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Monotone Iterative Technique for Partial Differential Equations of First Order

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1. INTRODUCTION

In view of the fact that first-order partial differential equations arise naturally in modelling growth population of cells which constanctly change their properties, a study of stable and chaotic solutions of such equations was initiated in [3]. In this paper we develop monotone iterative technique for first-order partial differential equations and for this purpose we need to discuss existence, uniqueness and comparison results. We also indicate how the monotone sequences obtained may be employed as candidates for Lyapunov functions in stability theory.

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2. Existence, Uniqueness and Comparison Results

We consider the initial value problem for first-order partial differential equation

$$u_t + f(t, x)u_x = g(t, x, u), \qquad u(0, x) = \phi(x), \tag{2.1}$$

where $f \in C[\Omega, R^n]$, $g \in C[\Omega \times R, R]$, $\Omega = [(t, x): 0 \leqslant t \leqslant T$ and $a \leqslant x \leqslant b$, $a, b, x \in R^n]$, $fu_x = \sum_{i=1}^n f_i(t, x) u_{x_i}$ and $\phi \in C^1[[a, b], R]$.

We begin by proving the following comparison result which is needed in our discussion.

THEOREM 2.1. *Assume that*

(A_0) $\alpha, \beta \in C^1[\Omega, R]$, $\alpha_t + f(t, x)\alpha_x \leqslant g(t, x, \alpha)$, $\alpha(0, x) \leqslant \phi(x)$, *and* $\beta_t + f(t, x)\beta_x \geqslant g(t, x, \beta)$, $\beta(0, x) \geqslant \phi(x)$ *for* $(t, x) \in \Omega$;

(A_1) $f(t, x)$ *is quasimonotone nonincreasing in* x *for each* t *and* $0 \geqslant f(t, a)$, $0 \leqslant f(t, b)$;

(A_2) $g(t, x, u_1) - g(t, x, u_2) \leqslant L(u_1 - u_2)$ *whenever* $u_1 \geqslant u_2$ *for some* $L \geqslant 0$. *Then* $\alpha(t, x) \leqslant \beta(t, x)$ *on* Ω.

Proof. Let us first prove the theorem for strict inequalities. For example, we suppose that $\alpha_t + f(t, x)\, \alpha_x < g(t, x, \alpha)$ and $\alpha(0, x) < \phi(x)$ on Ω and prove that $\alpha(t, x) < \beta(t, x)$ on Ω. If this conclusion is not true, then consider the set

$$Z = [(t, x) \in \Omega: \alpha(t, x) \geqslant \beta(t, x)].$$

Let Z_t be the projection of Z on the t-axis and let $t_0 = \inf Z_t$. Clearly $t_0 > 0$ and there exists an $x_0 \in [a, b]$ such that

$$\alpha(t_0, x_0) = \beta(t_0, x_0)$$

$$\alpha(t_0 - h, x_0) < \beta(t_0 - h, x_0) \qquad \text{for sufficiently small } h > 0$$

and

$$\alpha(t_0, x) \leqslant \beta(t_0, x) \qquad \text{for} \quad a \leqslant x \leqslant b.$$

It then follows that $\alpha_t(t_0, x_0) \geqslant \beta_t(t_0, x_0)$ and if $a < x_0 < b$, we also have $\alpha_{x_i}(t_0, x_0) = \beta_{x_i}(t_0, x_0)$, $i = 1, 2, \ldots, n$. In this case,we are led to the following contradiction

$$g(t_0, x_0, \alpha(t_0, x_0)) > \alpha_t + f(t_0, x_0)\alpha_x \geqslant \beta_t + f(t_0, x_0)\beta_x \geqslant g(t_0, x_0, \beta(t_0, x_0)).$$

If, on the other hand, for some j, $x_{0,j} = b_j$ and $x_{0i} < b_i$, $i \neq j$, and $a < x_0$, then we have $\alpha_{x_i}(t_0, x_0) = \beta_{x_i}(t_0, x_0)$, $i \neq j$, and $\alpha_{x_j}(t_0, x_0) \geqslant \beta_{x_j}(t_0, x_0)$. Hence using the assumption (A_1), we obtain $f_j(t_0, x_0) \geqslant f_j(t_0, b) \geqslant 0$ and $f_j(t_0, x_0)\alpha_{x_j}(t_0, x_0) \geqslant f_j(t_0, x_0)\beta_{x_j}(t_0, x_0)$. Consequently, we arrive at the inequality

$$g(t_0, x_0, \alpha(t_0, x_0)) > \alpha_t + f(t_0, x_0)\alpha_x \geqslant \beta_t + f(t_0, x_0)\beta_x \geqslant g(t_0, x_0, \beta(t_0, x_0))$$

which is a contradiction as before. This proves that $\alpha(t, x) < \beta(t, x)$ on Ω. If one of the inequalities in (A_0) is not strict, we set $\tilde{\alpha}(t, x) = \alpha(t, x) - \varepsilon e^{2Lt}$ and note that $\tilde{\alpha} > \alpha$. Then using (A_0) and (A_2), we see that

$$\begin{aligned} \tilde{\alpha}_t + f(t, x)\tilde{\alpha}_x &= \alpha_t + f(t, x)\,\alpha_x - 2L\varepsilon\, e^{2Lt} \\ &\leqslant g(t, x, \alpha) - 2L\varepsilon\, e^{2Lt} \\ &\leqslant g(t, x, \tilde{\alpha}) + L\varepsilon\, e^{2Lt} - 2L\varepsilon\, e^{2Lt} \\ &< g(t, x, \tilde{\alpha}), \end{aligned}$$

and $\tilde{\alpha}(0, x) < \phi(x)$ on Ω. Thus, the foregoing arguments imply that $\tilde{\alpha}(t, x) < \beta(t, x)$ on Ω. Taking limit as $\varepsilon \to 0$, we then get $\alpha(t, x) \leqslant \beta(t, x)$ on Ω and the proof is complete.

For other types of comparison results see [2].

We next prove an existence and uniqueness result for the problem (2.1). See also [1] for a similar result.

THEOREM 2.2. *Assume that* (A_1) *and* (A_2) *hold. Suppose further that*

(A_3) *for each* $(t_0, x_0) \in \Omega$, *there exists a unique solution* $x(t_0, x_0)$ *of*

$$x' = f(t, x), \qquad x(t_0) = x_0, \tag{2.2}$$

on $0 \leqslant t \leqslant T$, $x(t, t_0, x_0)$ *is continuously differentiable with respect to* (t_0, x_0) *and the relation* $(\partial x/\partial t_0)(t, t_0, x_0) + (\partial x/\partial x_0)(t, t_0, x_0)\, f(t_0, x_0) = 0$ *holds*;

(A_4) *for each* $x_0 \in [a, b]$ *and* $y_0 \in R$, *there exists a unique solution* $y(t, 0, y_0; x_0)$ *of*

$$y' = g(t, x(t, 0, x_0), y), \qquad y(0) = y_0, \tag{2.3}$$

on $0 \leqslant t \leqslant T$, *where* $x(t, 0, x_0)$ *is the unique solution of* (2.2), *and* $y(t, 0, y_0; x_0)$ *is continuously differentiable with respect to* (y_0, x_0). *Then there exists a unique solution* $u(t, x)$ *for the problem* (2.1) *on* Ω.

Proof. By (A_3) and (A_4), $x(t, t_0, x_0)$, $y(t, 0, y_0; x_0)$ are unique solutions of (2.2) and (2.3), respectively, on $0 \leqslant t \leqslant T$. Choose $y_0 = \phi(x_0)$ and note that if $x = x(t, 0, x_0)$, then, because of uniqueness, $x_0 = x(0, t, x)$. Also, the solution $(x(t,0,x_0), y(t,0,y_0;x_0))$ of the systems (2.2) and (2.3) is a characteristic equation of (2.1). Hence for each solution of (2.2) and (2.3) we have

$$u(t, x(t,0,x_0)) = y[t, 0, \phi(x_0); x_0] \tag{2.4}$$

and consequently

$$u(t,x) = y[t, 0, \phi(x(0,t,x)); x(0,t,x)], \qquad (t,x) \in \Omega. \tag{2.5}$$

Now using the assumptions (A_3) and (A_4), it is easy to show that $u(t,x)$ defined by (2.5) satisfies (2.1).

To show uniqueness of solutions of (2.1), suppose, if possible, that $u_1(t,x)$, $u_2(t,x)$ are two solutions of (2.1) on Ω. Then setting $\alpha = u_1$, $\beta = u_2$ and applying Theorem 2.1 it follows that $u_1(t,x) \leqslant u_2(t,x)$ on Ω. Similarly we can show that $u_2(t,x) \leqslant u_1(t,x)$ on Ω, proving uniqueness of solutions. Hence the proof is complete.

3. Monotone Iterative Technique

We are now in a position to describe the monotone iterative technique which yields monotone sequences. Specifically we prove the following result.

THEOREM 3.1. *Assume that* (A_0), (A_1) *and* (A_3) *hold with* $\alpha \leqslant \beta$ *on* Ω. *Suppose further that*

(A_5) *for some* $M > 0$, $g(t,x,u_1) - g(t,x,u_2) \geqslant -M(u_1 - u_2)$ *whenever* $\alpha \leqslant u_1 \leqslant u_2 \leqslant \beta$ *on* Ω *and* g_x *exists and is continuous on* $\Omega \times R$. *Then there exist monotone sequences* $\{\alpha_n(t,x)\}$, $\{\beta_n(t,x)\}$ *and the functions* $\rho(t,x)$, $r(t,x)$ *such that if* u *is any solution of* (2.1) *we have*

$$\alpha \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_n \leqslant \rho \leqslant u \leqslant r \leqslant \beta_n \leqslant \cdots \leqslant \beta_1 \leqslant \beta \qquad \text{on } \Omega.$$

Proof. Consider the linear IVP

$$u_t + f(t,x)u_x = G(t,x,u;\eta), \qquad u(0,x) = \phi(x), \tag{3.1}$$

where $G(t,x,u;\eta) = g(t,x,\eta) - M(u-\eta)$ and $\eta \in C[\Omega, R]$ is such that $\alpha \leqslant \eta \leqslant \beta$ on Ω.

Since $g(t,x,\alpha) - G(t,x,\alpha;\eta) \leqslant M(\eta - \alpha) + M(\alpha - \eta) = 0$ by (A_5), it follows that $\alpha_t + f(t,x)\alpha_x \leqslant G(t,x,\alpha;\eta)$, $\alpha(0,x) \leqslant \phi(x)$ on Ω. Similarly, we see that $\beta_t + f(t,x)\beta_x \geqslant G(t,x,\beta;\eta)$, $\beta(0,x) \geqslant \phi(x)$ on Ω. Hence (A_0) holds relative to (3.1). Also, if $u_1 \geqslant u_2$,

$$G(t,x,u_1;\eta) - G(t,x,u_2;\eta) = -M(u_1 - u_2) \leqslant M(u_1 - u_2)$$

and, therefore, (A_2) is satisfied for G. Furthermore, G is linear in u and

$\partial g/\partial x$ is assumed to exist and is continuous, it follows that (see [1, pp. 95–99]) (A_4) is satisfied relative to

$$y' = G(t, x(t, 0, x_0), y; \eta(t, x(t, 0, x_0))), \qquad y(0) = y_0. \tag{3.2}$$

As a result, by Theorem 2.2, there exists a unique solution $u(t, x)$ of (3.1) on Ω for every $\eta \in C[\Omega, R]$ such that $\alpha \leqslant \eta \leqslant \beta$ on Ω.

We define a mapping A by $A\eta = u$ where u is the unique solution of (3.1) corresponding to η. Concerning this mapping A, we shall show that (i) $\alpha \leqslant A\alpha$, $\beta \geqslant A\beta$, and (ii) A is monotone on the sector $[\alpha, \beta]$, that is, if $\alpha \leqslant \eta_1 \leqslant \eta_2 \leqslant \beta$, then $A\eta_1 \leqslant A\eta_2$.

Let $\eta = \alpha$ and let $A\alpha = \alpha_1$ where α_1 is the unique solution of (3.1). Then we have $\alpha_t + f(t, x)\alpha_x \leqslant G(t, x, \alpha; \alpha)$, $\alpha(0, x) \leqslant \phi(x)$, and $\alpha_{1t} + f(t, x)\alpha_{1x} = G(t, x, \alpha_1; \alpha)$, $\alpha_1(0, x) = \phi(x)$ on Ω. By Theorem 2.1, this implies that $\alpha \leqslant \alpha_1 = A\alpha$. Similarly we can show that $\beta \geqslant A\beta$ proving (i). To prove (ii), let $\eta_1, \eta_2 \in C[\Omega, R]$ be such that $\alpha \leqslant \eta_1 \leqslant \eta_2 \leqslant \beta$ and let $A\eta_1 = u_1$, $A\eta_2 = u_2$ where u_1, u_2 are the unique solutions of (3.1) corresponding to $\eta = \eta_1$, $\eta = \eta_2$, respectively. Then $u_{1t} + f(t, x)u_{1x} = G(t, x, u_1; \eta_1) \leqslant G(t, x, u_1; \eta_2)$ and $u_{2t} + f(t, x)u_{2x} = G(t, x, u_2; \eta_2)$. Also $u_1(0, x) = u_2(0, x)$. Hence by Theorem 2.1 we have $u_1 \leqslant u_2$ on Ω. This proves $A\eta_1 \leqslant A\eta_2$.

We now define the sequences $\{\alpha_n(t, x)\}$, $\{\beta_n(t, x)\}$ with $\alpha = \alpha_0$, $\beta = \beta_0$ such that $\alpha_{n+1} = A\alpha_n$, $\beta_{n+1} = A\beta_n$. Because of the properties (i), (ii) of A, it is easy to conclude that

$$\alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_n \leqslant \beta_n \leqslant \cdots \leqslant \beta_1 \leqslant \beta_0 \qquad \text{on } \Omega.$$

Consider the sequence $\{\alpha_n(t, x)\}$ and note that $\alpha_n(t, x) = \mathbf{y}_n[t, 0, \phi(x(0, t, x)); x(0, t, x)]$ on Ω where $\mathbf{y}_n = \mathbf{y}_n(t, 0, y_0; x_0)$ is the unique solution of (3.2) such that $y_0 = \phi(x_0)$. (See (2.4) and (2.5).) Thus $\alpha_n(t, x(t, 0, x_0)) = \mathbf{y}_n(t, 0, \phi(x_0); x_0)$ and $\alpha \leqslant \mathbf{y}_n \leqslant \beta$. Since $\{\mathbf{y}_n\}$ is monotone sequence, it is easy to conclude that $\mathbf{y}_n(t, 0, \phi(x_0); x_0)$ converges uniformly and monotonically as $n \to \infty$. Suppose that $\lim_{n\to\infty} \mathbf{y}_n(t, 0, \phi(x_0); x_0) = \mathbf{y}(t, 0, \phi(x_0); x_0)$ on $0 \leqslant t \leqslant T$. Then it is clear that $\mathbf{y}'(t, 0, \phi(x_0); x_0) = g(t, x(t, 0, x_0), \mathbf{y}(t, 0, \phi(x_0); x_0))$, $\mathbf{y}(0) = \phi(x_0)$. Consequently, we can now define $\rho(t, x) = \mathbf{y}[t, 0, \phi(x(0, t, x)); x(0, t, x)]$ on Ω. Similar arguments hold relative to the sequence $\{\beta_n(t, x)\}$ and one defines $r(t, x) = \bar{y}[t, 0, \phi(x(0, t, x)); x(0, t, x)]$ on Ω.

Finally we show that $\alpha \leqslant \rho \leqslant u \leqslant r \leqslant \beta$ on Ω where u is any solution of (2.1) such that $\alpha \leqslant u \leqslant \beta$ on Ω. For this it is enough to show that $\alpha_n \leqslant u \leqslant \beta_n$ on Ω and this we do by induction. Suppose that $\alpha_k \leqslant u \leqslant \beta_k$ for some k on Ω. Then we have

$$\alpha_{k+1,t} + f(t, x)\alpha_{k+1,x} = G(t, x, \alpha_{k+1}; \alpha_k), \qquad \alpha_{k+1}(0, x) = \phi(x),$$
$$u_t + f(t, x)u_x = g(t, x, u) \geqslant G(t, x, u; \alpha_k), \qquad u(0, x) = \phi(x).$$

This implies by Theorem 2.1 that $\alpha_{k+1} \leqslant u$ on Ω. We can show similarly that $u \leqslant \beta_{k+1}$. Hence it follows that $\alpha_n \leqslant u \leqslant \beta_n$ on Ω for all n, which proves the claim.

Remark. We note that the limit functions ρ, r need not be solutions of (2.1) in general. However, if (A_4) holds, then it follows that they are actually solutions of (2.1). Furthermore, this assumption also implies by Theorem 2.2 that the solutions of (2.1) are unique and hence we obtain from Theorem 3.1

$$\alpha \leqslant \rho = u = r \leqslant \beta \qquad \text{on } \Omega.$$

Usually in several situations, the constructed monotone sequences converge to extremal solutions of the given problem. Unfortunately, such a claim cannot be made in the present case since ρ, r need not even be continuously differentiable. Nonetheless, the sequences offer bounds on solutions which is of practical value.

4. Application to Construction of Lyapunov Functions

Consider the case $a = -b$, $b > 0$, $g(t, x, u) = g(t, u)$ and $\alpha \equiv 0$. Suppose that $\phi(x)$ is a positive definite function. Then Theorem 3.1 gives

$$\alpha_{nt} + f(t, x)\alpha_{nx} = g(t, \alpha_{n-1}) - M(\alpha_n - \alpha_{n-1}), \qquad \alpha_n(0, x) = \phi(x).$$

Setting $V(t, x) = \alpha_n(t, x)$ for some fixed n and noting that α_n is nondecreasing, we can conclude by (A_5) that

$$V_t + f(t, x) V_x \leqslant g(t, V) \qquad \text{on } \Omega.$$

If, on the other hand, we use β_n and set $V = \beta_n$, we get the reversed inequality. Since α_n, β_n are computable, this observation shows that appropriate members of these sequences can be chosen as candidates for Lyapunov functions in the theory of stability. As a trivial example, choose $n = 1$, $g(t, 0) \equiv 0$ and $\alpha_1(t, x) = V(t, x)$. Then $V(t, x) = e^{-Mt}\phi[x(0, t, x)]$. For stability theory in this setup see [2].

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