JOURNAL OF ALGEBRA 141, 484–504 (1991)

Relative Projectivity, Graded Clifford Theory, and Applications

J. L. GÓMEZ PARDO*

Departamento de Matemáticas, Universidad de Murcia, 30001 Murcia, Spain

AND

C. NĂSTĂSESCU

Facultatea de Matematica, Universitatea Bucuresti, R-70109 Bucharest, Romania

Communicated by Kent R. Fuller

Received August 5, 1989

INTRODUCTION

The notions of relative projectivity and relative injectivity have been extensively developed during the last years and have proved to be useful in a wide range of situations (see, e.g., [1, 2, 7, 9, 10], etc.). In this paper we focus our attention on these two concepts in the context of the category R-*gr* of all G-graded R-modules, where G is a group with identity element 1 and $R = \bigoplus_{\sigma \in G} R_\sigma$ a G-graded ring, and apply the results obtained to the study of graded Clifford theory and the structure of *gr*-simple modules.

In the first part of the paper we determine, in Theorem 1.1, the behaviour of relative projectivity and injectivity with respect to adjoint functors. This works well in our context, since the functor Ind: R_1-*mod* $\to$ R-*gr* (Coind: R_1-*mod* $\to$ R-*gr*) is a left (resp. right) adjoint of the exact functor $(-)_1: R\text{-}gr \to R_1\text{-}mod$ which assigns to each graded module its homogeneous 1-component. Then, in Section 2 we use the concept of a closed subcategory, i.e., a subcategory of a Grothendieck category which is closed under subobjects, quotient objects, and direct sums, to investigate how relative projectivity and injectivity behave via the forgetful functor U: R-*gr* $\to$ R-*mod*. In particular, we show that the *gr*-semisimple modules M are projective in the smallest closed subcategory of R-*mod*

* This author has been partially supported by the DGICYT (PB87-0703).

0021-8693/91 $3.00

which contains M and that they are M'-projective in *R-mod* for any *gr*-semisimple module M'. When one adds the hypotheses of R having finite support and M being finitely generated, then similar results hold for injectivity. This gives, for any *gr*-semisimple module M, an equivalence between the full subcategory $(R|M)$-*mod* of *R-mod*, whose objects are all the modules generated by M, and a quotient category of Δ-*mod*, where $\Delta = \operatorname{End}_R(M)^{\text{opp}}$ is the ring of endomorphisms acting as right operators. This equivalence is induced by the functor $\operatorname{Hom}_R(M, -)$ and when $M = \Sigma$ is a *gr*-simple module, it reduces to the "direct Clifford theory" given by Dade in [4, 5], i.e., to an equivalence between $(R|\Sigma)$-*mod* and Δ-*mod*.

The direct Clifford theorem proves to be a very powerful tool for studying *gr*-simple modules. In Section 3, we pay attention to the problem of determining the structure of these modules. Given a *gr*-simple module $\Sigma \in R$-*gr*, we try to answer the following questions:

(QI) What is the structure of Σ as R_1-module?

(QII) What is the structure of Σ regarded as an object of *R-mod*?

(QI) has been answered by Dade [5], but to illustrate our methods we include somewhat different proofs of his results. Regarding (QII), we get satisfactory answers for graded rings with finite support, essentially due to the fact that in this case Σ is quasi-injective in *R-mod*. In the general case we only find partial anwers, assuming some additional conditions on the group G.

After this paper was written, we have received the preprint [20], where our Corollary 2.11 is also proved.

0. Notation and Preliminaries

Throughout this paper, all rings R will be associative and with identity and all modules will be left R-modules. The category of left R-modules will be denoted by *R-mod*.

If G is a (multiplicative) group with identity element 1 and $R = \bigoplus_{\sigma \in G} R_\sigma$ a G-graded ring, the category of G-graded R-modules will be denoted by *R-gr*. If $M = \bigoplus_{\sigma \in G} M_\sigma$ and $N = \bigoplus_{\sigma \in G} N_\sigma$ are two G-graded modules, then $\operatorname{Hom}_{R\text{-}gr}(M, N)$ consists of the R-homomorphisms $f: M \to N$ such that $f(M_\sigma) \subseteq N_\sigma$ for every $\sigma \in G$. As it is well known [16], *R-gr* is a Grothendieck category. In particular, *R-gr* has enough injective objects and if $M \in R$-*gr*, we denote by $E^g(M)$ the injective envelope of M in *R-gr*, and by $E(M)$ the injective evelope of M in *R-mod*.

If M is a graded R-module, $h(M)$ will stand for the set of all homogeneous elements of M, i.e., $h(M) = \bigcup_{\sigma \in G} M_\sigma - \{0\}$. If $m \in M$, $m \neq 0$, we can write $m = \sum_{\sigma \in G} m_\sigma$, where $m_\sigma \in M_\sigma$; the finite set $\{m_\sigma | \sigma \in G,$

$m_\sigma \neq 0\}$ is called the set of homogeneous components of m. If $M = \bigoplus_{\lambda \in G} M_\lambda$ is a graded R-module and $\sigma \in G$, then the σ-suspension of M is defined as the graded module $M(\sigma)$ obtained from M by setting $M(\sigma)_\lambda = M_{\lambda\sigma}$. The σ-suspension functor T_σ: $R\text{-}gr \to R\text{-}gr$ defined by $T_\sigma(M) = M(\sigma)$ is an equivalence of categories.

Let M and N be graded R-modules. For each $\sigma \in G$ we set $\mathrm{HOM}_R(M, N)_\sigma = \{f: M \to N \mid f$ is R-linear and $f(M_\lambda) \subseteq N_{\lambda\sigma}$, $\forall \lambda \in G\} = \mathrm{Hom}_{R\text{-}gr}(M, N(\sigma)) = \mathrm{Hom}_{R\text{-}gr}(M(\sigma^{-1}), N)$. $\mathrm{HOM}_R(M, N)_\sigma$ is an additive subgroup of the group $\mathrm{Hom}_R(M, N)$ of all R-homomorphisms from M to N and $\mathrm{HOM}_R(M, N) = \bigoplus_{\sigma \in G} \mathrm{HOM}_R(M, N)_\sigma$ is a subgroup of $\mathrm{Hom}_R(M, N)$ and it is, in fact, a G-graded abelian group. Clearly, $\mathrm{HOM}_R(M, N)_1$ is just $\mathrm{Hom}_{R\text{-}gr}(M, N)$. It is well known that if M is finitely generated or G is a finite group, then $\mathrm{HOM}_R(M, N) = \mathrm{Hom}_R(M, N)$ [16]. If $N = M$, we denote $\mathrm{END}_R(M) = \mathrm{HOM}_R(M, M)$; then $\Lambda = \mathrm{END}_R(M)^{\mathrm{opp}}$ is a G-graded subring of $\Delta = \mathrm{End}_R(M)^{\mathrm{opp}}$.

A nonzero graded module Σ is called gr-simple if 0 and Σ are its only graded submodules, i.e., Σ is a simple object of the category R-gr. If $\Sigma = \bigoplus_{\sigma \in G} \Sigma_\sigma$ and $x_\sigma \in \Sigma_\sigma$ is a nonzero homogeneous element, then $Rx_\sigma = \Sigma$ and so Σ is a finitely generated R-module. Also, a gr-semisimple module is just a semisimple object of R-gr.

A G-graded ring $R = \bigoplus_{\sigma \in G} R_\sigma$ is called strongly graded if $R_\sigma R_\tau = R_{\sigma\tau}$ for every σ, $\tau \in G$. This is equivalent to $R_\sigma R_{\sigma^{-1}} = R_1$ for all $\sigma \in G$. On the other hand, R is called a crossed product if, for any $\sigma \in G$, R_σ contains an invertible element. It is clear that in this case R is strongly graded. We refer to [16] for all the definitions and basic properties of graded rings and modules.

1. Adjoint Functors and Relative Projectivity

Let $\mathscr{A}$ be a Grothendieck category [19, Chap. 17] and U an object of $\mathscr{A}$. If $M \in \mathscr{A}$, then U is said to be projective relative to M (or M-projective for short) if, for any epimorphism $u: M \to M'$ in $\mathscr{A}$, the induced homomorphism $\mathrm{Hom}_{\mathscr{A}}(U, M) \to \mathrm{Hom}_{\mathscr{A}}(U, M')$ is an epimorphism. Dually, U is injective relative to M (M-injective) if for any monomorphism $u: M' \to M$ in $\mathscr{A}$, $\mathrm{Hom}_{\mathscr{A}}(M, U) \to \mathrm{Hom}_{\mathscr{A}}(M', U)$ is an epimorphism. *If* U is U-projective (U-injective), then U is called quasi-projective (resp. quasi-injective). Obviously, U is projective (resp. injective) in $\mathscr{A}$ if and only if it is M-projective (resp. M-injective) for each object $M \in \mathscr{A}$.

Following [1, 2], we denote

$$\mathscr{P}\imath^{-1}(U) = \{M \in \mathscr{A} \mid U \text{ is } M\text{-projective}\}$$

$$\mathscr{I}\imath^{-1}(U) = \{M \in \mathscr{A} \mid U \text{ is } M\text{-injective}\}$$

We recall that an object $U \in \mathscr{A}$ is said to be small if the functor $\mathrm{Hom}_{\mathscr{A}}(U, -): \mathscr{A} \to \mathscr{A}\ell$ (where $\mathscr{A}\ell$ denotes the category of abelian groups) preserves direct sums.

By [1, 2], the classes $\mathscr{P}\imath^{-1}(U)$ and $\mathscr{I}\kern-1pt n^{-1}(U)$ are both closed under subobjects and epimorphic images. Also, $\mathscr{P}\imath^{-1}(U)$ is closed under finite direct sums and $\mathscr{I}\kern-1pt n^{-1}(U)$ under arbitrary direct sums. Furthermore, if U is a finitely generated object of $\mathscr{A}$, then $\mathscr{P}\imath^{-1}(U)$ is also closed under arbitrary direct sums.

Recall now that if $F: \mathscr{A} \to \mathscr{B}$ and $G: \mathscr{B} \to \mathscr{A}$ are additive functors between Grothendieck categories, F is a left adjoint of G (or G is a right adjoint of F) if there is a natural equivalence:

$$\phi: \mathrm{Hom}_{\mathscr{B}}(F(-), -) \to \mathrm{Hom}_{\mathscr{A}}(-, G(-)).$$

It is well known that in this case F is right exact and G is left exact.

If $T: \mathscr{A} \to \mathscr{B}$ is a functor and $\mathscr{C}$ a class of objects of $\mathscr{B}$, then we denote $T^{-1}(\mathscr{C}) = \{M \in \mathscr{A} \mid T(M) \in \mathscr{C}\}$.

We begin with a general result which is rather straightforward but will be useful in the sequel.

THEOREM 1.1. *Let $F: \mathscr{A} \to \mathscr{B}$ and $G: \mathscr{B} \to \mathscr{A}$ be functors between Grothendieck categories such that F is left adjoint of G. Then the following statements hold*:

(i) *If $U \in \mathscr{A}$ and G is exact, then $F(U)$ is Y-projective for every $Y \in G^{-1}(\mathscr{P}\imath^{-1}(U))$.*

(ii) *If $V \in \mathscr{B}$ and F is exact, then $G(V)$ is X-injective for any $X \in F^{-1}(\mathscr{I}\kern-1pt n^{-1}(V))$.*

Proof. Let us prove (i). Consider an epimorphism in $\mathscr{B}$, $u: Y \to Y''$. Since G preserves epimorphisms, $G(u): G(Y) \to G(Y'')$ is an epimorphism in $\mathscr{A}$. The hypothesis $Y \in G^{-1}(\mathscr{P}\imath^{-1}(U))$ says that U is $G(Y)$-projective and hence the induced sequence $\mathrm{Hom}_{\mathscr{A}}(U, G(Y)) \to \mathrm{Hom}_{\mathscr{A}}(U, G(Y'')) \to 0$ is exact. Applying the natural transformation ϕ we get that the sequence

$$\mathrm{Hom}_{\mathscr{B}}(F(U), Y) \to \mathrm{Hom}_{\mathscr{B}}(F(U), Y'') \to 0$$

is also exact, which shows that $F(U)$ is Y-projective.

Assertion (ii) is proved in a dual manner.

Remarks 1.2. In the situation of Theorem 1.1, if G is exact and P is assumed to be projective in $\mathscr{A}$, one gets that $F(P)$ is projective in $\mathscr{B}$ (since $\mathscr{P}\imath^{-1}(P) = \mathscr{A}$). Similarly, if F is exact and Q is injective in $\mathscr{B}$, then $G(Q)$ is injective in $\mathscr{A}$. These results are well known (see, e.g., [19, Proposition VI.9.5]).

COROLLARY 1.3. *With the notations of Theorem* 1.1, *the following assertions hold*:

(i) *Assume that G is exact and $U \in \mathscr{A}$. If $GF(U) \in \mathscr{P}r^{-1}(U)$ (e.g., if U is quasi-projective and $GF(U)$ is isomorphic to a quotient object of a finite direct sum of copies of U), then $F(U)$ is quasi-projective in $\mathscr{B}$.*

(ii) *Assume that F is exact and $V \in \mathscr{B}$. Then, if $FG(V) \in \mathscr{I}n^{-1}(V)$ (e.g., if V is quasi-injective and $FG(V)$ is isomorphic to a subobject of a direct sum of copies of V), then $G(V)$ is quasi-injective.*

We now apply the above results to quasi-projective and quasi-injective objects of R-*gr*. Let R be a G-graded ring. We are going to consider several functors between R_1-*mod* and R-*gr* (see [15] for the details).

For each $\sigma \in G$ there is an exact functor $(-)_\sigma$: R-*gr* $\to R_1$-*mod*, given by $M \to M_\sigma$, where $M = \bigoplus_{\lambda \in G} M_\lambda \in R$-*gr*. On the other hand, if $M \in R_1$-*mod*, the left R-module $R \otimes_{R_1} M$ has the natural grading $(R \otimes_{R_1} M)_\sigma = R_\sigma \otimes_{R_1} M$ and the mapping $M \to R \otimes_{R_1} M$ defines a functor Ind$(-)$: R_1-*mod* $\to R$-*gr*, which is called the induced functor.

Since R is an $R_1 - R$-bimodule, we can consider the left R-module $M' = \operatorname{Hom}_{R_1}(R, M)$ for each $M \in R_1$-*mod*. Defining, for each $\sigma \in G$

$$M'_\sigma = \{f \in \operatorname{Hom}_{R_1}(R, M) \mid f(R_{\sigma'}) = 0 \text{ for any } \sigma' \neq \sigma^{-1}\}$$

it is obvious that M'_σ is a subgroup of M and the sum $M^* = \sum_{\sigma \in G} M'_\sigma$ is direct. Since $R_\sigma M'_\tau \subseteq M'_{\sigma\tau}$, M^* is an object of R-*gr* which is called the module coinduced by M and is denoted by Coind(M). The mapping $M \to$ Coind(M) defines a functor Coind$(-)$: R_1-*mod* $\to R$-*gr* which is called the coinduced functor.

The basic properties of these functors are given in [15, Theorem 1.1] and can be summarized as follows: For each $\sigma \in G$, $T_{\sigma^{-1}} \circ \operatorname{Ind}$ is a left adjoint of $(-)_\sigma$ and $T_{\sigma^{-1}} \circ \operatorname{Coind}$ is a right adjoint of $(-)_\sigma$ so that, in particular, Ind is a left adjoint of $(-)_1$ and Coind is a right adjoint of $(-)_1$. Moreover, $(-)_\sigma \circ T_{\sigma^{-1}} \circ \operatorname{Ind} \cong 1_{R_1\text{-}mod}$ and $(-)_\sigma \circ T_{\sigma^{-1}} \circ \operatorname{Coind} \cong 1_{R_1\text{-}mod}$.

COROLLARY 1.4. *Let R be a G-graded ring, $U \in R_1$-mod, and $\sigma \in G$. Then the following assertions hold*:

(i) *If U is $R_\sigma \otimes_{R_1} U$-projective, then* Ind(U) *is* Ind$(U)(\sigma)$*-projective in R-gr.*

(ii) *If U is* $\operatorname{Hom}_{R_1}(R_\sigma, U)$*-injective, then* Coind$(U)$ *is* Coind$(U)(\sigma)$*-injective in R-gr.*

(iii) *If U is quasi-projective, then* Ind(U) *is quasi-projective in R-gr.*

(iv) *If U is quasi-injective, then* Coind(U) *is quasi-injective in R-gr.*

In particular, if U is a semisimple R_1-module, then $\mathrm{Ind}(U)$ $(\mathrm{Coind}(U))$ *is a quasi-projective (resp. quasi-injective) object of R-gr.*

Proof. (i) Since $T_{\sigma^{-1}} \circ \mathrm{Ind}$ is left adjoint of the exact functor $(-)_\sigma$ and $\mathrm{Ind}(U)_\sigma = R_\sigma \otimes_{R_1} U$, it follows from Theorem 1.1(i) that in our hypotheses $(T_{\sigma^{-1}} \circ \mathrm{Ind})(U)$ is $\mathrm{Ind}(U)$-projective. But, since T_σ is an isomorphism of categories we see that $\mathrm{Ind}(U)$ is $\mathrm{Ind}(U)(\sigma)$-projective.

The proof of (ii) is similar to that of (i) using now that $T_{\sigma^{-1}} \circ \mathrm{Coind}$ is a right adjoint of $(-)_\sigma$. Parts (iii) and (iv) follow from (i) and (ii) taking $\sigma = 1$, or, alternatively, Corollary 1.3 can be used.

2. Relative Projectivity in *R-gr* and Graded Clifford Theory

Let $\mathscr{A}$ be a Grothendieck category. A full subcategory $\mathscr{C}$ of $\mathscr{A}$ is called a *closed subcategory* (see [8, p. 395]) if $\mathscr{C}$ is closed under subobjects, quotient objects, and direct sums. If $\mathscr{C}$ is, furthermore, closed under extensions, then $\mathscr{C}$ is called a localizing subcategory of $\mathscr{A}$. It may be easily seen that a closed subcategory of a Grothendieck category is also a Grothendieck category.

If $\mathscr{C}$ is closed, then the sum $t_\mathscr{C}(M)$ of all the subobjects of $M \in \mathscr{A}$ which belong to $\mathscr{C}$ defines a left exact subfunctor $t_\mathscr{C}: \mathscr{A} \to \mathscr{A}$ of the identity of $\mathscr{A}$, which is called the preradical functor associated to $\mathscr{C}$.

Examples. (1) If $U \in \mathscr{A}$, then $\mathscr{I}n^{-1}(U)$ is a closed subcategory of $\mathscr{A}$ and if, furthermore, U is a finitely generated object of $\mathscr{A}$, then $\mathscr{P}r^{-1}(U)$ is also a closed subcategory.

(2) If $\mathscr{C}$ is the class of all the semisimple objects of $\mathscr{A}$, then $\mathscr{C}$ is a closed subcategory of $\mathscr{A}$, but $\mathscr{C}$ is not localizing in general.

(3) If $M \in \mathscr{A}$ is an arbitrary object, we denote by $\boldsymbol{\sigma}[M]$ the class of all the objects of $\mathscr{A}$ subgenerated by M (i.e., isomorphic to subobjects of quotient objects of direct sums of copies of M). Then $\boldsymbol{\sigma}[M]$ is a closed subcategory of $\mathscr{A}$ and is, in fact, the smallest closed subcategory of $\mathscr{A}$ containing M.

Assume now that $\mathscr{A} = R\text{-}mod$, with R an arbitrary ring. The closed subcategories of *R-mod* are also called hereditary pretorsion classes and they are in bijective correspondence with the left linear topologies of R, i.e., with the filters $\mathscr{F}$ of left ideals of R satisfying that if $I \in \mathscr{F}$ and $r \in R$, then $(I:r) \in \mathscr{F}$ (here, $(I:r) = \{a \in R \mid ar \in I\}$). This correspondence is given by $\mathscr{C} \to \mathscr{F}_\mathscr{C}$ with $\mathscr{F}_\mathscr{C} = \{I \subseteq R \mid R/I \in \mathscr{C}\}$, with inverse $\mathscr{F} \to \mathscr{C}_\mathscr{F}$ given by $\mathscr{C}_\mathscr{F} = \{X \in R\text{-}mod \mid l_R(x) \in \mathscr{F} \text{ for all } x \in X\}$ (here, $l_R(x) = \{r \in R \mid rx = 0\}$ denotes the annihilator of x) [19, Proposition VI.4.2].

Let R be a G-graded ring. If $\mathscr{C}$ is a cosed subcategory of R-gr, then $\mathscr{C}$ is called *rigid* if for any M of $\mathscr{C}$, $M(\sigma) \in \mathscr{C}$ for every $\sigma \in G$. If moreover $\mathscr{C}$ is a localizing subcategory of R-gr, then we obtain the concept of rigid localizing subcategory as given in [16].

EXAMPLES. (1) If $\mathscr{C}$ is the class of all the semisimple objects of R-gr, then it is clear that $\mathscr{C}$ is a rigid closed subcategory (if M is semisimple, then $M(\sigma)$ is also semisimple, for T_σ is an equivalence of categories).

(2) If $M \in R$-gr is graded G-invariant, i.e., $M \cong M(\sigma)$ in R-gr for every $\sigma \in G$, then it is easy to see that $\boldsymbol{\sigma}[M]$ is a rigid closed subcategory of R-gr. Now, if $M \in R$-gr, it is obvious that $\bigoplus_{\sigma \in G} M(\sigma)$ is a G-invariant graded module and so the smallest closed subcategory of R-gr which contains this module, $\boldsymbol{\sigma}^{gr}[M] = \boldsymbol{\sigma}[\bigoplus_{\sigma \in G} M(\sigma)]$ is rigid. In fact, it is the smallest rigid closed subcategory of R-gr containing M.

(3) There exist closed subcategories of R-gr which are not rigid. For example, take $\sigma \in G$ and let $\mathscr{C}_\sigma = \{M = \bigoplus_{\lambda \in G} M_\lambda \in R\text{-}gr \mid M_\sigma = 0\}$. Then $\mathscr{C}_\sigma$ is obviously a closed subcategory of R-gr (in fact, it is a localizing subcategory) but is not rigid unless $\mathscr{C}_\sigma = 0$.

We denote by $L(R)$ ($L^{gr}(R)$) the lattice of all left ideals (resp. of all graded left ideals) of the graded ring R. We will say that a nonempty subset H of $L^{gr}(R)$ is a graded linear topology on R it it is a filter in $L^{gr}(R)$ and satisfies the following additional condition: If $I \in H$ and $r \in h(R)$, then $(I : r) \in H$. Now, in a way similar to the correspondence between closed subcategories of R-mod and linear topologies on R, it can be shown that there is a bijective correspondence between rigid closed subcategories of R-gr and graded linear topologies on R, given by

$$\mathscr{C} \to H_{\mathscr{C}} = \{I \in L^{gr}(R) \mid R/I \in \mathscr{C}\}$$

$$H \to \mathscr{C}_H = \{M \in R\text{-}gr \mid l_R(x) \in H \text{ for all } x \in h(M)\}.$$

If H is a graded linear topology on R, then the set $\bar{H} = \{I \in L(R) \mid \exists J \in H, J \subseteq I\}$ is a left linear topology on R. Actually, it is easily seen that $\bar{H}$ is the smallest linear topology on R such that $H \subseteq \bar{H}$.

Let $\mathscr{C}$ be a rigid closed subcategory of R-gr. We denote by $\bar{\mathscr{C}}$ the smallest closed subcategory of R-mod such that $\mathscr{C} \subseteq \bar{\mathscr{C}}$. We then have:

PROPOSITION 2.1. *Let $\mathscr{C}$ be a rigid closed subcategory of R-gr. Then an R-module M belongs to $\bar{\mathscr{C}}$ if and only if there exists $N \in \mathscr{C}$ such that M is isomorphic to a quotient module of N.*

Proof. Let H be the graded linear topology of R associated to $\mathscr{C}$ and $\bar{H}$ the smallest linear topology on R such that $H \subseteq \bar{H}$. It is clear that

$\bar{C} = \mathscr{C}_{\bar{H}} = \{M \in R\text{-}mod \mid l_R(x) \in \bar{H},\ \forall x \in M\}$. Now, if $M \in R\text{-}mod$ is such that there exists an exact sequence in $R\text{-}mod$, $N \to M \to 0$ with $N \in \mathscr{C}$, then we obviously have that $M \in \bar{\mathscr{C}}$.

Conversely, assume that $M \in \bar{\mathscr{C}}$. Then for any $x \in M$ we have that $l_R(x) \in \bar{H}$ and hence there exists $I_x \in H$ such that $I_x \subseteq l_R(x)$. Thus we have an exact sequence in $R\text{-}mod$,

$$R/I_x \to R/l_R(x) \to 0$$

and, since $Rx \cong R/l_R(x)$, we get an exact sequence:

$$\bigoplus_{x \in M} R/I_x \to \bigoplus_{x \in M} Rx \to 0.$$

Therefore, setting $N = \bigoplus_{x \in M} R/I_x$, which obvisously belongs to $\mathscr{C}$, we see that M, being a quotient of $\bigoplus_{x \in M} Rx$, is also a quotient of N in $R\text{-}mod$.

Remarks. Observe that if $\mathscr{C} = R\text{-}gr$, then it follows from Proposition 2.1 that $\bar{\mathscr{C}} = R\text{-}mod$. Also, if $M \in R\text{-}gr$, then it is clear that $\overline{\boldsymbol{\sigma}^{gr}[M]}$ is the smallest closed subcategory of $R\text{-}mod$ containing M.

PROPOSITION 2.2. *Let $\mathscr{C}$ and $\bar{\mathscr{C}}$ be as above and $t_{\mathscr{C}}$ and $t_{\bar{\mathscr{C}}}$ the corresponding left exact preradicals. If $M \in R\text{-}gr$, then $t_{\bar{\mathscr{C}}}(M) = t_{\mathscr{C}}(M)$.*

Proof. Since $\mathscr{C} \subseteq \bar{\mathscr{C}}$, it is clear that $t_{\mathscr{C}}(M) \subseteq t_{\bar{\mathscr{C}}}(M)$. On the other hand, if $x \in t_{\bar{\mathscr{C}}}(M)$, then there exists $J \in \bar{H}$ such that $Jx = 0$ and thus there exists $I \in H$ with $I \subseteq J$, so that $Ix = 0$. If $x = \sum_{\sigma \in G} x_\sigma$ with $x_\sigma \in M_\sigma$, then $Ix_\sigma = 0$ for any $\sigma \in G$ (for I is a graded left ideal) and so $x_\sigma \in t_{\mathscr{C}}(M)$. Hence $x \in t_{\mathscr{C}}(M)$ and therefore $t_{\mathscr{C}}(M) = t_{\bar{\mathscr{C}}}(M)$.

Let $U: R\text{-}gr \to R\text{-}mod$ be the forgetful functor. Whenever we want to emphasize the distinction between M and $U(M)$, we shall write $U(M) = \underline{M}$, but if there is no danger of confusion we will also write $U(M) = M$ in order to make the notation less cumbersome. U has a right adjoint F (cf. [16, p. 4]) which is defined as follows: If $M \in R\text{-}mod$, then $F(M)$ is the additive group $\bigoplus_{\sigma \in G} {}^{\sigma}M$ (where each ${}^{\sigma}M$ is a copy of M, ${}^{\sigma}M = \{{}^{\sigma}x \mid x \in M\}$), with the R-module structure given by $a * {}^{\sigma}x = {}^{\tau\sigma}(ax)$ for $a \in R_\tau$. Obviously, the gradation of $F(M)$ is given by $F(M)_\sigma = {}^{\sigma}M$, $\sigma \in G$, and if $f \in \mathrm{Hom}_R(M, N)$, then $F(f) \in \mathrm{Hom}_{R\text{-}gr}(F(M), F(N))$ is given by $F(f)({}^{\sigma}x) = {}^{\sigma}f(x)$. We remark that F is an exact functor. Note also that $U(F(M))$ need not be a direct sum of copies of M, since the component ${}^{\sigma}M$ is not an R-submodule of $F(M)$, but just an R_1-submodule. On the other hand, it is easy to see [17, Lemma 3.1] that if $M \in R\text{-}gr$, then $F(M) \cong \bigoplus_{\lambda \in G} M(\lambda)$.

PROPOSITION 2.3. *Let $\mathscr{C}$ be a rigid closed subcategory of R-gr, and $\overline{\mathscr{C}}$ the smallest closed subcategory of R-mod containing $\mathscr{C}$. If $M \in \overline{\mathscr{C}}$, then $F(M) \in \mathscr{C}$.*

Proof. Since $M \in \overline{\mathscr{C}}$, by Proposition 2.1 there exists $N \in \mathscr{C}$ and an epimorphism $N \to M \to 0$. The exactness of F gives an exact sequence in R-gr, $F(N) \to F(M) \to 0$. Since $N \in R$-gr, we have that $F(N) = \bigoplus_{\sigma \in G} N(\sigma)$ and the fact that $\mathscr{C}$ is rigid entails that $F(N) \in \mathscr{C}$, so that $F(M) \in \mathscr{C}$ too.

If $M \in R$-gr, $M = \bigoplus_{\sigma \in G} M_\sigma$, we recall that the support of M is the set $\mathrm{Supp}(M) = \{\sigma \in G \mid M_\sigma \neq 0\}$. If $\mathrm{Supp}(M)$ is finite, we say that M has finite support and we write $\mathrm{Supp}(M) < \infty$.

One of the main results of this section is the following.

THEOREM 2.4. *Let $\mathscr{C}$ be a rigid closed subcategory of R-gr and $\overline{\mathscr{C}}$ the smallest closed subcategory of R-mod such that $\mathscr{C} \subseteq \overline{\mathscr{C}}$. Then the following statements hold:*

(i) *If $P \in R$-gr is a projective object of the category $\mathscr{C}$, then $\underline{P}$ is a projective object of the category $\overline{\mathscr{C}}$.*

(ii) *Assume that R has finite support. If $Q \in R$-gr has finite support and is an injective object of $\mathscr{C}$, then $\underline{Q}$ is an injective object of $\overline{\mathscr{C}}$.*

Proof. (i) By Proposition 2.3 we have functors $U' : \mathscr{C} \to \overline{\mathscr{C}}$, $F' : \overline{\mathscr{C}} \to \mathscr{C}$, where U' (F') is the restriction of the functor U (resp. F). Since F' remains a right adjoint of U' and, moreover, F' is exact, it follows from 1.2 that $\underline{P}$ ($= U'(P)$) is projective in $\overline{\mathscr{C}}$.

(ii) Let $E^g(Q)$ be the injective envelope of Q in R-gr. From [15, Theorem 2.1] it follows that $E^g(Q)$ is, in this case, also injective in R-mod. By Proposition 2.2 we have that $t_{\overline{\mathscr{C}}}(E^g(Q)) = t_{\mathscr{C}}(E^g(Q))$ and if we call this module Q', then it is clear that Q' is injective in $\overline{\mathscr{C}}$ and also in $\mathscr{C}$. Since $Q \in \mathscr{C}$, we have $Q \subseteq Q' \subseteq E^g(Q)$ and as $Q \subseteq Q'$ is an essential extension we get that $Q = Q'$ and so $\underline{Q}$ is injective in $\overline{\mathscr{C}}$.

Remarks. If $\mathscr{C} = R$-gr, then $\overline{\mathscr{C}} = R$-$mod$ and so part (i) of Theorem 2.4 reduces to the well known (and easy) result that if $P \in R$-gr is projective in R-gr, then $\underline{P}$ is projective in R-mod (see [16, Corollary I.2.3]). Similarly, part (ii) gives that if R has finite support and $Q \in R$-gr is injective in R-gr and has finite support, then $\underline{Q}$ is injective in R-mod (see [15, Theorem 2.1]).

The following corollary will be very useful in the sequel.

COROLLARY 2.5. *Let R be a G-graded ring and $M \in R$-gr. Then the following assertions hold:*

(i) *$\underline{M}$ is projective in $\boldsymbol{\sigma}[\underline{M}]$ if and only if M is projective in $\boldsymbol{\sigma}^{gr}[M]$.*

(ii) *If $\underline{M}$ is finitely generated, then $\underline{M}$ is quasi-projective in R-mod if and only if M is $M(\sigma)$-projective in R-gr for each $\sigma \in G$ (in particular, M is quasi-projective in R-gr).*

(iii) *Assume that R and M have finite support. Then the following conditions are equivalent:*

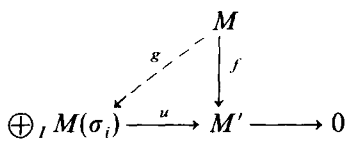
(a) *$\underline{M}$ is quasi-injective in R-mod (i.e., $\underline{M}$ is injective in $\boldsymbol{\sigma}[\underline{M}]$).*

(b) *M is $M(\sigma)$-injective in R-gr for each $\sigma \in G$ (in particular, M is quasi-injective in R-gr).*

(c) *M is injective in $\boldsymbol{\sigma}^{gr}[M]$.*

(iv) *If M is gr-semisimple, then $\underline{M}$ is projective in $\boldsymbol{\sigma}[\underline{M}]$ (in particular, $\underline{M}$ is quasi-projective in R-mod) and, furthermore, $\underline{M}$ is $\underline{M}'$-projective in R-mod for every gr-semisimple module M'. If moreover R has finite support and $\underline{M}$ is finitely generated, then $\underline{M}$ is $\underline{M}'$-injective in R-mod and, in particular, $\underline{M}$ is quasi-injective in R-mod.*

Proof. (i) The sufficiency follows from Theorem 2.4, since $\overline{\boldsymbol{\sigma}^{gr}[M]} = \boldsymbol{\sigma}[\underline{M}]$. Conversely, if $\underline{M}$ is projective in $\boldsymbol{\sigma}[\underline{M}]$, consider the diagram in *R-gr*,



where $(\sigma_i)_I$ is any family of elements of G. Since $M(\sigma_i) = M$ as R-modules, we see that there exists an R-homomorphism g making the diagram commutative. Now, Lemma I.2.1 of [16] entails the existence of $g' \in \mathrm{Hom}_{R\text{-}gr}(M, \bigoplus_I M(\sigma_i))$ such that $u \circ g' = f$ and hence M is projective in $\boldsymbol{\sigma}^{gr}[M]$.

(ii) As it is well known, if $\underline{M}$ is finitely generated, $\underline{M}$ is quasi-projective in *R-mod* if and only if it is projective in $\boldsymbol{\sigma}[\underline{M}]$. Similarly, since M is in this case a small object of $\boldsymbol{\sigma}^{gr}[M]$, the class $\mathscr{P}\imath^{-1}(M) \subseteq \boldsymbol{\sigma}^{gr}[M]$ is closed under direct sums and so M is $M(\sigma)$-projective in *R-gr* for every $\sigma \in G$ if and only if it is projective in $\boldsymbol{\sigma}^{gr}[M]$. Therefore, assertion (ii) follows from (i).

(iii) The implication (a) $\Rightarrow$ (b) follows in a way similar to the proof of the necessity in (i). (b) $\Rightarrow$ (*c*) is a consequence of the fact that the class $\mathscr{I}n^{-1}(M)$ of $\boldsymbol{\sigma}^{gr}[M]$ is closed under direct sums, subobjects, and quotient objects. Finally, (c) $\Rightarrow$ (a) follows from Theorem 2.4.

(iv) If M is gr-semisimple, it is clear that all objects of $\sigma^{gr}[M]$ are gr-semisimple and so M is projective in $\sigma^{gr}[M]$. Now (i) entails that $\underline{M}$ is also projective in $\sigma[\underline{M}]$ (and, in particular, quasi-projective in *R-mod*). On the other hand, if $\mathscr{C}$ is the rigid subcategory of *R-gr* consisting of all the gr-semisimple modules, it is obvious that M is both projective and injective in $\mathscr{C}$. Then it follows from Theorem 2.4 that $\underline{M}$ is projective in $\overline{\mathscr{C}}$ and so we see that $\underline{M}$ is $\underline{M}'$-projective in *R-mod* for each gr-semisimple module M'.

Finally, if R has finite support and $\underline{M}$ is finitely generated, then M has finite support too [15, Proposition 2.1] and, using again Theorem 2.4, we get that $\underline{M}$ is injective in $\overline{\mathscr{C}}$; i.e., $\underline{M}$ is $\underline{M}'$-injective for every gr-semisimple module M'.

Remarks. While the implications (a) ⇒ (b) and (b) ⇒ (c) in part (iii) of the above corollary are true even if the graded ring R does not have finite support, the implication (c) ⇒ (a) does not hold in general. For example, conider the $\mathbb{Z}$-graded ring $R = k[X, X^{-1}]$, where k is a field and X an indeterminate. Then R is an injective object in *R-gr* but is not quasi-injective (= injective) in *R-mod.*

COROLLARY 2.6. *Let R be a G-graded ring and Σ a gr-simple module. Then $\underline{\Sigma}$ is $\underline{\Sigma}'$-projective in R-mod for any gr-simple module Σ' and if, furthermore,* $\operatorname{Supp}(R) < \infty$, *then $\underline{\Sigma}$ is $\underline{\Sigma}'$-injective in R-mod.*

COROLLARY 2.7. *Let R be a graded ring and $N \in R_1$-mod. Then the following assertions hold:*

(i) *If N is finitely generated and $R_\sigma \otimes_{R_1} N$-projective for every $\sigma \in G$, then $\underline{\operatorname{Ind}(N)}$ is quasi-projective in R-mod.*

(ii) *If R has finite support and N is* $\operatorname{Hom}_{R_1}(R_\sigma, N)$*-injective for every $\sigma \in G$, then $\underline{\operatorname{Coind}(N)}$ is quasi-injective in R-mod.*

Proof. It is a direct consequence of Corollary 1.4 and Corollary 2.5.

We will use the following classical result of B. Mitchell [8, 19]. If $\mathscr{A}$ is a Grothendieck category with a small projective generator U, then $\mathscr{A}$ is equivalent to the category of modules *A-mod*, where $A = \operatorname{End}_{\mathscr{A}}(U)^{opp}$.

COROLLARY 2.8. *Let R be a G-graded ring, $\mathscr{C}$ a rigid closed subcategory of R-gr, and $\overline{\mathscr{C}}$ the smallest closed subcategory of R-mod containing $\mathscr{C}$. If $\mathscr{C}$ is equivalent to a module category, then so is $\overline{\mathscr{C}}$.*

Proof. Assume that $\mathscr{C}$ is equivalent to *A-mod* for some ring A; i.e., there exists an equivalence of categories T: *A-mod* $\to \mathscr{C}$. Then, calling $U = T({}_AA)$

we see that U is a small projective generator, so that U is a finitely generated R-module. Now, using Proposition 2.1 we see that U is a generator of $\overline{\mathscr{C}}$ and from Theorem 2.4 it follows that U is projective in $\overline{\mathscr{C}}$. Thus the aforementioned theorem of Mitchell implies that $\overline{\mathscr{C}}$ is equivalent to *B-mod*, where $B = \mathrm{End}_R(U)^{opp}$.

Remarks. Observe that, in general, $B \neq A$, for $A \cong \mathrm{End}_{\mathscr{C}}(U)^{opp} = \mathrm{End}_{R\text{-}gr}(U)^{opp}$. Also, taking $\mathscr{C} = R\text{-}gr$, we have that $\overline{\mathscr{C}} = R\text{-}mod$. Since as it is well known, *R-gr* is not in general equivalent to a category of modules [12, Remark 2.4], we see that the converse of Corollary 2.8 does not hold.

An object N of a Grothendieck category $\mathscr{A}$ is said to be M-generated (where $M \in \mathscr{A}$) if it is a quotient of a direct sum $M^{(I)}$ of copies of M. If each subobject of M is M-generated, then we say that M is a self-generator. It is easy to see that M is a self-generator if and only if, for any subobject $M' \subseteq M$, there exists a family $(f_i)_{i \in I}$ of elements of $\mathrm{End}_{\mathscr{A}}(M)$ such that $M' = \sum_{i \in I} f_i(M)$.

PROPOSITION 2.9. *Let M be a graded R-module and assume that M is projective in $\boldsymbol{\sigma}^{gr}[M]$. Then the following conditions are equivalent*:

(i) *Every graded submodule of M is $\bigoplus_{\sigma \in G} M(\sigma)$-generated in R-gr.*

(ii) *If M' is a graded submodule of M, then there exists a family $(f_i)_{i \in I}$ of elements of $\mathrm{END}_R(M)$ such that $M' = \sum_{i \in I} f_i(M)$.*

(iii) *$\bigoplus_{\sigma \in G} M(\sigma)$ is a projective generator of $\boldsymbol{\sigma}^{gr}[M]$.*

(iv) *$\underline{M}$ is a projective generator of $\boldsymbol{\sigma}[\underline{M}]$.*

Proof. (i)$\Leftrightarrow$(ii) follows in a straightforward way from the definitions. Now, bearing in mind that the σ-suspension functor induces an equivalence of $\boldsymbol{\sigma}^{gr}[\underline{M}]$ with itself, it is clear that in our hypotheses $\bigoplus_{\sigma \in G} M(\sigma)$ is projective in $\boldsymbol{\sigma}^{gr}[M]$ and so the proof of (i)$\Rightarrow$(iii) is the same as that of [7, Lemma 2.2]. On the other hand, (iii)$\Rightarrow$(i) is clear. Now, since $\overline{\boldsymbol{\sigma}^{gr}[M]} = \boldsymbol{\sigma}[\underline{M}]$, (iii)$\Rightarrow$(iv) follows from Proposition 2.1 and Corollary 2.5. Finally, if F denotes the right adjoint of the forgetful functor U: *R-gr* $\to$ *R-mod*, then by [17, Lemma 3.1], $FU(M) \cong \bigoplus_{\sigma \in G} M(\sigma)$ and using this, together with the exactness of F, it is easy to see that (iv)$\Rightarrow$(iii).

If $M \in R\text{-}gr$, we denote as in [5] by $(R|M)$-*mod* the full subcategory of *R-mod* whose objects are all the R-modules lying over M, i.e., all the modules which are $\underline{M}$-generated in *R-mod*. If $\Delta = \mathrm{End}_R(M)^{opp}$ and $\underline{M}$ is projective in $\boldsymbol{\sigma}[\underline{M}]$, then the class $\mathscr{C}_M = \{X \in \Delta\text{-}mod \mid Ml_{\Delta}(x) = M$ for all $x \in X\}$ is a localizing subcategory of Δ-*mod* [9, Theorem 1.3]. The quotient category of Δ-*mod* modulo $\mathscr{C}_M$ will be denoted by $\Delta\text{-}mod/\mathscr{C}_M$.

THEOREM 2.10. *Let $M \in R$-gr be projective in $\sigma^{gr}[M]$ and such that every graded submodule of M is $\bigoplus_{\sigma \in G} M(\sigma)$-generated in R-gr. Then $(R \mid M)$-mod is a closed subcategory of R-mod and if $\Delta = \operatorname{End}_R(M)^{opp}$, there are inverse equivalences of categories*:

$$\operatorname{Hom}_R(M, -) \colon (R \mid M)\text{-}mod \to \Delta\text{-}mod/\mathscr{C}_M$$

$$M \otimes_\Delta - \colon \Delta\text{-}mod/\mathscr{C}_M \to (R \mid M)\text{-}mod.$$

If moreover $\underline{M}$ is finitely generated, then Δ-mod$/\mathscr{C}_M = \Delta$-mod, so that $\operatorname{Hom}_R(M, -)$: $(R \mid M)$-mod $\to \Delta$-mod is an equivalence of categories.

Proof. By Proposition 2.9 we have that $\underline{M}$ is a projective generator of $\sigma[\underline{M}]$ and $(R \mid M)$-*mod* $= \sigma[\underline{M}]$. Now, the result follows applying [9, Theorem 1.3].

As a consequence of Theorem 2.10 we obtain the following extension of the "direct Clifford theory" given by Dade in [4, 5].

COROLLARY 2.11. *Let M be a gr-semisimple module. Then $(R \mid M)$-mod is a closed subcategory of R-mod and if $\Delta = \operatorname{End}_R(M)^{opp}$, there is an equivalence of categories $\operatorname{Hom}_R(M, -)$: $(R \mid M)$-mod $\to \Delta$-mod$/\mathscr{C}_M$.*

Proof. It is clear that if M is *gr*-semisimple, M is projective in $\sigma^{gr}[M]$ (for all the objects of this category are semisimple) and also every graded submodule of M is M-generated, so that we are in the hypotheses of Theorem 2.10 and the result follows.

In the finitely generated case we get the direct Clifford theorem:

COROLLARY 2.12. *Let M be a finitely generated gr-semisimple module (for instance, a gr-simple module). If $\Delta = \operatorname{End}_R(M)^{opp} = \operatorname{END}_R(M)^{opp}$, then $\operatorname{Hom}_R(M, -)$: $(R \mid M)$-mod $\to \Delta$-mod and $M \otimes_\Delta -$: Δ-mod $\to (R \mid M)$-mod are inverse equivalences of categories.*

3. STRUCTURE OF *gr*-SIMPLE MODULES

In order to simplify the notation we will henceforth write $U(M) = M$, so that, in particular, $\sigma[\underline{M}]$ becomes $\sigma[M]$.

Let $\Sigma = \bigoplus_{\sigma \in G} \Sigma_\sigma$ be a *gr*-simple module, i.e., a simple object of the category *R-gr*. We denote $G\{\Sigma\} = \{\sigma \in G \mid \Sigma(\sigma) \cong \Sigma\}$. It is clear that $G(\Sigma\}$ is a subgroup of G. If we set $\Delta = \operatorname{End}_R(\Sigma)^{opp}$ then, since Σ is finitely generated we have that $\Delta = \mathrm{END}_R(\Sigma)^{opp}$ and therefore Δ is a G-graded ring with the grading

$$\Delta_\sigma = (\mathrm{END}_R(\Sigma)^{opp})_\sigma = \mathrm{Hom}_{R\text{-}gr}(\Sigma, \Sigma(\sigma))$$

for any $\sigma \in G$. Since Σ is gr-simple, we have that $\Delta_\sigma = 0$ for any $\sigma \notin G\{\Sigma\}$, so that $\Delta = \bigoplus_{\sigma \in G\{\Sigma\}} \Delta_\sigma$. Since every nonzero homogeneous element of Δ is invertible, Δ is in fact a crossed product of $\Delta_1 = \mathrm{End}_{R\text{-}gr}(\Sigma)^{opp}$ by the subgroup $G(\Sigma)$.

Given a gr-simple module Σ, we will be concerned with the following two questions about Σ:

(QI) What is the structure of Σ as an R_1-module?

(QII) What is the structure of Σ when regarded without grading, i.e., as an object of *R-mod*?

We will start by answering question (QI). The results concerning the structure of Σ as R_1-module are known and they are due to Dade [5]; however, we present new proofs of them, making use of the functor Coind. These results will be contained in assertions (I1)–(I4) below.

First of all, we know that if $\Sigma = \bigoplus_{\sigma \in G} \Sigma_\sigma$ is gr-simple, then for any $\sigma \in G$, either $\Sigma_\sigma = 0$ or Σ_σ is R_1-simple. Indeed, if $0 \neq x \in \Sigma_\sigma$, then $Rx \neq 0$ and therefore $Rx = \Sigma$. Thus $R_1 x = \Sigma_\sigma$ and it follows that Σ_σ is simple as an R_1-module. Hence the first result:

(I1) Σ is semisimple as an R_1-module.

Let now $\sigma \in G\{\Sigma\}$ and $0 \neq u_\sigma \in \Delta_\sigma$. Then $u_\sigma : \Sigma \to \Sigma(\sigma)$ is an isomorphism in *R-gr* and so $u_\sigma(\Sigma_\lambda) = \Sigma(\sigma)_\lambda = \Sigma_{\lambda\sigma}$. Thus we have:

(I2) If $\sigma \in G\{\Sigma\}$ and $\lambda \in G$, then $\Sigma_\lambda \cong \Sigma_{\lambda\sigma}$ as R_1-modules.

Recall that if $M = \bigoplus_{\tau \in G} M_\tau$ is a graded module, then M is σ-faithful if, for every $0 \neq x_\tau \in M_\tau$, we have $R_{\sigma\tau^{-1}} x_\tau \neq 0$ (see, e.g., [15]).

(I3) For any $\sigma \in G$ such that $\Sigma_\sigma \neq 0$ we have $\mathrm{End}_{R_1}(\Sigma_\sigma) \cong \mathrm{End}_{R\text{-}gr}(\Sigma)$.

For the proof we will use the functor Coind: R_1-*mod* $\to$ *R-gr*. By [15, Proposition 1.2], since $\Sigma_\sigma \neq 0$, Σ is σ-faithful. Further, we have a canonical monomorphism in *R-gr*,

$$0 \longrightarrow \Sigma \xrightarrow{\alpha} \mathrm{Coind}\,(\Sigma_\sigma)(\sigma^{-1})$$

which is also an essential monomorphism [15, Proposition 1.1]. Now, if

$f \in \mathrm{End}_{R\text{-}gr}(\Sigma)$, then $f(\Sigma_\sigma) \subseteq \Sigma_\sigma$. If f_σ is the restriction of f to Σ_σ, then we have the ring homomorphism

$$\phi\colon \mathrm{End}_{R\text{-}gr}(\Sigma) \to \mathrm{End}_{R_1}(\Sigma_\sigma),\ \phi(f) = f_\sigma.$$

Let f, $g \in \mathrm{End}_{R\text{-}gr}(\Sigma)$ such that $\phi(f) = \phi(g)$, i.e., $f_\sigma = g_\sigma$. If $0 \neq x \in \Sigma_\sigma$, then $Rx = \Sigma$ and therefore, if $y \in \Sigma$ there exists $a \in R$ such that $y = ax$. Thus $f(y) = f(ax) = af(x) = af_\sigma(x) = ag_\sigma(x) = ag(x) = g(ax) = g(y)$ and so $f = g$; i.e., ϕ is a monomorphism. On the other hand, if $h \in \mathrm{End}_{R_1}(\Sigma_\sigma)$, consider the canonical morphism $\bar{h} \in \mathrm{End}_{R\text{-}gr}(\mathrm{Coind}(\Sigma_\sigma)(\sigma^{-1}))$. Since Σ is gr-simple and α is an essential monomorphism, we have that $\bar{h}(\alpha(\Sigma)) \subseteq \alpha(\Sigma)$. If we denote by f the unique gr-endomorphism of Σ such that $\bar{h} \circ \alpha = \alpha \circ f$, then we clearly have that $\phi(f) = h$, since $\mathrm{Coind}(\Sigma_\sigma)(\sigma^{-1})_\sigma = \mathrm{Coind}(\Sigma_\sigma)_1 = \mathrm{Hom}_{R_1}(R_1, \Sigma_\sigma) \cong \Sigma_\sigma$.

Next we look at the isotypic components of the R_1-semisimple module Σ. We may write $\Sigma = \bigoplus_{\omega \in \Omega} \Sigma_\omega$ where the Σ_ω denote the nonzero ω-isotypic components of Σ as R_1-module; i.e., $\Sigma_\omega = \bigoplus_{i \in I_\omega} \Sigma_i$ is the sum of all the simple R_1-submodules of Σ in the same isomorphism class ω. Then we have, denoting by $|\ |$ the cardinality of a set:

(I4) $\quad |I_\omega| = |G\{\Sigma\}|,\ |\Omega| \leqslant [G : G\{\Sigma\}]$.

To see this, let $\{\sigma_i\}_{i \in I}$ be a left transversal for $G\{\Sigma\}$ in G. Then $\Sigma = \bigoplus_{\sigma \in G} \Sigma_\sigma = \bigoplus_{i \in I} \bigoplus_{h \in G\{\Sigma\}} \Sigma_{\sigma_i h}$. By (I2) we have that $\sum_{h \in G\{\Sigma\}} \Sigma_{\sigma_i h}$ is contained in some isotypic component of Σ as R_1-module. On the other hand, if $i \neq j$, then $\Sigma_{\sigma_i} \ncong \Sigma_{\sigma_j}$. Indeed, if $\Sigma_{\sigma_i} \cong \Sigma_{\sigma_j}$ then $\mathrm{Coind}(\Sigma_{\sigma_i}) \cong \mathrm{Coind}(\Sigma_{\sigma_j})$ and hence $\mathrm{Coind}(\Sigma_{\sigma_i})(\sigma_i^{-1}) \cong \mathrm{Coind}(\Sigma_{\sigma_j})(\sigma_j^{-1})(\sigma_i^{-1}\sigma_j)$. Since the canonical monomorphism

$$0 \to \Sigma \to \mathrm{Coind}(\Sigma_{\sigma_i})(\sigma_i^{-1})$$

is essential, it follows that $\Sigma \cong \Sigma(\sigma_i^{-1}\sigma_j)$, i.e., $\sigma_i^{-1}\sigma_j \in G\{\Sigma\}$, a contradiction. Therefore, if $\Sigma_{\sigma_i} \neq 0$, then $\bigoplus_{h \in G\{\Sigma\}} \Sigma_{\sigma_i h}$ is exactly an isotypic component of Σ as R_1-module and assertion (I4) is proved.

The rest of the paper will be devoted to looking for answers to question (QII). In order to state the next result, we introduce the following notation: If $M \in R\text{-}mod$, we denote by $\mathrm{Spec}_R(M)$ the set of (isomorphism types $[S]$ of) simple R-modules S such that $S \cong P/Q$, where $Q \subseteq P \subseteq M$. $\mathrm{Spec}_{R_1}(M)$ is defined similarly.

PROPOSITION 3.1. *Let R be a graded ring. If Σ and Σ' are gr-simple modules, then the following assertions are equivalent*:

(i) *There exists $\sigma \in G$ such that $\Sigma' \cong \Sigma(\sigma)$.*

(ii) $\mathrm{Spec}_R(\Sigma) \cap \mathrm{Spec}_R(\Sigma') \neq \varnothing$.

(iii) $\mathrm{Spec}_R(\Sigma) = \mathrm{Spec}_R(\Sigma')$.

(iv) $\mathrm{Spec}_{R_1}(\Sigma) \cap \mathrm{Spec}_{R_1}(\Sigma') \neq \varnothing$.

(v) $\mathrm{Spec}_{R_1}(\Sigma) = \mathrm{Spec}_{R_1}(\Sigma')$.

Proof. Clearly assertion (i) implies every one of assertions (ii)–(v). Thus it will suffice to show that (ii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i) to complete the proof. To prove (ii) $\Rightarrow$ (iv) observe that if $S \in \mathrm{Spec}_R(\Sigma) \cap \mathrm{Spec}_R(\Sigma')$, then any $N \in \mathrm{Spec}_{R_1}(S)$ belong to $\mathrm{Spec}_{R_1}(\Sigma) \cap \mathrm{Spec}_{R_1}(\Sigma')$ and hence condition (iv) holds.

Finally, to prove (iv) $\Rightarrow$ (i) we make use of a result fo Dade which shows that two *gr*-simple modules are isomorphic in *R-gr* whenever there exists some $\tau \in G$ such that $0 \neq \Sigma_\tau \cong \Sigma'_\tau$, in R_1-*mod* [4, p. 62]. Now, if $N \in \mathrm{Spec}_{R_1}(\Sigma) \cap \mathrm{Spec}_{R_1}(\Sigma')$, then the simple R_1-module N must be isomorphic to both a component Σ'_τ of Σ' and a component $\Sigma_{\tau\sigma} = \Sigma(\sigma)_\tau$ of Σ, so that $\Sigma' \cong \Sigma(\sigma)$.

Remark. It follows from [13, 3.8] that if the graded ring R has finite support and S is a simple R-module, then there exists a *gr*-simple module Σ such that S is isomorphic to an R-submodule of Σ. Now, Proposition 3.1 shows that this Σ is unique up to a σ-translation, i.e., if S embeds in another *gr*-simple module Σ', then $\Sigma' \cong \Sigma(\sigma)$ for some $\sigma \in G$.

A very useful tool in the study of question (QII) will be the direct Clifford theorem. It will allow us to get a satisfactory answer when R is a ring of finite support.

THEOREM 3.2. *Let R be a graded ring such that* $\mathrm{Supp}(R) < \infty$. *If Σ is a gr-simple module, then the following assertions hold*:

(i) *Σ has finite length in R-mod.*

(ii) *$G\{\Sigma\}$ is a finite subgroup of G.*

(iii) *$\Delta = \mathrm{End}_R(\Sigma)^{opp}$ is a quasi-Frobenius ring.*

(iv) *If $n = |G\{\Sigma\}|$ and Σ is n-torsionfree, then Σ is semisimple of finite length in R-mod.*

(v) *If G is a torsionfree group, then Σ is a simple R-module. Moreover, every simple R-module is gradable.*

(vi) *If $S \in \mathrm{Spec}_R(\Sigma)$, then S is isomorphic to a minimal R-submodule of Σ.*

Proof. (i) Since Σ is finitely generated and R has finite support, Σ has finite support too, i.e., $\Sigma = \bigoplus_{i=1}^{s} \Sigma_{\sigma_i}$ for elements $\sigma_1, ..., \sigma_s \in G$. Thus Σ is an R_1-module of finite length, i.e., it is noetherian and artinian as R_1-module. Therefore, Σ also has finite length as R-module.

(ii) This follows from (I4) and the proof of (i) above.

(iii) This follows from the known fact that a crossed product of a QF ring by a finite group is QF, since in this case $G\{\Sigma\}$ is a finite by (ii).

(iv) If Σ is n-torsionfree, then n is invertible in Δ. Thus, by Maschke's theorem [14, Corollary 2.3], Δ is a semisimple artinian ring. Now, by the direct Clifford theorem, Σ is a semisimple R-module.

(v) If G is torsionfree, then $G\{\Sigma\}=\{1\}$ and thus $\Delta=\Delta_1$; i.e., Δ is a division ring. Now, by the direct Clifford theorem we have that Σ is a simple R-module. On the other hand, if S is a simple R-module, then S embeds in a *gr*-simple module Σ. Since Σ is also simple as R-module, we get that $S \cong \Sigma$.

(vi) By the direct Clifford theorem, if $S \in \mathrm{Spec}_R(\Sigma)$, then there exists a nonzero morphism $f\colon \Sigma \to S$ which must be an epimorphism. Now, since Δ is a QF ring, every simple Δ-module is isomorphic to a minimal left ideal of Δ. Then, using again the direct Clifford theorem we see that S is isomorphic to a minimal submodule of Σ.

Some of the results obtained in Theorem 3.2 can be extended without difficulty to finitely generated *gr*-semisimple modules. For example, we prove, with a technique different from the used in Theorem 3.2 and in a slightly more general way, that Δ is also in this case a QF ring. We have:

PROPOSITION 3.3. *Let R be a graded ring and M a gr-semisimple R-module, with $\Delta=\mathrm{End}_R(M)^{opp}$. Then the following conditions are equivalent*:

(i) *Δ is a quasi-Frobenius ring.*

(ii) *M is quasi-injective and noetherian in R-mod.*

(iii) *M is quasi-injective and artinian in R-mod.*

Proof. By Corollary 2.5, M is projective in $\sigma[M]$ and by Proposition 2.9, M is a generator of $\sigma[M]$, so that each submodule of M is M-generated. Thus the result follows from [10, Corollary 2.9].

COROLLARY 3.4. *Let R be a graded ring of finite support, M a gr-semisimple module, and $\Delta=\mathrm{End}_R(M)^{opp}$. Then the following conditions are equivalent*:

(i) *Δ is a quasi-Frobenius ring.*

(ii) *M has finitely generated essential socle in R-mod.*

(iii) *M is finitely generated in R-mod.*

(iv) *M has finite length in R-mod.*

Moreover, if these equivalent conditions hold, then every $S \in \operatorname{Spec}_R(M)$ *is isomorphic to a minimal R-submodule of M.*

Proof. If Δ is QF, then M has finite essential socle by Proposition 3.3, so that (i) $\Rightarrow$ (ii). If M has finite essential socle, then M is a finite direct sum of *gr*-simple modules, so that (ii) $\Rightarrow$ (iii). Again, if M is finitely generated, then it is a finite direct sum of *gr*-simple modules, so that M has finite length in *R-mod* by Theorem 3.2(i). The implication (iv) $\Rightarrow$ (i) is now a consequence of Proposition 3.3 bearing in mind that in this case M is quasi-injective by Corollary 2.5. Finally, the last assertion is proved exactly as part (vi) of Theorem 3.2.

We are now going to consider the general case in which R is no longer assumed to have finite support. In exchange for this added generality we must impose some condition on the group G. Thus we will say that G is a poly-{infinite cyclic} group if there exists a finite subnormal series $\{1\} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$, such that each factor G_{i+1}/G_i is an infinite cyclic group. The number n is an invariant of G (it does not depend on the particular series chosen) and is called the Hirsch number of G (and denoted by $h(G)$).

If $H \leqslant G$ is a subgroup of G, then

$$\{1\} = H \cap G_0 \lhd H \cap G_1 \lhd \cdots \lhd H \cap G_n = H$$

is a subnormal series of H and $H \cap G_{i+1}/H \cap G_i$ is canonically isomorphic to a subgroup of G_{i+1}/G_i. Hence, either $H \cap G_{i+1}/H \cap G_i = \{1\}$ or it is an infinite cyclic group. Therefore H is also a poly-{infinite cyclic} group.

We will make use of the following result:

PROPOSITION 3.5. *Let* $R = \bigoplus_{\sigma \in G} R_\sigma$ *be a crossed product such that* R_1 *is a (not necessarily commutative) domain. If G is a poly-{infinite cyclic} group, then R is a domain.*

Proof. Let $\{1\} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ be a subnormal series of G such that $G_{i+1}/G_i \cong \mathbb{Z}$ for every $0 \leqslant i \leqslant n$. We proceed by induction on n.

If $n = 1$, then $G \cong \mathbb{Z}$ and hence $R = \bigoplus_{i \in \mathbb{Z}} R_i$ is $\mathbb{Z}$-graded. Since R is a crossed product, for every $i \in \mathbb{Z}$ there exists an invertible element $u_i \in R_i$. Thus $Ru_i = R = u_i R$ and therefore $R_i = u_i R_0 = R_0 u_i$. If $a, b \in R$, $a \neq 0$, $b \neq 0$, we can write $a = a_1 + a_2 + \cdots + a_k$, $b = b_1 + b_2 + \cdots + b_m$, where $a_1, ..., a_k, b_1, ..., b_m$, are nonzero homogeneous elements. We may assume that $\deg a_1 < \deg a_2 < \cdots < \deg a_k$, $\deg b_1 < \deg b_2 < \cdots < \deg b_m$. The homogeneous component of maximum degree of the product ab is $a_k b_m$ and if $\deg a_k = r$, $\deg b_m = s$, then $a_k \in R_r$, $b_m \in R_s$, so that $a_k = \lambda u_r$, $b_m = \mu u_s$ where λ and μ are nonzero elements of R_0. Thus $a_k b_m = \lambda u_r \mu u_s = \lambda \mu' u_r u_s$, where $\mu' \in R_0$ is such that $\mu' u_r = u_r \mu$, and hence

$\mu' \neq 0$. Since R_0 is a domain, we then have that $\lambda\mu' \neq 0$ and thus, bearing in mind that u_r and u_s are invertible, we see that $a_k b_m \neq 0$ and so that $ab \neq 0$.

The general case reduces to the case $n = 1$ by considering the subgroup $H = G_{n-1}$ and R with the grading over $G/H \cong \mathbb{Z}$ given by $R \bigoplus_{c \in G/H} R_c$, where $R_c = \bigoplus_{\lambda \in c} R_\lambda$. Clearly, R is also a crossed product with this grading.

If $M \in R\text{-}mod$, then K.dim M will denote the Krull dimension of M (see [11]). If α is an ordinal, then M is said to be α-critical if K.dim $M = \alpha$ and K.dim $M/M' < \alpha$ for any nonzero submodule M' of M.

THEOREM 3.6. *Let R be a G-graded ring and Σ a gr-simple R-module. If G is a poly-$\{$infinite cyclic$\}$ group, then Σ has Krull dimension in R-mod. Moreover, Σ is k-critical, with $0 \leqslant k \leqslant h(G\{\Sigma\}) \leqslant h(G)$.*

Proof. Since $G\{\Sigma\}$ is a subgroup of G, $G\{\Sigma\}$ is also a poly-$\{$infinite cyclic$\}$ group. Since the ring $\Delta = \mathrm{End}_R(\Sigma)^{opp}$ is a strongly graded ring of type $G(\Sigma)$ and Δ_1 is a division ring, it follows from [16, Theorem II.5.24] that Δ has Krull dimension (on the left) and K.dim $\Delta \leqslant h(G\{\Sigma\})$. Now, by [11, Theorem 2.1] we have that Δ contains a k-critical left ideal I, with $k \leqslant h(G\{\Sigma\})$. Further, Δ is a domain by Proposition 3.5. Then, if $0 \neq a \in I$, the map $\phi: \Delta \to I$ given by $\phi(\lambda) = \lambda a$ is a monomorphism and so Δ is also k-critical. Now it follows from the direct Clifford theorem that Σ is k-critical.

Remark. Theorem 3.6 generalizes the well known fact that if R is a $\mathbb{Z}$-graded ring and Σ is a *gr*-simple R-module, then Σ is either simple in *R-mod* or 1-critical [16, Theorem II.6.4].

Recall from [18, p. 586] that a group G is said to be a *right ordered group* or an *RO-group* if the elements of G are linearly ordered with respect to the relation $<$ and if, for all $x, y, z \in G$, $x < y$ implies $xz < yz$. By [18, Lemma 1.6, p. 587], if the group G has a finite subnormal series

$$\{1\} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

with quotients G_{i+1}/G_i which are torsionfree abelian, then G is an RO-group. Thus, in particular, poly-$\{$infinite cyclic$\}$ groups are RO-groups. The following result [18, Lemma 1.7, p. 588] will be very useful: If G is an RO-group and A and B are finite nonempty subsets of G, then there exist b' and $b'' \in B$ such that the products $a_{\max} b'$ and $a_{\min} b''$ are uniquely represented in AB (here, $a_{\max}$ and $a_{\min}$ denote, respectively, the largest and the smallest element in A).

We will denote by $J(M)$ the Jacobson radical in *R-mod* of a module M. We have:

THEOREM 3.7. *Let R be a G-graded ring, with G an RO-group. If Σ is a gr-simple module, then $J(\Sigma)=0$.*

Proof. Using the direct Clifford theorem, it is enough to prove that $J(\Delta)=0$, where $\Delta=\operatorname{End}_R(\Sigma)^{opp}$. But $\Delta=\bigoplus_{\sigma\in G\{\Sigma\}}\Delta_\sigma$ and every nonzero homogeneous element of Δ is invertible. If $G\{\Sigma\}=\{1\}$, then $\Delta=\Delta_1$ is a division ring and so $J(\Delta)=0$. On the other hand, if $G\{\Sigma\}\neq\{1\}$, then $G\{\Sigma\}$ is infinite since an RO-group is torsionfree. First we prove that if $a\in\Delta$ is an invertible element, then a is homogeneous. Indeed, assume that there exists $b\in\Delta$ such that $ab=1$. We can write

$$a=a_{\sigma_1}+a_{\sigma_2}+\cdots+a_{\sigma_n},\qquad \text{where}\quad 0\neq a_{\sigma_i}\in R_{\sigma_i}\quad\text{and}\quad \sigma_1<\sigma_2<\cdots<\sigma_n$$

and, similarly,

$$b=b_{\tau_1}+b_{\tau_2}+\cdots+b_{\tau_m},\qquad \text{where}\quad 0\neq b_{\tau_i}\in R_{\tau_i}\quad\text{and}\quad \tau_1<\tau_2<\cdots<\tau_m.$$

Then, if $n\geqslant 2$ it follows from the above cited Lemma 1.7 of [18] that the product ab has at least two nonzero homogeneous components, which contradicts the fact that $ab=1$. Therefore, we have that $n=1$ and a is homogeneous.

Assume now that a is a nonzero element of $J(\Delta)$, so that $1-ba$ is invertible and hence homogeneous for any $b\in\Delta$. If $a=a_{\sigma_1}+a_{\sigma_2}+\cdots+a_{\sigma_n}$ with $a_{\sigma_i}\in R_{\sigma_i}$ then, since $G\{\Sigma\}$ is an infinite group, there exists an homogeneous element $0\neq b\in\Delta_\sigma$ with $\sigma\in G\{\Sigma\}$, such that $1\notin\{\sigma\sigma_1,...,\sigma\sigma_n\}$ and hence $1-ba$ has at least two nonzero homogeneous components. But on the other hand, we have seen that $1-ba$ is homogeneous and this contradiction shows that $a=0$, so that $J(\Delta)=0$.

The *gr*-Jacobson radical of a G-graded ring R will be denoted by $J^g(R)$. Recall that $J^g(R)$ is the intersection of all *gr*-maximal left (or right) ideals of R [16, p. 52].

COROLLARY 3.8. *Let R be a G-graded ring with G an RO-group. Then $J(R)\subseteq J^g(R)$.*

Proof. This follows from Theorem 3.7.

COROLLARY 3.9. *Let R be a G-graded ring, with G a poly-{infinite cyclic} group. Then $J(R)\subseteq J^g(R)$.*

COROLLARY 3.10. *Let R be a strongly G-graded ring with G an RO-group. Then $J(R)\subseteq RJ(R_1)$.*

Proof. We have $J^g(R)\cap R_1=J(R_1)$ (see [3]). Since R is strongly graded, $J^g(R)=RJ(R_1)$ and from Corollary 3.8 we get that $J(R)\subseteq RJ(R_1)$.

Remark. Corollary 3.10 generalizes a classical result of Zaleskii (see [18, Theorem 2.12, p. 602]).

ACKNOWLEDGMENT

We thank the referee for several helpful suggestions which have significantly contributed to improve the paper.

REFERENCES

1. T. ALBU AND C. NĂSTĂSESCU, "Relative Finiteness in Module Theory," Dekker, New York, 1984.
2. F. W. ANDERSON AND K. R. FULLER, "Rings and Categories of Modules," Springer-Verlag, New York, 1974.
3. M. BEATTIE, A generalization of the smash product of a graded ring, *J. Pure Appl. Algebra* **52** (1988), 219–226.
4. E. C. DADE, Clifford theory for group-graded rings, *J. Reine Angew. Math.* **369** (1986), 40–86.
5. E. C. DADE, Clifford theory for group-graded rings, II, *J. Reine Angew. Math.* **387** (1988), 148–181.
6. C. FAITH, "Algebra: Rings, Modules and Categories, I," Springer-Verlag, Berlin, 1973.
7. K. R. FULLER, Density and equivalence, *J. Algebra* **29** (1974), 528–550.
8. P. GABRIEL, Des catégories abéliennes, *Bull. Soc. Math. France* **90** (1962), 323–448.
9. J. L. GARCÍA HERNÁNDEZ AND J. L. GÓMEZ PARDO, On endomorphism rings of quasiprojective modules, *Math. Z.* **196** (1987), 87–108.
10. J. L. GARCÍA HERNÁNDEZ AND J. L. GÓMEZ PARDO, Self-injective and *PF* endomorphism rings, *Israel J. Math.* **58** (1987), 324–350.
11. R. GORDON AND J. C. ROBSON, Krull dimension, *Mem. Amer. Math. Soc.* **133** (1973).
12. C. MENINI AND C. NĂSTĂSESCU, When is *R-gr* equivalent to the category of modules?, *J. Pure Appl. Algebra* **51** (1988), 277–291.
13. C. MENINI AND C. NĂSTĂSESCU, *Gr*-simple modules and *gr*-Jacobson radical, preprint.
14. C. NĂSTĂSESCU, Strongly graded rings of finite groups, *Comm. Algebra* **11** (1983), 1033–1071.
15. C. NĂSTĂSESCU, Some constructions over graded rings: Applications, *J. Algebra* **120** (1989), 119–138.
16. C. NĂSTĂSESCU AND F. VAN OYSTAEYEN, "Graded Ring Theory," North-Holland, Amsterdam, 1982.
17. C. NĂSTĂSESCU, M. VAN DEN BERGH, AND F. VAN OYSTAEYEN, Separable functors applied to graded rings, *J. Algebra* **123** (1989), 397–413.
18. D. S. PASSMAN, "The Algebraic Structure of Group Rings," Wiley, New York, 1977.
19. B. STENSTRÖM, "Rings of Quotients," Springer-Verlag, Berlin, 1975.
20. B. R. ZHOU, Morita context functors and their applications over graded rings, preprint.