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Conjugate gradient-boundary element solution for distributed elliptic optimal control problems

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Abstract

An optimality system of equations for the optimal control problem governed by Helmholtz-type equations is derived. By the associated first-order necessary optimality condition, we obtain the conjugate gradient method (CGM) in the continuous case. Introducing the sequence of higher-order fundamental solutions, we propose an iterative algorithm based on the conjugate gradient-boundary element method using the multiple reciprocity method (CGM+MRBEM) for solving the discrete control input. This algorithm has an advantage over that of the existing literatures because the main attribute (the reduced dimensionality) of the boundary element method is fully utilized. Finally, the local error estimates for this scheme are obtained, and a test problem is given to illustrate the efficiency of the proposed method.

Keywords: Optimal control; Conjugate gradient method; Helmholtz-type equation; Boundary element method; Sequence of higher-order fundamental solutions; Error estimate

1. Introduction

In recent years, much attention has been paid to research of the distributed elliptic optimal control problems. The need for accurate and efficient solution method for these problems has become an important issue. There have been extensive theoretical and numerical studies for finite element approximation and finite difference technique for these problems (see [2,3,6,7,9–11,17]),

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but much less research is available for boundary element method. In general, the boundary element method has many advantages over the finite element and finite difference method in saving computational work for linear elliptic boundary value problems. One of the main attributes of the boundary element method is the reduced dimensionality due to the boundary integral formulation (see [4]). These features, however, do not seem to be convenient for some domain integrals in integral equations because of nonhomogeneous state equation associated with the control function. But we found that the conjugate gradient-boundary element method using the multiple reciprocity method (CGM+MRBEM) is available for solving this kind of optimal control problem. Some previous works about CGM and CGM+BEM are as follows. In the article by A.M. Ramos (see [14]), the author looks for the Nash equilibria for the multi-objective control of linear differential equations. Then, to combine finite difference methods for the time discretization and finite element methods for the space discretization, conjugate gradient algorithms for the iterative solutions of the discrete control problems are developed. In the article by L. Marin (see [12]), the author proposes the CGM+BEM solution to the Cauchy problem for Helmholtz-type equations. This method produces a convergent and stable numerical solution. But the CGM+BEM proposed by L. Marin is not available for distributed elliptic optimal control problem because the Cauchy problem considered in this article is homogeneous.

In this paper, we are interested in the optimal control problem

$$\min_u J(u,y) = \frac{1}{2}\|y - y_z\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u\|^2_{L^2(\Omega)}, \tag{1.1}$$

$$\Delta y + k^2 y = f + u, \quad \text{in } \Omega \subseteq R^2, \tag{1.2}$$

$$y = 0, \quad \text{on } \Gamma, \tag{1.3}$$

where Δ is Laplace operator, $\Omega \subseteq R^2$ is a bounded domain and Γ is its boundary. $y_z \in L^2(\Omega)$, $f \in L^2(\Omega)$ are given functions, and γ is the weight of the cost of the control function u. This optimal control system can be used to study the vibration of a structure, the acoustic cavity problem, the radiation wave, the scattering of a wave, and the problems of heat conduction (see [2,6,8,12,13,17]). It is well known that the fundamental solution of Eq. (1.2) can be expressed in terms of the zero-order Hankel function of the second kind as follows (see [8,13])

$$y^*(r) = -\frac{i}{4} H_0^{(2)}(kr), \tag{1.4}$$

where r stands for the distance between the source and field points, and $H_0^{(2)}$ is the zero-order Hankel function of the second kind, which leads to the consumption of a huge amount of computational time for computing the domain integration using the conventional BEM formulation. In order to avoid this shortcoming, an alternative method is the multiple reciprocity method (MRM) for the conversion of the domain integral to the corresponding boundary one by means of the sequence of higher-order fundamental solutions, in which all the integrals are expressed by the real-valued fundamental solutions (see [8,13,15,16]).

In our work, inspired by the literatures [2,3,8,9,12–14], we propose the CGM via the associated first-order necessary optimality condition, and then introduce the sequence of higher-order fundamental solutions of (1.2) (see [8]) to derive the CGM+MRBEM for finding the solution of the discrete control. In our algorithm, we are simply to constitute the finite element approximations for some unknown boundary functions by partitioning Γ. We are not to constitute the finite element approximations for functions on the domain Ω though a finite element mesh of Ω is given. By partitioning Ω, we simply obtain the numerical integration formulae for computing some domain integrals of known functions. The numerical solutions of functions are obtained by the boundary integral formulations. This is an advantage over much of the existing literature because the main attribute (the reduced dimensionality) of the boundary element method is fully utilized in the CGM+MRBEM.

This paper is organized as follows. In Section 2, after our efforts to analyze our problem, we constitute the necessary and sufficient optimality condition for (1.1)–(1.3). In Section 3, the conjugate gradient algorithm of the optimality system is derived. In Section 4, we describe the boundary integral equations and the boundary variational equations using multiple reciprocity method (MRM) for some of boundary value problems in the CGM, and give the CGM+MRBEM for the problem (1.1)–(1.3). Section 5 focuses on the error estimates for the CGM+MRBEM. In Section 6, a numerical example will be given to support our method.

2. Optimality system

In this section, we describe the optimality system of the problem (1.1)–(1.3).

Theorem 2.1. *Assume that Ω is convex, then there exists a unique solution $(y^*, u^*) \in (H_0^1(\Omega) \cap H^2(\Omega))^2$ to the optimal control problem* (1.1)–(1.3) *such that*

$$\begin{cases} \Delta y + k^2 y = f + \frac{1}{\gamma}\lambda, & \text{in } \Omega \subseteq R^2, \\ y = 0, & \text{on } \Gamma, \\ \Delta\lambda + k^2\lambda = -(y - y_z), & \text{in } \Omega \subseteq R^2, \\ \lambda = 0, & \text{on } \Gamma, \\ \gamma u = \lambda, & \text{in } \Omega. \end{cases} \tag{2.1}$$

Proof. For any $u \in L^2(\Omega)$, there exists the solution $y \in H_0^1(\Omega) \cap H^2(\Omega)$ of the problem (1.2)–(1.3) by the convexity of Ω. Let $y = y(u)$, where $y(u)$ denotes the solution of (1.2)–(1.3) as a function of u, then the mapping $u \to y(u)$ form $L^2(\Omega)$ to $H_0^1(\Omega) \cap H^2(\Omega)$ is affine and continuous. Let $y'(u, \delta u)$ be its first derivative at $u \in L^2(\Omega)$ for $y(u)$ in the direction δu, where δu is variation for u, then we have

$$\begin{cases} \Delta y'(u, \delta u) + k^2 y'(u, \delta u) = \delta u, & \text{in } \Omega \subseteq R^2, \\ y'(u, \delta u) = 0, & \text{on } \Gamma. \end{cases} \tag{2.2}$$

Let $\hat{J}(u) = J(y(u), u)$, then

$$\hat{J}'(u, \delta u) = \int_\Omega \big(y(u) - y_z\big) y'(u, \delta u)\, d\Omega + \gamma \int_\Omega u\delta u\, d\Omega,$$

$$\hat{J}''(u)(\delta u, \delta u) = \big\|y'(u, \delta u)\big\|_{L^2(\Omega)}^2 + \int_\Omega \big(y(u) - y_z\big) y''(u)(\delta u, \delta u)\, d\Omega + \gamma \|\delta u\|_{L^2(\Omega)}^2.$$

Obviously, the functional $\hat{J}$ is uniformly convex because the second derivative of $y(u)$ is zero. This implies existence of a unique solution u^* to (1.1)–(1.3).

Suppose that (y^*, u^*) is a solution of (1.1)–(1.3), then the solution is characterized by the first order necessary optimality condition $\hat{J}'(u^*, \delta u) = 0$ for all δu

$$\hat{J}'(u^*, \delta u) = \big(y^* - y_z, y'(u^*, \delta u)\big)_{L^2(\Omega)} + \gamma (u^*, \delta u)_{L^2(\Omega)} = 0, \quad \forall \delta u \in L^2(\Omega). \tag{2.3}$$

Introduce a costate function λ^* as the solution to the problem

$$\begin{cases} \Delta\lambda^* + k^2\lambda^* = -(y^* - y_z), & \text{in } \Omega \subseteq R^2, \\ \lambda^* = 0, & \text{on } \Gamma. \end{cases} \tag{2.4}$$

Then there exists the solution $\lambda^* \in H_0^1(\Omega) \cap H^2(\Omega)$ for any $y^* \in H_0^1(\Omega) \cap H^2(\Omega)$.

By (2.2) and (2.4), we have

$$\int_\Omega \lambda^* \Delta y'(u^*, \delta u)\, d\Omega + k^2 \int_\Omega \lambda^* y'(u^*, \delta u)\, d\Omega = \int_\Omega \lambda^* \delta u\, d\Omega,$$

$$\int_\Omega y'(u^*, \delta u)\Delta\lambda^*\, d\Omega + k^2 \int_\Omega y'(u^*, \delta u)\lambda^*\, d\Omega = -\int_\Omega y'(u^*, \delta u)(y^* - y_z)\, d\Omega.$$

Consequently, utilizing the Green formula and the boundary conditions of (2.2) and (2.4), we have

$$\int_\Omega \lambda^* \delta u\, d\Omega + \int_\Omega (y^* - y_z) y'(u^*, \delta u)\, d\Omega = 0.$$

By the equation above and (2.3), there holds

$$\hat{J}'(u^*, \delta u) = -(\lambda^*, \delta u)_{L^2(\Omega)} + \gamma (u^*, \delta u)_{L^2(\Omega)} = 0, \quad \forall \delta u \in L^2(\Omega). \tag{2.5}$$

Obviously, the optimality system (2.1) and the result $(y^*, u^*) \in (H_0^1(\Omega) \cap H^2(\Omega))^2$ hold. $\square$

3. Conjugate gradient algorithm

In the following we propose the conjugate gradient algorithm of the optimality system (2.1). For any $u \in L^2(\Omega)$, let $y = y(u)$ be the solution for (1.2)–(1.3), then the optimal control u^* is determined by the optimality condition

$$\nabla J\big(y(u), u\big) = 0, \tag{3.1}$$

where ∇ is gradient operator. It is obvious that $\nabla J : u \in L^2(\Omega) \to \nabla J(y(u), u) \in L^2(\Omega)$ is an affine mapping of $L^2(\Omega)$. Therefore, there exist a linear continuous mapping $\pounds(u)$ (dependent on u) and a function $\tilde{p}$ (independent of u) such that

$$\nabla J\big(y(u), u\big) = \pounds(u) - \tilde{p}. \tag{3.2}$$

Let us identify the linear mapping $\pounds(u)$. For any $u \in L^2(\Omega)$, by (2.5), the linear part of the affine mapping in the relation $\nabla J : u \to \nabla J(y(u), u)$ is defined by

$$\pounds(u) = \gamma u - \bar{p}, \tag{3.3}$$

where $\bar{p}$ is the solution of

$$\begin{cases} \Delta\bar{p} + k^2\bar{p} = -\bar{y}, & \text{in } \Omega, \\ \bar{p} = 0, & \text{on } \Gamma, \end{cases} \tag{3.4}$$

and $\bar{y}$ is the solution of

$$\begin{cases} \Delta\bar{y} + k^2\bar{y} = u, & \text{in } \Omega \\ \bar{y} = 0, & \text{on } \Gamma. \end{cases} \tag{3.5}$$

Let us identify $\tilde{p}$ (independent of u). The constant part of the affine mapping is the function $\tilde{p} \in L^2(\Omega)$ defined by the solution of

$$\begin{cases} \Delta\tilde{p} + k^2\tilde{p} = -(\tilde{y} - y_z), & \text{in } \Omega, \\ \tilde{p} = 0, & \text{on } \Gamma, \end{cases} \tag{3.6}$$

and $\tilde{y}$ is the solution of

$$\begin{cases} \Delta\tilde{y} + k^2\tilde{y} = f, & \text{in } \Omega, \\ \tilde{y} = 0, & \text{on } \Gamma. \end{cases} \tag{3.7}$$

Theorem 3.1. *The mapping $\pounds$ is linear, continuous, symmetric, and strongly positive.*

Proof. It is obvious that $\pounds$ is a linear mapping and it is easy to prove that it is a continuous mapping. Let us prove that $\pounds$ is symmetric and strongly positive. Let us consider $u, v \in L^2(\Omega)$, then we have

$$\big(\pounds(u), v\big)_{L^2(\Omega)} = (\gamma u - \bar{p}, v)_{L^2(\Omega)} = \gamma \int_\Omega uv\, d\Omega - \int_\Omega \bar{p}v\, d\Omega.$$

Let $\bar{y}(v), \bar{y}(u)$ denote the solutions of (3.5) as functions of the nonhomogeneous terms v and u, respectively. Let us focus on the integral $\int_\Omega \bar{p}v\, d\Omega$, using the Green formula and the boundary conditions of (3.4)–(3.5), we have

$$\begin{aligned} \int_\Omega \bar{p}v\, d\Omega &= k^2 \int_\Omega \bar{p}\bar{y}(v)\, d\Omega + \int_\Omega \bar{p}\Delta\bar{y}(v)\, d\Omega \\ &= -\int_\Omega \big(\Delta\bar{p} + \bar{y}(u)\big)\bar{y}(v)\, d\Omega + \int_\Omega \bar{p}\Delta\bar{y}(v)\, d\Omega \\ &= -\int_\Omega \bar{y}(u)\bar{y}(v)\, d\Omega + \int_\Omega \nabla\bar{y}(v)\cdot\nabla\bar{p}\, d\Omega - \int_\Gamma \bar{y}(v)\frac{\partial\bar{p}}{\partial n}\, d\Gamma \\ &\quad - \int_\Omega \nabla\bar{p}\cdot\nabla\bar{y}(v)\, d\Omega + \int_\Gamma \bar{p}\frac{\partial\bar{y}(v)}{\partial n}\, d\Gamma \\ &= -\int_\Omega \bar{y}(u)\bar{y}(v)\, d\Omega, \end{aligned}$$

$$\big(\pounds(u), v\big)_{L^2(\Omega)} = \gamma \int_\Omega uv\, d\Omega + \int_\Omega \bar{y}(u)\bar{y}(v)\, d\Omega.$$

This proves that $\pounds$ is a symmetric mapping. Furthermore, we have

$$\big(\pounds(u), u\big)_{L^2(\Omega)} = \gamma \int_\Omega uu\, d\Omega + \int_\Omega \bar{y}(u)\bar{y}(u)\, d\Omega \geqslant C\|u\|^2_{L^2(\Omega)},$$

which proves that $\pounds$ is strongly positive. □

Now, if we define $a(\cdot,\cdot): L^2(\Omega) \times L^2(\Omega) \to R$ by $a(u, v) = (\pounds(u), v)_{L^2}$, $\forall u, v \in L^2(\Omega)$, and $L: L^2(\Omega) \to R$ by $L(v) = (\tilde{p}, v)_{L^2}$, $\forall v \in L^2(\Omega)$, then the mapping $a(\cdot,\cdot)$ is linear, continuous, and symmetric. Obviously, the mapping L is also linear and continuous. Thus the optimal control u^* can be computed by the conjugate gradient algorithm using the character of quadratic functional.

An arbitrary function $u^{(0)}$ may be specified as an initial guess for the control input (in a general way, set $u^{(0)}=0$). Firstly, we solve problems

$$\begin{cases} \Delta y^{(0)} + k^2 y^{(0)} = f + u^{(0)}, & \text{in } \Omega, \\ y^{(0)} = 0, & \text{on } \Gamma, \end{cases} \tag{3.8}$$

$$\begin{cases} \Delta p^{(0)} + k^2 p^{(0)} = -(y^{(0)} - y_z), & \text{in } \Omega, \\ p^{(0)} = 0, & \text{on } \Gamma, \end{cases} \tag{3.9}$$

to get the gradient direction $g^{(0)} = \gamma u^{(0)} - p^{(0)}$ at $u^{(0)}$, and simultaneously let the first conjugate direction be the gradient direction, namely $s^{(0)} = g^{(0)}$. Then for $i \geqslant 0$, assuming that the iterative control $u^{(i)}$, the gradient direction $g^{(i)}$, and the conjugate direction $s^{(i)}$ are known, we update $u^{(i)}$ via $u^{(i+1)} = u^{(i)} - \rho_i s^{(i)}$, where ρ_i is the optimum iterative step at $u^{(i)}$ such that

$$\min_{\rho \geqslant 0} J\big(y\big(u^{(i)} - \rho s^{(i)}\big), u^{(i)} - \rho s^{(i)}\big) = J\big(y\big(u^{(i)} - \rho_i s^{(i)}\big), u^{(i)} - \rho_i s^{(i)}\big).$$

Obviously, ρ_i is characterized as $(\gamma(u^{(i)} - \rho_i s^{(i)}) - \bar{p}_i - \tilde{p}, s^{(i)})_{L^2(\Omega)} = 0$, where $\bar{p}_i$ is the solution of

$$\begin{cases} \Delta \bar{p}_i + k^2 \bar{p}_i = -\bar{y}_i, & \text{in } \Omega, \\ \bar{p}_i = 0, & \text{on } \Gamma, \end{cases}$$

and $\bar{y}_i$ is the solution of

$$\begin{cases} \Delta \bar{y}_i + k^2 \bar{y}_i = u^{(i)} - \rho_i s^{(i)}, & \text{in } \Omega, \\ \bar{y}_i = 0, & \text{on } \Gamma. \end{cases}$$

Assume that $y_1^{(i)}$ and $p_1^{(i)}$ are respectively the solutions of

$$\begin{cases} \Delta y_1^{(i)} + k^2 y_1^{(i)} = s^{(i)}, & \text{in } \Omega, \\ y_1^{(i)} = 0, & \text{on } \Gamma, \end{cases} \tag{3.10}$$

$$\begin{cases} \Delta p_1^{(i)} + k^2 p_1^{(i)} = -y_1^{(i)}, & \text{in } \Omega, \\ p_1^{(i)} = 0, & \text{on } \Gamma, \end{cases} \tag{3.11}$$

$y_2^{(i)}$ and $p_2^{(i)}$ are respectively the solutions of

$$\begin{cases} \Delta y_2^{(i)} + k^2 y_2^{(i)} = u^{(i)}, & \text{in } \Omega, \\ y_2^{(i)} = 0, & \text{on } \Gamma, \end{cases}$$

$$\begin{cases} \Delta p_2^{(i)} + k^2 p_2^{(i)} = -y_2^{(i)}, & \text{in } \Omega, \\ p_2^{(i)} = 0, & \text{on } \Gamma, \end{cases}$$

then $\bar{p}_i = p_2^{(i)} - \rho_i p_1^{(i)}$ holds. We can see that ρ_i is characterized as

$$\big(\gamma\big(u^{(i)} - \rho_i s^{(i)}\big) - p_2^{(i)} + \rho_i p_1^{(i)} - \tilde{p}, s^{(i)}\big)_{L^2(\Omega)} = 0.$$

Obviously, $\gamma u^{(i)} - p_2^{(i)} - \tilde{p} = g^{(i)}$ holds. Let $\tilde{g}^{(i)} = \gamma s^{(i)} - p_1^{(i)}$, then we have

$$\rho_i = \frac{(g^{(i)}, s^{(i)})_{L^2(\Omega)}}{(\tilde{g}^{(i)}, s^{(i)})_{L^2(\Omega)}}.$$

It is easy to prove that $g^{(i+1)} = g^{(i)} - \rho_i \tilde{g}^{(i)}$ holds. Because the conjugate coefficient is given by

$$\alpha_i = \frac{\|g^{(i+1)}\|^2_{L^2(\Omega)}}{\|g^{(i)}\|^2_{L^2(\Omega)}},$$

we can update the conjugate direction via $s^{(i+1)} = g^{(i+1)} + \alpha_i s^{(i)}$. Using the result

$$\left(g^{(i)}, s^{(i-1)}\right)_{L^2(\Omega)} = 0,$$

we improve the optimum iterative step ρ_i as follows

$$\rho_i = \frac{(g^{(i)}, s^{(i)})_{L^2(\Omega)}}{(\tilde{g}^{(i)}, s^{(i)})_{L^2(\Omega)}} = \frac{(g^{(i)}, g^{(i)} + \alpha_{i-1} s^{(i-1)})_{L^2(\Omega)}}{(\tilde{g}^{(i)}, s^{(i)})_{L^2(\Omega)}} = \frac{\|g^{(i)}\|^2_{L^2(\Omega)}}{(\tilde{g}^{(i)}, s^{(i)})_{L^2(\Omega)}}.$$

From above, the conjugate gradient algorithm can be given as follows.

Algorithm 1. Conjugate gradient algorithm (CGM):

Step 1. Choose $\varepsilon \geqslant 0$, and the initial control function $u^{(0)}$ is given.
Step 2. Solve (3.8)–(3.9), set $g^{(0)} = \gamma u^{(0)} - p^{(0)}$, $s^{(0)} = g^{(0)}$. Set $i = 0$.
Step 3. Solve (3.10)–(3.11), set $\tilde{g}^{(i)} = \gamma s^{(i)} - p_1^{(i)}$, $\rho_i = \frac{\|g^{(i)}\|^2_{L^2(\Omega)}}{(\tilde{g}^{(i)}, s^{(i)})_{L^2(\Omega)}}$.
Step 4. $u^{(i+1)} = u^{(i)} - \rho_i s^{(i)}$, $g^{(i+1)} = g^{(i)} - \rho_i \tilde{g}^{(i)}$.
Step 5. If $\frac{\|g^{(i+1)}\|^2_{L^2(\Omega)}}{\|g^{(0)}\|^2_{L^2(\Omega)}} \leqslant \varepsilon$, then take $u^* = u^{(i+1)}$; else, go to Step 6.
Step 6. $\alpha_i = \frac{\|g^{(i+1)}\|^2_{L^2(\Omega)}}{\|g^{(i)}\|^2_{L^2(\Omega)}}$.
Step 7. $s^{(i+1)} = g^{(i+1)} + \alpha_i s^{(i)}$, set $i = i + 1$, and go to Step 3.

4. Conjugate gradient-boundary element algorithm

In the following we derive the conjugate gradient-boundary element algorithm (CGM+MRBEM). Firstly, we address the MRBEM formulations for (3.8)–(3.11).

4.1. MRBEM formulations

It is well known that the fundamental solution of Eqs. (3.8)–(3.11) can be expressed in terms of the zero-order Hankel function of the second kind. This complex fundamental solution leads to some inconveniences for numerical algorithm, while these disadvantages of the conventional BEM formulation can be removed by using the MRBEM formulation, which all the computation can be carried out by the real-valued higher-order fundamental solutions.

We define the following sequence of the higher-order fundamental solutions for the problems (3.8)–(3.11) (see [8,13])

$$y_0^*(r) = -\frac{1}{2\pi} \ln r, \tag{4.1}$$

$$y_j^*(r) = -\frac{1}{2\pi}\frac{r^{2j}}{4^j(j!)^2}\left(\ln r - \sum_{l=1}^{j}\frac{1}{l}\right), \quad j = 1, 2, \ldots, \tag{4.2}$$

satisfying the equations

$$\Delta y_{j+1}^*(r) = y_j^*(r), \qquad \Delta y_0^*(r) + \delta(r) = 0.$$

We firstly consider the problem (3.8). The weighted residual formulation for the problem with the weighting function $y_0^*(r)$ yields the integral equation for $x \in \Omega \cup \Gamma$,

$$\begin{aligned} C(x)y^{(0)}(x) = &-\int_\Gamma \left\{ y^{(0)}(\eta)\frac{\partial}{\partial n_\eta}y_0^*(r) - y_0^*(r)\frac{\partial}{\partial n_\eta}y^{(0)}(\eta)\right\} d\Gamma_\eta \\ &+ \int_\Omega \big(f(\xi) + u^{(0)}(\xi)\big) y_0^*(r_\xi)\, d\Omega_\xi + k^2 \int_\Omega y^{(0)}(\xi) y_0^*(r_\xi)\, d\Omega_\xi, \end{aligned} \tag{4.3}$$

where $r_\xi = |\xi - x|$, $\xi \in \Omega$, $r = |\eta - x|$, $\eta \in \Gamma$. The normal vector n_η always points into the exterior of Ω.

$$C(x) = \begin{cases} 1, & x \in \Omega, \\ \frac{\alpha(x)}{2\pi}, & x \in \Gamma, \end{cases}$$

where $\alpha(x)$ denotes the interior angle in $x \in \Gamma$.

After the application of the higher-order fundamental solutions (4.2) repeatedly (the multiple reciprocity method (MRM)), we can obtain the following integral equation

$$\begin{aligned} C(x)y^{(0)}(x) = &-\sum_{j=0}^{n}\left(-k^2\right)^j \int_\Gamma \left\{ y^{(0)}(\eta)\frac{\partial}{\partial n_\eta}y_j^*(r) - y_j^*(r)\frac{\partial}{\partial n_\eta}y^{(0)}(\eta)\right\} d\Gamma_\eta \\ &+ \sum_{j=0}^{n}\left(-k^2\right)^j \int_\Omega \big(f(\xi) + u^{(0)}(\xi)\big) y_j^*(r_\xi)\, d\Omega_\xi + R_n(x), \end{aligned} \tag{4.4}$$

where $R_n(x) = (-1)^n (k^2)^{n+1}\int_\Omega y^{(0)}(\xi)y_n^*(r_\xi)\, d\Omega_\xi$. We note that the domain integral $R_n(x)$ approaches zero as $n \to \infty$, if $k < \infty$, and Ω is the finite region. Consequently, this domain integral becomes negligible for sufficiently large n. Let $y^*(r) = \sum_{j=0}^{n}(-k^2)^j y_j^*(r)$, then, as can be seen in [8,13–15], $y^*(r)$ is not the fundamental solution of the problem (3.8). It is the main part of the real part of the fundamental solution. For sufficiently large n, noting the boundary condition of (3.8), we are allowed to describe the boundary integral equation for $x \in \Gamma$ as

$$-\int_\Omega \big(f(\xi) + u^{(0)}(\xi)\big) y^*(r_\xi)\, d\Omega_\xi = \int_\Gamma y^*(r)\frac{\partial}{\partial n_\eta}y^{(0)}(\eta)\, d\Gamma_\eta. \tag{4.5}$$

For $x \in \Omega$ or $x \in \Gamma$, let $F^{(0)}(x) = -\int_\Omega (f(\xi) + u^{(0)}(\xi))y^*(r_\xi)\, d\Omega_\xi$, and for $\eta \in \Gamma$, let $\sigma^{(0)}(\eta) = \frac{\partial}{\partial n_\eta}y^{(0)}(\eta)$, then we have the boundary integral equation

$$F^{(0)}(x) = \int_\Gamma y^*(r)\sigma^{(0)}(\eta)\, d\Gamma_\eta. \tag{4.6}$$

Similarly, for $x \in \Omega$ or $x \in \Gamma$, let

$$\tilde{F}^{(0)}(x) = \int_{\Omega} \big(y^{(0)}(\xi) - y_z(\xi)\big) y^*(r_\xi)\, d\Omega_\xi,$$

$$F^{(i)}(x) = -\int_{\Omega} s^{(i)}(\xi) y^*(r_\xi)\, d\Omega_\xi,$$

$$\tilde{F}^{(i)}(x) = \int_{\Omega} y_1^{(i)}(\xi) y^*(r_\xi)\, d\Omega_\xi,$$

and for $\eta \in \Gamma$, let $\tilde{\sigma}^{(0)}(\eta) = \frac{\partial}{\partial n_\eta} p^{(0)}(\eta)$, $\sigma^{(i)}(\eta) = \frac{\partial}{\partial n_\eta} y_1^{(i)}(\eta)$, $\tilde{\sigma}^{(i)}(\eta) = \frac{\partial}{\partial n_\eta} p_1^{(i)}(\eta)$, then we can obtain the boundary integral equations of the problems (3.9)–(3.11) as follows

$$\tilde{F}^{(0)}(x) = \int_{\Gamma} y^*(r)\tilde{\sigma}^{(0)}(\eta)\, d\Gamma_\eta, \tag{4.7}$$

$$F^{(i)}(x) = \int_{\Gamma} y^*(r)\sigma^{(i)}(\eta)\, d\Gamma_\eta, \tag{4.8}$$

$$\tilde{F}^{(i)}(x) = \int_{\Gamma} y^*(r)\tilde{\sigma}^{(i)}(\eta)\, d\Gamma_\eta. \tag{4.9}$$

We consider the boundary integral equation (4.6). For $f \in L^2(\Omega)$, $u^{(0)} \in L^2(\Omega)$, an isomorphic mapping from $\sigma^{(0)} \in H^{-\frac{1}{2}}(\Gamma)$ to $F^{(0)} \in H^{\frac{1}{2}}(\Gamma)$ is defined by (4.6).

The boundary integral equation (4.6) is equivalent to the following variational problem: find $\sigma^{(0)} \in H^{-\frac{1}{2}}(\Gamma)$, such that

$$b\big(\sigma^{(0)}, \sigma'\big) = \big\langle F^{(0)}, \sigma'\big\rangle, \quad \forall \sigma' \in H^{-\frac{1}{2}}(\Gamma), \tag{4.10}$$

where

$$b\big(\sigma^{(0)}, \sigma'\big) = \int_{\Gamma}\int_{\Gamma} \sigma^{(0)}(\eta)\sigma'(x) y^*(r)\, d\Gamma_\eta\, d\Gamma_x,$$

$$\big\langle F^{(0)}, \sigma'\big\rangle = \int_{\Gamma} F^{(0)}(x)\sigma'(x)\, d\Gamma_x.$$

It is easy to prove by means of Lax–Milgram theorem that there exists a unique solution $\sigma^{(0)}$ of the variational equation (4.10) in $H^{-\frac{1}{2}}(\Gamma)$ (see [2,3,10]).

For any $x \in \Omega$, $y^{(0)}(x)$ can be expressed in the following integral formulation

$$y^{(0)}(x) = -F^{(0)}(x) + \int_{\Gamma} y^*(r)\sigma^{(0)}(\eta)\, d\Gamma_\eta. \tag{4.11}$$

In a similar way, we can further consider the boundary integral equations (4.7)–(4.9) which are equivalent to the following variational problems, respectively

$$b\big(\tilde{\sigma}^{(0)}, \sigma'\big) = \big\langle \tilde{F}^{(0)}, \sigma'\big\rangle, \quad \forall \sigma' \in H^{-\frac{1}{2}}(\Gamma), \tag{4.12}$$

$$b\big(\sigma^{(i)}, \sigma'\big) = \big\langle F^{(i)}, \sigma'\big\rangle, \quad \forall \sigma' \in H^{-\frac{1}{2}}(\Gamma), \tag{4.13}$$

$$b\big(\tilde{\sigma}^{(i)}, \sigma'\big) = \big\langle \tilde{F}^{(i)}, \sigma'\big\rangle, \quad \forall \sigma' \in H^{-\frac{1}{2}}(\Gamma). \tag{4.14}$$

We can also prove by means of Lax–Milgram theorem the existence of a unique solution to the variational problems (4.12)–(4.14), respectively.

For any $x \in \Omega$, $p^{(0)}(x)$, $y_1^{(i)}(x)$ and $p_1^{(i)}(x)$ can be expressed in the following integral formulations

$$p^{(0)}(x) = -\tilde{F}^{(0)}(x) + \int_\Gamma y^*(r)\tilde{\sigma}^{(0)}(\eta)\,d\Gamma_\eta, \quad x \in \Omega, \tag{4.15}$$

$$y_1^{(i)}(x) = -F^{(i)}(x) + \int_\Gamma y^*(r)\sigma^{(i)}(\eta)\,d\Gamma_\eta, \quad x \in \Omega, \tag{4.16}$$

$$p_1^{(i)}(x) = -\tilde{F}^{(i)}(x) + \int_\Gamma y^*(r)\tilde{\sigma}^{(i)}(\eta)\,d\Gamma_\eta, \quad x \in \Omega. \tag{4.17}$$

4.2. *Conjugate gradient-boundary element algorithm*

We assume, for the sake of simplicity, that Ω is a polygonal domain. We will make comments on a domain with a piecewise smooth boundary at the end of the paper. Let there be given a finite element mesh of the boundary Γ, and Ne nodal points in Γ, $\Gamma = \bigcup \Gamma_h$. Let Σ_h be a finite element mesh of Ω (in a general way, triangle or quadrangle partition), and N nodal points inside of Ω, $\Omega = \bigcup \Omega_h$, where $\Omega_h \in \Sigma_h$, $h = \max\{\mathrm{diam}(\Omega_h), \mathrm{meas}(\Gamma_h)\}$. It is worthwhile to emphasize that it is not necessary to constitute the finite element approximations on the domain Ω for functions $y^{(0)}(x)$, $p^{(0)}(x)$, $y_1^{(i)}(x)$ and $p_1^{(i)}(x)$. By partitioning Ω, we can obtain the numerical integration formula for computing $F^{(0)}(x)$, $\tilde{F}^{(0)}(x)$, $F^{(i)}(x)$ and $\tilde{F}^{(i)}(x)$ (see Section 6 in this paper). The numerical solutions of $y^{(0)}(x)$, $p^{(0)}(x)$, $y_1^{(i)}(x)$ and $p_1^{(i)}(x)$ on the domain Ω are obtained by the boundary integral formulations (4.11) and (4.15)–(4.17). For the sake of simplifying our discussion, we use the same mesh size for discretization approximations of $y^{(0)}(x)$, $p^{(0)}(x)$, $y_1^{(i)}(x)$ and $p_1^{(i)}(x)$, and the same partitioning of the boundary Γ for discretization approximations of $\sigma^{(0)}$, $\tilde{\sigma}^{(0)}$, $\sigma^{(i)}$ and $\tilde{\sigma}^{(i)}$.

We consider the discretization of the variational equations (4.10), (4.12)–(4.14). Let $\Phi_h = \{\sigma_h \in C(\Gamma);\ \sigma_h|_{\Gamma_j} \in P_l(x),\ \Gamma = \bigcup \Gamma_j\}$ be finite-dimensional subspace of $H^{-\frac{1}{2}}(\Gamma)$. Φ_h denotes the set of piecewise polynomials of degree l on the mesh boundary. The approximations of the variational problems (4.10), (4.12)–(4.14) are determined by: find $\sigma_h^{(0)}$, $\tilde{\sigma}_h^{(0)}$, $\sigma_h^{(i)}$ and $\tilde{\sigma}_h^{(i)} \in \Phi_h$, respectively, such that

$$b\big(\sigma_h^{(0)}, \sigma_h'\big) = \big\langle F_h^{(0)}, \sigma_h'\big\rangle, \quad \forall \sigma_h' \in \Phi_h, \tag{4.18}$$

$$b\big(\tilde{\sigma}_h^{(0)}, \sigma_h'\big) = \big\langle \tilde{F}_h^{(0)}, \sigma_h'\big\rangle, \quad \forall \sigma_h' \in \Phi_h, \tag{4.19}$$

$$b\big(\sigma_h^{(i)}, \sigma_h'\big) = \big\langle F_h^{(i)}, \sigma_h'\big\rangle, \quad \forall \sigma_h' \in \Phi_h, \tag{4.20}$$

$$b\big(\tilde{\sigma}_h^{(i)}, \sigma_h'\big) = \big\langle \tilde{F}_h^{(i)}, \sigma_h'\big\rangle, \quad \forall \sigma_h' \in \Phi_h, \tag{4.21}$$

where

$$F_h^{(0)}(x) = -\int_{\Omega_h} \big(f(\xi) + u_h^{(0)}(\xi)\big) y^*(r_\xi)\,d\Omega_{h,\xi},$$

$$\tilde{F}_h^{(0)}(x) = \int_{\Omega_h} \big(y_h^{(0)}(\xi) - y_z(\xi)\big) y^*(r_\xi)\, d\Omega_{h,\xi},$$

$$F_h^{(i)}(x) = -\int_{\Omega_h} s_h^{(i)}(\xi) y^*(r_\xi)\, d\Omega_{h,\xi}, \qquad \tilde{F}_h^{(i)}(x) = \int_{\Omega_h} y_{1,h}^{(i)}(\xi) y^*(r_\xi)\, d\Omega_{h,\xi}.$$

For any $x \in \Omega$, $y_h^{(0)}(x)$, $p_h^{(0)}(x)$, $y_{1,h}^{(i)}(x)$ and $p_{1,h}^{(i)}(x)$ can be expressed, respectively,

$$y_h^{(0)}(x) = -F_h^{(0)}(x) + \int_\Gamma y^*(r)\sigma_h^{(0)}(\eta)\, d\Gamma_\eta, \tag{4.22}$$

$$p_h^{(0)}(x) = -\tilde{F}_h^{(0)}(x) + \int_\Gamma y^*(r)\tilde{\sigma}_h^{(0)}(\eta)\, d\Gamma_\eta, \tag{4.23}$$

$$y_{1,h}^{(i)}(x) = -F_h^{(i)}(x) + \int_\Gamma y^*(r)\sigma_h^{(i)}(\eta)\, d\Gamma_\eta, \tag{4.24}$$

$$p_{1,h}^{(i)}(x) = -\tilde{F}_h^{(i)}(x) + \int_\Gamma y^*(r)\tilde{\sigma}_h^{(i)}(\eta)\, d\Gamma_\eta. \tag{4.25}$$

By the discrete variational problem (4.18), a system of linear equations with boundary unknown $\sigma_h^{(0)}$ can be written as

$$A\sigma_h^{(0)} = b^{(0)}, \tag{4.26}$$

where A is the stiffness matrix which solely depends on the geometry of the boundary Γ, and the vector $\sigma_h^{(0)}$ consists of the discretized values of the boundary unknown $\sigma_h^{(0)}$, and the vector $b^{(0)}$ consists of the discretized values of the known quantity which depends on $F_h^{(0)}$.

In a similar way, by the discrete variational problems (4.19)–(4.21), the system of linear equations with boundary unknowns $\tilde{\sigma}_h^{(0)}$, $\sigma_h^{(i)}$, $\tilde{\sigma}_h^{(i)}$ can be, respectively, written as

$$A\tilde{\sigma}_h^{(0)} = \tilde{b}^{(0)}, \tag{4.27}$$

$$A\sigma_h^{(i)} = b^{(i)}, \tag{4.28}$$

$$A\tilde{\sigma}_h^{(i)} = \tilde{b}^{(i)}. \tag{4.29}$$

Algorithm 2. Conjugate gradient-boundary element algorithm (CGM+MRBEM):

Step 1. $\varepsilon \geqslant 0$ is given, and the initial control vector $u_h^{(0)} = (u_1^{(0)}, u_2^{(0)}, \ldots, u_N^{(0)})^T$ is given, where N is equal to the number of the nodal points inside of Ω. In a general way, we set $u_h^{(0)} = 0$.

Step 2a. Solve the system of linear equations (4.26) to determine the $\sigma_h^{(0)}$. Compute $y_h^{(0)}(x) = (y_1^{(0)}, y_2^{(0)}, \ldots, y_N^{(0)})^T$ by (4.22).

Step 2b. Solve the system of linear equation (4.27) to determine $\tilde{\sigma}_h^{(0)}$. Compute $p_h^{(0)}(x) = (p_1^{(0)}, p_2^{(0)}, \ldots, p_N^{(0)})^T$ by (4.23). Set $g_h^{(0)} = \gamma u_h^{(0)} - p_h^{(0)}$, $s_h^{(0)} = g_h^{(0)}$. Set $i = 0$.

Step 3a. Solve the system of linear equations (4.28) to determine $\sigma_h^{(i)}$. Compute $y_{1,h}^{(i)}(x) = (y_{11}^{(i)}, y_{12}^{(i)}, \ldots, y_{1,N}^{(i)})^T$ by (4.24).

Step 3b. Solve the system of linear equations (4.29) to determine $\tilde{\sigma}_h^{(i)}$. Compute $p_{1,h}^{(i)}(x) = (p_{11}^{(i)}, p_{12}^{(i)}, \dots, p_{1,N}^{(i)})^T$ by (4.25). Set $\tilde{g}_h^{(i)} = \gamma s_h^{(i)} - p_{1,h}^{(i)}$, $\rho_{i,h} = \frac{(g_h^{(i)})^T g_h^{(i)}}{(\tilde{g}_h^{(i)})^T s_h^{(i)}}$.

Step 4. $u_h^{(i+1)} = u_h^{(i)} - \rho_{i,h} s_h^{(i)}$, $g_h^{(i+1)} = g_h^{(i)} - \rho_{i,h} \tilde{g}_h^{(i)}$.

Step 5. If $\frac{(g_h^{(i+1)})^T g_h^{(i+1)}}{(g_h^{(0)})^T g_h^{(0)}} \leqslant \varepsilon$, then take $u_h^* = u_h^{(i+1)}$; else, go to Step 6.

Step 6. $\alpha_{i,h} = \frac{(g_h^{(i+1)})^T g_h^{(i+1)}}{(g_h^{(i)})^T g_h^{(i)}}$.

Step 7. $s_h^{(i+1)} = g_h^{(i+1)} + \alpha_{i,h} s_h^{(i)}$, set $i = i + 1$, and go to Step 3a.

5. Local error estimates

Let the linear operator $\pounds_h$ be finite-dimensional discrete form of $\pounds$, and $\tilde{p}_h$ be the discrete form of $\tilde{p}$, apparently Algorithm 2 is equivalent to solving the discretized linear system $\pounds_h u_h = \tilde{p}_h$ which is derived from the optimality condition (3.1) by means of the MRBEM. We need the following lemmas to derive error estimates.

Lemma 5.1. *(See [5,14].) If the conjugate gradient method* (CGM) *is used to solve the system of linear equations $\pounds_h u_h = \tilde{p}_h$, then the iterate point $u_h^{(i)}$, $i = 1, 2, \dots$, satisfy the error bound*

$$\left\| u_h^{(i)} - u_h \right\|_{\pounds_h} \leqslant C \left(\frac{\sqrt{C_h} - 1}{\sqrt{C_h} + 1} \right)^i \left\| u_h^{(0)} - u_h \right\|_{\pounds_h}, \tag{5.1}$$

where $\|u_h\|_{\pounds_h} = (u_h^T \pounds_h u_h)^{\frac{1}{2}}$ is the norm induced by $\pounds_h$, and the condition number C_h is defined by $\|\pounds_h\| \cdot \|\pounds_h^{-1}\|$.

Because the linear operator $\pounds_h$ is bounded and $\|\pounds_h\| \cdot \|\pounds_h^{-1}\| \geqslant 1$, the numerical results obtained by the CGM+MRBEM is stable, see [12,14]. Hence, we are only to consider the local convergence between Algorithm 1 and Algorithm 2 at every iteration.

In order to investigate the convergence of the proposed CGM+MRBEM algorithm, at every iteration we evaluate the errors defined by energy-error estimator, L^2 error estimator, and L^∞ error estimator, respectively,

$$e_1^{(i+1)} = \left\| u_h^{(i+1)} - u^{(i+1)} \right\|_{H^1(\Omega)}, \tag{5.2}$$

$$e_2^{(i+1)} = \left\| u_h^{(i+1)} - u^{(i+1)} \right\|_{L^2(\Omega)}, \tag{5.3}$$

$$e_3^{(i+1)} = \left\| u_h^{(i+1)} - u^{(i+1)} \right\|_{L^\infty(\Omega)}, \tag{5.4}$$

where $u^{(i+1)}$, $u_h^{(i+1)}$ respectively come from Algorithms 1 and 2 after $i + 1$ iterations.

It is well known that $\|y_0^*(r)\|_{H^{\frac{1}{2}}(\Gamma)}$ is bounded. Because the series $\sum_{j=0}^{\infty} (-k^2)^j y_j^*(r)$ uniformly converges to the real part of the fundamental solution of the state equation (1.2) as long as r and k are bounded, it is clear that for the sequence of the higher-order fundamental solutions $y_j^*(r)$, $j = 1, 2, \dots$, and sufficiently large n, $y^*(r) = y^*(|\eta - x|) = \sum_{j=0}^{n} (-k^2)^j y_j^*(|\eta - x|)$ such that

$$\left\| y^*\left(|\eta - x|\right) \right\|_{H^{\frac{1}{2}}(\Gamma)} \leqslant C. \tag{5.5}$$

For simplicity of notation, assume that $u_h^{(0)} = u^{(0)} = 0$, then we have the conclusions as follows

Lemma 5.2. *Let $\sigma^{(0)}$ and $\sigma_h^{(0)}$ be respectively the solutions of* (4.10) *and* (4.18)*, and $\sigma^{(0)} \in H^{l+1}(\Gamma) \cap H^{-\frac{1}{2}}(\Gamma)$, then we have*:

$$\left\|\sigma^{(0)} - \sigma_h^{(0)}\right\|_{H^{-\frac{1}{2}}(\Gamma)} \leqslant Ch^{l+\frac{3}{2}}\left\|\sigma^{(0)}\right\|_{H^{l+1}(\Gamma)}, \tag{5.6}$$

$$\left\|\sigma^{(0)} - \sigma_h^{(0)}\right\|_{L^2(\Gamma)} \leqslant Ch^{l+1}\left\|\sigma^{(0)}\right\|_{H^{l+1}(\Gamma)}, \tag{5.7}$$

$$\left\|\sigma^{(0)} - \sigma_h^{(0)}\right\|_{L^\infty(\Gamma)} \leqslant Ch^{l+\frac{3}{2}}\left\|\sigma^{(0)}\right\|_{H^{l+1}(\Gamma)}, \tag{5.8}$$

where C is positive constant, independent of h.

Proof. We first estimate the error $\|\sigma^{(0)} - \sigma_h^{(0)}\|_{H^{-\frac{1}{2}}(\Gamma)}$. It is easy to prove that the bilinear form $b(\sigma^{(0)}, \sigma')$ on $H^{-\frac{1}{2}}(\Gamma)$ is bounded, continuous and coercive. We define the orthographic projection operator $\chi_h : H^{-\frac{1}{2}}(\Gamma) \to \Phi_h$, and the interpolation operator Π_h from $C(\Gamma) \subset H^{-\frac{1}{2}}(\Gamma) \to \Phi_h$, respectively, where $C(\Gamma)$ is a smooth subspace of $H^{-\frac{1}{2}}(\Gamma)$, then by the projective theorem, we have (see [1,2])

$$\begin{aligned} \inf_{\sigma_h^{(0)} \in \Phi_h} \left\|\sigma^{(0)} - \sigma_h^{(0)}\right\|_{H^{-\frac{1}{2}}(\Gamma)} &= \left\|\sigma^{(0)} - \chi_h \sigma^{(0)}\right\|_{H^{-\frac{1}{2}}(\Gamma)} \\ &\leqslant \left\|\sigma^{(0)} - \Pi_h \sigma^{(0)}\right\|_{H^{-\frac{1}{2}}(\Gamma)}. \end{aligned} \tag{5.9}$$

Using the boundedness, continuity and coerciveness of the bilinear form $b(\sigma^{(0)}, \sigma')$ on $H^{-\frac{1}{2}}(\Gamma)$ and the interpolation approximation theorem, we can conclude the following inequality:

$$\left\|\sigma^{(0)} - \sigma_h^{(0)}\right\|_{H^{-\frac{1}{2}}(\Gamma)} \leqslant C \inf_{\sigma_h^{(0)} \in \Phi_h} \left\|\sigma^{(0)} - \sigma_h^{(0)}\right\|_{H^{-\frac{1}{2}}(\Gamma)}.$$

We can also arrive at

$$\left\|\sigma^{(0)} - \sigma_h^{(0)}\right\|_{H^{-\frac{1}{2}}(\Gamma)} \leqslant C\left\|\sigma^{(0)} - \Pi_h \sigma^{(0)}\right\|_{H^{-\frac{1}{2}}(\Gamma)} \leqslant Ch^{l+\frac{3}{2}}\left\|\sigma^{(0)}\right\|_{H^{l+1}(\Gamma)}.$$

Similarly, we can conclude the asserted estimates (5.7) and (5.8). □

Theorem 5.1. *For any $x \in \Omega$, let $y^{(0)}(x)$ be given by* (4.11)*, and $y_h^{(0)}(x)$ be given by* (4.22)*, then we have*:

$$\left\|y^{(0)}(x) - y_h^{(0)}(x)\right\|_{H^1(\Omega)} \leqslant Ch^{l+\frac{3}{2}}\left\|\sigma^{(0)}\right\|_{H^{l+1}(\Gamma)}, \tag{5.10}$$

$$\left\|y^{(0)}(x) - y_h^{(0)}(x)\right\|_{L^2(\Omega)} \leqslant Ch^{l+1}\left\|\sigma^{(0)}\right\|_{H^{l+1}(\Gamma)}, \tag{5.11}$$

$$\left\|y^{(0)}(x) - y_h^{(0)}(x)\right\|_{L^\infty(\Omega)} \leqslant Ch^{l+\frac{3}{2}}\left\|\sigma^{(0)}\right\|_{H^{l+1}(\Gamma)}. \tag{5.12}$$

Proof. By the assumption $u_h^{(0)} = u^{(0)} = 0$, and by noting Lemma 5.2, we get the error estimate as follows

$$\begin{aligned}\left\|y^{(0)}(x)-y_h^{(0)}(x)\right\|_{H^1(\Omega)} &\leqslant \left\|y^*\big(|\eta-x|\big)\right\|_{H^{\frac{1}{2}}(\Gamma)}\left\|\sigma^{(0)}-\sigma_h^{(0)}\right\|_{H^{-\frac{1}{2}}(\Gamma)}\\ &\leqslant C\left\|\sigma^{(0)}-\sigma_h^{(0)}\right\|_{H^{-\frac{1}{2}}(\Gamma)} \leqslant Ch^{l+\frac{3}{2}}\left\|\sigma^{(0)}\right\|_{H^{l+1}(\Gamma)}.\end{aligned}$$

Similarly, we can conclude the asserted estimates (5.11)–(5.12). □

Theorem 5.2. *Let $u^{(i+1)}$ be the control function after $i+1$ iterations in Algorithm* 1, *and $u_h^{(i+1)}$ be the discrete control expressed by means of* (4.24) *and* (4.25) *in Algorithm* 2, *assume that $u^{(i)}=u_h^{(i)}$, $g^{(i)}=g_h^{(i)}$ and $s^{(i)}=s_h^{(i)}$ hold at the nodal points inside of Ω after i iterations in Algorithms* 1 *and* 2, *and the energy-error estimates $\|u^{(i)}-u_h^{(i)}\|_{H^1(\Omega)}=0$, $\|g^{(i)}-g_h^{(i)}\|_{H^1(\Omega)}=0$, and $\|s^{(i)}-s_h^{(i)}\|_{H^1(\Omega)}=0$ hold yet, then we obtain the local energy-error estimate as follows*

$$\left\|u^{(i+1)}-u_h^{(i+1)}\right\|_{H^1(\Omega)} \leqslant Ch^{l+\frac{3}{2}}\left\{\left\|\sigma^{(i)}\right\|_{H^{l+1}(\Gamma)}+\left\|\tilde\sigma^{(i)}\right\|_{H^{l+1}(\Gamma)}\right\}. \tag{5.13}$$

Proof. By Step 2 in Algorithm 1 and Step 2b in Algorithm 2, similar to Lemma 5.2 and Theorem 5.1, we can conclude the error estimates

$$\left\|\tilde\sigma^{(0)}-\tilde\sigma_h^{(0)}\right\|_{H^{-\frac{1}{2}}(\Gamma)} \leqslant C\left\{\left\|y^{(0)}-y_h^{(0)}\right\|_{H^1(\Omega)}+h^{l+\frac{3}{2}}\left\|\tilde\sigma^{(0)}\right\|_{H^{l+1}(\Gamma)}\right\}, \tag{5.14}$$

$$\left\|p^{(0)}-p_h^{(0)}\right\|_{H^1(\Omega)} \leqslant C\left\{\left\|y^{(0)}-y_h^{(0)}\right\|_{H^1(\Omega)}+h^{l+\frac{3}{2}}\left\|\tilde\sigma^{(0)}\right\|_{H^{l+1}(\Gamma)}\right\}. \tag{5.15}$$

If $i\geqslant 0$, then

$$\left\|y_1^{(i)}-y_{1,h}^{(i)}\right\|_{H^1(\Omega)} \leqslant C\left\{\left\|s^{(i)}-s_h^{(i)}\right\|_{H^1(\Omega)}+h^{l+\frac{3}{2}}\left\|\sigma^{(i)}\right\|_{H^{l+1}(\Gamma)}\right\}, \tag{5.16}$$

$$\begin{aligned}\left\|p_1^{(i)}-p_{1,h}^{(i)}\right\|_{H^1(\Omega)} &\leqslant C\left\{\left\|y_1^{(i)}-y_{1,h}^{(i)}\right\|_{H^1(\Omega)}+h^{l+\frac{3}{2}}\left\|\tilde\sigma^{(i)}\right\|_{H^{l+1}(\Gamma)}\right\}\\ &\leqslant C\left\{\left\|s^{(i)}-s_h^{(i)}\right\|_{H^1(\Omega)}+h^{l+\frac{3}{2}}\left(\left\|\sigma^{(i)}\right\|_{H^{l+1}(\Gamma)}+\left\|\tilde\sigma^{(i)}\right\|_{H^{l+1}(\Gamma)}\right)\right\}.\end{aligned} \tag{5.17}$$

Because the gradient operator $\nabla J(w(u),u)$ is linear, continuous, symmetrical, and strongly positive, there exists a constant C such that

$$\left\|u^{(i+1)}-u_h^{(i+1)}\right\|_{H^1(\Omega)} \leqslant C\left\|g^{(i+1)}-g_h^{(i+1)}\right\|_{H^1(\Omega)}. \tag{5.18}$$

Because the optimum iterative steps ρ_i, $\rho_{i,h}$ are bounded, we have

$$\left\|g^{(i+1)}-g_h^{(i+1)}\right\|_{H^1(\Omega)} \leqslant \left\|g^{(i)}-g_h^{(i)}\right\|_{H^1(\Omega)}+\max\{\rho_i,\rho_{i,h}\}\left\|\tilde g^{(i)}-\tilde g_h^{(i)}\right\|_{H^1(\Omega)}, \tag{5.19}$$

$$\left\|\tilde g^{(i)}-\tilde g_h^{(i)}\right\|_{H^1(\Omega)} \leqslant \gamma\left\|s^{(i)}-s_h^{(i)}\right\|_{H^1(\Omega)}+\left\|p_1^{(i)}-p_{1,h}^{(i)}\right\|_{H^1(\Omega)}. \tag{5.20}$$

We can conclude the asserted estimate (5.13) by noting the assumption conditions of this theorem. □

Similarly, we can obtain the local L^2 error estimator and L^∞ error estimator as follows

$$\left\|u^{(i+1)}-u_h^{(i+1)}\right\|_{L^2(\Omega)} \leqslant Ch^{l+1}\left(\left\|\sigma^{(i)}\right\|_{H^{l+1}(\Gamma)}+\left\|\tilde\sigma^{(i)}\right\|_{H^{l+1}(\Gamma)}\right), \tag{5.21}$$

$$\left\|u^{(i+1)}-u_h^{(i+1)}\right\|_{L^\infty(\Omega)} \leqslant Ch^{l+\frac{3}{2}}\left(\left\|\sigma^{(i)}\right\|_{H^{l+1}(\Gamma)}+\left\|\tilde\sigma^{(i)}\right\|_{H^{l+1}(\Gamma)}\right). \tag{5.22}$$

6. Numerical experiment

It is well known that in practice the optimal control problem (1.1)–(1.3) can rarely be solved analytically. But in order to present the performance of CGM+MRBEM algorithm proposed, we designedly solve an example with analytical solution in the domain $\Omega = (0,1) \times (0,1)$, where $k = 1$, $\gamma = 1$. In this example we have

$$f(x_1,x_2) = \left(1 - 2\pi^2\right) Z_1(x_1,x_2) - Z_2(x_1,x_2),$$
$$y_z(x_1,x_2) = \left(1 - 8\pi^2\right) Z_2(x_1,x_2) + Z_1(x_1,x_2),$$

where $Z_1(x_1,x_2) = \sin \pi x_1 \sin \pi x_2$, $Z_2(x_1,x_2) = \sin 2\pi x_1 \sin 2\pi x_2$. Then there exist the exact solutions $y^* = \sin \pi x_1 \sin \pi x_2$ and $u^* = \sin 2\pi x_1 \sin 2\pi x_2$ to the optimal control problem (1.1)–(1.3). In Figs. 1 and 2, the exact solutions u^* and w^* are plotted by Matlab 6.0.

The space discretization step Δx defined by $\Delta x = \frac{1}{N}$, where N is a positive integer ($(N-1)^2 =$ number of the nodal points in Ω). There are $4N$ nodal points on Γ. Then, for every $m_1, m_2 \in \{1,2,\ldots,N\}$, we take the quadrilateral Ω_h with nodal points $x_{m_1 m_2} = (m_1 \Delta x, m_2 \Delta x)$, $m_1, m_2 = 1,2,\ldots,N$.

For simplicity we assume that the finite element approximations for the unknown boundary functions $\sigma^{(0)}$, $\tilde{\sigma}^{(0)}$, $\sigma^{(i)}$, $\tilde{\sigma}^{(i)}$ are made up of the piecewise constant functions. We com-

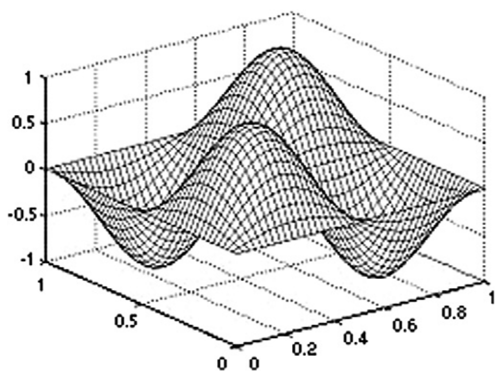

Fig. 1. The exact solution u^*.

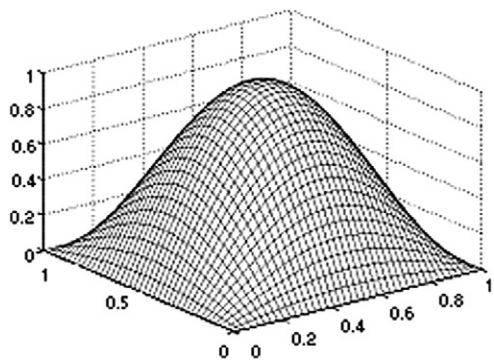

Fig. 2. The exact solution y^*.

pute some domain integrals of known functions using the numerical integration formula according to the discrete L_h^2-scalar product. For example, let us compute the domain integral $\tilde{F}_h^{(i)}(x) = \int_{\Omega_h} y_{1,h}^{(i)}(\xi) y^*(r_\xi)\, d\Omega_{h,\xi}$, where $y_{1,h}^{(i)}(\xi)$ is known vector after i iterations (see Step 3a of Algorithm 2), then we have

$$\tilde{F}_h^{(i)}(x) = \int\limits_{\Omega_h} y_{1,h}^{(i)}(\xi) y^*(r_\xi)\, d\Omega_{h,\xi} = (\Delta x)^2 \sum_{m_1,m_2=1}^{N-1} y_{1,h}^{(i)}(x_{m_1 m_2}) y^*\bigl(|x - x_{m_1 m_2}|\bigr).$$

From above, we can derive the system of linear equations (4.26)–(4.29). Let $N = 10$, Algorithm 2 is carried out by Matlab 6.0 (here we set $\varepsilon = 10^{-4}$, and set 20 terms in the expansion series of $y^*(r) = \sum_{j=0}^{n} (-k^2)^j y_j^*(r)$). We can obtain the numerical solution for the control function u. In Table 1, we present the comparison of the numerical solution by Algorithm 2 with the analytical solution for u^* on the part of the domain Ω (namely $(0, 0.5] \times (0, 1)$) due to the symmetry of the control function in Ω (the numerical results obtained in our algorithm are also symmetrical). For terseness of depiction, in Table 1 the error $u^* - u_h^*$ is denoted by e_h^* and the nodal point $(0.1m_1, 0.1m_2)$ is denoted by $x_{m_1 m_2}$ (for instance, the nodal point $(0.2, 0.3)$ is denoted by x_{23}).

We visualize the graph of the numerical solutions of the control function u_h^* and the graph of the errors $u^* - u_h^*$ at the nodal points on the domain Ω in Fig. 3 and Fig. 4, respectively. It is observed that the maximum errors are distributed along the boundary Γ, especially nearby the four vertices.

Let $N = 20$, Algorithm 2 is also carried out. We obtain the numerical solution for u. We can also visualize the graph of the numerical solutions of the control function u_h^* and the graph of

Table 1
Analytical solution, numerical solution and error for u, with $N = 10$

	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	x_{17}	x_{18}	x_{19}
u^*	0.3454	0.5590	0.5590	0.3454	0	−0.3454	−0.5590	−0.5590	−0.3454
u_h^*	0.4555	0.6379	0.6005	0.3501	0	−0.3501	−0.6005	−0.6379	−0.4566
e_h^*	−0.1101	−0.0789	−0.0415	−0.0047	0	0.0047	0.0415	0.0789	0.1112
	x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	x_{26}	x_{27}	x_{28}	x_{29}
u^*	0.5590	0.9045	0.9045	0.5590	0	−0.5590	−0.9045	−0.9045	−0.5590
u_h^*	0.6489	0.9505	0.9200	0.5580	0	−0.5580	−0.9210	−0.9511	−0.6493
e_h^*	−0.0899	−0.0460	−0.0155	0.0010	0	−0.0010	0.0165	0.0466	0.0903
	x_{31}	x_{32}	x_{33}	x_{34}	x_{35}	x_{36}	x_{37}	x_{38}	x_{39}
u^*	0.5590	0.9045	0.9045	0.5590	0	−0.5590	−0.9045	−0.9045	−0.5590
u_h^*	0.6159	0.9370	0.9231	0.5590	0	−0.5591	−0.9234	−0.9381	−0.6163
e_h^*	−0.0569	−0.0325	−0.0186	0	0	0.0001	0.0189	0.0336	0.0573
	x_{41}	x_{42}	x_{43}	x_{44}	x_{45}	x_{46}	x_{47}	x_{48}	x_{49}
u^*	0.3454	0.5590	0.5590	0.3454	0	−0.3454	−0.5590	−0.5590	−0.3454
u_h^*	0.3600	0.5610	0.5599	0.3455	0	−0.3457	−0.5600	−0.5627	−0.3616
e_h^*	−0.0146	−0.0020	−0.0009	−0.0001	0	0.0003	0.001	0.0037	0.0162
	x_{51}	x_{52}	x_{53}	x_{54}	x_{55}	x_{56}	x_{57}	x_{58}	x_{59}
u^*	0	0	0	0	0	0	0	0	0
u_h^*	0	0	0	0	0	0	0	0	0
e_h^*	0	0	0	0	0	0	0	0	0

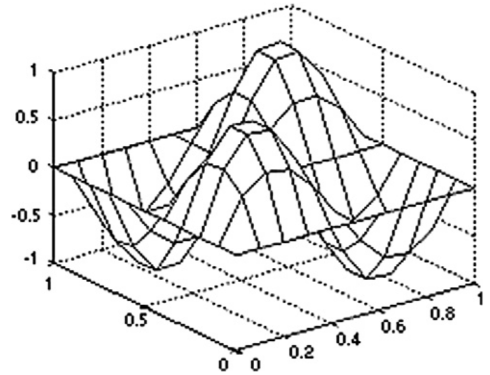

Fig. 3. The numerical solutions u_h^*, with $N = 10$.

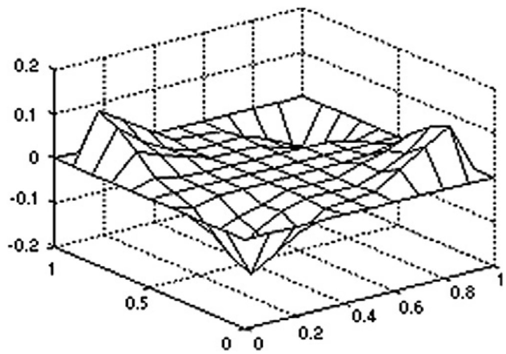

Fig. 4. The error $u^* - u_h^*$, with $N = 10$.

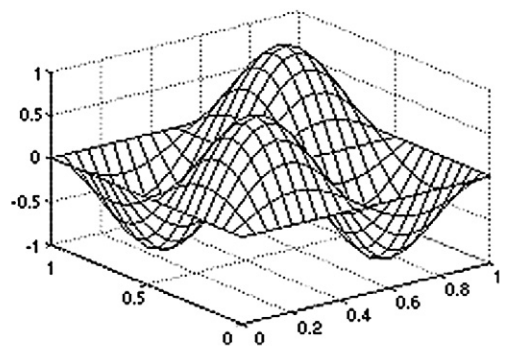

Fig. 5. The numerical solutions u_h^*, with $N = 20$.

the errors $u^* - u_h^*$ in Figs. 5 and 6, respectively. We also found that the maximum errors are distributed along the boundary Γ, especially nearby the four vertices.

If we set $N = 40$, then the numerical solution fully approaches the exact solution, and the error $|u^* - u_h^*|$ is less than 10^{-4}. From Figs. 4 and 6 it can be seen that the errors $|u^* - u_h^*|$ at nodal points keep decreasing gradually and the proposed CGM+MRBEM algorithm produces

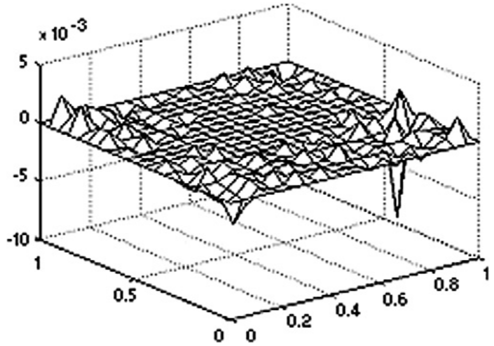

Fig. 6. The error $u^* - u_h^*$, with $N = 20$.

an accurate and convergent numerical solution with the increase in the number of boundary elements.

Furthermore, as shown in Figs. 4 and 6, we can find that the rate of convergence for the numerical solutions at the nodal points away from the boundary Γ is superior to that of the nodal points nearby the boundary Γ, especially superior to that of the four vertices of the square. This is due to the boundary integral equation with singular integral kernel from the higher-order fundamental solution y_0^* and the discontinuity of the normal derivative of the state function at the four vertices. In order to overcome this drawback, we can employ the h–p version of the boundary element method to improve our algorithm by simultaneously reducing the mesh size nearby the boundary Γ (especially that of the four vertices of the square) and by increasing the polynomial degrees of the finite element toward the vertices. Here we do not discuss this method in detail because it is not our main motive in this paper.

In the previous sections we assume that Ω is a polygonal domain. If Ω is a domain with a piecewise smooth boundary, we have to approximate the geometry shape of the boundary Γ using boundary element (for instance, linear element or quadratic element). Meanwhile, the errors due to approximating boundary have to be considered to add to the error estimates (5.13), (5.21) and (5.22). However, the proposed CGM+MRBEM algorithm remains valid.

7. Conclusions

In this paper, we have proposed the CGM+MRBEM algorithm for the optimal control (1.1)–(1.3). The system of optimality equations consisting of state and costate function is derived, which constitutes the necessary and sufficient optimality condition for (1.1)–(1.3). Introducing the sequence of the higher-order fundamental solutions, we have formulated the boundary integral equations and the boundary variational equations for relevant boundary value problems, and derived the integral representations of the solution for them in Ω. The CGM+MRBEM algorithm for solving the system of optimality equations is developed. The numerical results obtained in the numerical example are consistent with what we can expect from CGM+MRBEM algorithm, more precisely:

(1) The CGM+MRBEM algorithm requires the constitution of the finite element approximations for boundary functions only, which actualizes the reduced dimensionality due to the boundary integral formulation.

(2) The same stiffness matrix A in the linear equations (4.26)–(4.29) solely depends on the geometry of the boundary Γ in iterative process, which is of some convenience to ours algorithm and can save substantial computational work. Moreover, in iterative process, the other matrixes associated with our algorithm for computing the numerical solution (4.22)–(4.25) make no difference, because they solely depend on the geometry of the domain Ω. Hence these matrixes are simply computed once in iterative process.

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