NOTE

A Note on Convergence of Linear Positive Operators

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Recently Shisha and Mond [5, 6] and DeVore [1] determined a quantitative estimate for the degree of convergence of linear positive operators to a given continuous function on a closed and bounded interval from the degrees of convergence to the test functions $x^k$, $k = 0, 1, 2$. Ditzian [2] modified these results to operators defined for functions on $[0, \infty)$ or $(-\infty, \infty)$.

Following Ditzian [2], we define operators of the type $\mathcal{K}(T, S, \mu)$.

DEFINITION 1.1. Let $T \subset (-\infty, \infty)$ be closed, let $-\infty < a < b < \infty$, and set $S = T \cap [a, b]$. Let $\mu(t)$ be a real-valued function on $T$ satisfying $\mu(t) \geq 1$, $t \in T$. A sequence $\{L_n\}$ of linear positive operators is said to be of type $\mathcal{K}(T, S, \mu)$ if the domain of each $L_n$ consists of all functions (or all measurable functions) $f$ on $T$ satisfying there

$$|f(t)| \leq M(f)(t^2 + 1) \mu(t)$$

and if

$$\|L_n t^k(x) - x^k\|_{C^{1+\delta}} = O(1), \quad n \to \infty, k = 0, 1, 2.$$

and

$$\|L_n(t-x)^2 \mu(t)(x)\|_{C^{1+\delta}} \leq K \|L_n(t-x)^2(x)\|_{C^{1+\delta}} = O(1),$$

$K$ being a constant.

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The purpose of this note is to sharpen Theorem 2.1(C) of [2]. We prove

**Theorem 2.1.** Let $A$ be a positive number and let $S_1 = [a_1, b_1] \subset [a, b]$ so that for some $\eta > 0$, $[a_1 - \eta, b_1 + \eta] \cap T \cap \{(-\infty, \infty) - [a, b]\} = \emptyset$. Let $\{L_n\}$ be a sequence of linear positive operators of type $\mathfrak{N}(T, S, \mu)$ and let $f \in C^1[a, b]$. Then, for $n \geq 1$,

$$
\|L_n f - f\|_{C(S_1)} \leq \left(1 + \frac{1}{2A}\right)(\mu_n f^+; A\mu_n) + L\mu_n^2.
$$

where $\omega$ is the modulus of continuity on $[a, b]$, and $\mu_n = \|L_n(t - x)^2(x)\|^{1/2}$, $\|\|$ being the sup-norm on $S_1$ and $L$ a constant.

If $(L_n 1)(x) = 1$ and $(L_n t)(x) \equiv x$, then (2.1) reduces to

$$
\|L_n f - f\|_{C(S_1)} \leq \left(1 + \frac{1}{2A}\right)(\mu_n f^+; A\mu_n) + L\mu_n^2.
$$

**Proof.** For $x \in S_1$, $t \in [a_i - \eta, b_i + \eta] \cap T$ we write

$$
f(t) - f(x) = (t - x) f'(x) + \int_x^t (f'(\xi) - f'(x)) d\xi.
$$

Using the proof in [7] and the inequality

$$
|f'(\xi) - f'(x)| \leq \left(1 + \frac{|\xi - x|}{\delta}\right) \omega(f'; \delta), \quad \delta > 0.
$$
we get

$$
|L_n f(x) - f(x)(L_n 1)(x)|
\leq \left|L_n(t - x)(x)\right| |f'(x)| + \omega(f'; \delta) L_n \left[\int_x^t \left(1 + \frac{|\xi - x|}{\delta}\right) d\xi\right](x),
$$

Choosing $\delta = A\mu_n$, we pursue a slight modification of the proof given by Ditzian [2], details of which we may omit.

**Example.** The positive linear operators obtained from the inversion of...
the Weierstrass transform for measurable functions \( f \) on \( (-\infty, \infty) \) are given by

\[
(L_n f)(x) = \left( \frac{n}{4\pi} \right)^{1/2} \int_{-\infty}^{\infty} \exp \left( -\left( t - x \right)^2 \frac{n}{4} \right) f(t) \, dt, \quad n \geq 1.
\]

From [2] we have

\[
(L_n 1)(x) = 1, \quad (L_n t)(x) = x, \quad (L_n t^2)(x) = x^2 + \frac{2}{n},
\]

so that \( (L_n (t - x)^2)(x) = 2/n \). Also \( L_n \in \mathcal{L}(T, \mu) \), where \( T = (-\infty, \infty) \) and \( \mu(t) = e^{t^2/4} \). Choosing \( A = 1/\sqrt{2} \) in (2.2), we get

\[
\|L_n f - f\|_{L^1(S)} \leq \left( \frac{1 + \sqrt{2}}{\sqrt{n}} \right) \omega \left( f'; \frac{1}{\sqrt{n}} \right) + L_1(a, b, \eta) n^{-1},
\]

where \( L_1 \) being a constant, which is sharper than the corresponding estimate due to Ditzian [2].

It is worthwhile to point out that Theorem 4 of Mohapatra [3] and the result of Mond and Vasudevan [4] can be improved similarly.

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**REFERENCES**