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Equating decomposition numbers for different primes

Gordon James [a] and Andrew Mathas [b,*]

[a] *Department of Mathematics, Imperial College, 180 Queen's Gate, London SW7 2BZ, UK*
[b] *School of Mathematics F07, University of Sydney, Sydney NSW 2006, Australia*

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Abstract

This paper shows that certain decomposition numbers for the Iwahori–Hecke algebras of the symmetric groups and the q-Schur algebras at different roots of unity in characteristic zero are equal. To prove our results we first establish the corresponding theorem for the canonical basis of the level-one Fock space and then apply deep results of Ariki and Varagnolo–Vasserot.

1. Introduction

Throughout this note we adopt the standard notation for the modular representation theory of the symmetric groups, as can be found in [7,16].

Consider the following two submatrices of the p-modular decomposition matrices of the symmetric groups $\mathfrak{S}_n$:

* Corresponding author.
E-mail addresses: g.james@ic.ac.uk (G. James), a.mathas@maths.usyd.edu.au (A. Mathas).
URLs: http://www.ma.ic.ac.uk/~gdj/, http://www.maths.usyd.edu.au/u/mathas/.

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$$
\begin{array}{l|llllllll}
10,1 & 1 \\
9,2 & 1 & 1 \\
7,4 & . & 1 & 1 \\
7,2^2 & 1 & 1 & 1 & 1 \\
6,5 & . & . & 1 & . & 1 \\
6,2^2,1 & 1 & 1 & 1 & 1 & 1 & 1 \\
4^2,3 & . & . & 1 & 1 & 1 & . & 1 \\
4^2,2,1 & . & 1 & 1 & 1 & \mathbf{2} & 1 & 1 & 1
\end{array}
\qquad\qquad
\begin{array}{l|llllllll}
18,3 & 1 \\
17,4 & 1 & 1 \\
13,8 & . & 1 & 1 \\
13,4^2 & 1 & 1 & 1 & 1 \\
12,9 & . & . & 1 & . & 1 \\
12,4^2,1 & 1 & 1 & 1 & 1 & 1 & 1 \\
8^2,5 & . & . & 1 & 1 & 1 & . & 1 \\
8^2,4,1 & . & 1 & 1 & 1 & \mathbf{1} & 1 & 1 & 1
\end{array}
$$

$n = 11$ and $p = 3$ $\qquad$ $n = 21$ and $p = 5$

The two matrices are identical except for the labeling and the two bold faced entries (omitted entries are zero).

This paper was motivated by our attempts to compute the bold faced entry in the matrix above for $n = 21$ and $p = 5$. At the outset we knew that this number was either 1 or 2; however, we were unable to determine which of these possibilities was correct. Lübeck and Müller [13] have shown that this multiplicity is equal to 1 using computer calculations; see [14, Section 5.3] for details.

Note that the partitions $(4^2, 2, 1)$ and $(8^2, 4, 1)$ both have p-weight 3 and p-core $(p - 2, p - 2)$ for $p = 3$ and $p = 5$, respectively. Martin and Russell [15] have claimed that all of the p-modular decomposition numbers of the symmetric group of p-weight 3 are 0 or 1 when $p > 3$; unfortunately, their proof contains a gap when dealing with partitions with p-core $(p - 2, p - 2)$. This particular case is still open when $p > 5$; as a consequence, the claim in [15] that the decomposition numbers are always 0 or 1 for partitions of p-weight 3 when $p > 3$ is in doubt.

In this paper we prove a general theorem which indicates why the matrices above are very similar. This result is not about the decomposition matrices of the symmetric groups but rather about the closely related decomposition matrices of the q-Schur algebras $\mathcal{S}_{\mathbb{C},q}(n)$ at a complex root of unity. Our result shows that certain decomposition numbers of $\mathcal{S}_{\mathbb{C},q}(n)$ and $\mathcal{S}_{\mathbb{C},q'}(m)$ are equal for specified $m > n$. The decomposition matrix for the Iwahori–Hecke algebra $\mathcal{H}_{\mathbb{C},q}(\mathfrak{S}_n)$ is a submatrix of the decomposition matrix of $\mathcal{S}_{\mathbb{C},q}(n)$; so, in particular, our result shows that for the Iwahori–Hecke algebras all of the decomposition multiplicities above for $(n, e) = (11, 3)$ and $(m, e') = (21, 5)$ are equal (in the Hecke algebra case the multiplicities $d_{(4^2,2,1),(6,5)}$, when $e = 3$, and $d_{(8^2,4,1),(12,9)}$, when $e = 5$, are both equal to 1).

2. Abacuses and the q-Schur algebra

In order to state our results we recall the abacus notation for partitions introduced in [8]. Fix an integer $e \geqslant 2$. An e-abacus is an abacus with e runners, which we label from left to right as $\rho_0, \dots, \rho_{e-1}$. Number the bead positions on the abacus by $0, 1, 2, \dots$ reading from left to right and then top to bottom; so the bead positions on ρ_r are numbered $r + me$ for $m \geqslant 0$. We order the beads on a given e-abacus according to their bead positions.

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition. The length of λ is the smallest integer $\ell(\lambda)$ such that $\lambda_i = 0$ for all $i > \ell(\lambda)$. If $k \geqslant \ell(\lambda)$ then λ has a (unique) e-abacus representation with k beads; namely, the e-abacus with beads at positions

$$\lambda_k, \lambda_{k-1} + 1, \ldots, \lambda_2 + k - 2, \lambda_1 + k - 1.$$

The bead positions on an abacus for λ encode the first column hook lengths, so this gives a natural bijection between the abacuses with k beads and the partitions of length at most k. For our purposes it is important that the mth bead on an abacus for λ corresponds to row $l = k - m + 1$ of λ (row l of λ is empty if $l > \ell(\lambda)$).

In what follows we fix $k \geqslant 0$ and (with few exceptions) consider only abacuses with k beads; or, equivalently, partitions of length at most k.

We want to compare e-abacuses with $(e+1)$-abacuses. Fix an integer α with $0 \leqslant \alpha \leqslant e$. If λ is a partition with $\ell(\lambda) \leqslant k$ then λ can be represented on an e-abacus with k beads. Let λ^+ be the partition corresponding to the $(e+1)$-abacus obtained by inserting an *empty* runner before ρ_α in the e-abacus for λ (if $\alpha = e$ we insert an empty eth runner). Let $\rho_0^+, \ldots, \rho_e^+$ be the runners of the $(e+1)$-abacus of λ^+; then $\rho_r^+ = \rho_r$ if $r < \alpha$, ρ_α^+ is empty, and $\rho_r^+ = \rho_{r-1}$ if $r > \alpha$.

Although our notation does not reflect this the partition λ^+ does depend upon both the choice of α and the choice of k.

2.1. Example. Suppose that $e = 3$, $k = 4$, and $\alpha = 2$. Let $\lambda = (4^2, 3)$. Then the abacuses (with 4 beads) for λ, λ^+, and $\lambda^{++} = (\lambda^+)^+$ are as follows:

$$\begin{array}{ccc}
\begin{array}{ccc}
\multicolumn{3}{c}{e=3} \\ \hline
2 & 0 & 1 \\ \hline
\bullet & \cdot & \cdot \\
\cdot & \bullet & \cdot \\
\bullet & \bullet & \cdot \\
\cdot & \cdot & \cdot \\
\multicolumn{3}{c}{\lambda = (4^2,3)} \\ \hline
\end{array}
&
\begin{array}{cccc}
\multicolumn{4}{c}{e'=4} \\ \hline
0 & 1 & 2 & 3 \\ \hline
\bullet & \cdot & \cdot & \cdot \\
\cdot & \bullet & \cdot & \cdot \\
\bullet & \bullet & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\multicolumn{4}{c}{\lambda^+ = (6^2,4)} \\ \hline
\end{array}
&
\begin{array}{ccccc}
\multicolumn{5}{c}{e''=5} \\ \hline
1 & 2 & 3 & 4 & 0 \\ \hline
\bullet & \cdot & \cdot & \cdot & \cdot \\
\cdot & \bullet & \cdot & \cdot & \cdot \\
\bullet & \bullet & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\multicolumn{5}{c}{\lambda^{++} = (8^2,5)} \\ \hline
\end{array}
\end{array}$$

We have labeled the runners by their residues which will be introduced below.

The reader is invited to check that the partitions which label the decomposition matrix for $n = 21$ and $p = 5$ in the introduction are precisely the partitions λ^{++} as λ runs over the corresponding partitions of 11. We emphasize that the empty runner can be inserted anywhere in the abacus.

We are now almost ready to describe our main result. As in the introduction let q be a primitive eth root of unity in $\mathbb{C}$ and let $\mathcal{S}_{\mathbb{C},q}(n)$ be the q-Schur algebra defined over the complex numbers with parameter q; so $\mathcal{S}_{\mathbb{C},q}(n) = \mathcal{S}_{\mathbb{C},q}(n,n)$ in the notation of Dipper and James [3].

For each partition λ of n Dipper and James [4] (see also [16]), defined a right $\mathcal{S}_{\mathbb{C},q}(n)$-module W_q^λ, called a Weyl module. There is a natural bilinear form $\langle\,,\rangle$ on W_q^λ and $\operatorname{Rad} W_q^\lambda = \{x \in W_q^\lambda \mid \langle x, y\rangle = 0 \text{ for all } y \in W_q^\lambda\}$ is an $\mathcal{S}_{\mathbb{C},q}(n)$-submodule of W_q^λ; set $L_q^\lambda = W_q^\lambda / \operatorname{Rad} W_q^\lambda$. Dipper and James showed that L_q^λ is an absolutely irreducible $\mathcal{S}_{\mathbb{C},q}(n)$-module and, further, that every irreducible $\mathcal{S}_{\mathbb{C},q}(n)$-module arises uniquely in this way. Let $[W_q^\lambda : L_q^\mu]$ be the multiplicity of the simple module L_q^μ as a composition factor of W_q^λ.

Let q' be a primitive $(e+1)$st root of unity in $\mathbb{C}$. Then we also have the q'-Schur algebra $\mathcal{S}_{\mathbb{C},q'}(m)$ and modules $W_{q'}^\nu$ and $L_{q'}^\nu$, for ν a partition of m. We shall prove the following theorem.

2.2. Theorem. *Suppose that λ and μ are partitions of n of length at most k. Then*

$$\left[W_q^\lambda : L_q^\mu\right] = \left[W_{q'}^{\lambda^+} : L_{q'}^{\mu^+}\right].$$

It may not be clear to the reader that this result is really saying that the decomposition matrices of the blocks containing W_q^λ and $W_{q'}^{\lambda^+}$ are equal on those rows indexed by partitions of length at most k, when we order the rows of these matrices in a way compatible with dominance. This follows from Lemma 3.3 below.

Let $\mathcal{H}_{\mathbb{C},q}(\mathfrak{S}_n)$ be the Iwahori–Hecke algebra of $\mathfrak{S}_n$ [2,16]. Then for each partition λ there is an $\mathcal{H}_{\mathbb{C},q}(\mathfrak{S}_n)$-module S_q^λ, called a Specht module, which carries an associative bilinear form. Let $D_q^\lambda = S_q^\lambda / \operatorname{Rad} S_q^\lambda$; then D_q^λ is either zero or absolutely irreducible and every irreducible $\mathcal{H}_{\mathbb{C},q}(\mathfrak{S}_n)$-module arises uniquely in this way. Moreover, $D_q^\lambda \neq 0$ if and only if λ is e-regular; that is, if and only if no e non-zero parts of λ are equal.

There is a q-analogue of the Schur functor which maps W_q^λ to S_q^λ and L_q^λ to D_q^λ for each λ; in particular, this shows that $[W_q^\lambda : L_q^\mu] = [S_q^\lambda : D_q^\mu]$ whenever μ is e-regular. Hence, from Theorem 2.2 we obtain the following corollary.

2.3. Corollary. *Suppose that λ and μ are partitions of n of length at most k such that μ is e-regular. Then $[S_q^\lambda : D_q^\mu] = [S_{q'}^{\lambda^+} : D_{q'}^{\mu^+}]$.*

It is tempting to speculate that there is some form of category equivalence underpinning these results. However, in general, there are a different number of simple modules in the blocks for λ and λ^+, so these blocks are certainly not Morita equivalent.

Rather than prove our comparison theorem for decomposition numbers directly we prove a stronger result relating the LLT-polynomials [10,12]. We now recall the notation needed to describe this.

3. The Fock space and $U_v(\widehat{\mathfrak{sl}}_e)$—the regular case

Let v be an indeterminate over $\mathbb{C}$. The Fock space is the infinite rank free $\mathbb{C}[v, v^{-1}]$-module $\mathcal{F} = \bigoplus_{n\geqslant 0}\bigoplus_{\lambda\vdash n}\mathbb{C}[v, v^{-1}]\lambda$. The Fock space has a natural structure as a module for the affine quantum group $U_v(\widehat{\mathfrak{sl}}_e)$; we will describe how the negative part $U_v^-(\widehat{\mathfrak{sl}}_e)$ of $U_v(\widehat{\mathfrak{sl}}_e)$ acts on $\mathcal{F}$ since this is all we shall need (full details can be found in [10,16]).

The diagram of a partition λ is the set $[\lambda] = \{(c, d) \in \mathbb{N}^2 \mid d \leqslant \lambda_c\}$. A node is any ordered pair $(c, d) \in \mathbb{N}^2$; in particular, all of the elements of $[\lambda]$ are nodes. The e-residue of the node $x = (c, d)$ is $\operatorname{res}_e(x) = d - c \pmod e$; x is an i-node if $\operatorname{res}_e(x) = i$.

A node x is an addable node of λ if $[\lambda] \cup \{x\}$ is the diagram of a partition (and $x \notin [\lambda]$); similarly, x is removable if $[\lambda] \setminus \{x\}$ is the diagram of a partition (and $x \in [\lambda]$). For $i = 0, \dots, e-1$ let $A_i(\lambda)$ be the set of addable i-nodes for λ and $R_i(\lambda)$ be the set of removable i-nodes. Given two nodes $x = (c, d)$ and $y = (a, b)$ say that y is above x if $c > a$; if y is above x we write $y \succ x$.

In order to define the action of $U_v^-(\widehat{\mathfrak{sl}}_e)$ on $\mathcal{F}$ for $i = 0, \dots e-1$ write $\lambda \xrightarrow{i} \nu$ if ν is a partition of $n+1$ and $[\nu] = [\lambda] \cup \{x\}$ for some addable i-node x. Finally, we set $N_i(\lambda, \nu) = \#A_i(\lambda, \nu) - \#R_i(\lambda, \nu)$ where $A_i(\lambda, \nu) = \{y \in A_i(\lambda) \mid y \succ x\}$ and $R_i(\lambda, \nu) = \{y \in R_i(\lambda) \mid y \succ x\}$.

Let $F_0, \dots, F_{e-1}$ be the Chevalley generators of $U_v^-(\widehat{\mathfrak{sl}}_e)$. Then the action of $U_v^-(\widehat{\mathfrak{sl}}_e)$ on $\mathcal{F}$ is determined by

$$F_i\lambda = \sum_{\lambda \xrightarrow{i} \nu} v^{N_i(\lambda,\nu)}\nu, \tag{3.1}$$

for $0, 1, \dots, e-1$.

Let $\Lambda_0, \dots, \Lambda_{e-1}$ be the fundamental weights of $U_v(\widehat{\mathfrak{sl}}_e)$ and let $L(\Lambda_0)$ be the irreducible integrable highest weight module of high weight Λ_0. Then $L(\Lambda_0) \cong U_v(\widehat{\mathfrak{sl}}_e)\varnothing = U_v^-(\widehat{\mathfrak{sl}}_e)\varnothing$ as a $U_v(\widehat{\mathfrak{sl}}_e)$-module [10], where $\varnothing \in \mathcal{F}$ is the empty partition.

Let $\bar{\ }$ be the bar involution on $U_{\mathcal{A}}(\widehat{\mathfrak{sl}}_e)$, the Kostant–Lusztig $\mathcal{A}$-form of $U_v(\widehat{\mathfrak{sl}}_e)$ (where $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$). Then Lascoux, Leclerc, and Thibon [10] showed that $L(\Lambda_0)$ has a basis $\{B_\mu \mid \mu\ e\text{-regular}\}$ which is uniquely determined by the requirements that $\overline{B}_\mu = B_\mu$ and

$$B_\mu = \sum_{\substack{\lambda\vdash n\\ \mu \trianglerighteq \lambda}} b_{\lambda\mu}(v)\lambda$$

for some polynomials $b_{\lambda\mu}(v) \in \mathbb{Z}[v]$ such that $b_{\mu\mu}(v) = 1$ and $b_{\lambda\mu}(v) \in v\mathbb{Z}[v]$ whenever $\lambda \neq \mu$. This basis is the Kashiwara–Lusztig canonical basis of $L(\Lambda_0)$.

In particular, note that Lascoux, Leclerc, and Thibon [10] showed that $b_{\lambda\mu}(v) = 0$ if either $|\lambda| \neq |\mu|$ or if λ and μ have different e-cores.

We want to compare the actions of $U_v(\widehat{\mathfrak{sl}}_e)$ and $U_v(\widehat{\mathfrak{sl}}_{e+1})$ on the Fock space $\mathcal{F}$. In order to distinguish between these two algebras let $F_0^+, \dots, F_e^+$

be the Chevalley generators of $\boldsymbol{U}_v^-(\widehat{\mathfrak{sl}}_{e+1})$, let $\Lambda_0^+,\dots,\Lambda_e^+$ be its fundamental weights, and let $\mathcal{F}^+\cong\mathcal{F}$ be the Fock space for the $\boldsymbol{U}_v(\widehat{\mathfrak{sl}}_{e+1})$-action. Then $L(\Lambda_0^+)\cong\boldsymbol{U}_v^-(\widehat{\mathfrak{sl}}_{e+1})\varnothing$ as a $\boldsymbol{U}_v(\widehat{\mathfrak{sl}}_{e+1})$-module. Given an $(e+1)$-regular partition ν, let

$$B_\nu^+=\sum_{\nu\trianglerighteq\sigma}b_{\sigma\nu}^+(v)\sigma$$

be the corresponding canonical basis element of $L(\Lambda_0^+)\subseteq\mathcal{F}^+$.

We can now state a stronger version of Corollary 2.3.

3.2. Theorem. *Suppose that λ and μ are partitions of n of length at most k and suppose that μ is e-regular. Then $b_{\lambda\mu}(v)=b_{\lambda^+\mu^+}^+(v)$.*

Ariki [1, Proposition 4.3(2)] has shown that the polynomials $b_{\lambda\mu}(v)$ at $v=1$ compute the decomposition multiplicities; explicitly, $[S_q^\lambda:D_q^\mu]=b_{\lambda\mu}(1)$ and $[S_{q'}^{\lambda^+}:D_{q'}^{\mu^+}]=b_{\lambda^+\mu^+}^+(1)$. Consequently, Theorem 3.2 implies Corollary 2.3. The result also hints at additional structure because, conjecturally, the polynomials $b_{\lambda\mu}(v)$ and $b_{\lambda^+\mu^+}^+(v)$ also describe the Jantzen filtrations of S_q^λ and $S_{q'}^{\lambda^+}$; see [9,10].

We prove Theorem 3.2 directly using the LLT algorithm; in the next section we will extend this argument to cope with the case where μ is not necessarily e-regular.

Fred Goodman has pointed out that Theorem 3.2 can also be deduced from [6, Theorem 5.3]. We remark that the origin of our results, and those of Goodman and Wenzl, is that the $b_{\lambda\mu}(v)$ are parabolic Kazhdan–Lusztig polynomials for the parabolic subgroup $\mathfrak{S}_k$ of the extended affine Weyl group $\widehat{\mathfrak{S}}_k$ [6,12,17]; in turn, the parabolic Kazhdan–Lusztig polynomials are naturally indexed by the alcoves and, generically, the alcove geometry does not depend on k or e.

We begin the proof of Theorem 3.2 with the following lemma which is largely book keeping. For example, the result implicitly assumes that μ^+ is $(e+1)$-regular.

3.3. Lemma. *Let λ and μ be partitions of n of length at most k. Then*

(i) *μ^+ is $(e+1)$-regular;*
(ii) *λ and μ have the same e-core if and only if λ^+ and μ^+ have the same $(e+1)$-core; and*
(iii) *if λ and μ have the same e-core then $|\lambda^+|=|\mu^+|$.*

Proof. A partition is $(e+1)$-regular if and only if its $(e+1)$-abacus does not contain a string of $e+1$ consecutive beads. Hence, μ^+ is $(e+1)$-regular since the runner ρ_α^+ is empty; this proves (i). (In fact, if μ is e-regular then so is μ^+.)

Next, recall that the e-abacus for the e-core of λ is obtained by rearranging the beads on each runner of the e-abacus for λ in such a way that no bead has an empty bead position above it. Hence, if κ is the e-core of λ then κ^+ is the $(e+1)$-core of λ^+, so (ii) follows.

For (iii) define w by $|\lambda| = |\kappa| + we$; in other words, w is the e-weight of λ. Now, w can be read off the e-abacus for λ by adding up, for each bead β, the number of empty bead positions which are above β and also on the same runner. Consequently, w is also the $(e+1)$-weight of λ^+; hence, $|\lambda^+| = |\kappa^+| + w(e+1)$. Lastly, since κ is also the e-core of μ it follows that μ is also a partition of e-weight w and that $|\mu^+| = |\kappa^+| + w(e+1) = |\lambda^+|$, as required. □

We remark that if λ and μ are partitions of n with different e-cores then, in general, it is *not* true that $|\lambda^+| = |\mu^+|$.

Let $\mathcal{F}_{>k}$ be the $\mathbb{C}[v, v^{-1}]$-submodule of $\mathcal{F}$ spanned by the partitions of length strictly greater than k. By (3.1) $\mathcal{F}_{>k}$ is a $\boldsymbol{U}_v^-(\widehat{\mathfrak{sl}}_e)$-submodule of $\mathcal{F}$; it is not, however, a $\boldsymbol{U}_v(\widehat{\mathfrak{sl}}_e)$-submodule. Therefore, $\mathcal{F}_k = \mathcal{F}/\mathcal{F}_{>k}$ is a $\boldsymbol{U}_v^-(\widehat{\mathfrak{sl}}_e)$-module. We abuse notation and identify the elements of $\mathcal{F}$ with their images in $\mathcal{F}_k$; with this understanding, $\{\lambda \mid \ell(\lambda) \leqslant k\}$ is a basis of $\mathcal{F}_k$.

Similarly, $\mathcal{F}_k^+ = \mathcal{F}^+/\mathcal{F}_{>k}^+$ is a $\boldsymbol{U}_v^-(\widehat{\mathfrak{sl}}_{e+1})$-module. We want to compare the action of $\boldsymbol{U}_v^-(\widehat{\mathfrak{sl}}_e)$ on $\mathcal{F}_k$ with the action of $\boldsymbol{U}_v^-(\widehat{\mathfrak{sl}}_{e+1})$ on $\mathcal{F}_k^+$; to do this we reinterpret (3.1) in terms of abacuses.

Suppose λ is a partition with $\ell(\lambda) \leqslant k$ and consider the e-abacus of λ. For $0 \leqslant r < e$ define the e-residue of the runner ρ_r to be the integer $\mathrm{res}_e(\rho_r)$ determined by the following two conditions.

(i) The e-residue of the runner which holds the last bead is $\lambda_1 - 1 \pmod e$.
(ii) Modulo e, the e-residues of the runners increase by 1 from left to right.

In Example 2.1 the runners are labeled by their f-residues for $f = 3, 4$, and 5, respectively. Similarly, we define the $(e+1)$-residues $\mathrm{res}_{e+1}(\rho_r^+)$, for $0 \leqslant r \leqslant e$, of the runners of the $(e+1)$-abacus for λ^+.

The e-residue of a bead β is defined to be the e-residue of the corresponding runner. As we have seen, the k beads on the e-abacus correspond to the first k rows of λ (in reverse order); it is easy to see that the e-residue of a bead is equal to the e-residue of the node at the end of the corresponding row of λ. In particular, this implies that e-residues of the runners depend only on the e-core of λ.

The operator F_i acts on a partition λ by adding nodes of e-residue i. Because the e-residues of the runners correspond to the e-residues of nodes at the end of the rows of λ, this is the same as moving a bead on the e-abacus of λ from the runner with e-residue $i-1$ to an adjacent empty position on the runner with e-residue i.

Recall that in the definition of λ^+ we have fixed an integer α with $0 \leqslant \alpha < e$. We now introduce a $\mathbb{C}[v, v^{-1}]$-linear map ${}^{\alpha}F_i : \mathcal{F}_k \to \mathcal{F}_k^+$ for $i = 0, \ldots, e-1$.

To define ${}^{\alpha}F_i$ it is enough to describe ${}^{\alpha}F_i\lambda$ for each partition λ with $\ell(\lambda) \leqslant k$. As above, let $\rho_0, \dots, \rho_{e-1}$ be the runners of the e-abacus of λ (with k beads) and let $\rho_0^+, \dots, \rho_e^+$ be the runners of the $(e+1)$-abacus for λ^+. There is a unique r such that $i = \mathrm{res}_e(\rho_r)$ (and $0 \leqslant r < e$); set $j = \mathrm{res}_{e+1}(\rho_r^+)$. Define

$$ {}^{\alpha}F_i\lambda = \begin{cases} F_j^+\lambda^+, & \text{if } 0 \leqslant r < \alpha, \\ F_{j+1}^+F_j^+\lambda^+, & \text{if } r = \alpha, \\ F_{j+1}^+\lambda^+, & \text{if } \alpha < r < e, \end{cases} $$

where $j+1$ is understood modulo e. Similarly, we define the divided powers ${}^{\alpha}F_i^{(a)}$ for $a \geqslant 1$; for example, when $r = \alpha$ we set ${}^{\alpha}F_i^{(a)}\lambda = F_{j+1}^{+(a)}F_j^{+(a)}\lambda^+$.

For our final piece of notation observe that $\ell(\lambda^+) \geqslant \ell(\lambda)$ for any partition λ (and if λ is a partition of length at most k then $\ell(\lambda) \leqslant \ell(\lambda^+) \leqslant k$). Therefore, we have a well-defined $\mathbb{C}[v, v^{-1}]$-linear map $\Theta : \mathcal{F}_k \to \mathcal{F}_k^+$ determined by $\Theta(\lambda) = \lambda^+$ for $\ell(\lambda) \leqslant k$. As with λ^+, we emphasize that Θ depends upon both α and k. The map Θ is injective but not surjective, having image the span of those partitions of length at most k which have an $(e+1)$-abacus with k beads and with an empty runner ρ_α^+.

3.4. Lemma. *Suppose that $a \geqslant 1$ and that $0 \leqslant i < e$. Then the following diagram commutes*:

$$ \begin{tikzcd} \mathcal{F}_k \arrow[r, "F_i^{(a)}"] \arrow[dr, "{}^{\alpha}F_i^{(a)}"'] & \mathcal{F}_k \arrow[d, "\Theta"] \\ & \mathcal{F}_k^+. \end{tikzcd} $$

Proof. We give the proof only for $a = 1$; the proof of the general case is almost identical. It suffices to verify the lemma for a partition $\lambda \in \mathcal{F}_k$. As above, let ρ_r be the runner in the e-abacus for λ for which $i = \mathrm{res}_e(\rho_r)$ and set $j = \mathrm{res}_{e+1}(\rho_r^+)$.

First consider $F_i\lambda = \sum_\nu v^{N_i(\lambda,\nu)}\nu^+$. Recall that the beads on the e-abacus for λ are naturally indexed by the rows of λ and that the e-residue of a bead is defined to be the e-residue of the node which is at the end of the corresponding row. Therefore, an addable i-node of λ corresponds to a node on runner $r-1$ of the e-abacus which can be moved to the adjacent position on runner r (which must therefore be empty). Similarly, a removable i-node corresponds to a node on runner r which can be moved back to the adjacent position on runner $r-1$; here, $r \pm 1$ is to be understood modulo e. The addable and removable nodes of λ^+ have analogous descriptions.

Fix a partition ν with $\lambda \xrightarrow{i} \nu$ and write $[\nu] = [\lambda] \cup \{x\}$. Then there exists a node at position β_x on the runner $r-1$ of the e-abacus for λ which can be moved to the adjacent position on runner r so as to give the e-abacus for ν. Then $N_i(\lambda,\nu) = A - B$, where $A = \#\{y \in A_i(\lambda) \mid y \succ x\}$ and $B = \#\{y \in R_i(\lambda) \mid y \succ x\}$. If y is an addable or removable node of λ then $y \succ x$ if and only if it corresponds to a bead at position β_y with $\beta_y > \beta_x$. Hence, A is equal to the number of beads on runner ρ_{r-1} which come after β_x such that the adjacent position on ρ_r is vacant; similarly, B is equal to the number of beads on ρ_r which are after β_x and for which the adjacent position on runner $r-1$ is vacant.

Now consider the $(e+1)$-abacuses for λ^+ and ν^+. Assume first that $r < \alpha$. Then the runners ρ_{r-1} and ρ_r for λ are the same as the runners ρ^+_{r-1} and ρ^+_r for λ^+ and so the last paragraph shows that the addable and removable i-nodes for λ correspond exactly to the addable and removable j-nodes for λ^+. Hence, $N_j(\lambda^+, \nu^+) = A - B = N_i(\lambda, \nu)$. Similarly, when $r > \alpha$ the addable and removable i-nodes for λ correspond to the addable and removable $(j+1)$-nodes for λ^+ and $N_i(\lambda, \nu) = N_{j+1}(\lambda^+, \nu^+)$.

Finally, consider the case when $r = \alpha$. This time runner ρ_{r-1} is equal to ρ^+_{r-1} and runner ρ_r is equal to ρ^+_{r+1}; whereas runner $\rho^+_r = \rho^+_\alpha$ of λ^+ is empty. Therefore, the addable and removable i-nodes of λ again correspond to addable and removable j-nodes of λ^+ except that this time there are additional addable j-nodes of λ^+ corresponding to the adjacent pairs of beads on the runners ρ_{r-1} and ρ_r of the e-abacus of λ. Let σ be the partition such that $\lambda^+ \xrightarrow{j} \sigma \xrightarrow{j+1} \nu^+$. Since ρ^+_r is empty, λ^+ has no removable j-nodes. Therefore, if we let l be the number of pairs of adjacent beads on runners ρ_{r-1} and ρ_r which are below β_x then $N_j(\lambda^+, \sigma) = A + l$. Next observe that σ has a single addable $(j+1)$-node (corresponding to the bead which we just moved), and that the removable $(j+1)$-nodes of σ correspond to the removable i-nodes of λ together with the l beads on runner ρ_r which we have already paired with an adjacent bead on ρ_{r-1}; therefore, $N_{j+1}(\sigma, \nu^+) = -(B + l)$. Consequently, $N_j(\lambda^+, \sigma) + N_{j+1}(\sigma, \nu^+) = A - B = N_i(\lambda, \nu)$ and so we have

$$F^+_{j+1} F^+_j \lambda^+ = \sum_{\nu^+} v^{N_i(\lambda,\nu)} \nu^+ = \Theta(F_i \lambda),$$

where the sum is over those partitions ν^+ for which there exists a partition σ such that $\lambda^+ \xrightarrow{j} \sigma \xrightarrow{j+1} \nu^+$. Note that there are additional terms in the expansion of $F_j \lambda^+$ (corresponding to the pairs of adjacent beads on runners ρ_{r-1} and ρ_r of the e-abacus for λ); however, they all disappear when F_{j+1} is applied because these extra partitions do not have any addable $(j+1)$-nodes. This completes the proof. □

Now consider $L(\Lambda_0)_k = L(\Lambda_0)/\big(L(\Lambda_0) \cap \mathcal{F}_k\big)$. If μ is an e-regular partition with $\ell(\mu) \leqslant k$ let $\widetilde{B}_\mu = B_{\widetilde{\mu}} + \mathcal{F}_{>k}$. As noted by Goodman and Wenzl [6, Lemma 4.1], the elements $\{\widetilde{B}_\mu \mid \mu$ is e-regular and $\ell(\mu) \leqslant k\}$ give a basis of $L(\Lambda_0)_k$.

The bar involution induces a well-defined map on $\mathcal{F}_k$ via $\overline{a + \mathcal{F}_{>k}} = \bar{a} + \mathcal{F}_{>k}$ for all $a \in \mathcal{F}$. It is easy to see that $\widetilde{B}_\mu$ is the unique element of $\mathcal{F}_k$ which is bar invariant and of the form $\mu + \sum_\lambda b_{\lambda\mu}(v)\lambda$ for some polynomials $b_{\lambda\mu}(v) \in v\mathbb{Z}[v]$, the sum being over the partitions of length at most k.

Similarly, $\{\widetilde{B}^+_\nu \mid \nu$ is $(e+1)$-regular and $\ell(\nu) \leqslant k\}$, where $\widetilde{B}^+_\nu = B^+_\nu + \mathcal{F}^+_{>k}$, is a basis of the $\boldsymbol{U}^-_v(\widehat{\mathfrak{sl}}_{e+1})$-module $L(\Lambda^+_0)_k = L(\Lambda^+_0)/(L(\Lambda^+_0) \cap \mathcal{F}^+_k)$.

3.5. Proposition. *Suppose that μ is an e-regular partition with at most k rows. Then $\widetilde{B}^+_{\mu^+} = \Theta(\widetilde{B}_\mu)$.*

Proof. Looking at the definitions, $\Theta(\widetilde{B}_\mu) = \mu^+$ plus a $v\mathbb{Z}[v]$-linear combination of other terms. As we will see, it is enough to show that $\Theta(\widetilde{B}_\mu)$ is a bar invariant element of $\mathcal{F}^+_k$.

Let $\varnothing \in \mathcal{F}_k$ be the image of the empty partition in $\mathcal{F}_k$. Following Lascoux, Leclerc, and Thibon [10, Lemma 6.4] let $(r_1^{a_1}, \dots, r_s^{a_s})$ be the e-residue sequence of μ corresponding to the e-ladders in the diagram of μ. Then $A_\mu = F^{(a_s)}_{r_s} \dots F^{(a_1)}_{r_1}\varnothing$ is a bar invariant element of $\mathcal{F}_k$ of the form $A_\mu = \mu + \sum_\lambda a_{\lambda\mu}(v)\lambda$ where $a_{\lambda\mu}(v) \in \mathbb{Z}[v, v^{-1}]$ and the sum is over partitions λ of length at most k such that $\mu \rhd \lambda$. Therefore, there exist uniquely determined polynomials $\alpha_{\sigma\mu}(v) \in \mathbb{Z}[v]$ such that $\widetilde{B}_\mu = A_\mu - \sum_\sigma \alpha_{\sigma\mu}(v)\widetilde{B}_\sigma$, where the sum is over e-regular partitions σ such that $\mu \rhd \sigma$ and $\ell(\sigma) \leqslant k$.

Now consider the element $A^+_\mu = \Theta(A_\mu) = \mu + \sum_\lambda a_{\lambda\mu}(v)\lambda^+$ in $\mathcal{F}^+_k$. By the Lemma, $A^+_\mu = {}^\alpha F^{(a_s)}_{r_s} \dots {}^\alpha F^{(a_1)}_{r_1}\varnothing$; hence, A^+_μ is bar invariant. By induction on dominance $\widetilde{B}^+_\sigma = \Theta(\widetilde{B}_\sigma)$ for $\mu \rhd \sigma$. Therefore, the element $\Theta(\widetilde{B}_\mu) = A^+_\mu - \sum_\sigma \alpha_{\sigma\mu}(v)\widetilde{B}_{\sigma^+}$ is also bar invariant. Consequently, $\Theta(\widetilde{B}_\mu) - \widetilde{B}^+_{\mu^+}$ is a bar invariant element of $\bigoplus_\lambda v\mathbb{Z}[v]\lambda$; hence, $\Theta(\widetilde{B}_\mu) - \widetilde{B}^+_{\mu^+} = 0$ as we wished to show. $\square$

Proof of Theorem 3.2. It is easy to see [10] that the polynomials $b^+_{\sigma\tau}(v)$ are nonzero only if σ and τ have the same $(e+1)$-core. Therefore, if μ is an e-regular partition with $\ell(\mu) \leqslant k$ then $\widetilde{B}^+_{\mu^+} = \sum_{\mu \trianglerighteq \lambda} b^+_{\lambda^+\mu^+}(v)\lambda^+$; on the other hand, $\Theta(B_\mu) = \sum_{\mu \trianglerighteq \lambda} b_{\lambda\mu}(v)\lambda^+$, where in both sums $\ell(\lambda) \leqslant k$. Hence, Theorem 3.2 follows from the proposition. $\square$

4. The main theorem

In this section we extend Theorem 3.2 to the case where μ is not necessarily e-regular; this will prove Theorem 2.2.

The Fock space also admits an action from a Heisenberg algebra $\mathcal{H}_e$ [12]. The action of $\mathcal{H}_e$ on $\mathcal{F}$ commutes with the action of $\boldsymbol{U}_v(\widehat{\mathfrak{sl}}_e)$ and it is useful because $\mathcal{F}$ is irreducible when considered as a module for the algebra generated by the actions of $\boldsymbol{U}_v(\widehat{\mathfrak{sl}}_e)$ and $\mathcal{H}_e$ on $\mathcal{F}$. In addition, the action of $\mathcal{H}_e$ allowed

Leclerc and Thibon [12, §7.9] to extend the bar involution to the whole of $\mathcal{F}$; in turn, this enabled them to extend the canonical basis of $L(\Lambda_0)$ to give a basis $\{B_\mu \mid \mu \text{ a partition}\}$ of $\mathcal{F}$, where the element B_μ is again uniquely determined by the two conditions that $\overline{B}_\mu = B_\mu$ and

$$B_\mu = \sum_{\substack{\lambda \vdash n \\ \mu \trianglerighteq \lambda}} b_{\lambda\mu}(v)\lambda$$

for some polynomials $b_{\lambda\mu}(v) \in \mathbb{Z}[v]$ such that $b_{\mu\mu}(v) = 1$ and $b_{\lambda\mu}(v) \in v\mathbb{Z}[v]$ whenever $\lambda \neq \mu$. We will show that Theorem 3.2 generalizes to the non-regular case.

As in the previous sections we are only interested in the action of a subalgebra $\mathcal{H}_e^-$ of $\mathcal{H}_e$; for the full story see [12, Section 7.5]. The algebra $\mathcal{H}_e^-$ is generated by elements $\mathcal{V}_m$ for $m \geqslant 0$; before we can describe how $\mathcal{V}_m$ acts on $\mathcal{F}$ we need some more notation.

An e-ribbon is a connected strip of e-nodes which does not contain a 2×2 square; more precisely, an e-ribbon is a set of e nodes $R = \{(a_1, b_1), \dots, (a_e, b_e)\}$ such that (a_{i+1}, b_{i+1}) is either $(a_i + 1, b_i)$ or $(a_i, b_i - 1)$, for $i = 1, \dots, e-1$. The head of R is the node $\operatorname{head}(R) = (a_1, b_1)$ and $\operatorname{spin}_e(R) = \#\{1 \leqslant i < e \mid a_{i+1} = a_i + 1\}$ is the e-spin of R.

If λ and ν are partitions then we write $\lambda \xrightarrow{m:e} \nu$ if $[\lambda] \subseteq [\nu]$ and $[\nu] \setminus [\lambda]$ is a disjoint union of m e-ribbons such that the head of each ribbon is either in the first row of λ or is of the form (i, j) where $(i-1, j) \in [\lambda]$. Lascoux, Leclerc and Thibon (see [12, Section 4.1]), call ν/λ an e-ribbon tableau of weight (m) and they note that there is a unique way of writing $[\nu] \setminus [\lambda]$ as a disjoint union of ribbons; we will see this below when we reinterpret ribbons in terms of abacuses. Finally, if $\lambda \xrightarrow{m:e} \nu$ then $\operatorname{spin}_e(\nu/\lambda)$, the e-spin of ν/λ, is the sum of the e-spins of the ribbons in $[\nu] \setminus [\lambda]$.

For example, if $\lambda = (3)$ and $e = 2$ then the partitions ν with $\lambda \xrightarrow{2:2} \nu$ are

with spins 0, 0, 1, and 2, respectively.

The algebra $\mathcal{H}_e^-$ is the subalgebra of $\mathcal{H}_e$ generated by elements $\mathcal{V}_m$ for $m \geqslant 1$. For each m, $\mathcal{V}_m$ acts on the Fock space as the $\mathbb{C}[v, v^{-1}]$-linear map determined by

$$\mathcal{V}_m\lambda = \sum_{\lambda \xrightarrow{m:e} \nu} (-v)^{-\operatorname{spin}_e(\nu/\lambda)}\nu$$

for all partitions λ. Observe that $\mathcal{F}_{>k}$ is a $\mathcal{H}_e^-$-module; hence, there is a well-defined action of $\mathcal{V}_m$ on the quotient space $\mathcal{F}_k$.

Similarly, there is an action of the negative Heisenberg algebra $\mathcal{H}_{e+1}^-$ on the Fock space $\mathcal{F}^+$ and this induces an action on $\mathcal{F}_k^+$. We denote the generators of $\mathcal{H}_{e+1}^-$ by $\mathcal{V}_m^+$ for $m \geqslant 1$.

4.1. Lemma. *Suppose that λ and ν are partitions of n of length at most k.*

(i) *If $\lambda \overset{m:e}{\rightsquigarrow} \nu$ then λ and ν have the same e-core.*
(ii) *We have $\lambda \overset{m:e}{\rightsquigarrow} \nu$ if and only if $\lambda^+ \overset{m:e+1}{\rightsquigarrow} \nu^+$; moreover, if $\lambda \overset{m:e}{\rightsquigarrow} \nu$ then* $\operatorname{spin}_e(\nu/\lambda) = \operatorname{spin}_{e+1}(\nu^+/\lambda^+)$.

Proof. The lemma will follow once we reinterpret the condition $\lambda \overset{m:e}{\rightsquigarrow} \nu$ in terms of abacuses. Suppose that $\lambda \overset{m:e}{\rightsquigarrow} \nu$. Then $[\nu] \setminus [\lambda]$ is a disjoint union of e-ribbons. Extend the partial order $\succ$ on the set of nodes to a total order by defining $(a,b) \succ (c,d)$ if either $c > a$ or $c = a$ and $b > d$. Totally order the ribbons $R_1, \dots, R_m$ in $[\nu] \setminus [\lambda]$ so that $i > j$ whenever $\operatorname{head}(R_i) \succ \operatorname{head}(R_j)$. Then the condition that the head of R_i is of the form (a,b) with either $a = 1$ or $(a-1,b) \in [\lambda]$ is equivalent to saying that $[\nu] \setminus (R_1 \cup \dots \cup R_i)$ is the diagram of a partition for $i = 1, \dots, m$. Hence, it is enough to treat the case $m = 1$. So let $R = R_1$ where $[\nu] = [\lambda] \cup R$.

Now the ribbon R is a rim hook and it is well known that removing a rim hook of length e from ν is the same as moving a bead β on an e-abacus for ν to the (empty) bead position on the same runner which is in the preceding row. Further, by definition, $\operatorname{spin}_e(\nu/\lambda)$ is the leg length of R minus one and, in terms of the e-abacus, the leg length of R is equal to the number of beads on the abacus which are between the old and new positions of β. Similarly, the condition $\lambda^+ \overset{m:e+1}{\rightsquigarrow} \mu^+$ depends only on the $(e+1)$-abacuses of ν^+ and λ^+. As the e and $(e+1)$ abacuses differ only by the insertion of an empty runner, all of the assertions of the lemma now follow. $\square$

4.2. Corollary. *Suppose that $m \geqslant 1$. Then the following diagram commutes*:

$$\begin{tikzcd}
\mathcal{F}_k \arrow[r, "\mathcal{V}_m"] \arrow[d, "\Theta"'] & \mathcal{F}_k \arrow[d, "\Theta"] \\
\mathcal{F}_k^+ \arrow[r, "\mathcal{V}_m^+"'] & \mathcal{F}_k^+
\end{tikzcd}$$

Proof. As with Lemma 4.1 it suffices to check the result for a partition λ of length at most k. By the definitions and the previous lemma,

$$\Theta(\mathcal{V}_m\lambda) = \sum_{\lambda \overset{m:e}{\rightsquigarrow} \nu} (-v)^{-\operatorname{spin}_e(\nu/\lambda)}\Theta(\nu)$$

$$= \sum_{\lambda^+ \xrightarrow{m:e+1} \nu^+} (-v)^{-\operatorname{spin}_{e+1}(\nu^+/\lambda^+)} \nu^+$$
$$= \mathcal{V}_m^+ \lambda^+.$$

Therefore, $\mathcal{V}_m^+\big(\Theta(\lambda)\big) = \Theta\big(\mathcal{V}_m\lambda\big)$ and we are done. □

Let μ be a partition. As in the last section let $\widetilde{B}_\mu$ and $\widetilde{B}_\mu^+$ denote the image of $B_\mu \in \mathcal{F}$ and $B_\mu^+ \in \mathcal{F}^+$, respectively, in $\mathcal{F}_k$ and $\mathcal{F}_k^+$. Then a basis of $\mathcal{F}_k$ is given by the $\widetilde{B}^\mu$ as μ runs over all partitions of length at most k and similarly for $\mathcal{F}_k^+$.

By Proposition 3.5 we know that $\widetilde{B}_{\mu^+}^+ = \Theta(B_\mu)$ whenever μ is an e-regular partition with $\ell(\mu) \leqslant k$. We can now drop the requirement that μ should be e-regular.

4.3. Proposition. *Suppose that μ is a partition with $\ell(\mu) \leqslant k$. Then $\widetilde{B}_{\mu^+}^+ = \Theta(\widetilde{B}_\mu)$.*

Proof. As before, the element $\Theta(\widetilde{B}_\mu)$ is equal to μ^+ plus a $v\mathbb{Z}[v]$-linear combination of other terms. As in the proof of Proposition 3.5, it is enough to show that $\Theta(\widetilde{B}_\mu)$ is invariant under the bar involution.

By [12, Proposition 7.6] the bar involution on $\mathcal{F}$ is completely determined by the conditions $\overline{\varnothing} = \varnothing$, $\overline{F_i^{(m)}x} = F_i^{(m)}\bar{x}$, and $\overline{\mathcal{V}_m x} = \mathcal{V}_m \bar{x}$, for all $x \in \mathcal{F}$, $0 \leqslant i < e$, and $m \geqslant 1$. For each partition $\tau = (\tau_1, \ldots, \tau_s)$ let $\mathcal{V}_\tau = \mathcal{V}_{\tau_1} \ldots \mathcal{V}_{\tau_s}$. Then

$$\mathcal{F} = \bigoplus_\tau \boldsymbol{U}_v^-(\widehat{\mathfrak{sl}}_e)\mathcal{V}_\tau \varnothing$$

is a decomposition of $\mathcal{F}$ into a direct sum of irreducible $\boldsymbol{U}_v(\widehat{\mathfrak{sl}}_e)$-modules (where τ runs over all partitions of all integers); see [12, §7.5]. Moreover, the modules $\boldsymbol{U}_v^-(\widehat{\mathfrak{sl}}_e)\mathcal{V}_\tau\varnothing$, for different τ, are all isomorphic as $\boldsymbol{U}_v^-(\widehat{\mathfrak{sl}}_e)$-modules. Therefore, there exists a bar invariant basis of $\mathcal{F}$ of the form $A_{\sigma\tau} = F_\sigma \mathcal{V}_\tau \varnothing$ where $F_\sigma \in \boldsymbol{U}_{\mathcal{A}}^-(\widehat{\mathfrak{sl}}_e)$, σ is an e-regular partition and τ is an arbitrary partition (the elements F_σ are defined in terms of e-residue sequences as in the proof of Proposition 3.5). Consequently, we can write $\widetilde{B}_\mu = \sum_{\sigma,\tau} a_{\sigma\tau}(v) A_{\sigma\tau}$ for some bar invariant Laurent polynomials $a_{\sigma\tau}(v) \in \mathbb{Z}[v, v^{-1}]$. Now, $A_{\sigma\tau}^+ = \Theta(A_{\sigma\tau}) = \Theta(F_\sigma \mathcal{V}_\tau \varnothing) = {}^\alpha F_\sigma \mathcal{V}_\tau^+ \varnothing$ by Lemma 3.4 and Corollary 4.2; therefore, $A_{\sigma\tau}^+$ is a bar invariant element of $\mathcal{F}_k^+$. Hence, $\Theta(\widetilde{B}_\mu) = \sum_{\sigma,\tau} a_{\sigma\tau}(v) A_{\sigma\tau}^+$ is a bar invariant element of $\mathcal{F}_k^+$, as we needed to show. □

4.4. Remark. For each composition $\tau = (\tau_1, \ldots, \tau_s)$ Leclerc and Thibon [12] show that the action of the element $\mathcal{V}_\tau = \mathcal{V}_{\tau_1} \ldots \mathcal{V}_{\tau_s}$ upon $\mathcal{F}$ is described by certain polynomials associated with the ribbon tableaux of weight τ. This is completely analogous to the way in which the action of $F_{r_s}^{(a_s)} \ldots F_{r_1}^{(a_1)}$ on $\mathcal{F}$ can be described in terms of polynomials associated with standard tableaux.

Note that the Proposition also implies the following. Suppose that λ and μ are partitions of length at most k which have the some e-core and that $d_{\lambda\mu} \neq 0$. Then $\lambda \trianglerighteq \mu$ if and only if $\lambda^+ \trianglerighteq \mu^+$. A direct combinational proof of this seems to be difficult (and the assumption that $d_{\lambda\mu} \neq 0$ is surely extraneous).

Comparing the coefficient of λ^+ in $\widetilde{B}^+_{\mu^+}$ and $\Theta(\widetilde{B}_\mu)$ we obtain the following generalization of our main theorem.

4.5. Theorem. *Suppose that λ and μ are partitions of length at most k. Then*

$$b_{\lambda\mu}(v) = b^+_{\lambda^+\mu^+}(v).$$

In order to compute the polynomials $b_{\lambda\mu}(v)$ when μ is not e-regular it is necessary to first invert the "R-matrix" which describes the bar involution on the basis of $\mathcal{F}$ given by the set of partitions. Computationally, this is quite time consuming; in comparison the regular case is much easier, being essentially Gaussian elimination. Corollary 4.5 therefore gives a slightly more efficient way of computing the polynomials $b_{\lambda\mu}(v) = b^+_{\lambda^+\mu^+}(v)$ since μ^+ is an $(e+1)$-regular partition by Lemma 3.3(i).

Recall that W^λ_q and L^μ_q are the Weyl modules and simple modules, respectively, for the q-Schur algebra $\mathcal{S}_{\mathbb{C},q}(n,n)$. Varagnolo and Vasserot [17] have shown that $[W^\lambda_q : L^\mu_q] = b_{\lambda\mu}(1)$; similarly, $[W^{\lambda^+}_{q'} : L^{\mu^+}_{q'}] = b^+_{\lambda^+\mu^+}(1)$.

4.6. Corollary. *Suppose that λ and μ are partitions of length at most k. Then*

$$\bigl[W^\lambda_q : L^\mu_q\bigr] = \bigl[W^{\lambda^+}_{q'} : L^{\mu^+}_{q'}\bigr] = \bigl[S^{\lambda^+}_{q'} : D^{\mu^+}_{q'}\bigr].$$

Proof. That $[W^\lambda_q : L^\mu_q] = [W^{\lambda^+}_{q'} : L^{\mu^+}_{q'}]$ follows directly from Theorem 4.5 and the remarks above. For the second claim, observe that the partition μ^+ is $(e+1)$-regular by Lemma 3.3(i); therefore, $D^{\mu^+}_{q'} \neq 0$. Consequently, $[W^{\lambda^+}_{q'} : L^{\mu^+}_{q'}] = [S^{\lambda^+}_{q'} : D^{\mu^+}_{q'}]$ by Schur–Weyl reciprocity. □

Standard Schur functor arguments yield the corresponding statements for the q-Schur algebras $\mathcal{S}_{\mathbb{C},q}(n,r)$ and $\mathcal{S}_{\mathbb{C},q'}(n,r)$; we leave the details to the reader.

The last result is interesting because it shows that every decomposition number for $\mathcal{S}_{\mathbb{C},q}(n,n)$ is also a decomposition number for some Hecke algebra $\mathcal{H}_{\mathbb{C},q'}(\mathfrak{S}_m)$. In contrast Erdmann [5] has shown that in a given characteristic knowing all decomposition numbers for the classical Schur algebras (i.e. $q=1$) is equivalent to knowing all decomposition numbers for the symmetric groups (for all n and for a fixed p). Leclerc [11] has proved the analogous result relating the decomposition numbers of the q-Schur algebras $\mathcal{S}_{\mathbb{C},q}(n,n)$ and the Hecke algebras $\mathcal{H}_{\mathbb{C},q}(\mathfrak{S}_n)$ (for all n and for a fixed q). No such result is known in the cross characteristic case (i.e. positive characteristic with $q \neq 1$).

Finally, we remark that the full action of $\boldsymbol{U}_v(\widehat{\mathfrak{sl}}_e)$ on $\mathcal{F}_k$ and $\boldsymbol{U}_v(\widehat{\mathfrak{sl}}_{e+1})$ on $\mathcal{F}^+_k$ are compatible via the map Θ (in order to make the statement for $\boldsymbol{U}^+_v(\widehat{\mathfrak{sl}}_e)$

precise $\mathcal{F}_k$ must be considered as a submodule of $\mathcal{F}$, rather than a quotient). This can be proved using similar arguments or, more simply, by invoking [12, Proposition 7.9] which says that the actions of $\boldsymbol{U}_v^-(\widehat{\mathfrak{sl}}_e)$ and $\boldsymbol{U}_v^+(\widehat{\mathfrak{sl}}_e)$ on $\mathcal{F}$ are adjoint with respect to a natural scalar product on $\mathcal{F}$. The same argument also proves the corresponding statements for the Heisenberg algebras $\mathcal{H}_e$ and $\mathcal{H}_{e+1}$.

5. Examples

Below we give part of the "crystallized" decomposition matrices $(b_{\lambda\mu}(v))$ of the q-Schur algebras $\mathcal{S}_{\mathbb{C},q}(n)$ for $(e,n)=(2,6)$, $(3,11)$, $(4,16)$, and $(5,21)$. By our results, taking $k=4$ and $\alpha=2$, these submatrices are all the same.

$\cdots e=5$	$e=4$	$e=3$	$e=2$								
$18,3$	$14,2$	$10,1$	6	1							
$17,4$	$13,3$	$9,2$	$5,1$	v	1						
$\cdots 13,8$	$11,5$	$7,4$	$4,2$	.	v	1					
$13,4^2$	$10,3^2$	$7,2^2$	$4,1^2$	v	v^2	v	1				
$12,9$	$9,7$	$6,5$	3^2	.	.	v	.	1			
$12,4^2,1$	$9,3^2,1$	$6,2^2,1$	$3,1^3$	v^2	v	v^2	v	v	1		
$\cdots 8^2,5$	$6^2,4$	$4^2,3$	2^3	.	.	v^2	v	v	.	1	
$8^2,4,1$	$6^2,3,1$	$4^2,2,1$	$2^2,1^2$	.	v^2	v^3	v^2	v^2	v	v	1

Setting $v=1$ we recover the decomposition matrices of the corresponding q-Schur algebras. Note that when $e>2$ all of the rows are indexed by partitions which are e-regular; therefore, in these cases the matrix above is a submatrix of the decomposition matrix for the corresponding Iwahori–Hecke algebra $\mathcal{H}_{\mathbb{C},q}(\mathfrak{S}_n)$; in particular, setting $v=1$ we recover one of the decomposition matrices from the introduction.

To emphasize the dependence on k we again start with $(e,n)=(2,6)$ but now take $k=6$ (and $\alpha=2$); this yields the following matrices:

$e=5$	$e=4$	$e=3$	$e=2$										
$21,6,3^2$	$16,4,2^2$	$11,2,1^2$	6	1									
$20,7,3^2$	$15,5,2^2$	$10,3,1^2$	$5,1$	v	1								
$16,11,3^2$	$12,8,2^2$	$8,5,1^2$	$4,2$	.	v	1							
$16,7^2,3$	$12,5^2,2$	$8,3^2,1$	$4,1^2$	v	v^2	v	1						
$15,12,3^2$	$11,9,2^2$	$7,6,1^2$	$3,3$	.	.	v	.	1					
$15,7^2,4$	$11,5^2,3$	$7,3^2,2$	$3,1^3$	v^2	v	v^2	v	v	1				
$11^2,8,3$	$8^2,6,2$	$5^2,4,1$	2^3	.	.	v^2	v	v	.	1			
$11^2,7,4$	$8^2,5,3$	$5^2,3,2$	$2^2,1^2$	.	v^2	v^3	v^2	v^2	v	v	1		
$11,7^2,4^2$	$8,5^2,3^2$	$5,3^2,2^2$	$2,1^4$	v^2	v^3	.	v	.	v^2	.	v	1	
$10,7^2,4^2,1$	$7,5^2,3^2,1$	$4,3^2,2^2,1$	1^6	v^3	.	.	v^2	.	.	.	.	v	1

A consequence of Lemma 3.3 is that all of these matrices are the rows of decomposition matrices of the corresponding blocks which are indexed by partitions with at most k rows (where the partitions indexing the rows are ordered in a way compatible with dominance).

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References

[1] S. Ariki, On the decomposition numbers of the Hecke algebra of $G(m,1,n)$, J. Math. Kyoto Univ. 36 (1996) 789–808.
[2] R. Dipper, G. James, Representations of Hecke algebras of general linear groups, Proc. London Math. Soc. (3) 52 (1986) 20–52.
[3] R. Dipper, G. James, The q-Schur algebra, Proc. London Math. Soc. (3) 59 (1989) 23–50.
[4] R. Dipper, G. James, q-Tensor space and q-Weyl modules, Trans. Amer. Math. Soc. 327 (1991) 251–282.
[5] K. Erdmann, Decomposition numbers for symmetric groups and composition factors of Weyl modules, J. Algebra 180 (1996) 316–320.
[6] F.M. Goodman, H. Wenzl, Crystal bases of quantum affine algebras and Kazhdan–Lusztig polynomials, Int. Math. Res. Notices 5 (1999) 251–275.
[7] G.D. James, The Representation Theory of the Symmetric Groups, in: SLN, Vol. 682, Springer-Verlag, New York, 1978.
[8] G.D. James, Some combinatorial results involving Young diagrams, Proc. Cambridge Philos. Soc. 83 (1978) 1–10.
[9] G.D. James, A. Mathas, A q-analogue of the Jantzen–Schaper theorem, Proc. London Math. Soc. (3) 74 (1997) 241–274.
[10] A. Lascoux, B. Leclerc, J.-Y. Thibon, Hecke algebras at roots of unity and crystal bases of quantum affine algebras, Comm. Math. Phys. 181 (1996) 205–263.
[11] B. Leclerc, Decomposition numbers and canonical bases, Algebr. Represent. Theory 3 (2000) 277–287.
[12] B. Leclerc, J.-Y. Thibon, Littlewood–Richardson coefficients and Kazhdan–Lusztig polynomials, in: M. Kashiwara, et al. (Eds.), Combinatorial Methods in Representation Theory, in: Adv. Pure Math., Vol. 28, 2000, pp. 155–220.
[13] F. Lübeck, J. Müller, private communication.
[14] F. Lübeck, M. Neunhöffer, Enumerating large orbits and direct condensation, Experiment. Math. 10 (2001) 197–205.
[15] S. Martin, L. Russell, Defect 3 blocks of symmetric group algebras, J. Algebra 213 (1999) 304–339.
[16] A. Mathas, Hecke Algebras and Schur Algebras of the Symmetric Group, in: Univ. Lecture Notes, Vol. 15, AMS, 1999.
[17] M. Varagnolo, E. Vasserot, On the decomposition matrices of the quantized Schur algebra, Duke J. Math. 100 (1999) 267–297.