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Analysis of a Model for Imaging in Photolithography

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The governing equation of a model for imaging in photolithography is studied. The density ρ of the photoactive component of the resist, which is a function of time and position, decreases at a rate assumed proportional to the local light intensity. It satisfies a nonlinear differential equation

$$\frac{d\rho}{dt}(t) = F(\rho(t)), \qquad (*)$$

of which an evaluation of the right-hand side requires solving Maxwell's equations in a periodic 2D-configuration of dielectrics consisting of the resist and the substrate. The electric permittivity of the resist is a function of position which depends on ρ. The Maxwell problem is studied by applying the limiting absorption principle. It is proved using the contraction mapping theorem that for every exposure time and every initial density $(*)$ has a unique solution which is a smooth function of time and position when all data are smooth. © 1989 Academic Press, Inc.

1. INTRODUCTION

The most common pattern forming technique in the fabrication of integrated circuits is photolithography. In this technique monocromatic UV-light is used to image a pattern of apertures in a mask into a light sensitive film called photoresist. The light transmitted by the mask and the optical system induces a chemical bleaching of the resist. The rate at which the density ρ of the photoactive component (PAC) decreases is assumed proportional to the local light intensity. Furthermore, the electric permittivity of the resist and hence also the light intensity depend on ρ. The pattern of equiconcentration surfaces (curves in 2D) of the PAC after an exposure is commonly referred to as the latent image of the mask.

In this paper we study the governing equation of a mathematical model for latent image formation in which the light intensity is calculated using Maxwell's equations. We consider a periodic 2D-configuration of a substrate consisting of a number of electrically homogeneous and time-independent layers $\Omega_2, ..., \Omega_l$ with the photosensitive and in general

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inhomogeneous resist Ω_{pr} on top. The region Ω_1 above the resist contains the light source, the optical system and the 1-dimensional periodic mask. A Cartesian coordinate system (x_1, x_2) is chosen such that the configuration is periodic with respect to x_1 with period 1. In Fig. 1 one period of the configuration is shown. In the following $\Omega_1, ..., \Omega_l$ and Ω_{pr} will always be the intersection of the layers defined above and the region $(0, 1) \times \mathbb{R}$. The interfaces of the layers are in general not flat and they need not be representable as functions of x_1 as in Fig. 1. Furthermore, unless stated otherwise, no smoothness of the interfaces is assumed.

For simplicity we assume that the light source is coherent and that it emits polarized light. Then, the light transmitted by the mask and the optical system can be discribed by a single time-harmonic electromagnetic field which we shall refer to as the incident or incoming field because it is incident on the resist. This field should be calculated separately and is considered to be given in this paper. The total field in the region below the optical system and, in particular, in the resist is calculated as if the light

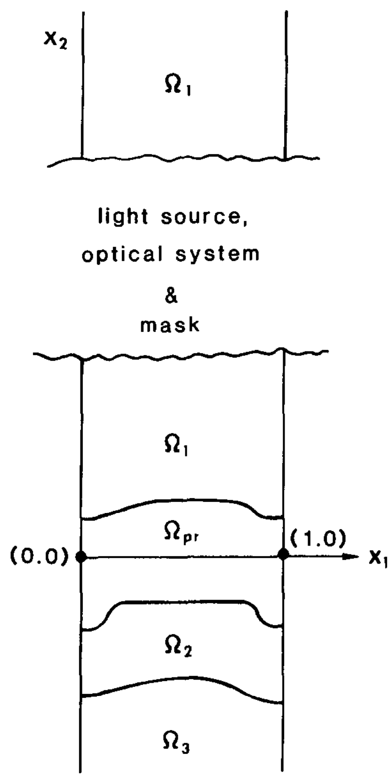

FIG. 1. One period of a periodic configuration with $l = 3$.

source, the optical system, and the mask are absent. Therefore, from now on Ω_1 will be considered to consist entirely of air.

Let the exposure begin at $t=0$ and let $\varepsilon(t)$: $(0, 1) \times \mathbb{R} \to \mathbb{C}$ be the relative electric permittivity at time t with real and imaginary parts $\varepsilon'(t)$ and $\varepsilon''(t)$, respectively. Then

$$\left.\begin{array}{ll} \text{(i)} & \text{for } j=1, ..., l\text{: } \varepsilon(t)|_{\Omega_j} = \varepsilon_j = \varepsilon_j' + i\varepsilon_j'', \text{ where } \varepsilon_j' \text{ and } \varepsilon_j'' \\ & \text{are constants and } \varepsilon_1 = 1, \\ \text{(ii)} & \varepsilon(t)|_{\Omega_{pr}} = \varepsilon_{pr}(t), \text{ where } \varepsilon_{pr}'(t) \text{ and } \varepsilon_{pr}''(t) \text{ are bounded} \\ & \text{measurable functions on } \Omega_{pr} \text{ with positive infimum.} \end{array}\right\} \quad (1.1)$$

The magnetic permeability of all regions is equal to the value in vacuum μ_0. For simplicity it is assumed that the substrate consists of dielectrics, but the presence of conducting layers requires only minor modifications of the analysis.

Let $x = (x_1, x_2)$ be a point of $(0, 1) \times \mathbb{R}$ and let $\mathbf{E}^i(t, x) = \mathrm{Re}[\mathscr{E}^i(x) e^{-i\omega t}]$ be the given incoming electric field. $\mathscr{E}^i$ is assumed to be a 1-periodic function with respect to x_1. This assumption is justified when the region $(0, 1) \times \mathbb{R}$ is close to the optical axis and when the light source is a point source on that axis. In case also $\varepsilon(t)$ is 1-periodic, the same will hold for the total electric field which we shall denote by $\mathbf{E}(t, x) = \mathrm{Re}[\mathscr{E}(t, x) e^{-i\omega t}]$.

To a good approximation the local rate of decrease of e.m. energy per unit volume of the resist is

$$\tfrac{1}{2}\omega\varepsilon_0\varepsilon_{pr}''(t, x)\,|\mathscr{E}(t, x)|^2, \quad (1.2)$$

where $|\cdot|$ denotes the norm on $\mathbb{C}^3$. Let $\rho(t, x)$ and $\tilde{\rho}(t, x)$ be the densities of the PAC and of the reaction product of the bleaching, respectively. Then

$$\rho(t, x) + \tilde{\rho}(t, x) = \rho(0, x) + \tilde{\rho}(0, x), \qquad \forall t \geqslant 0, \forall x \in \Omega_{pr}, \quad (1.3)$$

and we assume that the right-hand side has a positive infimum on Ω_{pr}. We adopt the following general relation between the electric permittivity of the resist and ρ,

$$\varepsilon_{pr}(t, x) = h'(x, \rho(t, x)) + ih''(x, \rho(t, x)), \qquad \forall t \geqslant 0, \forall x \in \Omega_{pr}, \quad (1.4)$$

where h', h'': $\Omega_{pr} \times (0, \infty)$, are given 1-periodic functions with respect to x_1 and are such that for every $s \geqslant 0$ there exists $r \geqslant 1$ with

$$\frac{1}{r} \leqslant h'(x, \rho) \leqslant r, \qquad \frac{1}{r} \leqslant h''(x, \rho) \leqslant r, \qquad \forall (x, \rho) \in \Omega_{pr} \times [0, s]. \quad (1.5)$$

We shall be more specific about the dependence of ε_{pr}'' on ρ. We assume that for some pair of strictly increasing fucntions g, $\tilde{g}$: $[0, \infty) \to [0, \infty)$ with $g(0) = \tilde{g}(0) = 0$ and for some $Q \geq 0$,

$$\varepsilon''_{pr}(t, x) = Q + g(\rho(t, x)) + \tilde{g}(\tilde{\rho}(t, x)). \tag{1.6}$$

Q represents the contribution to ε''_{pr} of the constituents of the resist other than the PAC and the reaction product. Using (1.2) and (1.6) it follows that ε''_{pr} can indeed be expressed in the form $h''(x, \rho(t, x))$ as stated in (1.4). Note that when the resist is homogeneous at $t = 0$, h'' does not depend explicitly on x.

Now the local rate of decrease of ρ is assumed to be proportional to $g(\rho(t, x))/\varepsilon''_{pr}(t, x)$ times the local rate of decrease of e.m. energy as given by (1.2). Hence, apart from a constant positive factor

$$\frac{\partial \rho}{\partial t}(t, x) = -g(\rho(t, x))\, |\mathscr{E}(t, x)|^2, \qquad \forall t \geq 0, \forall x \in \Omega_{pr}, \tag{1.7}$$

where the total electric field $\mathscr{E}(t, x)$ is at every $t \geq 0$, the solution of

$$\omega^2 \mu_0 \varepsilon_0 \varepsilon(t, x)\, \mathscr{E}(t, x) - \operatorname{curl} \operatorname{curl} \mathscr{E}(t, x) = 0 \qquad \text{on} \quad (0, 1) \times \mathbb{R}, \tag{1.8}$$

where

$$\left.\begin{array}{l} \mathscr{E}(t, x) \text{ is 1-periodic with respect to } x_1, \\ \mathscr{E}(t, x) - \mathscr{E}^i(t, x) \text{ satisfies the outgoing radiation condition} \\ \text{for } x_2 \to +\infty, \\ \mathscr{E}(t, x) \text{ vanishes or satisfies the outgoing radiation condi-} \\ \text{tion for } x_2 \to -\infty \text{ depending on whether } \Omega_l \text{ is lossy or} \\ \text{lossless, respectively,} \end{array}\right\} \tag{1.9}$$

and where $\varepsilon(t)$: $(0, 1) \times \mathbb{R} \to \mathbb{C}$ satisfies (1.1) for given $\varepsilon_j \in \mathbb{C}$, $j = 1, ..., l$, and $\varepsilon_{pr}(t)$: $\Omega_{pr} \to \mathbb{C}$ is given by (1.4).

The outgoing radiation condition will be formulated in Section 3. Furthermore, we remark that (1.8) is derived from Maxwell's equations using the quasi-static approximation.

Equations (1.4), (1.6), and (1.7) contain as a special case the well-known relations used in Dill's model [2].

It will be proved in Section 4 that when $\varepsilon(t)$: $(0, 1) \times \mathbb{R} \to \mathbb{C}$ satisfying (1.1) is specified, the momentary total field $\mathscr{E}(t)$ is uniquely determined by (1.8), (1.9). Since the ε_j, $j = 1, ..., l$, are known fixed constants, it follows from (1.4) that $\mathscr{E}(t)$ can be considered to be determined by $\rho(t)$:

$\Omega_{pr} \to (0, \infty)$. We shall therefore write $\mathscr{E}_{\rho(t)}$. Then it follows that (1.7) is a nonlinear differential equation for $\rho(t)$,

$$\frac{d\rho}{dt}(t) = F(\rho(t)), \qquad t \geq 0, \tag{1.10}$$

of which an evaluation of the right-hand side requires solving boundary value problem (1.8), (1.9). For every t, (1.10) is an equation in some space of functions: $\Omega_{pr} \to \mathbb{R}$.

The following additional assumptions on the smoothness of g, h', and h'' will be used:

(i) g: $[0, \infty) \to [0, \infty)$ is twice differentiable,

(ii) h', h'': $\Omega_{pr} \times [0, \infty)$ are continuous in (x, ρ) and continuously differentiable with respect to $\rho \in [0, \infty)$ with derivatives which are uniformly bounded with respect to $x \in \Omega_{pr}$.

Assumption (i) and $g(0) = 0$ imply that

$$G(\rho) \equiv \frac{g(\rho)}{\rho} \text{ has a continuous derivative on } [0, \infty). \tag{1.11}$$

Let ρ_0: $\Omega_{pr} \to (0, \infty)$ be the density of the PAC at the beginning of the exposure:

$$\rho(0, x) = \rho_0(x), \qquad \forall x \in \Omega_{pr}. \tag{1.12}$$

Then, using the definition of G, (1.7) and (1.12) imply

$$\rho(t, x) = \rho_0(x) \exp\left[-\int_0^t G(\rho(s, x)) \, |\mathscr{E}_{\rho(s)}(x)|^2 \, ds \right]. \tag{1.13}$$

We shall prove that when the incident field is TE-polarized, i.e., $\mathscr{E}^i$ is everywhere orthogonal to the (x_1, x_2)-plane, then for every exposure time $t_e > 0$ and every continuous ρ_0: $\Omega_{pr} \to (0, \infty)$ which is 1-periodic with respect to x_1, there exists a unique 1-periodic ρ which is a solution of (1.10), (1.12). Furthermore, when the interfaces between the resist and the adjacent layers are smooth, the functions g, h', h'', and ρ_0 are smooth; then ρ is a smooth function of both time and position.

The existence proof for initial value problem (1.10), (1.12) requires a thorough analysis of boundary value problem (1.8), (1.9), in particular with regard to the qualitative dependence of the electric field on the electric permitivity of the resist for the study of which we apply the limiting absorption method. This method is studied in a general, abstract context in [7]. In [8] Wilcox studies scattering theory for a periodic geometry similar to ours but consisting of homogeneous materials. The contradiction type argument which we use in order to obtain estimates for the field goes back to Eidus [4]. Although the existence proof for (1.10), (1.12) given below is valid only for the case of TE-polarized fields, the study of the boundary value problem is carried out for the general case.

Finally, some remarks about the assumptions for the incoming field are given. Because any real light source is extended, the effect of partial incoherence can not be neglected in general. The correct approach is to divide the light source into point sources and to solve at each time t boundary value problem (1.8), (1.9) for all incoming fields corresponding to all point sources. Instead of $|\mathscr{E}(t)|^2$, one should then use on the right-hand side of (1.7)

$$\int_{\text{source}} |\mathscr{E}^s(t)|^2 \, ds, \tag{1.14}$$

where $\mathscr{E}^s$ is the field in the resist due to point source s. For point sources which are not on the optical axis the incoming fields are quasi-periodic rather than periodic. This means that the boundary conditions for these fields differ from those in (1.9). However, the analysis required for this more general case is essentially the same as that for (1.8), (1.9). Furthermore the existence proof for ρ can be easily generalized to the case of integration over point sources as that in (1.14). We shall therefore retain the assumptions stated above and limit the analysis to the case of a single incoming field which is 1-periodic.

When the light used is unpolarized, the intensity of the light emitted by point source s is:

$$\tfrac{1}{2}\{|\mathscr{E}^s_{\text{TE}}|^2 + |\mathscr{E}^s_{\text{TM}}|^2\},$$

where $\mathscr{E}^s_{\text{TE}}$ and $\mathscr{E}^s_{\text{TM}}$ are the total fields corresponding to a TE- and TM-polarized incoming field, respectively. However, as mentioned above, the existence proof for ρ given below applies only to the TE-component.

Results of numerical simulations for the case of partial incoherent TE-polarized incoming fields are described in [6].

2. Notations

We shall use the Cartesian coordinate system (x_1, x_2, x_3) where the x_1- and x_2-axis are as in Fig. 1. $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the corresponding orthonormal basis. For $\mathbf{f}, \mathbf{g} \in \mathbb{C}^3$ we write

$$\mathbf{f} = \sum_{j=1}^{3} f_j \mathbf{e}_j, \quad \mathbf{g} = \sum_{j=1}^{3} g_j \mathbf{e}_j,$$

$$\mathbf{f} \cdot \mathbf{g} = \sum_{j=1}^{3} f_j g_j, \qquad \text{and} \qquad |\mathbf{f}|^2 = \sum_{j=1}^{2} f_j \bar{f}_j,$$

where the bar denotes complex conjugation. Furthermore, $\wedge$ will be the vector product and x will be a point $(x_1, x_2) \in (0, 1) \in \mathbb{R}$.

We shall next explain the notations used for some Sobolev spaces (see, e.g., [1, 3, 5]).

For open $\Omega \subset (0, 1) \times \mathbb{R}$ and $m \in \mathbb{N}$, $H^m(\Omega)$ is the space of functions u: $\Omega \to \mathbb{C}$ having derivatives of order $\leqq m$ which are in $L^2(\Omega)$. The norms of $L^2(\Omega)$ and $H^m(\Omega)$ are denoted by $\| \ \|_0^\Omega$ and $\| \ \|_m^\Omega$, respectively.

Bold type letters are used for spaces of vector fields $\mathbf{u}$: $\Omega \to \mathbb{C}^3$. For example, $\mathbf{H}^m(\Omega) = H^m(\Omega) \times H^m(\Omega) \times H^m(\Omega)$. For $\mathbf{u} = (u_1, u_2, u_3) \in \mathbf{H}^m(\Omega)$ we define $\|\mathbf{u}\|_m^\Omega = \{\sum_{j=1}^3 (\|u_j\|_m^\Omega)^2\}^{1/2}$ as the norm on $\mathbf{H}^m(\Omega)$.

For $-\infty \leqq a < b \leqq \infty$ we introduce

$$\mathbf{H}^1(\text{curl}; (0, 1) \times (a, b)) = \{\mathbf{u} \in \mathbf{L}^2((0, 1) \times (a, b)); \\ \text{curl}\, \mathbf{u} \in \mathbf{L}^2((0, 1) \times (a, b))\}$$

equipped with the norm

$$\|\mathbf{u}\|_{1,\text{curl}}^{(0,1)\times(a,b)} = \left\{\int_a^b \int_0^1 (|\mathbf{u}|^2 + |\text{curl}\, \mathbf{u}|^2)\, dx_1\, dx_2\right\}^{1/2};$$

$$\mathbf{H}^2(\text{curl}; (0, 1) \times (a, b)) = \{\mathbf{u} \in \mathbf{L}^2((0, 1) \times (a, b)); \\ \text{curl}\,\text{curl}\, \mathbf{u} \in \mathbf{L}^2((0, 1) \times (a, b))\}$$

with the norm

$$\|\mathbf{u}\|_{2,\text{curl}}^{(0,1)\times(a,b)} = \left\{\int_a^b \int_0^1 (|\mathbf{u}|^2 + |\text{curl}\,\text{curl}\, \mathbf{u}|^2)\, dx_1\, dx_2\right\}^{1/2}.$$

Let S^1 be the unit circle in the complex plane and let Φ: $[0, 1] \times \mathbb{R} \to S^1 \times \mathbb{R}$ be the map

$$\Phi(x_1, x_2) = (e^{2\pi i x_1}, x_2).$$

When $-\infty < a < b < \infty$ then $\mathscr{C}^\infty(S^1 \times [a, b])$ and $\mathscr{C}^\infty(S^1 \times \mathbb{R})$ are the spaces of all infinitely differentiable φ: $[0, 1] \times [a, b] \to \mathbb{C}$, respectively, φ: $[0, 1] \times \mathbb{R} \to \mathbb{C}$, such that φ and all its derivatives are 1-periodic with respect to x_1. The notation $\mathscr{C}^\infty(S^1 \times [a, b])$ is motivated by the fact that, using the map Φ, the space of smooth periodic functions $[0, 1] \times [a, b] \to \mathbb{C}$ can be mapped 1–1 onto the space of smooth functions defined on the cylinder $S^1 \times [a, b]$. For $-\infty \leqq a < b \leqq \infty$, $\mathscr{D}(S^1 \times (a, b))$ is the space of all $\varphi \in \mathscr{C}^\infty(S^1 \times [a. b])$ of which the support is a compact subset of $[0, 1] \times (a, b)$.

For $-\infty < a < b < \infty$, $H^m(S^1 \times (a, b))$ is the closure of $\mathscr{C}^\infty(S^1 \times [a, b])$ in $H^m((0, 1) \times (a, b))$ and $\mathbf{H}^m(S^1 \times (a, b))$, $\mathbf{H}^1(\text{curl}; S^1 \times (a, b))$, and

$\mathbf{H}^2(\text{curl}; S^1 \times (a, b))$ are the closures of $\mathscr{C}^\infty(S^1 \times [a, b]) \equiv \prod_{j=1}^3 \mathscr{C}^\infty(S^1 \times [a, b])$ in $\mathbf{H}^m((0, 1) \times (a, b))$, $\mathbf{H}^1(\text{curl}; (0, 1) \times (a, b))$, and $\mathbf{H}^2(\text{curl}; (0, 1) \times (a, b))$, respectively. Furthermore, $H^m(S^1 \times \mathbb{R})$ is the closure of $\mathscr{D}(S^1 \times \mathbb{R})$ in $H^m((0, 1) \times \mathbb{R})$ and $\mathbf{H}^m(S^1 \times \mathbb{R})$, $\mathbf{H}^1(\text{curl}; S^1 \times \mathbb{R})$, and $\mathbf{H}^2(\text{curl}; S^1 \times \mathbb{R})$ are the closures of $\mathscr{D}(S^1 \times \mathbb{R}) = \prod_{j=1}^3 \mathscr{D}(S^1 \times \mathbb{R})$ in $\mathbf{H}^m((0, 1) \times \mathbb{R})$, $\mathbf{H}^1(\text{curl}; (0, 1) \times \mathbb{R})$, and $\mathbf{H}^2(\text{curl}; (0, 1) \times \mathbb{R})$, respectively. It should be remarked that when $a = -\infty$ and $b = +\infty$ we have $\mathbf{H}^2(\text{curl}; S^1 \times (a, b)) \subset \mathbf{H}^1(\text{curl}; S^1 \times (a, b))$, but that when $a > -\infty$ or $b < \infty$ this inclusion is false.

There holds in particular

$$\int_{-\infty}^{\infty} \int_0^1 \text{curl curl } \mathbf{u} \cdot \bar{\mathbf{v}} \, dx_1 \, dx_2$$

$$= \int_{-\infty}^{\infty} \int_0^1 \text{curl } \mathbf{u} \cdot \text{curl } \bar{\mathbf{v}} \, dx_1 \, dx_2,$$

$$\forall \mathbf{u} \in \mathbf{H}^2(\text{curl}; S^1 \times \mathbb{R}), \forall \mathbf{v} \in \mathbf{H}^1(\text{curl}; S^1 \times \mathbb{R}).$$

All Sobolev spaces introduced above are Hilbert spaces. When $a = -\infty$ and $b = \infty$ we shall write $\| \ \|_m$, $\| \ \|_{1,\text{curl}}$, and $\| \ \|_{2,\text{curl}}$ for the norms defined above. For $-\infty \leqq a < b \leqq \infty$,

$$H^m_{\text{loc}}(S^1 \times (a, b)) \equiv \{u \in \mathscr{D}'((0, 1) \times (a, b)); \forall c, d \in \mathbb{R} \text{ with } a < c < d < b:$$
$$u|_{(0,1) \times (c,d)} \in H^m(S^1 \times (c, d))\},$$

where $\mathscr{D}'((0, 1) \times (a, b))$ is the space of distributions on $(0, 1) \times (a, b)$. $H^m_{\text{loc}}(S^1 \times (a, b))$ is equipped with the Fréchet topology generated by the seminorms:

$$\|u\|_m^{(0,1) \times (c,d)}, \qquad c, d \in \mathbb{R} \text{ with } a < c < d < b.$$

The space $\mathbf{H}^m_{\text{loc}}(S^1 \times (a, b))$ is defined analogously.

For $m = 1, 2$ and $-\infty \leqq a < b \leqq \infty$ we put

$$\mathbf{H}^m_{\text{loc}}(\text{curl}; S^1 \times (a, b)) = \{\mathbf{u} \in \mathscr{D}'((0, 1) \times (a, b)); \forall c, d \in \mathbb{R} \text{ with} a < c < d < b:$$
$$\mathbf{u}|_{(0,1) \times (c,d)} \in \mathbf{H}^m(\text{curl}; S^1 \times (c, d))\}$$

equipped with the Fréchet topology generated by

$$\|\mathbf{u}\|_{m,\text{curl}}^{(0,1) \times (c,d)}, \qquad c, d \in \mathbb{R}, a < c < d < b.$$

Let X and Y be Fréchet spaces with $\{p_i\}$ and $\{q_j\}$ as fundamental systems of seminorms. Then $B(X, Y)$ denotes the space of all continuous linear maps $L: X \to Y$ equipped with the Fréchet topology generated by the seminorms:

$$r_{ij}(L) \equiv \sup_{\substack{x \in X \\ x \neq 0}} \frac{q_j(L(x))}{p_i(x)}.$$

L will be called an isomorphism when L is a linear and topological isomorphism.

3. Preliminary Lemmas

In the present and the next section we shall study boundary value problem (1.8), (1.9). In the sequel ε will always be a function ε: $[0, 1] \times \mathbb{R} \to C$ which is 1-periodic with respect to x_1 and satisfies

$$\left.\begin{array}{ll} \text{(i)} & \varepsilon|_{\Omega_j} = \varepsilon_j = \varepsilon_j' + i\varepsilon_j'' \text{ for } j = 1, ..., l, \text{ where the } \varepsilon_j' \text{ are positive constants and the } \varepsilon_j'' \text{ are nonnegative constants with, in particular, } \varepsilon_1 = 1. \\ \text{(ii)} & \varepsilon|_{\Omega_{pr}} = \varepsilon_{pr} = \varepsilon_{pr}' + i\varepsilon_{pr}'', \text{ where } \varepsilon_{pr}', \varepsilon_{pr}'': \Omega_{pr} \to (0, \infty) \text{ are bounded measurable functions which satisfy} \\ & \operatorname*{essinf}_{\Omega_{pr}} \varepsilon_{pr}' > 0 \quad \text{and} \quad \operatorname*{essinf}_{\Omega_{pr}} \varepsilon_{pr}'' > 0. \end{array}\right\} \quad (3.1)$$

For the study of problem (1.10), (1.12) it is important to consider the influence of a perturbation of the electric permittivity on the field which is the corresponding solution of (1.8), (1.9). Since only ε_{pr}' and ε_{pr}'' are time dependent, only ε_{pr}' and ε_{pr}'' will be perturbed whereas the ε_j are considered fixed throughout the rest of this paper.

For general $\alpha \in L^\infty((0, 1) \times \mathbb{R})$ we define operator A_α: $\mathbf{H}^2(\text{curl}; S^1 \times \mathbb{R}) \to \mathbf{L}^2((0, 1) \times \mathbb{R})$ by

$$A_\alpha \mathbf{u} = \alpha \mathbf{u} - \text{curl curl } \mathbf{u}.$$

A_α is clearly continuous. Furthermore, it is easy to see:

Lemma 3.1. *When* A_α: $\mathbf{H}^2(\text{curl}; S^1 \times \mathbb{R}) \to \mathbf{L}^2((0, 1) \times \mathbb{R})$ *is* 1–1, *then the image of* $\mathbf{H}^2(\text{curl}; S^1 \times \mathbb{R})$ *is dense in* $\mathbf{L}^2((0, 1) \times \mathbb{R})$.

We shall next formulate a sufficient condition for α in order that A_α is an isomorphism. Put $\alpha' = \text{Re}\, \alpha$ and $\alpha'' = \text{Im}\, \alpha$. We have for every $\mathbf{u} \in \mathbf{H}^2(\text{curl}; S^1 \times \mathbb{R})$

$$\left|\int_{-\infty}^{\infty}\int_0^1 \bar{\mathbf{u}}\cdot A_\alpha(u)\,dx_1\,dx_2\right|^2 = \left|\int_{-\infty}^{\infty}\int_0^1 \{\alpha\,|\mathbf{u}|^2 - |\mathrm{curl}\,\mathbf{u}|^2\}\,dx_1\,dx_2\right|^2$$
$$= \left(\iint |\mathrm{curl}\,\mathbf{u}|^2\right)^2 - 2\left(\iint \alpha'\,|\mathbf{u}|^2\right)$$
$$\cdot\left(\iint |\mathrm{curl}\,\mathbf{u}|^2\right) + \left(\iint \alpha'\,|\mathbf{u}|^2\right)^2$$
$$+\left(\iint \alpha''\,|\mathbf{u}|^2\right)^2$$
$$= \xi^2 - 2a\xi + a^2 + b^2, \tag{3.2}$$

where we have written $\xi = \iint |\mathrm{curl}\,\mathbf{u}|^2$, $a = \iint \alpha'\,|\mathbf{u}|^2$, and $b = \iint \alpha''\,|\mathbf{u}|^2$. Furthermore, we put $\eta = \iint |\mathbf{u}|^2$.

Suppose $b > 0$. Then for every λ satisfying $0 \leqq \lambda \leqq b^2/((\eta + a)^2 + b^2)$

$$\xi^2 - 2a\xi + a^2 + b^2 \geqq \lambda(\xi + \eta)^2. \tag{3.3}$$

Now, suppose $L \equiv \mathrm{essinf}_{(0,1)\times\mathbb{R}}\,\alpha'' > 0$. Then

$$\frac{b^2}{(\eta + a)^2 + b^2} \geqq \frac{L^2}{(1 + \|\alpha'\|_\infty)^2 + \|\alpha''\|_\infty^2} \geqq \frac{1}{2}\left(\frac{L}{1 + \|\alpha\|_\infty}\right)^2, \tag{3.4}$$

Hence, when we choose λ equal to the right-hand side of (3.4), inequality (3.3) is satisfied.

We conclude therefore that when $L > 0$ and $b > 0$,

$$\left|\int_{-\infty}^{\infty}\int_0^1 \bar{\mathbf{u}}\cdot A_\alpha\,\mathbf{u}\right| \geqq \frac{1}{\sqrt{2}}\,\frac{L}{1 + \|\alpha\|_\infty}\,(\|\mathbf{u}\|_{1,\mathrm{curl}})^2. \tag{3.5}$$

But $b = 0$ and $L > 0$ imply $\mathbf{u} = 0$ and then (3.5) is evidently also true; hence, (3.5) holds for all $\mathbf{u} \in \mathbf{H}^2(\mathrm{curl}; S^1 \times \mathbb{R})$. Using (3.5) and $\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} = \alpha\mathbf{u} - A_\alpha(\mathbf{u})$ it follows, furthermore, that

$$\|\mathbf{u}\|_{2,\mathrm{curl}} \leqq \left\{1 + \frac{\sqrt{2}\,(1 + \|\alpha\|_\infty)^2}{\mathrm{essinf}_{(0,1)\times\mathbb{R}}\,\alpha''}\right\}\|A_\alpha(\mathbf{u})\|_0, \qquad \forall \mathbf{u} \in \mathbf{H}^2(\mathrm{curl}; S^1 \times \mathbb{R}). \tag{3.6}$$

This estimate and Lemma 3.1 imply:

LEMMA 3.2. *When* $\alpha \in L^\infty((0,1)\times\mathbb{R})$ *satisfies* $\mathrm{essinf}_{(0,1)\times\mathbb{R}}\,\alpha'' > 0$, *then* A_α: $\mathbf{H}^2(\mathrm{curl}; S^1 \times \mathbb{R}) \to \mathbf{L}^2((0,1)\times\mathbb{R})$ *is an isomorphism.*

Because $\varepsilon_1'' = \varepsilon''|_{\Omega_1} = 0$, ε does not satisfy the hypothesis for α in Lemma 3.2, and we will show that A_ε is 1–1 but not onto.

We shall first formulate the outgoing and incoming radiation condition. Let $\mathbf{v}(t, x) = \mathbf{u}(x)e^{-i\omega t}$ be a vector field which is 1-periodic with respect to x_1, of which the amplitude $\mathbf{u} \in \mathbf{L}^2_{\text{loc}}((0, 1) \times (b, \infty))$ and which satisfies the wave equation $\lambda(\partial^2 v/\partial t^2) - \Delta \mathbf{v} = 0$ on $(0, 1) \times (b, \infty)$, where $b \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > 0$ and $\operatorname{Im} \lambda \geqq 0$. Then $\mathbf{u}$ satisfies

$$\omega^2 \lambda u + \Delta \mathbf{u} = 0 \qquad \text{on} \quad (0, 1) \times (b, \infty). \tag{3.7}$$

By substituting a Fourier series $\mathbf{u}(x_1, x_2) = \sum_{n=-\infty}^{\infty} \hat{\mathbf{u}}(n, x_2) e^{2\pi i n x_1}$ it follows that for some $\mathbf{a}_n, \mathbf{b}_n \in \mathbb{C}^3$,

$$\left.\begin{aligned} \hat{\mathbf{u}}(n, x_2) &= \mathbf{a}_n e^{-ik_n x_2} + \mathbf{b}_n e^{ik_n x_2}, && \text{when} \quad n^2 \neq \frac{\omega^2 \lambda}{4\pi^2}, \\ \hat{\mathbf{u}}(n, x_2) &= \mathbf{a}_n x_2 + \mathbf{b}_n, && \text{when} \quad n^2 = \frac{\omega^2 \lambda}{4\pi^2}, \end{aligned}\right\} \tag{3.8}$$

where

$$k_n = (\omega^2 \lambda - 4\pi^2 n^2)^{1/2}. \tag{3.9}$$

Here and henceforth, the branch of the complex square root is used for which the cut is along the negative real axis, $\xi^{1/2} > 0$ and $(-\xi)^{1/2} = +i\xi^{1/2}$ for $\xi > 0$.

Now, let $\operatorname{Im} \lambda = 0$. Then $k_n > 0$ for $n^2 < \omega^2 \lambda / 4\pi^2$ and $k_n = i\,|k_n|$ when $n^2 \geqq \omega^2 \lambda / 4\pi^2$. We require that the vectors $\hat{\mathbf{u}}(n, x_2)$ are bounded for $x_2 \to \infty$. Then, $\mathbf{a}_n = 0$ when $n^2 \geqq \omega^2 \lambda / 4\pi^2$. Furthermore we define $\mathbf{v}(t, x) = \mathbf{u}(x) e^{-i\omega t}$ as an *outgoing wave* for $x_2 \to +\infty$ when for some $\mathbf{c}_n \in \mathbb{C}^3$ and for $(x_1, x_2) \in (0, 1) \times (b, \infty)$

$$\mathbf{u}(x_1, x_2) = \sum_{n=-\infty}^{\infty} \mathbf{c}_n e^{2\pi i n x_1 + i k_n x_2}, \tag{3.10}$$

and the amplitude $\mathbf{u}$ for which (3.10) applies is said to satisfy the *outgoing radiation condition* (orc) for $x_2 \to +\infty$. $\mathbf{v}(t, x)$ is called an *incoming wave* for $x_2 \to +\infty$ and $\mathbf{u}$ is said to satisfy the *incoming radiation condition* (irc) for $x_2 \to +\infty$ when for some $\mathbf{c}_n \in \mathbb{C}^3$ and for $(x_1, x_2) \in (0, 1) \times (b, \infty)$

$$\mathbf{u}(x_1, x_2) = \sum_{n^2 \leqslant \omega^2 \lambda / 4\pi^2} \mathbf{c}_n e^{2\pi i n x_1 - i k_n x_2} + \sum_{n^2 > \omega^2 \lambda / 4\pi^2} \mathbf{c}_n e^{2\pi i n x_1 + i k_n x_2}. \tag{3.11}$$

Next, consider the case $\operatorname{Im} \lambda > 0$ which means that $(0, 1) \times (b, \infty)$ is lossy. In this case we have $\operatorname{Im} k_n > 0$ for all n, and since we require that the vectors $\hat{\mathbf{u}}(n, x_2)$ are bounded for $x_2 \to +\infty$, it follows that $\mathbf{u}$ must satisfy (3.10). For brevity we shall again say that $\mathbf{u}$ satisfies the orc for $x_2 \to +\infty$

when (3.10) holds, although in case $\operatorname{Im}\lambda > 0$, $\mathbf{u}$ decreases exponentially when $x_2 \to \infty$.

Analogously, when $\mathbf{v}(t, x) = \mathbf{u}(x)e^{-i\omega t}$ satisfies the wave equation $\lambda(\partial^2\mathbf{v}/\partial t^2) - \Delta\mathbf{v} = 0$ on $(0, 1)\times(-\infty, -b)$, then $\mathbf{u}$ is said to satisfy the orc for $x_2 \to -\infty$ when (3.10) applies with x_2 replaced by $-x_2$. This terminology will again also be used when $\operatorname{Im}\lambda > 0$. In case $\operatorname{Im}\lambda = 0$, $\mathbf{u}$ is said to satisfy the irc for $x_2 \to -\infty$ when (3.11) holds with x_2 replaced by $-x_2$.

Now let $b > 0$ be a number sufficiently large in order that

$$(0, 1)\times(b, \infty)\subset\Omega_1 \quad \text{and} \quad (0, 1)\times(-\infty, -b)\subset\Omega_l. \tag{3.12}$$

Let $\mathbf{u}\in\mathbf{H}^2_{\mathrm{loc}}(\mathrm{curl};\ S^1\times\mathbb{R})$ satisfy $A_\varepsilon\mathbf{u} = 0$ on $(0, 1)\times(b, \infty)\cup(0, 1)\times(-\infty, -b)$. Then

$$\varepsilon_1\mathbf{u} - \operatorname{curl}\operatorname{curl}\mathbf{u} = 0 \quad \text{on} \quad (0, 1)\times(b, \infty),$$

$$\varepsilon_l\mathbf{u} - \operatorname{curl}\operatorname{curl}\mathbf{u} = 0 \quad \text{on} \quad (0, 1)\times(-\infty, -b).$$

Since ε_1 and ε_l are constants, $\operatorname{div}\mathbf{u} = 0$ on $(0, 1)\times(b, \infty)\cup(0, 1)\times(-\infty, -b)$; thus, using the identity, $\operatorname{curl}\operatorname{curl}\mathbf{w} = -\Delta\mathbf{w} + \operatorname{grad}\operatorname{div}\mathbf{w}$, it follows that

$$\left.\begin{aligned} \varepsilon_1\mathbf{u} + \Delta\mathbf{u} = 0 \quad &\text{on} \quad (0, 1)\times(b, \infty),\\ \varepsilon_l\mathbf{u} + \Delta\mathbf{u} = 0 \quad &\text{on} \quad (0, 1)\times(-\infty, -b).\end{aligned}\right\} \tag{3.13}$$

For the case where $\mathbf{u}$ satisfies the orc for $x_2 \to \pm\infty$ we shall derive two useful formulae. According to the definition of the orc, there exist $\mathbf{c}^+(n)$, $\mathbf{c}^-(n)\in\mathbb{C}^3$ such that

$$\mathbf{u}(x_1, x_2) = \sum_{n=-\infty}^{\infty} \mathbf{c}^+(n)e^{2\pi i n x_1 + i k_n^+ x_2}, \qquad \forall(x_1, x_2)\in(0, 1)\times(b, \infty), \tag{3.14}$$

$$\mathbf{u}(x_1, x_2) = \sum_{n=-\infty}^{\infty} \mathbf{c}^-(n)e^{2\pi i n x_1 - i k_n^- x_2}, \qquad (x_1, x_2)\in(0, 1)\times(-\infty, -b), \tag{3.15}$$

where

$$k_n^+ = (\varepsilon_1 - 4\pi^2 n^2)^{1/2}, \qquad k_n^- = (\varepsilon_l - 4\pi^2 n^2)^{1/2}.$$

Now, (3.13) implies in particular that u is smooth on $\{(x_1, x_2);\ x_1\in[0, 1], |x_2| > b\}$. Because

$$\operatorname{curl}\mathbf{u} = \frac{\partial u_3}{\partial x_2}\mathbf{e}_1 - \frac{\partial u_3}{\partial x_1}\mathbf{e}_2 + \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\mathbf{e}_3,$$

we have

$$(\mathbf{e}_2 \wedge \operatorname{curl} \mathbf{u}) \cdot \bar{\mathbf{u}} = \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \bar{u}_1 - \frac{\partial u_3}{\partial x_2} \bar{u}_3.$$

Hence, for $|x_2| > b$,

$$\begin{aligned} &\int_0^1 (\mathbf{e}_2 \wedge \operatorname{curl} \mathbf{u}(x_1, x_2)) \cdot \overline{\mathbf{u}(x_1, x_2)} \, dx_1 \\ &\quad = \int_0^1 \left(-\bar{u}_1 \frac{\partial u_1}{\partial x_2} + \bar{u}_1 \frac{\partial u_2}{\partial x_1} - \bar{u}_3 \frac{\partial u_3}{\partial x_2} \right) dx_1 \\ &\quad = \int_0^1 \left(-\bar{u}_1 \frac{\partial u_1}{\partial x_2} - u_2 \frac{\partial \bar{u}_1}{\partial x_1} - \bar{u}_3 \frac{\partial u_3}{\partial x_2} \right) dx_1 \\ &\quad = \int_0^1 \left(-\bar{u}_1 \frac{\partial u_1}{\partial x_2} + u_2 \frac{\partial \bar{u}_2}{\partial x_2} - \bar{u}_3 \frac{\partial u_3}{\partial x_2} \right) dx_1, \end{aligned} \tag{3.16}$$

where we used the periodicity of $\mathbf{u}$ and $\operatorname{div} \mathbf{u} = \partial u_1/\partial x_1 + \partial u_2/\partial x_2 = 0$ for $|x_2| > b$. By substitution of (3.14) into (3.16) we obtain

$$\begin{aligned} &\int_0^1 (\mathbf{e}_2 \wedge \operatorname{curl} \mathbf{u}(x_1, x_2)) \cdot \overline{\mathbf{u}(x_1, x_2)} \, dx_1 \\ &\quad = -i \sum_{n=-\infty}^{\infty} \{ k_n^+ \, |c_1^+(n)|^2 + \overline{k_n^+} \, |c_2^+(n)|^2 + k_n^+ \, |c_3^+(n)|^2 \} \\ &\qquad \times e^{2 \operatorname{Re}(i k_n^+ x_2)}, \qquad \forall x_2 > b. \end{aligned} \tag{3.17}$$

By substituting (3.15) into (3.16) we find

$$\begin{aligned} &\int_0^1 (\mathbf{e}_2 \wedge \operatorname{curl} \mathbf{u}(x_1, x_2)) \cdot \overline{\mathbf{u}(x_1, x_2)} \, dx_1 \\ &\quad = +i \sum_{n=-\infty}^{\infty} \{ k_n^- \, |c_1^-(n)|^2 + \overline{k_n^-} \, |c_2^-(n)|^2 + k_n^- \, |c_3^-(n)|^2 \} \\ &\qquad \times e^{-2 \operatorname{Re}(i k_n^- x_2)}, \qquad \forall x_2 < -b. \end{aligned} \tag{3.18}$$

LEMMA 3.3. *Let* $u \in \mathbf{H}^2_{\mathrm{loc}}(\operatorname{curl}; S^1 \times \mathbb{R})$ *satisfy* $A_\varepsilon \mathbf{u} = 0$ *on* $(0, 1) \times \mathbb{R}$. *Suppose* $\mathbf{u} = 0$ *on* Ω_{pr}; *then* $\mathbf{u} = 0$ *on* $(0, 1) \times \mathbb{R}$.

Proof. We have

$$\varepsilon \mathbf{u} - \operatorname{curl} \operatorname{curl} \mathbf{u} = 0 \qquad \text{on} \quad (0, 1) \times \mathbb{R}.$$

Let Ω_j be a layer adjacent to Ω_{pr}. Since $\varepsilon_j = \varepsilon|_{\Omega_j}$ and $\mathbf{u}$ vanishes on Ω_{pr} it follows that

$$\varepsilon_j \mathbf{u} - \operatorname{curl}\operatorname{curl}\mathbf{u} = 0 \qquad \text{on} \quad \Omega_{pr} \cup \Omega_j.$$

Hence, since ε_j is constant, $\operatorname{div}\mathbf{u} = 0$ on $\Omega_{pr} \cup \Omega_j$ and thus

$$\varepsilon_j \mathbf{u} + \Delta\mathbf{u} = 0 \qquad \text{on} \quad \Omega_{pr} \cup \Omega_j.$$

This implies that $\mathbf{u}$ is analytic on $\Omega_{pr} \cup \Omega_j$, hence, $\mathbf{u} = 0$ on $\Omega_{pr} \cup \Omega_j$. By repeating this argument until all regions are dealt with it follows that $\mathbf{u} = 0$ on $(0, 1) \times \mathbb{R}$.

LEMMA 3.4. *Let* $\mathbf{u} \in \mathbf{H}^2_{\mathrm{loc}}(\operatorname{curl}; S^1 \times \mathbb{R})$ *satisfy* $A_\varepsilon \mathbf{u} = 0$ *on* $(0, 1) \times \mathbb{R}$ *and the orc for* $x_2 \to \pm\infty$. *Then* $\mathbf{u} = 0$ *on* $(0, 1) \times \mathbb{R}$.

Proof. Let $b > 0$ satisfy (3.12) and let $a > b$. We have

$$\begin{aligned}\int_{-a}^{a} \int_0^1 \{-|\operatorname{curl}\mathbf{u}|^2 + \operatorname{curl}\operatorname{curl}\mathbf{u}\cdot\bar{\mathbf{u}}\}\, dx_1\, dx_2 \\ = \int_0^1 (\mathbf{e}_2 \wedge \operatorname{curl}\mathbf{u}(x_1, a))\cdot\overline{\mathbf{u}(x_1, a)}\, dx_1 \\ - \int_0^1 (\mathbf{e}_2 \wedge \operatorname{curl}\mathbf{u}(x_1, -a))\cdot\overline{\mathbf{u}(x_1, -a)}\, dx_1. \qquad (3.19)\end{aligned}$$

By substituting $\operatorname{curl}\operatorname{curl}\mathbf{u} = \varepsilon\mathbf{u}$ into the left-hand side of (3.19) and the series (3.17), (3.18) into the right-hand side, one obtains

$$\begin{aligned}\int_{-a}^{a} \int_0^1 \{-|\operatorname{curl}\mathbf{u}|^2 + \varepsilon\,|\mathbf{u}|^2\}\, dx_1\, dx_2 \\ = -i \sum_{n=-\infty}^{\infty} \{k_n^+\, |c_1^+(n)|^2 + \overline{k_n^+}\, |c_2^+(n)|^2 + k_n^+\, |c_3^+(n)|^2\} e^{2\operatorname{Re}[ik_n^+ a]} \\ -i \sum_{n=-\infty}^{\infty} \{k_n^-\, |c_1^-(n)|^2 + \overline{k_n^-}\, |c_2^-(n)|^2 + k_n^-\, |c_3^-(n)|^2\} e^{-2\operatorname{Re}[ik_n^- a]}.\end{aligned} \qquad (3.20)$$

Because $\varepsilon'' \geqq 0$ and $\operatorname{Re} k_n^+ \geqq 0$, $\operatorname{Re} k_n^- \geqq 0$ for every n, the imaginary part of the left-hand side of (3.20) is nonnegative whereas the imaginary part of the right-hand side is nonpositive. Hence $\varepsilon''\, |\mathbf{u}|^2 = 0$ on $(0, 1) \times \mathbb{R}$ which in view of $\operatorname{essinf}_{\Omega_{pr}} \varepsilon'' > 0$ yields

$$\mathbf{u} = 0 \qquad \text{on} \quad \Omega_{pr}.$$

Then Lemma 3.3 implies $\mathbf{u} = 0$ on $(0, 1) \times \mathbb{R}$.

Remark. In the proof of Lemma 3.4 the property $\operatorname{essinf}_{\Omega_{pr}} \varepsilon'' > 0$ is used. In fact, it suffices for the conclusion to remain valid that there is some lossy region. When all regions are lossless, i.e., when $\varepsilon'' = 0$ on $(0, 1) \times \mathbb{R}$, then Lemma 3.4 is false in general.

Lemma 3.4 implies in particular that A_ε: $\mathbf{H}^2(\text{curl}; S^1 \times \mathbb{R}) \to \mathbf{L}^2((0, 1) \times \mathbb{R})$ is 1–1. Indeed, when $\mathbf{u} \in \mathbf{H}^2(\text{curl}, S^1 \times \mathbb{R})$ satisfies $A_\varepsilon \mathbf{u} = 0$, then $\mathbf{u}$ decreases exponentially for $|x_2| \to \infty$; hence, $\mathbf{u}$ satisfies the orc for $|x_2| \to \infty$. Therefore, by Lemma 3.4, $\mathbf{u} = 0$.

However, A_ε: $\mathbf{H}^2(\text{curl}; S^1 \times \mathbb{R}) \to \mathbf{L}^2((0, 1) \times \mathbb{R})$ is not onto. To see this, choose $m \in \mathbb{N} \cup \{0\}$ such that $4\pi^2 m^2 < \varepsilon_1$ and define

$$\mathbf{g}(x_1, x_2) = e^{2\pi i m x_1 + i k_m^+ x_2} \mathbf{e}_1, \qquad \text{for} \quad (x_1, x_2) \in (0, 1) \times (b, \infty),$$

where b satisfies (3.12) again. Then $A_\varepsilon \mathbf{g} = 0$ on $(0, 1) \times (b, \infty)$. Extend $\mathbf{g}$ to a smooth periodic vector field $\mathbf{u}$: $(0, 1) \times \mathbb{R} \to \mathbb{C}$ such that $\mathbf{u}$ vanishes on $(0, 1) \times (-\infty, -b)$. Then $\mathbf{f} \equiv A_\varepsilon(\mathbf{u}) \in \mathbf{L}^2((0, 1) \times \mathbb{R})$. Suppose there exists $\mathbf{v} \in \mathbf{H}^2(\text{curl}; S^1 \times \mathbb{R})$ such that $A_\varepsilon(\mathbf{v}) = \mathbf{f}$. Then $\mathbf{u} - \mathbf{v} \in \mathbf{H}^2_{\text{loc}}(\text{curl}; S^1 \times \mathbb{R})$ and $\mathbf{u} - \mathbf{v}$ satisfies the orc for $|x_2| \to \infty$. Hence by Lemma 3.4, $\mathbf{v} = \mathbf{u}$. But $\mathbf{u} \notin \mathbf{H}^2(\text{curl}; S^1 \times \mathbb{R})$; hence, we have a contradiction. Therefore, A_ε: $\mathbf{H}^2(\text{curl}; S^1 \times \mathbb{R}) \to \mathbf{L}^2((0, 1) \times \mathbb{R})$ is not onto.

Put $Y = A_\varepsilon(\mathbf{H}^2(\text{curl}; S^1 \times \mathbb{R}))$. According to Lemma 3.1, Y is dense in $\mathbf{L}^2((0, 1) \times \mathbb{R})$. The inverse A_ε^{-1}: $Y \to \mathbf{H}^2(\text{curl}; S^1 \times \mathbb{R})$ is of course not continuous. However, according to Lemma 3.2 the mapping $A_{\varepsilon + i\lambda}^{-1}$: $\mathbf{L}^2(0, 1) \times \mathbb{R}) \to \mathbf{H}^2(\text{curl}; S^1 \times \mathbb{R})$ is continuous for every $\lambda > 0$. It will be proved in the next section that when $\mathbf{L}^2_b((0, 1) \times \mathbb{R})$ is the space of all $\mathbf{f} \in \mathbf{L}^2((0, 1) \times \mathbb{R})$ with $f(x_1, x_2) = 0$ for all $|x_2| \geqq b$ and when the operators $A_{\varepsilon + i\lambda}^{-1}$ are restricted to $\mathbf{L}^2_b((0, 1) \times \mathbb{R})$, then the limit $\lim_{\lambda \downarrow 0} A_{\varepsilon + i\lambda}^{-1}$ exists in $B(\mathbf{L}^2_b((0, 1) \times \mathbb{R}), \mathbf{H}^2_{\text{loc}}(\text{curl}; S^1 \times \mathbb{R}))$ for every b. In this way a continuous inverse of A_ε is obtained.

We conclude this section with a derivation of two useful identities involving certain Green's functions of the operator $\zeta + \Delta$, where $\zeta \in \mathbb{C}$ with $\operatorname{Im} \zeta \geqq 0$.

Let $\mathbb{R}^+ = (0, \infty)$, $\mathbb{R}^- = (-\infty, 0)$. There exist Green's functions G_ζ^+ : $((0, 1) \times \mathbb{R}^+)^2 \to \mathbb{C}$ and G_ζ^- : $((0, 1) \times \mathbb{R}^-)^2 \to \mathbb{C}$ of the operator $\zeta + \Delta$; i.e.,

$$\zeta G_\zeta^\pm(x, y) + \Delta_x G_\zeta^\pm(x, y) = \delta(x - y), \qquad \forall x = (x_1, x_2) \in (0, 1) \times \mathbb{R}^\pm,$$
$$\forall y = (y_1, y_2) \in (0, 1) \times \mathbb{R}^\pm,$$

such that

(a) $x_1 \to G_\zeta^\pm(x_1, x_2, y_1, y_2)$ is 1-periodic;
and if we write

$$\hat{G}_\zeta^\pm(n, x_2, y_1, y_2) = \int_0^1 G_\zeta^\pm(x_1, x_2, y_1, y_2) e^{-2\pi i n x_1}\, dx_1 \quad \text{and}$$

$$k_n = (\zeta - 4\pi^2 n^2)^{1/2}, \tag{3.21a}$$

(b) $\forall (y_1, y_2) \in (0, 1) \times \mathbb{R}^\pm$ and $\forall n \in \mathbb{Z}$,

$$\lim_{x_2 \to \pm\infty} \left\{ \frac{\partial}{\partial x_2} \hat{G}_\zeta^\pm(n, x_2, y_1, y_2) \mp i k_n \hat{G}_\zeta^\pm(n, x_2, y_1, y_2) \right\} = 0. \tag{3.21b}$$

(c) $\forall n \in \mathbb{Z}$, $\forall x_2 \in \mathbb{R}^\pm$ and uniformly in $y_1 \in (0, 1)$,

$$\left.\begin{aligned} &\lim_{y_2 \to \pm\infty} \left\{ \frac{\partial}{\partial y_2} \hat{G}_\zeta^\pm(n, x_2, y_1, y_2) \mp i k_n \hat{G}_\zeta^\pm(n, x_2, y_1, y_2) \right\} = 0 \\ &\lim_{y_2 \to \pm\infty} \left\{ \frac{\partial^2}{\partial x_2\, \partial y_2} \hat{G}_\zeta^\pm(n, x_2, y_1, y_2) \mp i k_n \frac{\partial}{\partial x_2} \hat{G}_\zeta^\pm(n, x_2, y_1, y_2) \right\} = 0. \end{aligned}\right\} \tag{3.21c}$$

(d) $\forall (x_1, x_2) \in (0, 1) \times \mathbb{R}^\pm$, $\forall y_2 \in \mathbb{R}^\pm$:

$$\int_0^1 G_\zeta^\pm(x_1, x_2, y_1, y_2) e^{-2\pi i n y_1}\, dy_1 = \hat{G}_\zeta^\pm(-n, x_2, x_1, y_2). \tag{3.21d}$$

(e) For every pair of disjoint subsets $K_1, K_2 \subset (0, 1) \times \mathbb{R}^\pm$ the set of functions $\{\zeta \to G_\zeta^\pm(x, y);\ x \in K_1,\ y \in K_2\}$ is equicontinuous on the upper half of the complex plane including the real axis. The same holds for derivatives of $G_\zeta^\pm$ with respect to x_1, x_2, y_1, and y_2. (3.21e)

Formulae for the functions $G_\zeta^\pm$ are given in the Appendix. Now, let $\mathbf{u} \in \mathbf{H}^2(S^1(b, \infty))$ satisfy $\zeta \mathbf{u} + \Delta \mathbf{u} = 0$ on $(0, 1) \times (b, \infty)$. Then $\mathbf{u}$ decreases exponentially for $x_2 \to +\infty$. Hence, if $a > b$, then for every $y = (y_1, y_2) \in (0, 1) \times (a, \infty)$,

$$\begin{aligned} \mathbf{u}(y) &= \int_a^\infty \int_0^1 \mathbf{u}(x)\, \Delta_x G_\zeta^+(x, y)\, dx_1\, dx_2 \\ &\quad - \int_a^\infty \int_0^1 \Delta \mathbf{u}(x)\, G_\zeta^+(x, y)\, dx_1\, dx_2 \\ &= \int_0^1 \frac{\partial \mathbf{u}}{\partial x_2}(x_1, a)\, G_\zeta^+(x_1, a, y_1, y_2)\, dx_1 \\ &\quad - \int_0^1 \mathbf{u}(x_1, a) \frac{\partial}{\partial x_2} G_\zeta^+(x_1, a, y_1, y_2)\, dx_1. \end{aligned} \tag{3.22}$$

Analogously, if $\mathbf{u} \in \mathbf{H}^2(S^1 \times (-\infty, -b))$ satisfies $\zeta \mathbf{u} + \Delta \mathbf{u} = 0$ on $(0, 1) \times (-\infty, -b)$, then for $a > b$ and every $y = (y_1, y_2) \in (0, 1) \times (-\infty, -a)$,

$$\mathbf{u}(y) = \int_0^1 \mathbf{u}(x_1, -a) \frac{\partial}{\partial x_2} G_\zeta^-(x_1, -a, y_1, y_2)\, dx_1 - \int_0^1 \frac{\partial \mathbf{u}}{\partial x_2}(x_1, -a)\, G_\zeta^-(x_1, -a, y_1, y_2)\, dx_1. \tag{3.23}$$

4. The Limiting Absorption Method

We consider the limit $\lim_{\lambda \downarrow 0} A_{\varepsilon + i\lambda}^{-1}$. For every $\lambda > 0$ the mapping $A_{\varepsilon + i\lambda}^{-1}$: $\mathbf{L}^2((0, 1) \to \mathbf{H}^2(\text{curl}; S^1 \times \mathbb{R})$ is continuous, and using (3.6) it follows that

$$\|A_{\varepsilon + i\lambda}^{-1}(\mathbf{f})\|_{2,\text{curl}} \leqq \left\{1 + \frac{\sqrt{2}(1 + \|\varepsilon\|_\infty + \lambda)^2}{\lambda}\right\} \times \|\mathbf{f}\|_0, \qquad \forall \mathbf{f} \in \mathbf{L}^2((0, 1) \times \mathbb{R}). \tag{4.1}$$

The mappings $A_{\varepsilon + i\lambda}^{-1}$, $\lambda > 0$, are evidently also continuous: $\mathbf{L}^2((0, 1) \times \mathbb{R}) \to \mathbf{H}^2_{\text{loc}}(\text{curl}; S^1 \times \mathbb{R})$.

For $b > 0$ let $\mathbf{L}_b^2((0, 1) \times \mathbb{R})$ be the space of all $\mathbf{f} \in \mathbf{L}^2((0, 1) \times \mathbb{R})$ for which $\mathbf{f}(x_1, x_2) = 0$ when $|x_2| \geqq b$. $\mathbf{L}_b^2((0, 1) \times \mathbb{R})$ is equipped with the $\mathbf{L}^2$-norm $\|\ \|_0$. It will be proved that $\lim_{\lambda \downarrow 0} A_{\varepsilon + i\lambda}^{-1}$ exists in $B(\mathbf{L}_b^2((0, 1) \times \mathbb{R}), \mathbf{H}^2_{\text{loc}}(\text{curl}; S^1 \times \mathbb{R}))$ and that this convergence is uniform for ε in the set

$$E_r \equiv \left\{\varepsilon \in L^\infty((0, 1) \times \mathbb{R});\ \varepsilon \text{ satisfies (3.1) and } \frac{1}{r} \leqq \varepsilon'(x) \leqq r,\ \frac{1}{r} \leqq \varepsilon''(x) \leqq r \text{ for a.e. } x \in \Omega_{pr}\right\}, \tag{4.2}$$

where $1 < r < \infty$. We prove first:

Theorem 4.1. *Let $b > 0$ satisfy (3.12). Then for every $r > 1$ and $\bar{\lambda} > 0$, the mappings*

$$A_{\varepsilon + i\lambda}^{-1}: \mathbf{L}_b^2((0, 1) \times \mathbb{R}) \to \mathbf{H}^2_{\text{loc}}(\text{curl}; S^1 \times \mathbb{R}), \qquad \varepsilon \in E_r,\ 0 < \lambda \leqq \bar{\lambda},$$

are equicontinuous, that is, $\forall a > 0\ \exists C_a > 0$, independent of $\varepsilon \in E_r$ and of $\lambda \in (0, \bar{\lambda}]$, such that

$$\|A_{\varepsilon + i\lambda}^{-1}(\mathbf{f})\|_{2,\text{curl}}^{(0,1) \times (-a,a)} \leqq C_a \|\mathbf{f}\|_0, \qquad \forall \mathbf{f} \in \mathbf{L}_b^2((0, 1) \times \mathbb{R}).$$

Proof. Suppose $(\exists a>0)(\forall m\in\mathbb{N})(\exists\varepsilon_m\in E_r)$ $(\exists\lambda_m\in(0,\bar{\lambda}])(\exists\mathbf{f}_m\in\mathbf{L}^2_b((0,1)\times\mathbb{R}))$ such that

$$\|\mathbf{f}_m\|_0\leqq\frac{1}{m}, \tag{4.3}$$

$$\|A^{-1}_{\varepsilon_m+i\lambda_m}(\mathbf{f}_m)\|^{(0,1)\times(-a,a)}_{2,\mathrm{curl}}=1. \tag{4.4}$$

We shall show that this assumption leads to a contradiction.

Without restricting the generality we may assume $a>b$. Let us write $\mathbf{u}_m=A^{-1}_{\varepsilon_m+i\lambda_m}(\mathbf{f}_m)$. Then $\mathbf{u}_m\in\mathbf{H}^2(\mathrm{curl};S^1\times\mathbb{R})$ and

$$(\varepsilon_m+i\lambda_m)\mathbf{u}_m-\mathrm{curl\,curl}\,\mathbf{u}_m=\mathbf{f}_m\qquad\text{on}\quad(0,1)\times\mathbb{R}. \tag{4.5}$$

We prove first:

LEMMA 4.2. *There exists a subsequence* $\{\mathbf{u}_{m_k}\}_{k=1}^{\infty}$ *and* $\mathbf{u}\in\mathbf{H}^2_{\mathrm{loc}}(\mathrm{curl};S^1\times(-a,a))$ *such that* $\lim_{k\to\infty}\mathbf{u}_{m_k}=\mathbf{u}$ *in* $\mathbf{L}^2_{\mathrm{loc}}((0,1)\times(-a,a))$. *Furthermore,* $\mathbf{u}=0$ *on* Ω_{pr}.

Proof. Because $\varepsilon_m\in E_r$ we have $\varepsilon''_m+\lambda_m\geqq\varepsilon''_m\geqq 1/r$. Hence

$$\begin{aligned}\iint_{\Omega_{pr}}|\mathbf{u}_m|^2&\leqq r\iint_{\Omega_{pr}}(\varepsilon''_m+\lambda_m)\,|\mathbf{u}_m|^2\\&\leqq r\,\mathrm{Im}\int_{-\infty}^{\infty}\int_0^1(\varepsilon_m+i\lambda_m)\,|\mathbf{u}_m|^2\\&=r\,\mathrm{Im}\int_{-\infty}^{\infty}\int_0^1\{(\varepsilon_m+i\lambda_m)\,|\mathbf{u}_m|^2-|\mathrm{curl}\,\mathbf{u}_m|^2\}\\&=r\,\mathrm{Im}\int_{-b}^{b}\int_0^1\mathbf{f}_m\cdot\overline{\mathbf{u}_m}\leqq\frac{r}{m}\,\|\mathbf{u}_m\|^{(0,1)\times(-a,a)}_0,\end{aligned} \tag{4.6}$$

where in the last equality we used $a>b$. Now (4.4) implies in particular that $\{\mathbf{u}_m\}_{m=1}^{\infty}$ is bounded in $\mathbf{L}^2((0,1)\times(-a,a))$. Hence (4.6) yields

$$\lim_{m\to\infty}\iint_{\Omega_{pr}}|\mathbf{u}_m|^2=0. \tag{4.7}$$

Let $H^1_0(S^1\times(-a,a))$ be the closure of $\mathscr{D}(S^1\times(-a,a))$ in $H^1(S^1\times(-a,a))$ and let $\psi_m\in H^1_0(S^1\times(-a,a))$ satisfy

$$\int_{-a}^{a}\int_0^1\nabla\psi_m\cdot\nabla\varphi=\int_{-a}^{a}\int_0^1\mathbf{u}_m\cdot\nabla\varphi,\qquad\forall\varphi\in H^1_0(S^1\times(-a,a)). \tag{4.8}$$

Then, for some constant $C_a>0$ independent of m,

$$\|\psi_m\|^{(0,1)\times(-a,a)}_1\leqq C_a\,\|\mathbf{u}_m\|^{(0,1)\times(-a,a)}_0. \tag{4.9}$$

Hence $\{\psi_m\}_{m=1}^{\infty}$ is bounded in $H_0^1(S^1\times(-a,a))$ and thus there exists a subsequence $\{\psi_{m_k}\}_{k=1}^{\infty}$ and $\psi\in H_0^1(S^1\times(-a,a))$ such that $\lim_{k\to\infty}\psi_{m_k}=\psi$ weakly in $H_0^1(S^1\times(-a,a))$. Because $H_0^1(S^1\times(-a,a))\hookrightarrow L^2((0,1)\times(-a,a))$ is compact we also have

$$\lim_{k\to\infty}\psi_{m_k}=\psi \qquad \text{in} \quad L^2((0,1)\times(-a,a)). \tag{4.10}$$

Let $\mathbf{v}_m=\mathbf{u}_m-\nabla\psi_m$ on $(0,1)\times(-a,a)$. Since by (4.8), $\Delta\psi_m=\nabla\cdot\mathbf{u}_m$, there holds $\nabla\cdot\mathbf{v}_m=0$; hence, $\Delta\mathbf{v}_m=-\operatorname{curl}\operatorname{curl}\mathbf{v}_m = -\operatorname{curl}\operatorname{curl}\mathbf{u}_m = \mathbf{f}_m-(\varepsilon_m+i\lambda_m)\mathbf{u}_m$ on $(0,1)\times(-a,a)$. Therefore, $\{\mathbf{v}_m\}_{m=1}^{\infty}$ and $\{\Delta\mathbf{v}_m\}_{m=1}^{\infty}$ are bounded in $\mathbf{L}^2((0,1)\times(-a,a))$; hence, $\{\mathbf{v}_m\}_{m=1}^{\infty}$ is bounded in $\mathbf{H}^2_{\text{loc}}(S^1\times(-a,a))$. Because $\mathbf{H}^2_{\text{loc}}(S^1\times(-a,a))\hookrightarrow\mathbf{H}^1_{\text{loc}}(S^1\times(-a,a))$ is compact, it follows that there exist $\mathbf{v}\in\mathbf{H}^2_{\text{loc}}(S^1\times(-a,a))$ and a subsequence such that

$$\lim_{k\to\infty}\mathbf{v}_{m_k}=\mathbf{v} \qquad \text{in} \quad \mathbf{H}^1_{\text{loc}}(S^1\times(-a,a)). \tag{4.11}$$

Without restricting the generality we may assume that for the indices $\{m_k\}_{k=1}^{\infty}$ both (4.10) and (4.11) apply.

Choose a_1, a_2 such that $b<a_1<a_2<a$, and $\chi\in\mathscr{D}(\mathbb{R})$ such that $\chi(x_2)=1$ when $|x_2|\leqq a_1$, $\chi(x_2)=0$ when $|x_2|\geqq a_2$. Then, using

$$\begin{aligned}&\nabla\cdot[(\varepsilon_m+i\lambda_m)\nabla(\psi_m-\psi_n)]\\&\quad=\nabla\cdot(\mathbf{f}_m-\mathbf{f}_n)-\nabla\cdot[(\varepsilon_m+i\lambda_m)(\mathbf{v}_m-\mathbf{v}_n)]\\&\qquad+\nabla\cdot[(\varepsilon_n-\varepsilon_m)\mathbf{u}_n]+i(\lambda_n-\lambda_m)\nabla\cdot\mathbf{u}_n,\end{aligned}$$

we find

$$\begin{aligned}&\int_{-a_2}^{a_2}\int_0^1(\varepsilon_m(x)+i\lambda_m)\,\chi(x_2)\,|\nabla(\psi_m(x)-\psi_n(x))|^2\,dx_1\,dx_2\\&\quad=\int_{-a_2}^{a_2}\int_0^1(\mathbf{f}_m-\mathbf{f}_n)\cdot\nabla[\chi\overline{(\psi_m-\psi_n)}]\\&\qquad-\int_{-a_2}^{a_2}\int_0^1(\varepsilon_m+i\lambda_m)(\mathbf{v}_m-\mathbf{v}_n)\cdot\nabla[\chi\overline{(\psi_m-\psi_n)}]\\&\qquad+\int_{-a_2}^{a_2}\int_0^1(\varepsilon_n-\varepsilon_m)\mathbf{u}_n\cdot\nabla[\chi\overline{(\psi_m-\psi_n)}]\\&\qquad+i(\lambda_n-\lambda_m)\int_{-a_2}^{a_2}\int_0^1\mathbf{u}_n\cdot\nabla[\chi\overline{(\psi_m-\psi_n)}]\\&\qquad+\int_{-a_2}^{a_2}\int_0^1(\varepsilon_m+i\lambda_m)\overline{(\psi_m-\psi_n)}\,\nabla(\psi_m-\psi_n)\cdot\nabla\chi.\end{aligned} \tag{4.12}$$

It follows from properties (3.1) that $\varepsilon_m - \varepsilon_n = 0$ on the complement of Ω_{pr} in $(0, 1) \times \mathbb{R}$; hence, the third integral on the right-hand side of (4.12) can be replaced by an integral over Ω_{pr}. Furthermore, since $\varepsilon_m \in E_r$, there exists a constant $p > 0$ such that $\varepsilon'_m(x) \geqq p$ for a.e. $x \in (0, 1) \times \mathbb{R}$ and for every m, and there exists a constant $q > 0$ such that $\|\varepsilon_m\|_\infty \leqq q$ for all m. Furthermore, $\{\psi_m\}_{m=1}^\infty$ is bounded in $H_0^1(S^1 \times (-a, a))$ and $\|\mathbf{u}_n\|_0^{(0,1)\times(-a,a)} \leqq 1$. With these remarks it follows from (4.12) that there exists a constant $C > 0$, depending on a_1, a_2 but not on m and n such that

$$p \int_{-a_1}^{a_1} \int_0^1 |\nabla(\psi_m - \psi_n)|^2 \leqq \operatorname{Re} \int_{-a_2}^{a_2} \int_0^1 (\varepsilon_m + i\lambda_m)\, \chi\, |\nabla(\psi_m - \psi_n)|^2$$

$$\leqq C\, \|\mathbf{f}_m - \mathbf{f}_n\|_0 + C(q + \bar{\lambda}) \left\{ \|\mathbf{v}_n - \mathbf{v}_m\|_0^{(0,1)\times(-a_2,a_2)} + \|\mathbf{u}_n\|_0^{\Omega_{pr}} + \frac{|\lambda_n - \lambda_m|}{q + \bar{\lambda}} + \|\psi_m - \psi_n\|_0^{(0,1)\times(-a,a)} \right\}.$$

Without restriction of the generality we may assume that $\{\lambda_{m_k}\}$ is a Cauchy sequence. Then, using (4.3), (4.7), (4.10), (4.11), and the fact that $a_1 \in (b, a)$ is arbitrary, it follows that $\lim_{k\to\infty} \nabla\psi_{m_k} = \nabla\psi$ in $\mathbf{L}^2_{\mathrm{loc}}((0, 1) \times (-a, a))$. Because $\mathbf{u}_{m_k} = \mathbf{v}_{m_k} + \nabla\psi_{m_k}$ we conclude that when $\mathbf{u} = \mathbf{v} + \nabla\psi$, $\lim_{k\to\infty} \mathbf{u}_{m_k} = \mathbf{u}$ in $\mathbf{L}^2_{\mathrm{loc}}((0, 1) \times (-a, a))$; and, because of (4.7), $\mathbf{u} = 0$ on Ω_{pr}. This proves Lemma 4.2.

We shall now complete the proof of Theorem 4.1. Because $\{\lambda_m\}_{m=1}^\infty$ is bounded and $\{\varepsilon_m\}_{m=1}^\infty$ is bounded in $L^\infty((0, 1) \times \mathbb{R})$, there exists a subsequence $\{m_k\}_{k=1}^\infty$ of indices for which Lemma 4.2 applies and, furthermore,

$$\lim_{k\to\infty} \lambda_{m_k} = \lambda,$$

$$\lim_{k\to\infty} \varepsilon_{m_k} = \varepsilon \qquad \text{in the weak dual topology of } L^\infty((0, 1) \times \mathbb{R}),$$

for some $\lambda \in [0, \bar{\lambda}]$ and some $\varepsilon \in L^\infty((0, 1) \times \mathbb{R})$. It is clear that ε has the properties (3.1) except perhaps $\operatorname{essinf}_{\Omega_{pr}} \varepsilon' > 0$ and $\operatorname{essinf}_{\Omega_{pr}} \varepsilon'' > 0$ which need not be satisfied. Furthermore, $\lambda = 0$ since otherwise (4.1), (4.3) contradict (4.4).

According to (3.1) the ε_{m_k} are constant on the complement of Ω_{pr} in $(0, 1) \times \mathbb{R}$. Furthermore, by Lemma 4.2, $\lim_{k\to\infty} \mathbf{u}_{m_k} = 0$ in $\mathbf{L}^2(\Omega_{pr})$. Therefore, $\lim_{k\to\infty} (\varepsilon_{m_k} + i\lambda_{m_k})\mathbf{u}_{m_k} = \varepsilon\mathbf{u}$ in $\mathbf{L}^2_{\mathrm{loc}}((0, 1) \times (-a, a))$. Then (4.5) yields $\lim_{k\to\infty} \operatorname{curl}\operatorname{curl} \mathbf{u}_{m_k} = \mathbf{u}$ in $\mathbf{L}^2_{\mathrm{loc}}((0, 1) \times (-a, a))$, hence,

$$\lim_{k\to\infty} \mathbf{u}_{m_k} = \mathbf{u} \qquad \text{in} \quad \mathbf{H}^2_{\mathrm{loc}}(\operatorname{curl}; S^1 \times (-a, a)). \tag{4.13}$$

Because $\mathbf{f}_{m_k}(x_1, x_2) = 0$ when $|x_2| \geqq b$ and since b satisfies (3.12) it follows that

$$\left.\begin{aligned} \varepsilon_1 \mathbf{u}_{m_k} + \Delta \mathbf{u}_{m_k} &= 0 \qquad \text{on} \quad (0, 1) \times (b, \infty), \\ \varepsilon_l \mathbf{u}_{m_k} + \Delta \mathbf{u}_{m_k} &= 0 \qquad \text{on} \quad (0, 1) \times (-\infty, -b), \end{aligned}\right\} \tag{4.14}$$

where $\varepsilon_1 = \varepsilon|_{\Omega_1}$ and $\varepsilon_l = \varepsilon|_{\Omega_l}$ are constants. Since $a > b$, (4.13) and (4.14) imply

$$\lim_{k \to \infty} \mathbf{u}_{m_k} = \mathbf{u} \qquad \text{in} \quad \mathbf{H}^2_{\text{loc}}(S^1 \times (b, a)) \text{ and in } \mathbf{H}^2_{\text{loc}}(S^1 \times (-a, -b));$$

so using a well-known trace map (see, e.g., [3]),

$$\left.\begin{aligned} \lim_{k \to \infty} \mathbf{u}_{m_k}(x_1, \pm a_2) &= \mathbf{u}(x_1, \pm a_2) \\ \lim_{k \to \infty} \frac{\partial}{\partial x_2} \mathbf{u}_{m_k}(x_1, \pm a_2) &= \frac{\partial}{\partial x_2} \mathbf{u}(x_1, \pm a_2) \end{aligned}\right\} \quad \text{in} \quad \mathbf{L}^2(0, 1), \tag{4.15}$$

for $a_2 \in (b, a)$. By using (3.22), (3.23) we obtain for $y_2 > a_2$

$$\begin{aligned} \mathbf{u}_{m_k}(y_1, y_2) = \int_0^1 \bigg\{ &\frac{\partial}{\partial x_2} \mathbf{u}_{m_k}(x_1, a_2)\, G^+_{\varepsilon_1}(x_1, a_2, y_1, y_2) \\ &- \mathbf{u}_{m_k}(x_1, a_2) \frac{\partial}{\partial x_2} G^+_{\varepsilon_1}(x_1, a_2, y_1, y_2) \bigg\}\, dx_1, \end{aligned} \tag{4.16}$$

and for $y_2 < -a_2$

$$\begin{aligned} \mathbf{u}_{m_k}(y_1, y_2) = \int_0^1 \bigg\{ &\mathbf{u}_{m_k}(x_1, -a_2) \frac{\partial}{\partial x_2} G^-_{\varepsilon_l}(x_1, -a_2, y_1, y_2) \\ &- \frac{\partial \mathbf{u}_{m_k}}{\partial x_2}(x_1, -a_2)\, G^-_{\varepsilon_l}(x_1, -a_2, y_1, y_2) \bigg\}\, dx_1. \end{aligned}$$

From these formulae and (4.15), one deduces that $\{u_{m_k}\}_{k=1}^{\infty}$ and all derivatives converge uniformly on compact subsets of $(0, 1) \times (a_2, \infty) \cup (0, 1) \times (-\infty, -a_2)$. If we define $\mathbf{u}(x_1, x_2) \equiv \lim_{k \to \infty} \mathbf{u}_{m_k}(x_1, x_2)$ for $|x_2| \geqq a_2$, then using (4.13) we conclude

$$\lim_{k \to \infty} \mathbf{u}_{m_k} = \mathbf{u} \qquad \text{in} \quad \mathbf{H}^2_{\text{loc}}(\text{curl}; S^1 \times \mathbb{R}).$$

By passing to the limit $m_k \to \infty$ in (4.5) we find

$$\varepsilon \mathbf{u} - \text{curl curl } \mathbf{u} = 0 \qquad \text{on} \quad (0, 1) \times \mathbb{R},$$

and since $\mathbf{u}=0$ on Ω_{pr}, Lemma 3.3 implies $\mathbf{u}=0$ on $(0,1)\times\mathbb{R}$. But then (4.20) is seen to contradict (4.4). Hence the proof of Theorem 4.1 is complete.

THEOREM 4.3. *Let $b>0$ satisfy* (3.12). *For every $\bar{\lambda}>0$ the map*

$$(0,\bar{\lambda}]\ni\lambda\to A^{-1}_{\varepsilon+i\lambda}\in B(\mathbf{L}^2_b((0,1)\times\mathbb{R}),\mathbf{H}^2_{\mathrm{loc}}(\mathrm{curl};S^1\times\mathbb{R}))$$

is uniformly continuous, i.e., $(\forall\eta>0)(\forall a>0)(\exists\delta>0)(\forall\lambda,\ \mu\in(0,\bar{\lambda}]$ *with* $|\lambda-\mu|<\delta)$,

$$\|A^{-1}_{\varepsilon+i\lambda}(\mathbf{f})-A^{-1}_{\varepsilon+i\mu}(\mathbf{f})\|^{(0,1)\times(-a,a)}_{2,\mathrm{curl}}$$
$$\leqslant\eta\ \|\mathbf{f}\|_0,\qquad\forall\mathbf{f}\in\mathbf{L}^2_b((0,1)\times\mathbb{R}).$$

Proof. Suppose $(\exists\eta>0)(\exists a>0)(\forall m\in\mathbb{N})(\exists\lambda_m,\ \mu_m\in(0,\bar{\lambda}]$ with $|\lambda_m-\mu_m|<1/m)$ $(\exists\mathbf{f}\in\mathbf{L}^2_b((0,1)\times\mathbb{R}))$ such that

$$\|\mathbf{f}_m\|_0=1, \tag{4.17}$$

$$\|A^{-1}_{\varepsilon+i\lambda_m}(\mathbf{f}_m)-A^{-1}_{\varepsilon+i\mu_m}(\mathbf{f}_m)\|^{(0,1)\times(-a,a)}_{2,\mathrm{curl}}\geqq\eta. \tag{4.18}$$

Without restricting the generality we may assume $a>b$. Write $\mathbf{u}_m=A^{-1}_{\varepsilon+i\lambda_m}(\mathbf{f}_m)$, $\mathbf{v}_m=A^{-1}_{\varepsilon+i\mu_m}(\mathbf{f}_m)$, and $\mathbf{w}_m=\mathbf{u}_m-\mathbf{v}_m$. Then

$$(\varepsilon+i\lambda_m)\mathbf{u}_m-\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}_m=\mathbf{f}_m,$$
$$(\varepsilon+i\mu_m)\mathbf{v}_m-\mathrm{curl}\,\mathrm{curl}\,\mathbf{v}_m=\mathbf{f}_m, \tag{4.19}$$

and

$$(\varepsilon+i\lambda_m)\mathbf{w}_m-\mathrm{curl}\,\mathrm{curl}\,\mathbf{w}_m=i(\mu_m-\lambda_m)\mathbf{v}_m. \tag{4.20}$$

Note that the right-hand side of (4.20) does not have bounded support in general. Theorem 4.1 implies that $\{\mathbf{u}_m\}_{m=1}^{\infty}$ and $\{\mathbf{v}_m\}_{m=1}^{\infty}$ are bounded in $\mathbf{H}^2_{\mathrm{loc}}(\mathrm{curl};S^1\times\mathbb{R})$. Furthermore, (4.19) implies

$$(\varepsilon_1+i\lambda_m)\mathbf{u}_m+\Delta\,\mathbf{u}_m=(\varepsilon_1+i\mu_m)\mathbf{v}_m+\Delta\,\mathbf{v}_m=0\qquad\text{on}\quad(0,1)\times(b,\infty),$$
$$(\varepsilon_l+i\lambda_m)\mathbf{u}_m+\Delta\,\mathbf{u}_m=(\varepsilon_l+i\mu_m)\mathbf{v}_m+\Delta\,\mathbf{v}_m=0\qquad\text{on}\quad(0,1)\times(-\infty,-b). \tag{4.21}$$

Therefore, there exist $\mathbf{u}$, $\mathbf{v}\in\mathbf{H}^2_{\mathrm{loc}}(\mathrm{curl},S^1\times\mathbb{R})$ and subsequences such that $\lim_{k\to\infty}\mathbf{u}_{m_k}=\mathbf{u}$ and $\lim_{k\to\infty}\mathbf{v}_{m_k}=\mathbf{v}$ in $\mathbf{H}^2_{\mathrm{loc}}(S^1\times(b,\infty))$ and in $\mathbf{H}^2_{\mathrm{loc}}(S^1\times(-\infty,-b))$ and furthermore also weakly in $\mathbf{H}^2(\mathrm{curl},S^1\times(c,d))$, $\forall c,\ d\in\mathbb{R}$ with $c<d$. Hence, when $\mathbf{w}=\mathbf{u}-\mathbf{v}$,

$$\left.\begin{array}{ll}\lim_{k\to\infty}\mathbf{w}_{m_k}=\mathbf{w} & \text{in }\mathbf{H}^2_{\mathrm{loc}}(S^1\times(b,\infty))\text{ and}\\ & \text{in }\mathbf{H}^2_{\mathrm{loc}}(S^1\times(-\infty,-b))\text{ and weakly}\\ & \text{in }\mathbf{H}^2(\mathrm{curl};S^1\times(c,d)),\ \forall c,\ d\in\mathbb{R}\text{ with }c<d.\end{array}\right\} \tag{4.22}$$

Furthermore, we may assume

$$\lim_{k\to\infty} \lambda_{m_k} = \lim_{k\to\infty} \mu_{m_k} = \lambda_0,$$

for some $\lambda_0 \in [0, \bar{\lambda}]$. Using (4.1) it follows easily that, when $\lambda_0 > 0$, (4.20) contradicts (4.18). Therefore $\lambda_0 = 0$.

Choose a_1, a_2 with $b < a_1 < a_2 < a$ and $\chi \in \mathscr{D}(\mathbb{R})$ such that $\chi(x_2) = 1$ when $|x_2| \leqq a_1$ and $\chi(x_2) = 0$ when $|x_2| \geqq a_2$. Then with (4.20),

$$\begin{aligned}(\varepsilon + i\lambda_{m_k})\, \chi \mathbf{w}_{m_k} &- \operatorname{curl}\operatorname{curl}(\chi \mathbf{w}_{m_k}) \\ &= i(\mu_{m_k} - \lambda_{m_k})\, \chi \mathbf{v}_{m_k} + \mathbf{h}_{m_k},\end{aligned} \tag{4.23}$$

where $\chi \mathbf{w}_{m_k}(x_1, x_2) = \chi(x_2)\, \mathbf{w}_{m_k}(x_1, x_2)$, $\chi \mathbf{v}_{m_k}(x_1, x_2) = \chi(x_2)\, \mathbf{v}_{m_k}(x_1, x_2)$, and $\{\mathbf{h}_{m_k}\}_{k=1}^{\infty}$ is a sequence which converges in $\mathbf{L}^2((0, 1) \times \mathbb{R})$. This is easily verified using (4.22). Hence the right-hand side of (4.23) converges in $\mathbf{L}^2((0, 1) \times \mathbb{R})$, and since all these functions have support contained in $[0, 1] \times [-a_2, a_2]$ it follows from Theorem 4.1, (4.23), and (4.22) that

$$\lim_{k\to\infty} \mathbf{w}_{m_k} = \mathbf{w} \qquad \text{in} \quad \mathbf{H}^2_{\mathrm{loc}}(\mathrm{curl}; S^1 \times \mathbb{R}), \tag{4.24}$$

and by passing to the limit $m_k \to \infty$ in (4.20)

$$\varepsilon \mathbf{w} - \operatorname{curl}\operatorname{curl} \mathbf{w} = 0 \qquad \text{on} \quad (0, 1) \times \mathbb{R}.$$

In order to complete the proof of Theorem 4.3, it suffices to show that $\mathbf{w}$ satisfies the orc for $|x_2| \to \infty$ because then Lemma 3.4 yields $\mathbf{w} = 0$ on $(0, 1) \times \mathbb{R}$ and thus (4.24) contradicts (4.18).

In order to deduce that $\mathbf{w}$ satisfies the orc for $|x_2| \to \infty$, we use formulae (3.22), (3.23) for $\mathbf{u}_{m_k}$ and $\mathbf{v}_{m_k}$. This is justified by (4.21). We find for $y_2 \geqq a_2$

$$\begin{aligned}\mathbf{w}_{m_k}(y_1, y_2) &= \mathbf{u}_{m_k}(y_1, y_2) - \mathbf{v}_{m_k}(y_1, y_2) \\ &= \int_0^1 \left\{ \frac{\partial}{\partial x_2} \mathbf{w}_{m_k}(x_1, a_2)\, G^+_{\varepsilon_1 + i\lambda_{m_k}}(x_1, a_2, y_1, y_2) \right. \\ &\quad \left. - \mathbf{w}_{m_k}(x_1, a_2) \frac{\partial}{\partial x_2} G^+_{\varepsilon_1 + i\lambda_{m_k}}(x_1, a_2, y_1, y_2) \right\} dx_1 \\ &\quad + \int_0^1 \frac{\partial}{\partial x_2} \mathbf{v}_{m_k}(x_1, a_2) \{ G^+_{\varepsilon_1 + i\lambda_{m_k}}(x_1, a_2, y_1, y_2) \\ &\quad - G^+_{\varepsilon_1 + i\mu_{m_k}}(x_1, a_2, y_1, y_2) \}\, dx_1 \\ &\quad - \int_0^1 \mathbf{v}_{m_k}(x_1, a_2) \left\{ \frac{\partial}{\partial x_2} G^+_{\varepsilon_1 + i\lambda_{m_k}}(x_1, a_2, y_1, y_2) \right. \\ &\quad \left. - \frac{\partial}{\partial x_2} G^+_{\varepsilon_1 + i\mu_{m_k}}(x_1, a_2, y_1, y_2) \right\} dx_1.\end{aligned} \tag{4.25}$$

Because $\{\mathbf{v}_{m_k}\}_{k=1}^{\infty}$ is bounded in $\mathbf{H}^2_{\mathrm{loc}}(S^1 \times (b, \infty))$ and because $\lim_{k \to \infty} \mathbf{w}_{m_k} = \mathbf{w}$ in $\mathbf{H}^2_{\mathrm{loc}}(S^1 \times (b, \infty))$, it follows from a trace theorem that

$$\{\mathbf{v}_{m_k}(x_1, a_2)\}_{k=1}^{\infty} \text{ and } \left\{\frac{\partial}{\partial x_2} \mathbf{v}_{m_k}(x_1, a_2)\right\}_{k=1}^{\infty} \text{ are bounded in } \mathbf{L}^2(0, 1),$$

and

$$\lim_{k \to \infty} \mathbf{w}_{m_k}(x_1, x_2) = \mathbf{w}(x_1, x_2),$$

$$\lim_{k \to \infty} \frac{\partial}{\partial x_2} \mathbf{w}_{m_k}(x_1, x_2) = \frac{\partial \mathbf{w}}{\partial x_2}(x_1, x_2) \text{ in } \mathbf{L}^2(0, 1), \ \forall x_2 \geqslant a_2.$$

Then, using property (3.21e) of G_ζ^+ we conclude from (4.25) by passing to the limit $k \to \infty$ that

$$\mathbf{w}(y_1, y_2) = \int_0^1 \left\{\frac{\partial \mathbf{w}}{\partial x_2}(x_1, a_2)\, G_{\varepsilon_1}^+(x_1, a_2, y_1, y_2) - \mathbf{w}(x_1, a_2) \frac{\partial}{\partial x_2} G_{\varepsilon_1}^+(x_1, a_2, y_1, y_2)\right\} dx_1, \quad \forall y_2 > a_2.$$

Hence, with (3.21d) we find for all $y_2 > a_2$ and all $n \in \mathbb{Z}$,

$$\hat{\mathbf{w}}(n, y_2) = \int_0^1 \left\{\frac{\partial \mathbf{w}}{\partial x_2}(x_1, a_2)\, \hat{G}_{\varepsilon_1}^+(-n, a_2, x_1, y_2) - \mathbf{w}(x_1, a_2) \frac{\partial}{\partial x_2} \hat{G}_{\varepsilon_1}^+(-n, a_2, x_1, y_2)\right\} dx_1,$$

$$\frac{\partial \hat{\mathbf{w}}}{\partial y_2}(n, y_2) = \int_0^1 \left\{\frac{\partial \mathbf{w}}{\partial x_2}(x_1, a_2) \frac{\partial}{\partial y_2} \hat{G}_{\varepsilon_1}^+(-n, a_2, x_1, y_2) - \mathbf{w}(x_2, a_2) \frac{\partial^2}{\partial x_2\, \partial y_2} \hat{G}_{\varepsilon_1}^+(-n, a_2, x_1, y_2)\right\} dx_1.$$

By applying (3.21c) we derive from these two equalities

$$\lim_{y_2 \to \infty} \left\{\frac{\partial \hat{\mathbf{w}}}{\partial y_2}(n, y_2) - ik_n\, \hat{\mathbf{w}}(n, y_2)\right\} = 0, \qquad \forall n \in \mathbb{Z},$$

where $k_n = (\varepsilon_1 - 4\pi^2 n^2)^{1/2}$. From this one deduces immediately that $\mathbf{w}$ satisfies the orc for $x_2 \to \infty$.

By using (3.23) instead of (3.22) we find analogously that $\mathbf{w}$ also satisfies the orc for $x_2 \to -\infty$. This completes the proof of Theorem 4.3.

THEOREM 4.4. *Let* $b>0$ *satisfy* (3.12). *Then for every* $\mathbf{f}\in \mathbf{L}^2_b((0,1)\times\mathbb{R})$ *there exists a unique* $\mathbf{u}_\varepsilon\in \mathbf{H}^2_{\mathrm{loc}}(\mathrm{curl}; S^1\times\mathbb{R})$ *such that*

$$\varepsilon \mathbf{u}_\varepsilon - \mathrm{curl}\,\mathrm{curl}\,\mathbf{u}_\varepsilon = \mathbf{f} \qquad \text{on} \quad (0,1)\times\mathbb{R}, \tag{4.26}$$

and $\mathbf{u}_\varepsilon$ *satisfies the orc for* $x_2\to\pm\infty$. *Furthermore, for every* $r>1$ *and* $a>0$ *there exists a number* $C(r,a,b)>0$ *such that*

$$\|\mathbf{u}_\varepsilon\|^{(0,1)\times(-a,a)}_{2,\mathrm{curl}} \leqq C(r,a,b)\,\|\mathbf{f}\|_0, \qquad \forall \mathbf{f}\in \mathbf{L}^2_b((0,1)\times\mathbb{R}),\ \forall\varepsilon\in E_r. \tag{4.27}$$

Proof. Uniqueness is implied by Lemma 3.4. Define $\mathbf{u}_\varepsilon \equiv \lim_{\lambda\downarrow 0} A^{-1}_{\varepsilon+i\lambda}(\mathbf{f})$; then Theorem 4.3 implies that (4.26) holds. Furthermore, according to Theorem 4.1 the set of operators $\{A^{-1}_{\varepsilon+i\lambda};\ \varepsilon\in E_r,\ 0<\lambda\leqq\bar{\lambda}\}$ is equicontinuous in $B(\mathbf{L}^2_b((0,1)\times\mathbb{R}),\ \mathbf{H}^2_{\mathrm{loc}}(\mathrm{curl}, S^1\times\mathbb{R}))$. Hence if we write $A^{-1}_\varepsilon \equiv \lim_{\lambda\downarrow 0} A^{-1}_{\varepsilon+i\lambda}$ then it follows that $\{A^{-1}_\varepsilon; \varepsilon\in E_r\}$ is also an equicontinuous subset of $B(\mathbf{L}^2_b((0,1)\times\mathbb{R}),\ \mathbf{H}^2_{\mathrm{loc}}(\mathrm{curl}; S^1\times\mathbb{R}))$. This proves (4.27).

We shall now apply the foregoing to boundary value problem (1.8), (1.9). The factor $\omega^2\varepsilon_0\mu_0$ which appears in (1.8) will be assumed 1.

The incoming electric field $\mathscr{E}^i$ is assumed to be defined on some set $(0,1)\times(b,\infty)\subset\Omega_1$ and satisfies Helmholtz's equation there:

$$\varepsilon_1\mathscr{E}^i + \Delta\mathscr{E}^i = 0 \qquad \text{on} \quad (0,1)\times(b,\infty)\subset\Omega_1.$$

Then we have

THEOREM 4.5. *For given incoming field* $\mathscr{E}^i\in\mathbf{H}^2_{\mathrm{loc}}(S^1\times(b,\infty))$, *where* $b>0$ *satisfies* (3.12), *there exists a unique* $\mathscr{E}_\varepsilon\in\mathbf{H}^2_{\mathrm{loc}}(\mathrm{curl}; S^1\times\mathbb{R})$ *such that*

(i) $\varepsilon\mathscr{E}_\varepsilon - \mathrm{curl}\,\mathrm{curl}\,\mathscr{E}_\varepsilon = 0$ *on* $(0,1)\times\mathbb{R}$,

(ii) $\mathscr{E}_\varepsilon - \mathscr{E}^i$ *satisfies the orc for* $x_2\to\infty$,

(iii) $\mathscr{E}_\varepsilon$ *satisfies the orc for* $x_2\to-\infty$.

Furthermore, for every $r>1$, $a>0$, *and* $b'>b$, *there exists a number* $C(r,a,b,b')>0$ *which is independent of* $\mathscr{E}^i$ *such that*

$$\|\mathscr{E}_\varepsilon\|^{(0,1)\times(-a,a)}_{2,\mathrm{curl}} \leqq C(r,a,b,b')\,\|\mathscr{E}^i\|^{(0,1)\times(b,b')}_2, \qquad \forall\varepsilon\in E_r, \tag{4.28}$$

where $\|\ \ \|^{(0,1)\times(b,b')}_2$ *is the norm of* $\mathbf{H}^2(S^1\times(b,b'))$.

Proof. Uniqueness follows again from Lemma 3.4. Let $\chi\in\mathscr{D}(\mathbb{R})$ satisfy $\chi(x_2)=1$, $\forall x_2\geqq b'$, $\chi(x_2)=0$, $\forall x_2\leqq b$ and define $\chi\mathscr{E}^i$ and $\mathbf{f}$ by

$$\begin{aligned} \chi\mathscr{E}^i(x_1,x_2) &\equiv \chi(x_2)\,\mathscr{E}^i(x_1,x_2) && \text{on} \quad (0,1)\times(b,\infty),\\ &\equiv 0 && \text{on} \quad (0,1)\times(-\infty,b),\\ \mathbf{f} &\equiv -\varepsilon\chi\mathscr{E}^i + \mathrm{curl}\,\mathrm{curl}\,\chi\mathscr{E}^i && \text{on} \quad (0,1)\times\mathbb{R}. \end{aligned} \tag{4.29}$$

Then $\mathbf{f} \in \mathbf{L}^2_{b'}((0,1) \times \mathbb{R})$ and there exists a number $C(r, b'-b) > 0$ independent of $\mathscr{E}^i$ such that

$$\|\mathbf{f}\|_0 \leqq C(r, b'-b) \, \|\mathscr{E}^i\|_2^{(0,1)\times(b,b')}, \qquad \forall \varepsilon \in E_r,$$

where $\| \ \|_2^{(0,1)\times(b,b')}$ is the norm on $\mathbf{H}^2(S^1 \times (b,b'))$. Let $\mathbf{u}_\varepsilon$ be as in Theorem 4.4; then for some number $C(r, a, b') > 0$, $C(r, a, b, b') > 0$,

$$\begin{aligned} \|\mathbf{u}_\varepsilon\|_{2,\mathrm{curl}}^{(0,1)\times(-a,a)} &\leqq C(r, a, b') \, \|\mathbf{f}\|_0 \\ &\leqq C(r, a, b, b') \, \|\mathscr{E}^i\|_2^{(0,1)\times(b,b')}, \qquad \forall \varepsilon \in E_r. \end{aligned}$$

Finally, take $\mathscr{E}_\varepsilon = \mathbf{u}_\varepsilon + \chi \mathscr{E}^i$; then $\mathscr{E}_\varepsilon \in \mathbf{H}^2_{\mathrm{loc}}(\mathrm{curl}; S^1 \times \mathbb{R})$ satisfies (i), (ii), and (iii). Furthermore, using the last estimate it is easy to see that (4.28) also is satisfied.

THEOREM 4.6. *Let $b > 0$ satisfy* (3.12) *and let $\mathbf{f} \in \mathbf{L}^2_b((0,1) \times \mathbb{R})$. We write $A_\varepsilon^{-1}(\mathbf{f}) \equiv \mathbf{u}_\varepsilon$ where $\mathbf{u}_\varepsilon \in \mathbf{H}^2_{\mathrm{loc}}(\mathrm{curl}; S^1 \times \mathbb{R})$ is the field satisfying* (4.26) *and the orc for $x_2 \to \pm\infty$. Then we have*

$$\bigcup_{r>1} E_r \ni \varepsilon \mapsto A_\varepsilon^{-1}(f) \in \mathbf{H}^2_{\mathrm{loc}}(\mathrm{curl}; S^1 \times \mathbb{R}) \tag{4.30}$$

is $\mathscr{C}^\infty$-Fréchet differentiable, where $\bigcup_{r>1} E_r$ carries the topology of $L^\infty((0,1) \times \mathbb{R})$. Furthermore, the first derivative is given by

$$\delta[A_\varepsilon^{-1}(\mathbf{f})](\varepsilon - \tilde{\varepsilon}) = A_\varepsilon^{-1}((\varepsilon - \tilde{\varepsilon}) \, A_\varepsilon^{-1}(\mathbf{f})), \qquad \forall \varepsilon, \tilde{\varepsilon} \in \bigcup_{r>1} E_r. \tag{4.31}$$

Proof. Put $\mathbf{u}_\varepsilon = A_\varepsilon^{-1}(\mathbf{f})$. We have $A_\varepsilon(\mathbf{u}_\varepsilon - \mathbf{u}_{\tilde{\varepsilon}}) = (\varepsilon - \tilde{\varepsilon})\mathbf{u}_{\tilde{\varepsilon}}$, hence,

$$\mathbf{u}_\varepsilon - \mathbf{u}_{\tilde{\varepsilon}} = A_\varepsilon^{-1}((\varepsilon - \tilde{\varepsilon})\mathbf{u}_{\tilde{\varepsilon}}).$$

This implies that (4.30) is Fréchet differentiable with the derivative given by (4.31). Using an induction argument it is easy to see that (4.30) is in fact infinitely Fréchet differentiable.

COROLLARY. *Let $\mathscr{E}^i \in \mathbf{H}^2_{\mathrm{loc}}(S^1 \times \mathbb{R})$ be a given incoming field and let $\mathscr{E}_\varepsilon$ be the field described in Theorem* 4.5. *Then we have*

$$\bigcup_{r>1} E_r \ni \varepsilon \mapsto \mathscr{E}_\varepsilon \in \mathbf{H}^2_{\mathrm{loc}}(\mathrm{curl}; S^1 \times \mathbb{R}) \tag{4.32}$$

is infinitely Fréchet differentiable with the first-order derivative given by

$$\delta \mathscr{E}_\varepsilon(\varepsilon - \tilde{\varepsilon}) = A_\varepsilon^{-1}((\varepsilon - \tilde{\varepsilon}) \, A_\varepsilon^{-1}(\mathbf{f})), \qquad \forall \varepsilon, \tilde{\varepsilon} \in E_r, \tag{4.33}$$

where $\mathbf{f}$ *is given by* (4.29).

5. Existence of $\rho(t)$

Let ρ_0: $\Omega_{pr} \to [0, \infty)$ be the given initial density of the PAC. ρ_0 is continuous on $\bar{\Omega}_{pr}$ and 1-periodic with respect to x_1. For $t \geqslant 0$, $\rho(t)$: $\bar{\Omega}_{pr} \to [0, \infty)$ will denote a continuous periodic function which satisfies $\rho(t, x) \leqslant \|\rho_0\|_\infty^{\Omega_{pr}}$, $\forall (t, x) \in [0, \infty) \times \Omega_{pr}$.

The function $\varepsilon(t)$: $(0, 1) \times \mathbb{R} \to \mathbb{C}$ is defined by

$$\varepsilon(t)|_{\Omega_j} = \varepsilon_j \qquad \text{for} \quad j = 1, ..., l, \tag{5.1a}$$

$$\begin{aligned} \varepsilon(t, x) &= \varepsilon_{pr}(t, x) \\ &= h'(x, \rho(t, x)) + ih''(x, \rho(t, x)), \qquad \forall x \in \Omega_{pr}, \end{aligned} \tag{5.1b}$$

where the ε_j are the (fixed) constants in (3.1) and h', h'' are given functions as in Section 1. It is clear that for every $\rho(t)$ as described above, $\varepsilon(t)$ defined by (5.1) satisfies

$$\varepsilon(t) \in E_r, \tag{5.2}$$

where $r > 1$ is such that (1.5) holds with $s = \|\rho_0\|_\infty^{\Omega_{pr}}$.

For given fixed incident field $\mathscr{E}^i$ let $\mathscr{E}_{\varepsilon(t)}$ be the total field described in Theorem 4.5. We shall also write $\mathscr{E}_{\rho(t)}$ for this field, where $\rho(t)$ and $\varepsilon(t)$ are always related by (5.1b).

In the following we assume that the incident field is TE-polarized, i.e.,

$$\mathscr{E}^i(x_1, x_2) = \mathscr{E}^i(x_1, x_2)\mathbf{e}_3,$$

where $\mathscr{E}^i$: $[0, 1] \times [b, \infty) \to \mathbb{C}$ is 1-periodic with respect to x_1. It is easy to see that in this problem (1.8), (1.9) can be separated into two uncoupled problems, one for the $\mathbf{e}_1$- and $\mathbf{e}_2$-components of the total field and the other for the $\mathbf{e}_3$-component. The first problem admits only the zero solution; hence, the total field is also TE-polarized:

$$\mathscr{E}_{\varepsilon(t)}(x_1, x_2) = \mathscr{E}_{\varepsilon(t)}(x_1, x_2)\mathbf{e}_3.$$

Then evidently, div $\mathscr{E}_{\varepsilon(t)} = 0$ and hence

$$\Delta \mathscr{E}_{\varepsilon(t)} = -\text{curl curl}\, \mathscr{E}_{\varepsilon(t)}. \tag{5.3}$$

Now Theorem 4.5 and (5.3) imply $\mathscr{E}_{\varepsilon(t)} \in \mathbf{H}^2_{\text{loc}}(S^1 \times \mathbb{R})$, and for every $a > 0$,

$$\|\mathscr{E}_{\varepsilon(t)}\|_2^{(0,1)\times(-a,a)} \leqq C(r, a, b, b')\, \|\mathscr{E}^i\|_2^{(0,1)\times(b,b')}, \tag{5.4}$$

for some constant $C(r, a, b, b')$. In the following, r, b, b', and the incident field are all fixed.

Let $\mathscr{C}(S^1 \times [-a, a])$ be the space of continuous vector fields $[0, 1] \times [-a, a] \to \mathbb{C}^3$ which are 1-periodic with respect to x_1. Sobolev's imbedding theorem $\mathbf{H}^2(S^1 \times [-a, a]) \subsetneq \mathscr{C}(S^1 \times [-a, a])$ implies, when $a > 0$ is chosen such that $\Omega_{pr} \subset (0, 1) \times (-a, a)$,

$$\|\mathscr{E}_{\varepsilon(t)}\|_{\infty}^{\Omega_{pr}} \leqq C \, \|\mathscr{E}_{\varepsilon(t)}\|_{2}^{(0,1)\times(-a,a)}, \tag{5.5}$$

for some constant C independent of $\mathscr{E}_{\varepsilon(t)}$.

After these preliminary remarks we now consider integral Eq. (1.13) for $\rho(t)$,

$$\rho(t, x) = \rho_0(x) \exp\left[-\int_0^t G(\rho(s, x)) \, |\mathscr{E}_{\rho(s)}(x)|^2 \, ds \right], \qquad t \geqq 0, \; x \in \Omega_{pr}, \tag{5.6}$$

where $\rho_0 \in \mathscr{C}(\overline{\Omega_{pr}})$ is the given nonnegative function which is 1-periodic with respect to x_1 and where $\mathscr{E}_{\rho(s)}$ is the total field $\mathscr{E}_{\varepsilon(s)}$ when $\varepsilon(s)$ is defined by (5.1). Let T be the operator

$$T(\rho)(t, x) = \rho_0(x) \exp\left[-\int_0^t G(\rho(s, x)) \, |\mathscr{E}_{\rho(s)}(x)|^2 \, ds \right], \qquad t \geqq 0, \; x \in \Omega_{pr}, \tag{5.7}$$

with domain

$$D_T = \{\rho \in \mathscr{C}_b([0, \infty) \times \overline{\Omega_{pr}}); \rho \text{ is 1-periodic with respect to } x_1 \text{ and } 0 \leqq \rho(t, x) \leqq \|\rho_0\|_{\infty}^{\Omega_{pr}}, \; \forall (t, x) \in [0, \infty) \times \overline{\Omega_{pr}}\},$$

where $\mathscr{C}_b([0, \infty) \times \overline{\Omega_{pr}})$ is the linear space of all bounded continuous functions $[0, \infty) \times \overline{\Omega_{pr}} \to \mathbb{R}$. Since G is a continuous function it is clear that T maps D_T into itself.

According to the corollary of Theorem 4.6 the map $E_r \ni \varepsilon \mapsto \mathscr{E}_\varepsilon \in \mathbf{H}^2_{\mathrm{loc}}(\mathrm{curl}; S^1 \times \mathbb{R})$ is Fréchet differentiable and, hence, using (5.3) which holds for the TE-polarized case which we consider, it follows that $E_r \ni \varepsilon \mapsto \mathscr{E}_\varepsilon \in \mathbf{H}^2_{\mathrm{loc}}(S^1 \times \mathbb{R})$ is Fréchet differentiable. From Theorem 4.4 and (4.33) we deduce that the derivative of this map is uniformly bounded on E_r; hence, this map is uniformly Lipschitz on E_r. Then, using (5.5) we conclude that for some $L_1 > 0$,

$$\|\mathscr{E}_{\varepsilon(t)} - \mathscr{E}_{\tilde{\varepsilon}(t)}\|_{\infty}^{\Omega_{pr}} \leqq L_1 \, \|\varepsilon(t) - \tilde{\varepsilon}(t)\|_{\infty}^{\Omega_{pr}}, \qquad \forall \varepsilon(t), \tilde{\varepsilon}(t) \in E_r. \tag{5.8}$$

Now let $\varepsilon(t)$ and $\tilde{\varepsilon}(t)$ be related by (5.1b) to $\rho(t)$, respectively $\tilde{\rho}(t)$. Then, if we write $\mathscr{E}_{\rho(t)}$ and $\mathscr{E}_{\tilde{\rho}(t)}$ instead of $\mathscr{E}_{\varepsilon(t)}$, respectively $\mathscr{E}_{\tilde{\varepsilon}(t)}$, it follows from

(5.8) and property (ii) of h' and h'' (see Section 1) that for some constant $L_2 > 0$,

$$\|\mathscr{E}_{\rho(t)} - \mathscr{E}_{\tilde{\rho}(t)}\|_\infty^{\Omega_{pr}} \leqq L_2 \|\rho(t) - \tilde{\rho}(t)\|_\infty^{\Omega_{pr}}, \quad \forall t \geqq 0, \forall \rho, \tilde{\rho} \in D_T. \tag{5.9}$$

Furthermore, (5.4) and (5.5) imply that for some constant $C_2 > 0$,

$$\|\mathscr{E}_{\rho(t)}\|_\infty^{\Omega_{pr}} \leqq C_2, \qquad \forall t \geqq 0, \forall \rho \in D_T. \tag{5.10}$$

Because G has by assumption a locally bounded derivative on $[0, \infty)$, we have for some constants L_3, $C_3 > 0$,

$$\begin{aligned} &\|G(\rho(t)) - G(\tilde{\rho}(t))\|_\infty^{\Omega_{pr}} \\ &\quad \leqq L_3 \|\rho(t) - \tilde{\rho}(t)\|_\infty^{\Omega_{pr}}, \qquad \forall t \geqq 0, \quad \forall \rho, \tilde{\rho} \in D_T, \end{aligned} \tag{5.11}$$

and

$$\|G(\rho(t))\|_\infty^{\Omega_{pr}} \leqq C_3, \qquad \forall t \geqq 0, \forall \rho \in D_T. \tag{5.12}$$

Put

$$q \equiv 2 \|\rho_0\|_\infty^{\Omega_{pr}} (C_2^2 L_3 + 2C_2 C_3 L_2)$$

and define the norm

$$\|\rho\|_{\infty,q}^{\Omega_{pr}} = \max_{t \geqq 0} e^{-qt} \|\rho(t)\|_\infty^{\Omega_{pr}},$$

on the space $\mathscr{C}_b([0, \infty) \times \overline{\Omega_{pr}})$. Then $\mathscr{C}_b([0, \infty) \times \overline{\Omega_{pr}})$ is a Banach space of which D_T is a closed subset. Using (5.9)–(5.12) it follows from ρ, $\tilde{\rho} \in D_T$ that

$$\begin{aligned} &e^{-qt} \|T(\rho)(t) - T(\tilde{\rho})(t)\|_\infty^{\Omega_{pr}} \\ &\quad \leqq \|\rho_0\|_\infty^{\Omega_{pr}} e^{-qt} \\ &\qquad \times \int_0^t \|G(\rho(s)) \, |\mathscr{E}_{\rho(s)}|^2 - G(\tilde{\rho}(s)) \, |\mathscr{E}_{\tilde{\rho}(s)}|^2\|_\infty^{\Omega_{pr}} \, ds \\ &\quad \leqq \|\rho_0\|_\infty^{\Omega_{pr}} (C_2^2 L_3 + 2C_2 C_3 L_2) e^{-qt} \\ &\qquad \times \int_0^t e^{qs} \, ds \max_{0 \leqq s \leqq t} e^{-qs} \|\rho(s) - \tilde{\rho}(s)\|_\infty^{\Omega_{pr}} \\ &\quad \leqq \tfrac{1}{2} \|\rho - \tilde{\rho}\|_{\infty,q}^{\Omega_{pr}}, \qquad \forall t \geqq 0. \end{aligned}$$

Hence,

$$\|T(\rho) - T(\tilde{\rho})\|_{\infty,q}^{\Omega_{pr}} \leqq \tfrac{1}{2} \|\rho - \tilde{\rho}\|_{\infty,q}^{\Omega_{pr}}, \qquad \forall \rho, \tilde{\rho} \in D_T,$$

which implies that $T: D_T \to D_T$ is a contraction. We conclude that T has a unique fixed point ρ. Hence we have proved:

THEOREM 5.1. *When the incident field is TE-polarized, there exists for every nonnegative function $\rho_0 \in \mathscr{C}(\overline{\Omega_{pr}})$ which is 1-periodic with respect to x_1, a unique solution ρ of* (1.2), (1.9) *on* $[0, \infty)$ *such that for every $t \geqq 0$ $\rho(t)$: $\Omega_{pr} \to (0, \infty)$ is a continuous 1-periodic function.*

6. Smoothness of the Solution

For simplicity we assume that there exists $a > 0$ such that (see Fig. 1)

$$\overline{\Omega_{pr}} \subset [0, 1] \times (-a, a) \subset \overline{\Omega_1 \cup \Omega_{pr} \cup \Omega_2} \tag{6.1}$$

and let for $k \in \mathbb{N}$, $\tilde{H}^k(S^1 \times (-a, a))$ be the space of all $w \in L^2((0, 1) \times (-a, a))$ such that $w|_{\Omega_{pr}} \in H^k(\Omega_{pr})$, $w|_{\mathcal{O}_a} \in H^k(\mathcal{O}_a)$ with $\mathcal{O}_a \equiv (0, 1) \times (-a, a) \backslash \Omega_{pr}$ and w and all its derivatives up to and including order $k-1$ are 1-periodic with respect to x_1. Analogously, $\tilde{\mathscr{C}}^k(S^1 \times [-a, a])$ is the space of all w: $(0, 1) \times (-a, a) \to \mathbb{C}$ such that $w|_{\Omega_{pr}} \in \mathscr{C}^k(\overline{\Omega_{pr}})$, $w|_{\mathcal{O}_a} \in \mathscr{C}^k(\overline{\mathcal{O}_a})$ and w and all its derivatives up to and including order k are 1-periodic with respect to x_1. $\tilde{\mathbf{H}}^k(S^1 \times (-a, a))$ and $\tilde{\mathscr{C}}^k(S^1 \times (-a, a))$ are the analogous spaces of vector fields defined on $(0, 1) \times (-a, a)$.

When the interfaces between Ω_{pr} and the adjacent layers Ω_1 and Ω_2 are smooth, the following result is valid:

LEMMA 6.1. *If $\mathbf{f} \in \tilde{\mathbf{H}}^k(S^1 \times (-a, a))$ and $\mathbf{u} \in \mathbf{H}^1(S^1 \times (-a, a))$ satisfies $\Delta \mathbf{u} = \mathbf{f}$ on $(0, 1) \times (-a, a)$, then for every a_1 satisfying* (6.1) *and $a_1 < a$, $\mathbf{u} \in \tilde{\mathbf{H}}^{k+2}(S^1 \times (-a_1, a_1))$.*

We assume again that $\mathscr{E}^i$ is TE-polarized and for simplicity that the density before exposure has positive infimum:

$$\inf_{x \in \Omega_{pr}} \rho_0(x) > 0. \tag{6.2}$$

Furthermore, we assume in addition to the assumptions mentioned in Section 1:

(i) $\rho_0 \in \mathscr{C}^\infty(\overline{\Omega_{pr}})$ and ρ_0 and all its derivatives are 1-periodic with respect to x_1.

(ii) $G \in \mathscr{C}^\infty((0, \infty))$.

(iii) $h', h'' \in \mathscr{C}^\infty(\overline{\Omega_{pr}} \times [0, \infty))$ and h', h'' and all its derivatives are 1-periodic with respect to x_1.

(iv) The interfaces between Ω_{pr} and its adjacent layers are smooth.

Let ρ satisfy

$$\rho(t, x) = \rho_0(x) \exp\left[-\int_0^t G(\rho(s, x)) \, |\mathscr{E}_{\rho(s)}(x)|^2 \, ds \right],$$
$$t \geqq 0, \; x \in \Omega_{pr}, \tag{6.3}$$

where $\mathscr{E}_{\rho(t)}$ satisfies

$$\varepsilon(t) \mathscr{E}_{\rho(t)} + \Delta \mathscr{E}_{\rho(t)} = 0 \qquad \text{on} \quad (0, 1) \times \mathbb{R}, \tag{6.4}$$

with

$$\varepsilon(t, x) = h'(x, \rho(t, x)) + ih''(x, \rho(t, x)), \qquad x \in \Omega_{pr}. \tag{6.5}$$

Equation (6.4) follows from (1.8) when units are chosen such that $\omega^2 \mu_0 \varepsilon_0 = 1$ and by using the fact that for TE-polarized light, $\operatorname{div} \mathscr{E}_{\rho(t)} = 0$.

Now, according to the existence theorem $\rho \in \mathscr{C}_b([0, \infty) \times \overline{\Omega_{pr}})$ and using (6.5) it follows that $\varepsilon(t) \mathscr{E}_{\rho(t)} \in \mathbf{L}^2(S^1 \times (-a, a))$. Then Lemma 6.1 implies $\mathscr{E}_{\rho(t)} \in \tilde{\mathbf{H}}^2(S^1(-a_1, a_1))$, $\forall a_1 \in (0, a)$, satisfying (6.1), and with Sobolev's imbedding theorem

$$\mathscr{E}_{\rho(t)} \in \tilde{\mathbf{H}}^2(S^1 \times (-a_1, a_1)) \cap \tilde{\mathscr{C}}(S^1 \times [-a_1, a_1]),$$
$$\forall a_1 \in (0, a) \text{ satisfying (6.1)}. \tag{6.6}$$

By differentiating (6.3) with respect to x_j, one finds

$$\frac{\partial \rho}{\partial x_j}(t, x) = -\rho(t, x) \int_0^t \frac{dG}{d\rho}(\rho(s, x))$$
$$\times |\mathscr{E}_{\rho(s)}(x)|^2 \frac{\partial \rho}{\partial x_j}(s, x) \, ds + f(t, x), \tag{6.7}$$

for some f: $[0, \infty) \times \Omega_{pr} \to \mathbb{R}$. Now, (6.2), (6.3), and (6.6), imply

$$\inf_{0 \leqq s \leqq t} \; \inf_{x \in \Omega_{pr}} \rho(s, x) > 0$$

and, thus, using (ii) and (6.6) it follows that the kernel of the linear integral equation (6.7) for $\partial \rho / \partial x_j$ is a bounded function. Furthermore, one can show that f is a continuous function of t with values in $L^2(\Omega_{pr})$. Then it follows from (6.7) that $\partial \rho / \partial x_j$ is continuous in t with values in $L^2(\Omega_{pr})$ and hence ρ is continuous in t with values in $H^1(\Omega_{pr})$. Then (6.5) implies $\varepsilon(t) \in \tilde{H}^1(S^1 \times (-a, a))$, $\forall t \geqq 0$; thus, with (6.4) and Lemma 6.7, $\mathscr{E}_{\rho(t)}$ is continuous in t with values in $\tilde{\mathbf{H}}^3(S^1 \times (-a_1, a_1))$ for every $a_1 \in (0, a)$ satisfying (6.1). Then, by Sobolev's imbedding theorem, $\mathscr{E}_{\rho(t)}$ is continuous in t

with values in $\mathscr{C}^1(S^1 \times [-a_1, a_1])$ for every $a_1 \in (0, a)$ satisfying (6.1) and from this one can deduce that $t \mapsto f(t) \in \mathscr{C}(\overline{\Omega_{pr}})$ is continuous. Then integral equation (6.7) for $\partial\rho/\partial x_j$ implies that

$$t \mapsto \frac{\partial \rho}{\partial x_j}(t) \in \mathscr{C}(\overline{\Omega_{pr}})$$

is continuous; hence, $t \mapsto \rho(t) \in \mathscr{C}^1(\overline{\Omega_{pr}})$ is continuous and by using (6.5) it follows that $t \mapsto \varepsilon(t) \in \tilde{\mathscr{C}}^1(S^1 \times [-a, a])$ is continuous. We conclude that

$$\left.\begin{array}{c} \rho(t) \in \mathscr{C}^1(\overline{\Omega_{pr}}), \qquad \varepsilon(t) \in \tilde{\mathscr{C}}^1(S^1 \times [-a, a]), \\ \mathscr{E}_{\rho(t)} \in \tilde{\mathsf{H}}^3(S^1 \times (-a_1, a_1)) \cap \tilde{\mathscr{C}}^1(S^1 \times [-a_1, a_1]), \\ \forall a_1 \in (0, a), \text{ satisfying (6.1)}, \end{array}\right\} \tag{6.8}$$

and all these mappings are continuous in t.

By differentiation of (6.7) with respect to x_1 and x_2, one can deduce analogously by using induction that for every $k \geqq 1$,

$$\left.\begin{array}{c} \rho(t) \in \mathscr{C}^k(\overline{\Omega_{pr}}), \qquad \varepsilon(t) \in \tilde{\mathscr{C}}^k(S^1 \times [-a, a]), \\ \mathscr{E}_{\rho(t)} \in \tilde{\mathsf{H}}^{k+2}(S^1 \times (-a_1, a_1)) \cap \tilde{\mathscr{C}}^k(S^1 \times [-a_1, a_1]), \\ \forall a_1 \in (0, a), \text{ satisfying (6.1)}, \end{array}\right\} \tag{6.9}$$

and all mappings depend continuously on t. Hence, in particular, $\rho(t) \in \mathscr{C}^\infty(\overline{\Omega_{pr}})$.

By using arguments similar to those used in the proof of Theorem 4.6, one can deduce from (6.3), (6.4) that for every $k \leqq 1$, $\mathscr{E}_\rho$ and ρ are infinitely differentiable functions of t with values in $\tilde{\mathsf{H}}^{k+2}(S^1 \times (-a_1, a_1))$ for every a_1 as above, respectively in $\mathscr{C}^k(\overline{\Omega_{pr}})$. Hence,

THEOREM 6.2. *When* $\mathscr{E}^i$ *is TE-polarized and* (6.2), (i), (ii), *and* (iii) *hold, then the solution* ρ *of* (1.10), (1.12) *is in* $\mathscr{C}^\infty([0, \infty) \times \overline{\Omega_{pr}})$.

APPENDIX

The Green's functions G_ζ^+, G_ζ^- with the properties (3.21) are given by

$$G_\zeta^\pm(x_1, x_2, y_1, y_2) = \sum_{n=-\infty}^{\infty} \hat{G}_\zeta^\pm(n, x_2, y_1, y_2) e^{2\pi i n x_1},$$

with

$$\hat{G}_\zeta^+(n, x_2, y_1, y_2) = -\frac{\sin(k_n x_2)}{k_n} e^{-2\pi i n y_1 + i k_n y_2} + H(x_2 - y_2)\frac{\sin(k_n(x_2 - y_2))}{k_n} \times e^{-2\pi i n y_1}, \quad \text{when} \quad 4\pi^2 n^2 \neq \zeta,$$

$$\hat{G}_\zeta^+(n, x_2, y_1, y_2) = -x_2 e^{-2\pi i n y_1} + (x_2 - y_2) \times H(x_2 - y_2) e^{-2\pi i n y_1} \quad \text{when} \quad 4\pi^2 n^2 = \zeta,$$

$$\hat{G}_\zeta^-(n, x_2, y_1, y_2) = \hat{G}_\zeta^+(n, -x_2, y_1, -y_2), \qquad \forall n \in \mathbb{Z},$$

where $k_n = (\zeta - 4\pi^2 n^2)^{1/2}$ and H is Heaviside's function.

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