

holds, where $(x_n)_{n=0}^{\infty}$ is defined by the recursion (1.5), and j is defined by the relation (1.7). Note that the above limit may be rewritten as

$$\lim_{n\to\infty} \langle Gx_n, x_n\rangle^{1/2n} = \mu_j. \tag{1.8}$$

Here, the sequence $(\mu^{-2n}\langle Gx_n, x_n\rangle)_{n=0}^{\infty}$ is monotone decreasing if $\mu > \|A\|_G$, where $\|A\|_G$ is defined by

$$\|A\|_G = \max_{0\neq x\in\mathbb{C}^r} \frac{\|Ax\|_G}{\|x\|_G}.$$

We now turn to the case in which G is negative definite. In this case, we do not have a positive definite norm; however, the limit (1.8) leads to

$$\lim_{n\to\infty} \left(-\langle Gx_n, x_n\rangle\right)^{1/2n} = \mu_j, \tag{1.9}$$

where j is defined by (1.7). Here also a monotonicity condition appears if A

is invertible after introducing a factor μ. In fact, for each μ satisfying

$$0 < \mu < \|A^{-1}\|_{-G}^{-1}$$

the sequence $(\mu^{-2n}\langle Gx_n, x_n\rangle)_{n=0}^{\infty}$ is monotone decreasing. To see this, note that the above conditions on μ imply

$$\begin{aligned}\langle -Gx_{n+1}, x_{n+1}\rangle &= \|x_{n+1}\|_{-G}^2 = \|Ax_n\|_{-G}^2 \\ &\geqslant \|A^{-1}\|_{-G}^{-2}\|x_n\|_{-G}^2 \geqslant \mu^2\|x_n\|_{-G}^2 = \mu^2\langle -Gx_n, x_n\rangle,\end{aligned}$$

whence,

$$\mu^{-2(n+1)}\langle Gx_{n+1}, x_{n+1}\rangle \leqslant \mu^{-2n}\langle Gx_n, x_n\rangle \qquad (n = 0, 1, \dots).$$

Consider now the case in which G is not assumed to be definite. In this case the limit analogue to (1.8) or (1.9) is false in general. For an example consider the case when x_0 is an eigenvector of A corresponding to λ_j and is also an isotropic vector for G. In this case $x_n = A^n x_0 = \lambda_j^n x_0$, whence

$$\langle Gx_n, x_n\rangle = |\lambda_j|^{2n}\langle Gx_0, x_0\rangle = 0.$$

This is clearly incompatible with limits of the form (1.8) or (1.9)

We now introduce a G-monotonicity condition for the general case.

G-Monotonicity. Let G be a self-adjoint matrix of order r and μ a positive number. We say that the system $x_{n+1} = Ax_n (n = 0, 1, \dots)$ is *G-monotone* with *parameter of monotonicity* μ if the condition

$$\mu^2\langle Gx_n, x_n\rangle \geqslant \langle Gx_{n+1}, x_{n+1}\rangle + \varepsilon\|x_n\|^2 \qquad (x_0 \in \mathbb{C}^r;\ n = 0, 1, \dots) \tag{1.10}$$

holds for some positive number ε and any initial vector x_0. This condition is equivalent to the matrix inequality

$$\mu^2 G - A^*GA \geqslant \varepsilon I, \tag{1.11}$$

which means that $\mu^{-1}A$ is a strict contraction in the metric defined by $\langle Gx, x\rangle$ $(x \in \mathbb{C}^r)$. Clearly, this implies that A does not have eigenvalues of magnitude equal to μ. Thus, there exists a well-defined index ν such that

$$\mu_\nu < \mu < \mu_{\nu+1},$$

where $\nu = 0$ if $\mu < \mu_1$ and $\nu = r$ if $\mu_r < \mu$. Moreover, by the well-known inertia theorem, ν is equal to the number of positive eigenvalues of G, counting multiplicities, and G is invertible.

Let us also remark that the G-monotonicity condition with suitable parameter of monotonicity occurs in the above examples where $G > 0$, or $G < 0$ and A invertible.

If the system $x_{n+1} = Ax_n (n = 0, 1, \dots)$ is G-monotone, then we can introduce a partition of $\mathbb{C}^r$ in a natural way. We define $\mathscr{P}$ to be the set of all vectors x_0 in $\mathbb{C}^r$ such that $\langle Gx_n, x_n\rangle \geqslant 0 (n = 0, 1, \dots)$, where $x_{n+1} = Ax_n (n = 0, 1, \dots)$. Note that $0 \in \mathscr{P}$. We also denote

$$\mathscr{P}^c = \mathbb{C}^r \setminus \mathscr{P}.$$

Some preliminary properties of this partition are given in the next result.

THEOREM 1.1. *Assume that the system $x_{n+1} = Ax_n$ is G-monotone with parameter of monotonicity μ, define ν to be the unique integer such that $\mu_\nu < \mu < \mu_{\nu+1}$ if $\mu_1 < \mu < \mu_r$, and let $\nu = 0$ if $\mu < \mu_1$ and $\nu = r$ if $\mu_r < \mu$. Then*

$$\mathscr{P} = \operatorname{Im} P_\nu, \tag{1.12}$$

and $\operatorname{Ker} P_\nu$ *is a maximal linear subspace of $\mathscr{P}^c \cup \{0\}$. Moreover,* $\operatorname{Im} P_\nu$ *(respectively* $\operatorname{Ker} P_\nu$*) is a maximal G-positive definite (respectively G-negative definite) subspace of $\mathbb{C}^r$.*

The G-monotone power method in indefinite metric is presented in the next theorem.

THEOREM 1.2 (G-MONOTONE POWER METHOD IN INDEFINITE METRIC). *Assume that the system $x_{n+1} = Ax_n$ is G-monotone with parameter of monotonicity μ, define ν to be the unique integer such that $\mu_\nu < \mu < \mu_{\nu+1}$ if $\mu_1 < \mu < \mu_r$, and let $\nu = 0$ if $\mu < \mu_1$ and $\nu = r$ if $\mu_r < \mu$. Then for each* *nonzero vector* $x_0 \in \mathbb{C}^r$ *such that* $\langle Gx_n, x_n\rangle \geqslant 0 (n = 0, 1, \dots)$, *the equality*

$$\lim_{n\to\infty} \left(\langle Gx_n, x_n\rangle\right)^{1/2n} = \mu_j \tag{1.13}$$

holds, where $j \in \{1, \dots, \nu\}$ *is uniquely defined by the relation* $x_0 \in \operatorname{Im} P_j \setminus \operatorname{Im} P_{j-1}$, *and for each vector* $x_0 \in \mathbb{C}^r$ *such that* $\langle Gx_k, x_k\rangle < 0$ *for some* k, *the equality*

$$\lim_{n\to\infty} \left(-\langle Gx_n, x_n\rangle\right)^{1/2n} = \mu_j \tag{1.14}$$

holds, where $j \in \{\nu + 1, \dots, r\}$ *is uniquely defined by the relation* $x_0 \in \operatorname{Im} P_j \setminus \operatorname{Im} P_{j-1}$.

We remark that although in (1.14) the numbers $-\langle Gx_n, x_n\rangle$ are not necessarily positive for all n, they are certainly positive if $n \geqslant k$. Therefore, the sequence $(-\langle Gx_n, x_n\rangle)^{1/2n}$, whose limit is given by (1.14), is considered here only for $n \geqslant k$.

The inertia theorem (namely the fact that the number of eigenvalues λ of A satisfying $|\lambda| < \mu$ (respectively $|\lambda| > \mu$) is equal to the number of positive (respectively negative) eigenvalues of G, counting multiplicities), as an immediate consequence of these theorems. For the inertia theorem see [DK, Hi, K, OS, S, T1–2, Wie, Wim, WZ]. See also the review in [C] and Chapter 13 of [LT].

Similar results hold if the system $x_{n+1} = A^h x_n$ is G-monotone for some positive integer h. Infinite-dimensional generalizations of the above results are presented separately.

2. PROOFS

In this section we consider a G-monotone system

$$x_{n+1} = Ax_n \qquad (n = 0, 1, \dots), \tag{2.1}$$

with parameter of monotonicity $\mu > 0$. Here G and A are $r \times r$ matrices with G self-adjoint. We always associate the vector x_0 with the sequence $(x_n)_{n=0}^{\infty}$ defined by the recursion (2.1) with the initial data x_0. We use the same notation as in the introduction. In particular, since μ is a parameter of monotonicity, the matrix A has no eigenvalues of magnitude equal to μ, whence

$$\mu \neq \mu_j \qquad (j = 1, \dots, r). \tag{2.2}$$

We define ν to be the unique integer such that

$$\mu_\nu < \mu < \mu_{\nu+1} \tag{2.3}$$

if $\mu_1 < \mu < \mu_r$ and let $\nu = 0$ if $\mu < \mu_1$ and $\nu = r$ if $\mu_r < \mu$.

We use the power method in its classical form. Namely, for each $x_0 \neq 0$ the limit

$$\lim_{n\to\infty} \|x_n\|^{1/n} = \mu_j \tag{2.4}$$

holds, where $j \in \{1, \dots, r\}$ is uniquely defined by the relation

$$x_0 \in \operatorname{Im} P_j \setminus \operatorname{Im} P_{j-1}. \tag{2.5}$$

Let us first show that $\operatorname{Im} P_\nu$ is a G-positive definite subspace of $\mathbb{C}^r$. Indeed, for each vector $x_0 \in \operatorname{Im} P_\nu$, inequality (1.10) leads to

$$\mu^{-2n}\langle Gx_n, x_n\rangle - \mu^{-2(n+1)}\langle Gx_{n+1}, x_{n+1}\rangle \geqslant \varepsilon\mu^{-2(n+1)}\|x_n\|^2.$$

Adding these inequalities for $n = 0, \dots, h-1$, where h is an arbitrary positive integer, yields

$$\langle Gx_0, x_0\rangle - \mu^{-2h}\langle Gx_h, x_h\rangle \geqslant \varepsilon\mu^{-2}\|x_0\|^2 \qquad (h = 1, 2, \dots), \tag{2.6}$$

after disregarding some nonnegative terms on the right-hand side. However,

$$|\mu^{-2h}\langle Gx_h, x_h\rangle| \leqslant \|G\|\,\mu^{-2h}\|x_h\|^2 \qquad (h = 1, 2, \dots), \tag{2.7}$$

and

$$x_h = A^h x_0 \qquad (h = 1, 2, \dots). \tag{2.8}$$

Since $x_0 \in \operatorname{Im} P_\nu$ and $|\lambda_\nu| = \mu_\nu < \mu$, the vector x_0 is a linear combination of eigenvectors and generalized eigenvectors corresponding to eigenvalues of A of magnitude less than μ. Hence, we have

$$\lim_{h\to\infty} \mu^{-h}\|A^h x_0\| = 0.$$

Combining this with (2.7) and (2.8), it follows that

$$\lim_{h\to\infty} |\mu^{-2h}\langle Gx_h, x_h\rangle| = 0.$$

Therefore, by taking the limit in (2.6) we obtain

$$\langle Gx_0, x_0\rangle \geqslant \varepsilon\mu^{-2}\|x_0\|^2.$$

This holds for each $x_0 \in \operatorname{Im} P_\nu$ showing $\operatorname{Im} P_\nu$ is G-positive definite.

Since $\operatorname{Im} P_\nu$ is G-positive definite and invariant under the system (2.1), it is clear from the definition of $\mathscr{P}$ that

$$\operatorname{Im} P_\nu \subset \mathscr{P}. \tag{2.9}$$

We now prove the first part of Theorem 1.2. Let $0 \neq x_0 \in \mathbb{C}^r$ be an arbitrary nonzero vector such that

$$\langle Gx_n, x_n\rangle \geqslant 0 \qquad (n = 0, 1, \dots). \tag{2.10}$$

Then also

$$\langle Gx_{n+1}, x_{n+1}\rangle \geqslant 0 \qquad (n = 0, 1, \dots),$$

and therefore, (1.10) implies

$$\mu^2\langle Gx_n, x_n\rangle \geqslant \varepsilon\|x_n\|^2 \qquad (n = 0, 1, \dots).$$

Hence, we obtain

$$\|G\|\,\|x_n\|^2 \geqslant \langle Gx_n, x_n\rangle \geqslant \varepsilon\mu^{-2}\|x_n\|^2 \qquad (n = 0, 1, \dots).$$

These inequalities mean that the norms $\|x_n\|$ and $\langle Gx_n, x_n\rangle^{1/2}$ are equivalent on the orbit $(x_n)_{n=0}^{\infty}$. Consequently, we obtain from the usual power method (2.4) that

$$\lim_{n\to\infty} \langle Gx_n, x_n\rangle^{1/2n} = \lim_{n\to\infty} \|x_n\|^{1/n} = \mu_j, \tag{2.11}$$

where $j \in \{1, \dots, r\}$ is defined by the relation (2.5).

Let us now remark that (1.10) also leads to

$$\mu^2 \langle Gx_n, x_n \rangle \geqslant \langle Gx_{n+1}, x_{n+1} \rangle \qquad (n = 0, 1, \dots).$$

Thus

$$\limsup_{n \to \infty} \langle Gx_n, x_n \rangle^{1/2n} \leqslant \mu.$$

Combining this with (2.11) we obtain $\mu_j \leqslant \mu$. This inequality and (2.2) lead to

$$\mu_j < \mu.$$

By the definition (2.3) of ν this means that

$$j \leqslant \nu. \tag{2.12}$$

This inequality and (2.11) prove equality (1.13) of Theorem 1.2.

Now let $0 \neq x_0 \in \mathscr{P}$ be an arbitrary nonzero vector in $\mathscr{P}$. By the definition of $\mathscr{P}$, inequalities (2.10) hold. Hence, by the last paragraph, the limit (2.11) holds where j is defined by the relation (2.5) and satisfies inequality (2.12). In particular, it follows from (2.5) that

$$x_0 \in \operatorname{Im} P_j.$$

Thus, inequality (2.12) leads to

$$x_0 \in \operatorname{Im} P_j \subset \operatorname{Im} P_\nu.$$

This holds for each $0 \neq x_0 \in \mathscr{P}$. Since $0 \in \operatorname{Im} P_\nu$, we obtain that

$$\mathscr{P} \subset \operatorname{Im} P_\nu.$$

Combining this with (2.9) yields

$$\mathscr{P} = \operatorname{Im} P_\nu. \tag{2.13}$$

Thus (1.12) of Theorem 1.1 holds.

Let us also remark that all the numbers $\{\mu_j\}_{j=1}^{\nu}$ actually occur in the right-hand side of (1.13) with suitable initial vectors x_0. In fact, for $j \in \{1, \dots, \nu\}$, denote

$$i = \min\{k \in \{1, \dots, \nu\} : \mu_k = \mu_j\}.$$

Then $\operatorname{Im} P_{i-1} \neq \operatorname{Im} P_i$, and therefore, we can take

$$0 \neq x_0 \in \operatorname{Im} P_i \setminus \operatorname{Im} P_{i-1}$$

to be an eigenvector of A corresponding to the eigenvalue λ_i. Then by (2.13) and $i \leqslant \nu$ we obtain

$$x_0 \in \operatorname{Im} P_\nu = \mathscr{P}.$$

Moreover, since $x_0 \in \operatorname{Im} P_i \setminus \operatorname{Im} P_{i-1}$ we have by (1.13)

$$\lim_{n \to \infty} \langle Gx_n, x_n \rangle^{1/2n} = \mu_i.$$

However, $\mu_j = \mu_i$ by the definition of i, and therefore,

$$\lim_{n \to \infty} \langle Gx_n, x_n \rangle^{1/2n} = \mu_j.$$

We now turn our attention to inequality (1.14) of Theorem 1.2. Let $x_0 \in \mathbb{C}^r$ be such that

$$\langle Gx_k, x_k \rangle < 0,$$

for some nonnegative integer k. Inequality (1.10) implies

$$\langle Gx_n, x_n \rangle < 0 \qquad (n = k, k+1, \dots), \tag{2.14}$$

and

$$|\langle Gx_{n+1}, x_{n+1} \rangle| \geqslant \mu^2 |\langle Gx_n, x_n \rangle| + \varepsilon \|x_n\|^2 \qquad (n = k, k+1, \dots). \tag{2.15}$$

This inequality shows in particular that

$$|\langle Gx_n, x_n\rangle| \geqslant \varepsilon \|x_{n-1}\|^2 \qquad (n = k+1, k+2, \dots). \tag{2.16}$$

However, $\|x_n\| = \|Ax_{n-1}\| \leqslant \|A\|\,\|x_{n-1}\|$, and therefore,

$$\|x_{n-1}\| \geqslant (1 + \|A\|)^{-1}\,\|x_n\|.$$

Combining this with (2.16) we obtain

$$\|G\|\,\|x_n\|^2 \geqslant |\langle Gx_n, x_n\rangle| \geqslant \varepsilon(1+\|A\|)^{-2}\,\|x_n\|^2 \quad (n = k+1, k+2, \dots). \tag{2.17}$$

It follows from these inequalities and the power method (2.4) that

$$\lim_{n\to\infty} |\langle Gx_n, x_n\rangle|^{1/2n} = \lim_{n\to\infty} \|x_n\|^{1/n} = \mu_j, \tag{2.18}$$

where $j \in \{1, \dots, r\}$ is defined by the relation (2.5). By (2.14), equality (2.18) implies (1.14) of Theorem 1.2., and we still have to show that $j > \nu$. To see this, note that (2.15) also leads to

$$|\langle Gx_{n+1}, x_{n+1}\rangle| \geqslant \mu^2\,|\langle Gx_n, x_n\rangle| \qquad (n = k, k+1, \dots),$$

whence,

$$|\langle Gx_n, x_n\rangle| \geqslant \mu^{2(n-k)}|\langle Gx_k, x_k\rangle| \qquad (n = k, k+1, \dots).$$

Since $\langle Gx_k, x_k\rangle \neq 0$ by (2.14) we obtain from these inequalities that

$$\liminf_{n\to\infty} |\langle Gx_n, x_n\rangle|^{1/2n} \geqslant \mu.$$

In view of (2.18), this means $\mu_j \geqslant \mu$. Recalling (2.2), we obtain

$$\mu_j > \mu.$$

By the definition (2.3) of ν this implies

$$j > \nu. \tag{2.19}$$

This completes the proof of the second part of Theorem 1.2 and equality (1.14). As for (1.13), by choosing x_0 to be suitable eigenvectors of A one concludes that all the numbers $\{\mu_j\}_{j=\nu+1}^{r}$ actually occur in the right-hand side of (1.14).

There remains to prove the second part of Theorem 1.1, namely, that $\ker P_\nu$ is a maximal linear subspace of $\mathscr{P}^c \cup \{0\}$, which is also G-negative definite. Note first that by (2.13), $\operatorname{Ker} P_\nu \setminus \{0\}$ is contained in the complement of $\mathscr{P} = \operatorname{Im} P_\nu$ in $\mathbb{C}^r$. Thus, $\operatorname{Ker} P_\nu \setminus \{0\} \subset \mathscr{P}^c$, and therefore

$$\operatorname{Ker} P_\nu \subset \mathscr{P}^c \cup \{0\}. \tag{2.20}$$

Denote by S the unit sphere of $\operatorname{Ker} P_\nu$

$$S = \{x \in \operatorname{Ker} P_\nu : \|x\| = 1\}.$$

Let x_0 be an arbitrary vector in S. Then x_0 is a nonzero vector in $\operatorname{Ker} P_\nu$, whence $x_0 \in \mathscr{P}^c$ by (2.20). Thus, the definitions of $\mathscr{P}$ and $\mathscr{P}^c$ imply that there exists a nonnegative integer $n_0 = n_0(x_0)$ such that

$$\langle Gx_{n_0}, x_{n_0} \rangle < 0,$$

whence

$$\langle GA^{n_0}x_0, A^{n_0}x_0 \rangle < 0.$$

By continuity, there exists a neighborhood $\mathscr{O}_{x_0}$ of x_0 in S such that

$$\langle GA^{n_0}x, A^{n_0}x \rangle < 0 \qquad (x \in \mathscr{O}_{x_0}),$$

where $n_0 = n_0(x_0)$. Condition (1.10) now leads to

$$\langle GA^{n}x, A^{n}x \rangle < 0 \qquad (x \in \mathscr{O}_{x_0};\, n = n_0(x_0), n_0(x_0) + 1, \dots). \tag{2.21}$$

Now let $x_0^{(1)}, \dots, x_0^{(l)}$ be a finite set of points in S such that

$$S = \bigcup_{i=1}^{l} \mathscr{O}_{x_0^{(i)}}.$$

Denote

$$N = \max\left(n_0\left(x_0^{(1)}\right), \dots, n_0\left(x_0^{(l)}\right)\right).$$

Then the last two equalities and (2.21) imply that

$$\langle GA^N x, A^N x\rangle < 0 \qquad (x \in S).$$

Since S is the unit sphere of $\operatorname{Ker} P_\nu$ this leads to

$$\langle GA^N x, A^N x\rangle < 0 \qquad (0 \neq x \in \operatorname{Ker} P_\nu). \tag{2.22}$$

Finally, recall that inequalities (1.2) combined with the definition (2.3) of ν imply that

$$\mu < \mu_{\nu+1} \leqslant \cdots \leqslant \mu_r .$$

From these inequalities and $\mu > 0$ we obtain that $\mu_j \neq 0$ for $(j = \nu + 1, \dots, r)$, and therefore (1.1) leads to $|\lambda_j| \neq 0$ for $(j = \nu + 1, \dots, r)$. Since the eigenvalues of $A|_{\operatorname{Ker} P_\nu}$ are $\lambda_{\nu+1}, \dots, \lambda_r$, $A|_{\operatorname{Ker} P_\nu}$ is invertible. Consequently, for each nonzero vector $y \in \operatorname{Ker} P_\nu$ there exists a nonzero vector $x \in \operatorname{Ker} P_\nu$ with $A^N x = y$, and therefore, (2.22) leads to $\langle Gy, y\rangle < 0$. This shows that $\operatorname{Ker} P_\nu$ is a G-negative definite subspace of $\mathbb{C}^r$, which is contained in $\mathscr{P}^c \cup \{0\}$ by (2.20).

If $\operatorname{Ker} P_\nu$ is a proper subspace of M, where M is a linear subspace of $\mathbb{C}^r$, then by (2.13)

$$M \cap \mathscr{P} = M \cap \operatorname{Im} P_\nu \neq \{0\}.$$

Thus, M is not contained in $\mathscr{P}^c \cup \{0\}$. This shows that $\operatorname{Ker} P_\nu$ is a maximal linear subspace of $\mathscr{P}^c \cup \{0\}$.

Finally, we have shown above that $\operatorname{Im} P_\nu$ is G-positive definite, and $\operatorname{Ker} P_\nu$ is G-negative definite. Therefore, these subspaces of $\mathbb{C}^r$ are also maximal G-positive definite and G-negative definite, respectively.

REFERENCES

[B] E. K. Blum, *Numerical Analysis and Computation: Theory and Practice*, Addison–Wesley, 1972.

[C] B. E. Cain, Inertia theory, *Linear Algebra Appl.* 30:211–240 (1980).

[DK] Ju.L. Daleckii and M. G. Krein, *Stability of Solutions of Differential Equations in Banach Space*, Transl. Math. Monographs, Vol. 43, Amer. Math. Soc., Providence, RI, 1974.

[F] C.-E. Fröberg, *Introduction to Numerical Analysis*, Addison–Wesley, 1965.

[Hi] R. D. Hill, Inertia theory for simultaneously triangulable complex matrices, *Linear Algebra Appl.* 2:131–142 (1969).

[K] M. G. Krein, *Stability Theory of Differential Equations in Banach Spaces* (in Russian), Kiev, 1984 (an expanded English translation of this book is [DK]).

[LT] P. Lancaster and M. Tismenetsky, *The Theory of Matrices*, Academic Press, New York, 1985.

[OS] A. Ostrowski and H. Schneider, Some theorems on the inertia of general matrices, *J. Math. Anal. Appl.* 4:72–84 (1962).

[S] P. Stein, Some general theorems on iterants, *J. Res. Natl. Bur. Standards* 48:82–83 (1952).

[T1] O. Taussky, A generalization of a theorem of Lyapunov, *J. Soc. Ind. Appl. Math.* 9:640–643 (1961).

[T2] O. Taussky, Matrices C with $C^n \to 0$, *J. Algebra* 1:5–10 (1964).

[Wie] H. Wielandt, On the eigenvalues of $A + B$ and AB, *J. Res. Natl. Bur. Standards B* 77:61–63 (1973).

[Wim] H. Wimmer, On the Ostrowski-Schneider inertia theorem, *J. Math. Anal. Appl.* 41:164–169 (1973).

[WZ] H. Wimmer and A. D. Ziebur, Remarks on inertia theorems for matrices, *Czech. Math. J.* 25:556–561 (1975).

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