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2-extendability and k-resonance of non-bipartite Klein-bottle polyhexes[☆]

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ABSTRACT

C. Thomassen classified Klein-bottle polyhexes into five classes [C. Thomassen, Tilings of the torus and the Klein bottle and vertex-transitive graphs on a fixed surface, Trans. Amer. Math. Soc. 323 (1991) 605–635]. In this paper, by implementing cutting and gluing operations, we reclassify the five classes of Klein-bottle polyhexes into two classes—the bipartite case $K(p, q, t)$ and the non-bipartite case $N(p, q, t)$. Further, we completely characterize 2-extendable and k-resonant non-bipartite Klein-bottle polyhexes, respectively.

1. Introduction

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. A set M of edges in a graph G is a *matching* if no two members of M share a common end vertex. A matching M is *perfect* if every vertex of G is incident with an edge in M. Let $n \geq 1$ be an integer. A connected graph G possessing at least $2n+2$ vertices is said to be *n-extendable* if there exists a matching of size n and every such matching $M \subseteq E(G)$ can be extended to a perfect matching of G. The concept of n-extendable graph was introduced by Plummer [10]. It is well known that n-extendable graphs are $(n + 1)$-connected [9,10].

A *fullerene graph* is a 3-connected cubic plane graph with only pentagonal and hexagonal faces. It corresponds to a spherical fullerene molecule in chemistry; for details, see [5]. Zhang and Zhang [22] found the 2-extendability of fullerene graphs and showed that fullerene graphs with p vertices have at least $\lceil \frac{3(p+2)}{4} \rceil$ perfect matchings. A graph is bipartite if the vertices of it can be divided into two independent sets. A graph is non-bipartite if and only if it contains a cycle of odd length. Then every fullerene graph is non-bipartite since it contains (exactly twelve) pentagons. In [10], it was shown that every 2-extendable non-bipartite graph is 3-connected and bicritical, and hence a brick. Bricks play an important role in the decomposition of bicritical graphs, the odd cycle property of 1-extendable graphs and Pfaffian orientation of graphs.

After the discovery of spherical fullerenes, the extension of fullerenes on other closed surfaces was considered in [4,7]. Precisely, it was shown that only four surfaces were possible: sphere, torus, projective plane and Klein bottle. The torus and Klein bottle are the only two surfaces that are tiled entirely with hexagons and the corresponding tilings are called toroidal polyhexes and Klein-bottle polyhexes, respectively. Many properties of toroidal polyhexes with chemical relevance were considered. A survey is referred to [7]. The classification of toroidal polyhexes in a dual form was given by Altschuler early in 1973 [1]. Later, Thomassen classified Klein-bottle polyhexes (respectively, toroidal polyhexes) with girth 6 into five (respectively, two) classes. In this paper, by the use of cutting and gluing operations which source from topology, we reclassify the five classes into two new classes $K(p, q, t)$ and $N(p, q, t)$ which are given as follows. (Note that by using the

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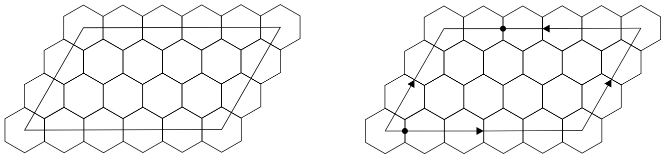

Fig. 1. A 5×3-parallelogram in a hexagonal lattice (left) and the Klein bottle boundary identification (right) ($t = 1$, both black dots are identified).

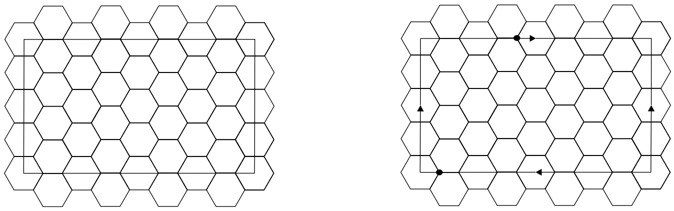

Fig. 2. An 8×8-rectangle in a hexagonal lattice (left) and the Klein bottle boundary identification with $t = 3$ (right).

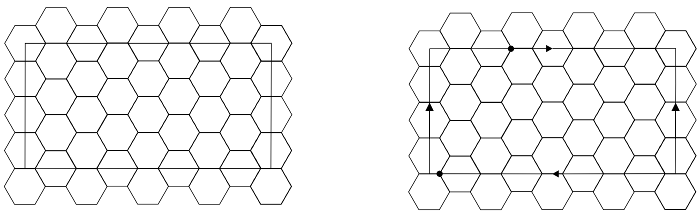

Fig. 3. An 8×7-rectangle in a hexagonal lattice (left) and the Klein bottle boundary identification with $t = 2$ (right).

same technique, one can reclassify the two classes of toroidal polyhexes given by Thomassen into one new class $H(p, q, t)$ whose definition can be found in [18]).

Given integers $p \geq 1, q \geq 1$ and $t \geq 0$. Let P be a $p \times q$-parallelogram in a hexagonal lattice illustrated in Fig. 1, every corner of P lies at the center of a hexagon. The top side (respectively, the bottom side) intersects p vertical edges and each of the two parallel lateral sides passes through q edges perpendicular to them. A *Klein-bottle polyhex* $K(p, q, t)$ is obtained from P by the following boundary identification: first identify two lateral sides along the same direction and then identify the bottom side with the top side along the reverse direction with a torsion t.

Given $q \geq 1, t \geq 0$ and even $p \geq 2$. Let R be a $p \times q$-rectangle in a hexagonal lattice illustrated in Figs. 2 and 3: every top corner of R lies at the center of a hexagon and every bottom corner of R lies at the center of a hexagon if q is even but at the center of a horizontal edge otherwise. The top side (respectively, the bottom side) covers $\frac{p}{2}$ horizontal edges and each of the two parallel lateral sides passes through $\lceil \frac{q}{2} \rceil$ edges perpendicular to them. A *Klein-bottle polyhex* $N(p, q, t)$ is obtained from R by the following boundary identification: first identify two lateral sides along the same direction and then identify the bottom side with the top side along the reverse direction with a torsion t (note that t must have the opposite parity with q).

It is obvious that all $K(p, q, t)$ are bipartite as declared in [18]. In Section 2, we will show that all $N(p, q, t)$ are non-bipartite not depending on the value of torsion t. In Section 3 we recall Thomassen's classification of Klein-bottle polyhexes and reclassify it into two classes $K(p, q, t)$ and $N(p, q, t)$. In Section 4 the matching extendability of non-bipartite Klein-bottle polyhex $N(p, q, t)$ is considered. For any $K(p, q, t)$, it has been proven that it is 1-extendable if and only if it has at least four vertices, and 2-extendable if and only if it is a strong embedding [18]. Here the strong embedding of an embedded graph means that every face of it is bounded by a cycle. For $N(p, q, t)$, we show that it is 1-extendable if and only if $q \geq 2$, and 2-extendable if and only if $p \geq 4$ and $q \geq 5$. Unlike $K(p, q, t)$, there are non-1-extendable graphs with arbitrarily large number of vertices and non-2-extendable graphs with strong embeddings in $N(p, q, t)$.

Finally, we characterize the k-resonance of $N(p, q, t)$. For an embedded graph G on some fixed surface, it is called *a map* if every face of it is homeomorphic to an open disk and bounded by a cycle. A map is called *k-resonant* if for any i ($1 \leq i \leq k$) disjoint faces with even length, the remainder of the map after deleting the i faces has a perfect matching or is empty. The concept of resonance originates from Clar's aromatic sextet theory [2] and Randić's conjugated circuit model [11]. The k-resonance was first introduced by Zheng in benzenoid systems in [23]. Hereafter, the k-resonance of many other graphs were researched extensively in [6–8,12–14,17,20,19,21,23]. The 1-resonance and k-resonance ($k \geq 3$) of these graphs are well determined and in general, 3-resonance implies k-resonance for any $k \geq 1$. But 2-resonance for benzenoid systems, open-ended nanotubes and fullerenes remains open. In [19], an unexpected result that every B–N fullerene graph is 2-resonant is given. For Klein-bottle polyhexes of the form $K(p, q, t)$, Shiu and Zhang [13] gave a complete characterization for its k-resonance. In Section 5, we characterize all the k-resonant non-bipartite Klein-bottle polyhexes $N(p, q, t)$. Up until now, the 2-extendability and k-resonance of all Klein-bottle polyhexes are solved completely.

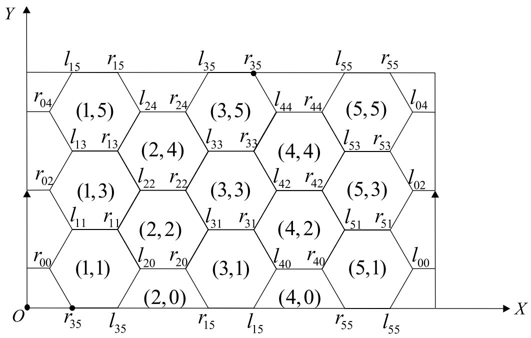

Fig. 4. Labels for hexagons and vertices of the Klein-bottle polyhex $N(6, 6, 3)$ and the same label points are identified.

2. Preliminaries

For convenience, we first establish a Cartesian coordinate system XOY of $N(p, q, t)$ in a similar way as described in [13] (see Fig. 4): take the bottom side as the x-axis and the left vertical side as the y-axis, their intersection as the origin O and R lying on the non-negative region. On the x-axis, half of the sum of the side length and the distance between two opposite vertices in a hexagon is a unit length, and on the y-axis, half of the distance between two parallel edges in a hexagon is a unit length. For any positive integer n, we use Z_n to denote the set $\{0, 1, 2, \ldots, n-1\}$ with arithmetic modulo n. Each hexagon is named by the coordinate (x, y) at its center, where $x \in Z_p$ and $y \in Z_q$. This hexagon is thus denoted by (x, y) ($h_{x,y}$ or h_{xy}). For each hexagon (x, y), the left top vertex is named by $l_{x,y}$ or l_{xy} and the right top vertex is denoted by $r_{x,y}$ or r_{xy}.

Under this notation, for $q \geq 2$, $r_{x,0}$ is adjacent to $l_{x,0}$, $l_{x+1,1}$ and $r_{t-x,q-1}$, $l_{x,0}$ is adjacent to $r_{x,0}$, $r_{x-1,1}$ and $l_{t-x+2,q-1}$, $r_{x,q-1}$ is adjacent to $l_{x,q-1}$, $l_{x+1,q-2}$ and $r_{t-x,0}$, and $l_{x,q-1}$ is adjacent to $r_{x,q-1}$, $r_{x-1,q-2}$ and $l_{t-x+2,0}$. For $y \neq 0, q-1$, r_{xy} is adjacent to l_{xy}, $l_{x+1,y-1}$ and $l_{x+1,y+1}$. Also, l_{xy} is adjacent to $r_{x,y}$, $r_{x-1,y+1}$ and $r_{x-1,y-1}$. But for $q = 1$, $r_{x,0}$ is adjacent to $l_{x,0}$, $r_{t-x,0}$ and $r_{t-x,0}$, and $l_{x,0}$ is adjacent to $r_{x,0}$, $l_{t-x+2,0}$ and $l_{t-x+2,0}$.

We define a mapping ϕ_{sd} from $N(p, q, t)$ to $N(p, q, t+2)$ as follows.

$$\phi_{sd}(r_{x,y}) = r_{x+1,y-1} \quad \text{and} \quad \phi_{sd}(l_{x,y}) = l_{x+1,y-1} \quad \text{for } 1 \leq y \leq q-1, \quad \text{and}$$
$$\phi_{sd}(r_{x,0}) = l_{(t+2)-x,q-1} \quad \text{and} \quad \phi_{sd}(l_{x,0}) = r_{(t+2)-x,q-1}.$$

An *isomorphism* between two simple graphs G and H is a bijection $\pi : V(G) \to V(H)$ such that for $x, y \in V(G)$, x and y are adjacent if and only if $\pi(x)$ and $\pi(y)$ are adjacent in H.

Theorem 2.1. *ϕ_{sd} is a hexagon-preserving isomorphism from $N(p, q, t)$ to $N(p, q, t+2)$.*

Proof. It can be seen that ϕ_{sd} is a bijection from $V(N(p, q, t))$ to $V(N(p, q, t+2))$. For the preserving of the adjacency, it is easy to see that

$$\phi_{sd}(r_{x,0})\phi_{sd}(r_{t-x,q-1}) = l_{(t+2)-x,q-1}r_{(t+2)-x-1,q-2}, \quad \text{and}$$
$$\phi_{sd}(l_{x,0})\phi_{sd}(l_{t-x+2,q-1}) = r_{(t+2)-x,q-1}l_{(t+2)-x+1,q-2},$$

which are edges of $N(p, q, t+2)$. (The preserving of the adjacency of the other edges is trivial). So we can extend ϕ_{sd} to be a bijection between the edge sets of $N(p, q, t)$ and $N(p, q, t+2)$.

Moreover, we can show that ϕ_{sd} preserves the hexagonal faces from $N(p, q, t)$ to $N(p, q, t+2)$. For any $x, y \geq 2$, it can easily be seen that the image of the hexagon $h_{x,y}$ in $N(p, q, t)$ is the hexagon $h_{x+1,y-1}$ in $N(p, q, t+2)$. For the hexagon $h_{x,1}$,

$$\phi_{sd}(h_{x,1}) = \phi_{sd}(l_{x,1}r_{x,1}l_{x+1,0}l_{t-x+1,q-1}r_{t-x+1,q-1}r_{x-1,0}) \quad (1)$$
$$= l_{x+1,0}r_{x+1,0}r_{(t+2)-x-1,q-1}l_{(t+2)-x,q-2}r_{(t+2)-x,q-2}l_{(t+2)-x+1,q-1}. \quad (2)$$

By verifying the adjacency relation, (2) is the hexagonal face $h_{x+1,0}$ of $N(p, q, t+2)$. For the hexagon $h_{x,0}$, if $q \geq 2$,

$$\phi_{sd}(h_{x,0}) = \phi_{sd}(l_f x,\ 0r_{x,0}r_{t-x,q-1}l_{t-x+1,q-2}r_{t-x+1,q-2}l_{t-x+2,q-1}) \quad (3)$$
$$= r_{(t+2)-x,q-1}l_{(t+2)-x,q-1}r_{(t+2)-x-1,q-2}l_{(t+2)-x,q-3}r_{(t+2)-x,q-3}l_{(t+2)-x+1,q-2}, \quad (4)$$

if $q = 1$,

$$\phi_{sd}(h_{x,0}) = \phi_{sd}(l_{x,0}r_{x,0}r_{t-x,0}r_{x,0}l_{x,0}l_{t-x+2,0}) \quad (5)$$
$$= r_{(t+2)-x,0}l_{(t+2)-x,0}l_{2+x,0}l_{(t+2)-x,0}r_{(t+2)-x,0}r_{x,0}, \quad (6)$$

where (4) or (6) is the hexagonal face $h_{(t+2)-x,q-1}$ of $N(p, q, t+2)$. □

By the above theorem, $N(p, q, t)$ can be abbreviated as $N(p, q)$.

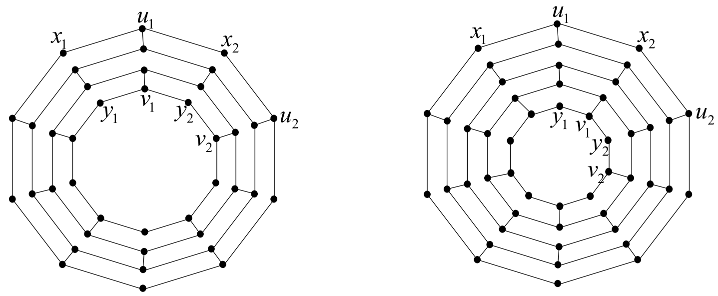

Fig. 5. Hexagonal cylinders $H_{5,3}$ and $H_{5,4}$, respectively.

Fig. 6. A hexagonal Möbius circuit of length 5 and a hexagonal Möbius double circuit of length 9. (The length here means the number of hexagons in the tilings.)

Theorem 2.2. *$N(p, q)$ is non-bipartite for any p and q.*

Proof. If q is even, by Theorem 2.1, it suffices to consider the case that $t = 1$. Then there is a $(q + 1)$-length cycle $r_{1,q-1}r_{0,0}l_{1,1}r_{0,2}l_{1,3} \cdots r_{0,q-2}l_{1,q-1}r_{1,q-1}$ in $N(p, q)$. If q is odd, similarly, we may consider $t = 0$ only. Then there is a q-length cycle $l_{1,q-1}l_{1,0}r_{0,1}l_{1,2}r_{0,3}l_{1,4} \cdots l_{1,q-3}\, r_{0,q-2}l_{1,q-1}$ in $N(p, q)$. It can be seen that there is a cycle of odd length in $N(p, q)$. Hence $N(p, q)$ is a non-bipartite graph. □

3. The reclassification of Klein-bottle polyhexes

We first recall the classification of Klein-bottle polyhexes given by Thomassen.

From the Cartesian product $C_{2k}\Box P_{m+1}$ of a $2k$-cycle C_{2k} $(x_1u_1x_2u_2 \cdots x_ku_kx_1)$ and an $(m + 1)$-path P_{m+1} $(12 \cdots (m + 1))$, by deleting the set of edges $\{\{(x_i, 2s + 1), (x_i, 2s + 2)\} \mid 0 \le s \le \lceil \frac{m}{2} - 1\rceil, 1 \le i \le k\} \cup \{\{(u_i, 2s), (u_i, 2s + 1)\} \mid 1 \le s \le \lfloor \frac{m}{2} \rfloor, 1 \le i \le k\}$, the resulting graph is called a *hexagonal cylinder of length k and breadth m*, and denoted by $H_{k,m}$ (see Fig. 5).

In the graph $H_{k,m}$, we call $C_{2k}\Box\{1\}$ and $C_{2k}\Box\{m + 1\}$ the two *peripheral cycles*. For convenience, we relabel the vertices of the two peripheral cycles. Replace $(x_i, 1)$ and $(u_i, 1)$ by x_i and u_i $(1 \le i \le k)$, respectively, and also replace $(x_i, m + 1)$ and $(u_i, m + 1)$ by v_{i-1} and y_i $(1 \le i \le k)$ if m is even but y_i and v_i $(1 \le i \le k)$ otherwise. (All subscripts here and in the next paragraph are taken modulo k.) For example, Fig. 5 shows the hexagonal cylinders $H_{5,3}$ and $H_{5,4}$, respectively with new vertex labels. A hexagonal cylinder circuit is a hexagonal cylinder of breadth 1. A hexagonal Möbius circuit and a hexagonal Möbius double circuit are defined as indicated in Fig. 6.

We now present the constructions of several new classes of graphs obtained by adding edges to the graph $H_{k,m}$. $H_{k,m,a}$ is obtained by adding the edges $y_1x_2, y_2x_1, y_3x_k, \ldots, y_ix_{k+3-i}, \ldots, y_kx_3$. For even k and odd m, $H_{k,m,b}$ is obtained by adding the edges $y_1x_1, y_2x_k, y_3x_{k-1}, \ldots, y_ix_{k+2-i}, \ldots, y_kx_2$. For even k, $H_{k,m,c}$ is obtained by adding all diagonals of the form $x_ix_{i+\frac{k}{2}}$ and $y_iy_{i+\frac{k}{2}}$ to the peripheral cycles. If k is odd, $H_{k,m,f}$ is obtained by adding a star cycle with respect to each peripheral cycle, that is, we add new vertices $w_1, w_2, \ldots, w_k$, and the edges w_ix_i for $i = 1, 2, \ldots, k$ and the cycle $w_1w_{1+\frac{k+1}{2}}w_2w_{2+\frac{k+1}{2}} \ldots w_{\frac{k+1}{2}}w_1$, and a similar star cycle is added to the other peripheral cycle. For odd k, $H_{k,d}$ (which is also the degenerate $H_{k,0,f}$) is obtained by pasting two Möbius double circuits together along their peripheral cycles such that the resulting graph is cubic.

Thomassen proved the following theorem.

Theorem 3.1 ([15]). *Let G be a connected cubic graph of girth 6 and $\mathscr{C}$ a collection of 6-cycles in G such that every 2-path in G is contained in precisely one cycle of $\mathscr{C}$. Then for some natural numbers k, m and r, G is isomorphic to one of*

$$H_{k,m,r}, H_{k,m,a}, H_{k,m,b}, H_{k,m,c}, H_{k,d}, H_{k,m,e}, H_{k,m,f}.$$

The definition of $H_{k,m,r}$ and $H_{k,m,e}$ omitted here can be found in [15].

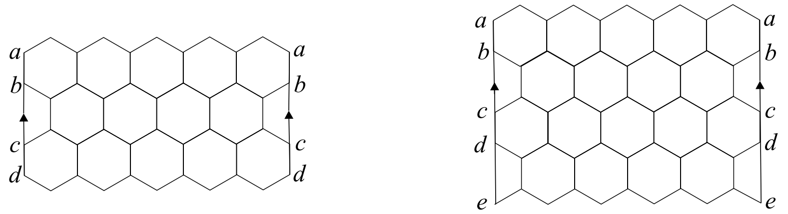

Fig. 7. The plane representation of $H_{5,3}$ and $H_{5,4}$, respectively.

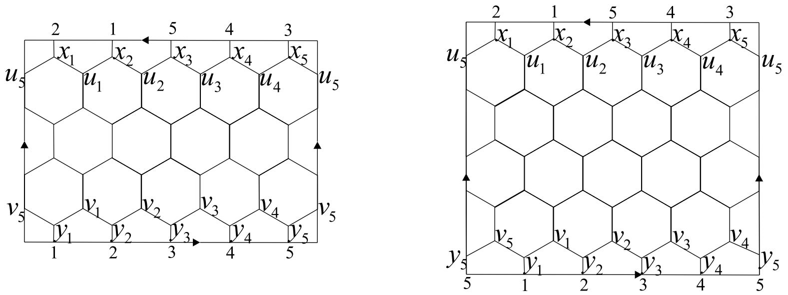

Fig. 8. The rectangle representation of $H_{5,3,a}$ (left) and $H_{5,4,a}$ (right).

Remark 3.2. As pointed out in [15], although the result and its proof in the above theorem are purely combinatorial, it actually may be convenient to think of the 6-cycles as polygons pasted together such that they form a torus or a Klein bottle. In order to avoid the degenerate cases, Thomassen imposed the "girth 6" condition on the tilings. If we drop the "girth 6" condition, by using the same proof technique we can verify that there is exactly one new class $N(2, k)$ with integer $k \geq 1$ obtained. For even k, $N(2, k)$ is precisely the degenerate $H_{k,0,c}$. But for odd k, it can be seen that $N(2, k)$ is not isomorphic to any one of $H_{k,m,a}$, $H_{k,m,b}$, $H_{k,m,c}$, $H_{k,d}$ and $H_{k,m,f}$ from the reclassification. Moreover, it can be seen that there are non-degenerate Klein-bottle polyhexes with girth less than 6: $K(p, q)$ with $\min(p, q) = 2$ and $N(p, q)$ with $q = 3, 4, 5$ or $p = 2$.

Note that $H_{k,m,r}$ and $H_{k,m,e}$ are toroidal polyhexes and $H_{k,d}$ is the degenerate $H_{k,0,f}$. Hence Theorem 3.1 can be reformulated as the following.

Theorem 3.3. *Let G be a Klein-bottle polyhex. Then it is isomorphic to one of* $H_{k,m,a}$, $H_{k,m,b}$, $H_{k,m,f}$, $H_{k,m,c}$ *and* $N(2, k)$ *for* $k \geq 1$ *and* $m \geq 0$.

Now it is time to present the detailed reclassification of the five classes of Klein-bottle polyhexes given by Thomassen.

First we embed the hexagonal cylinder $H_{k,m}$ on a cylinder regularly, that is, every hexagon is regular in the embedding. Then along a generatrix of the cylinder which covers the center of some hexagon, we cut the cylinder out and put it in the plane with the points on the generatrix identified. Finally, taking out the cylinder, we get a plane representation of the hexagonal cylinder with the same label points identified (see Fig. 7 for the illustration).

Lemma 3.4. $H_{k,m,a} \cong K(k, m+1, (p+1)k - \lfloor \frac{m}{2} \rfloor)$ *for* $p = \lfloor \frac{m}{2k} \rfloor$.

Proof. Since $H_{k,m,a}$ is obtained by adding edges to $H_{k,m}$, by using the plane representation of $H_{k,m}$ we can get a rectangle representation of $H_{k,m,a}$ illustrated in Fig. 8. Let l_1 and l_2 be the left side and the right side of the rectangle representation of $H_{k,m,a}$, respectively.

If m is odd, then let l_3 be the segment from the left most point at the bottom side to the top side such that the angle from the bottom side to l_3 in the anticlockwise direction is 60° (see l_3 in Fig. 9). Then we cut out $H_{k,m,a}$ along l_3, glue l_1 with l_2 and finally get a parallelogram representation of $H_{k,m,a}$.

It is obvious that there exists some integer t satisfying $H_{k,m,a} \cong K(k, m+1, t)$. If $m+1 \leq 2k$, then the subscript of the left most vertex among $\{x_i | 1 \leq i \leq k\}$ in the parallelogram is $\frac{m+1}{2} + 1$. So it is adjacent to $y_{k+3-(\frac{m+1}{2}+1)}$ and $t = k+3-(\frac{m+1}{2}+1)-1 = k - \frac{m-1}{2}$. If $m+1 > 2k$, then there exists a positive integer p satisfying $2pk \leq m < 4pk$. Then $H_{k,m-2pk,a} \cong K(k, m-2pk, t')$ (see Fig. 9) and $t' = (p+1)k - \frac{m-1}{2}$. From the process of the transformation, we know that $t = t' = (p+1)k - \frac{m-1}{2}$. Hence we conclude that $H_{k,m,a} \cong K(k, m+1, (p+1)k - \frac{m-1}{2})$ when m is odd.

If m is even, then let l_3 be the segment beginning at the center c of the hexagon at the left most corner of the bottom side and ending at the top side such that the angle from the bottom side to l_3 in the anticlockwise direction is 60°. Then we cut out $H_{k,m,a}$ along l_3, glue l_1 with l_2 and finally get a parallelogram representation of $H_{k,m,a}$. By a simple calculation, we get that $H_{k,m,a} \cong K(k, m+1, k - \frac{m}{2})$ if $m \leq 2k-1$ and $H_{k,m,a} \cong K(k, m+1, (p+1)k - \frac{m}{2})$ if there exists a positive integer p satisfying $2pk \leq m < 4pk$.

By the above arguments, we have that $H_{k,m,a} \cong K(k, m+1, (p+1)k - \lfloor \frac{m}{2} \rfloor)$ for $p = \lfloor \frac{m}{2k} \rfloor$. □

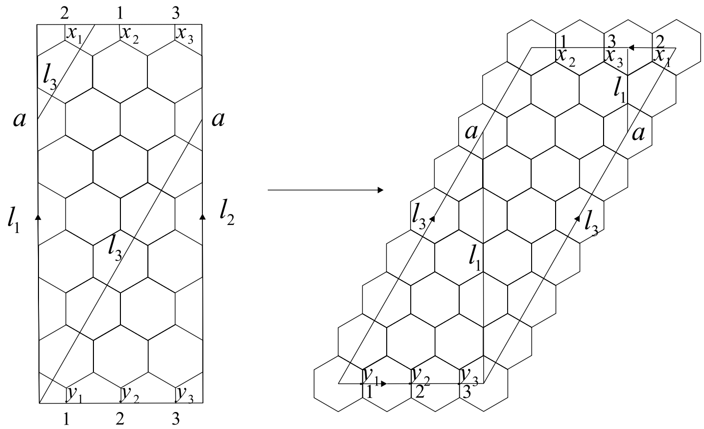

Fig. 9. The transformation from the rectangle representation to the parallelogram representation of $H_{3,7,a}$.

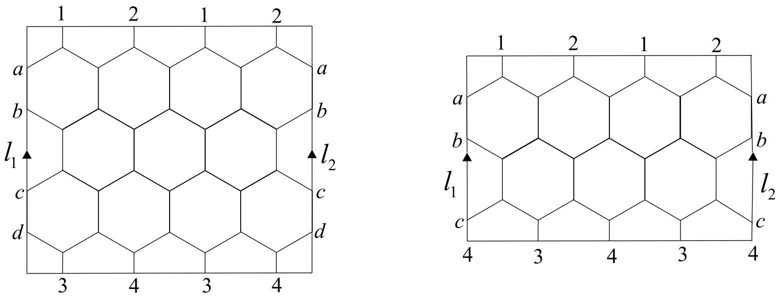

Fig. 10. The rectangle representation of $H_{4,3,c}$ (left) and $H_{4,2,c}$ (right).

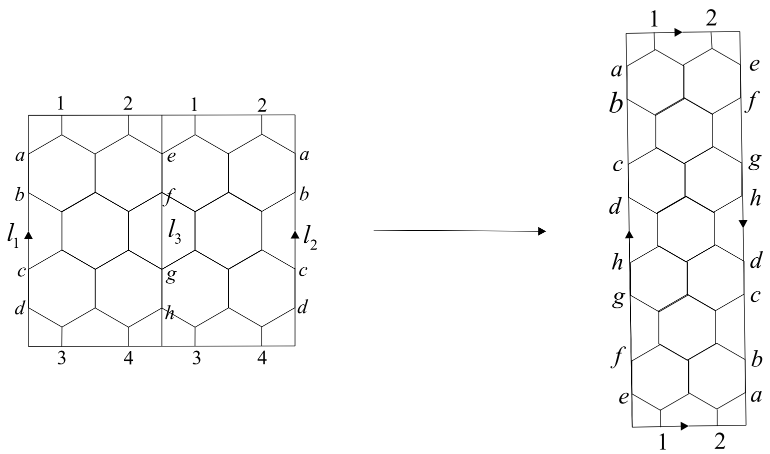

Fig. 11. The transformation of $H_{4,3,c}$ from the plane representation to the representation of the form $N(8, 4, 7)$.

Lemma 3.5. $H_{k,m,b} \cong K(k, m+1, (p+1)k - \frac{m+1}{2})$ *for* $p = \lfloor \frac{m}{2k} \rfloor$.

Proof. The proof is similar to that of Lemma 3.4. □

Lemma 3.6. $H_{k,m,c} \cong N(2m+2, k, 2m-1)$.

Proof. Similar to $H_{k,m,a}$, we can get a rectangle representation of $H_{k,m,c}$ illustrated in Fig. 10 (the same label points are identified).

First, let l_3 be the segment which is parallel to l_1 and l_2 and has the same distance at l_1 and l_2 as shown in Fig. 11. Then along l_3, we cut $H_{k,m,c}$ out and glue the same label points on the bottom side. Finally a rectangle representation of the form $N(2m+2, k, t)$ for some integer t is obtained. From such a process we know that $t = 2m+1$. Hence $H_{k,m,c} \cong N(2m+2, k, 2m+1)$. □

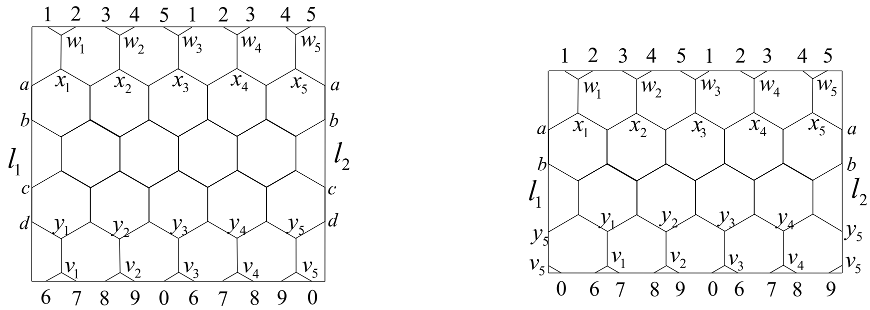

Fig. 12. The rectangle representation of $H_{5,3,f}$ (left) and $H_{5,2,f}$ (right).

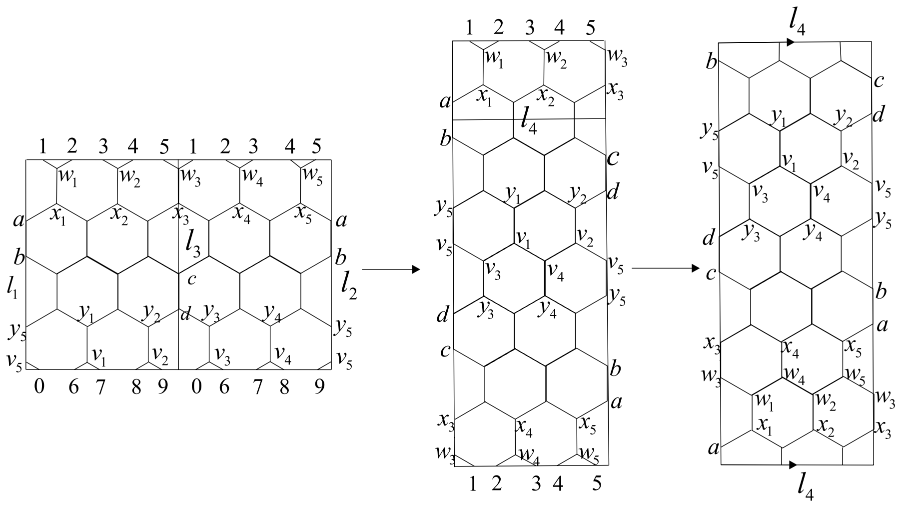

Fig. 13. The transformation of $H_{5,2,f}$ from the rectangle representation to the representation of the form $N(8, 5, 4)$.

Lemma 3.7. $H_{k,m,f} \cong N(2m + 4, k, 2m)$.

Proof. Similar to $H_{k,m,a}$, we also get a rectangle representation of $H_{k,m,f}$ illustrated in Fig. 12 (the same label points are identified).

In the rectangle representation of $H_{k,m,f}$, let l_3 be the segment which is parallel to l_1 and l_2 and has the same distance at them. Then we cut $H_{k,m,f}$ out along the segment l_3, and glue the same label points on the bottom side, resulting in a rectangle representation of $H_{k,m,f}$ as shown in Fig. 13.

Finally, let l_4 be the segment passing through the center of the hexagon with x_1 as the top vertex of it and parallel to the top (or the bottom) side. Then along l_4, cut the graph out and glue the top and the bottom sides. We then get a rectangle representation of the form $N(2m + 4, k, t)$ for some integer t. From this process we know that $t = 2m$. Hence $H_{k,m,f} \cong N(2m + 4, k, 2m)$. □

Since $H_{k,d}$ is the degenerate $H_{k,0,f}$, $H_{k,d} \cong H_{k,0,f} \cong N(4, k, 0)$.

From the above arguments, we conclude that any Klein-bottle polyhex is isomorphic to $K(p, q, t)$ or $N(p, q, t)$ for some specified t determined by p and q. For $K(p, q, t)$, Shiu and Zhang [13] showed previously that all $K(p, q, t)$ are equivalent for $0 \le t \le p - 1$. Combining the result with Theorem 2.1, we obtain the following theorem.

Theorem 3.8. *Let G be a Klein-bottle polyhex. Then it is isomorphic to either $K(p, q)$ or $N(p, q)$.*

Remark 3.9. The theorem also contains the degenerate Klein-bottle polyhexes: $K(p, q)$ with $\min(p, q) = 1$ and $N(p, q)$ with $q = 0, 1$, i.e., there is a hexagonal face not bounded by a cycle of length six.

4. 2-extendability of $N(p, q)$

In this section, we characterize all the 2-extendable graphs in $N(p, q)$.

First of all, some notations are required. For a vertex $v \in V(N(p, q))$ labeled as r_{x_v,y_v} or l_{x_v,y_v}, y_v is called the y-coordinate of it. For any edge e_i of $N(p, q)$, let the y-coordinates of its both end vertices be y_{i1} and y_{i2} with $y_{i1} \le y_{i2}$.

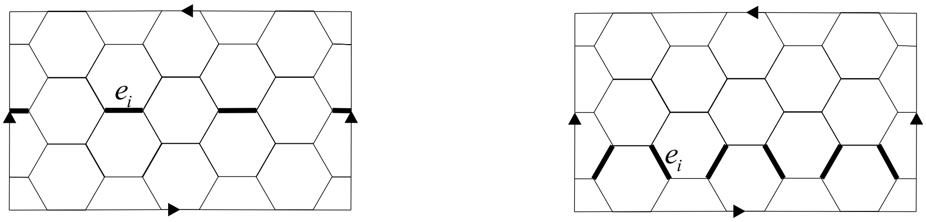

Fig. 14. Matchings $L_{y_{i1}}$ (left) and $B_{y_{i2}}$ (right) with edges in bold.

For convenience, two matchings L_{y_0} and B_{y_0} are defined. The edges of the form $r_{x,y}l_{x,y}$ for $x \in Z_p, y \in Z_q$ are parallel to the x-axis and they form a perfect matching of $N(p, q)$. We call the set of edges $r_{x,y_0}l_{x,y_0}$ with y_0 fixed the y_0-*layer* (denoted by L_{y_0}) and denote the set of edges with y-coordinates of its end vertices y_0 and $y_0 - 1$ by B_{y_0} (illustrated in Fig. 14). Note that L_{y_0} and B_{y_0} are matchings. Thus for an edge e_i, if $y_{i1} = y_{i2}$, then $e_i \in L_{y_{i1}}$ otherwise $e_i \in B_{y_{i2}}$.

Theorem 4.1. *$N(p, q)$ is 1-extendable if and only if $q \geq 2$.*

Proof. For $q \geq 2$, choose any edge e_i in $E(N(p, q))$. If $y_{i1} = y_{i2}$, the perfect matching $\bigcup_{y\in Z_q} L_y$ contains e_i, otherwise, $\bigcup_{y\in Z_q - Y} L_y \cup B_{y_{i2}}$ contains it, where $Y = \{y_{i1}, y_{i2}\}$.

For $q = 1$, $N(p, 1)$ has a loop by Theorem 2.2. Thus it is not 1-extendable. □

$N(p, q)$ $(q \geq 2)$ is not 3-extendable by the fact that a k-extendable graph is $(k + 1)$-connected. We now consider the 2-extendability of the non-bipartite Klein-bottle polyhexes.

We want to prove that $N(p, q)$ is not 2-extendable for $p = 2$ or $q \leq 4$. The following lemmas will be used.

Lemma 4.2 ([10]). *Let positive integers n and p be given. If G is an n-extendable graph on p vertices, then G is also $(n - 1)$-extendable.*

From the above lemma, $N(p, 1)$ is not 2-extendable by the evidence that $N(p, 1)$ is not 1-extendable.

Lemma 4.3 ([3]). *Let v be a vertex of degree $n + t$ in an n-extendable graph G. Then $G[N(v)]$ does not contain a matching of size t.*

Lemma 4.4. *$N(p, q)$ is not 2-extendable for $q = 2, 3$.*

Proof. It follows immediately from Lemma 4.3 that if G is a 3-regular 2-extendable graph, then G cannot contain a triangle. By Theorem 2.2, there are triangles in both $N(p, 2, 1)$ and $N(p, 3, 0)$. Hence $N(p, 2)$ and $N(p, 3)$ are not 2-extendable. □

For $S \subseteq V(G)$, let $o(G - S)$ denote the number of odd components in the graph $G - S$.

Lemma 4.5 ([16], Tutte's Theorem). *A graph G has a perfect matching if and only if $o(G - S) \leq |S|$ for every $S \subseteq V(G)$.*

Next we will prove that $N(p, 4)$ and $N(2, q)$ $(q \geq 4)$ are not 2-extendable by Tutte's Theorem although they are strong embeddings.

Lemma 4.6. *$N(p, 4)$ is not 2-extendable.*

Proof. By Theorem 2.1, we may choose $t = 1$. Let $e_1 = r_{0,2}l_{1,3}$, $e_2 = r_{1,1}l_{2,2}$ and $G' = N(p, 4) - r_{0,2} - l_{1,3} - r_{1,1} - l_{2,2}$. Then $G' - r_{0,0}$ has two isolated vertices $l_{1,1}$ and $r_{1,3}$. Hence $o(G' - r_{0,0}) \geq 2$. By Lemma 4.5, G' has no perfect matchings. Thus $N(p, 4)$ is not 2-extendable. □

Lemma 4.7. *$N(2, q)$ is not 2-extendable for $q \geq 4$.*

Proof. Let $e_1 = l_{1,q-1}r_{0,q-2}$, $e_2 = l_{0,q-2}r_{1,q-3}$. Let $S = \{r_{ij} \mid i + j \equiv q (\text{mod } 2), 0 \leq i \leq 1, 0 \leq j \leq q - 4\}$. Then $|S| = q - 3$. We can see that $N(2, q) - l_{1,q-1} - r_{0,q-2} - l_{0,q-2} - r_{1,q-3} - S$ is an independent set of size $q - 1$. By Lemma 4.5, $N(2, q) - l_{1,q-1} - r_{0,q-2} - l_{0,q-2} - r_{1,q-3}$ has no perfect matchings. Hence $N(2, q)$ is not 2-extendable. □

In the following, we will prove that $N(p, q)$ is 2-extendable for the remaining case $p \geq 4$ and $q \geq 5$.

For convenience, we give some other notations. Let $G[y - 2, y - 1, y]$ be the subgraph of $N(p, q)$ induced by the vertices with the y-coordinates $y-2, y-1$ or y. Let B'_y be a perfect matching of $G[y-2, y-1, y]$ which is alternating in every hexagon of $G[y - 2, y - 1, y]$. Similarly, let $G[y - 4, y - 3, y - 2, y - 1, y]$ be the subgraph of $N(p, q)$ induced by the vertices with the y-coordinates $y-4, y-3, y-2, y-1$ or y. Let B''_y be a perfect matching of $G[y-4, y-3, y-2, y-1, y]$ which is alternating in the ten-length cycles which are the boundary of a pair of adjacent hexagons of $G[y - 4, y - 3, y - 2, y - 1, y]$ with the line connecting the centers of the two hexagons being vertical as shown in Fig. 15. Note that B'_y and B''_y are not unique for any fixed y. In the following, we will choose them according to our needs.

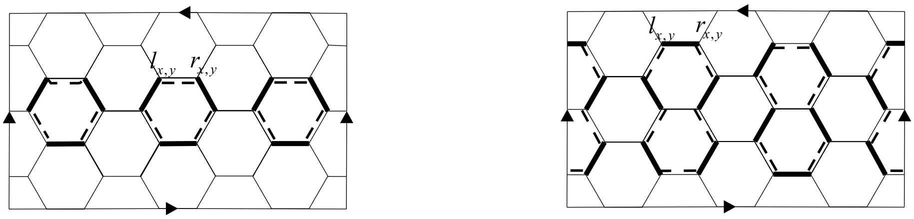

Fig. 15. A matching B'_y (left) and a matching B''_y (right) with edges in bold.

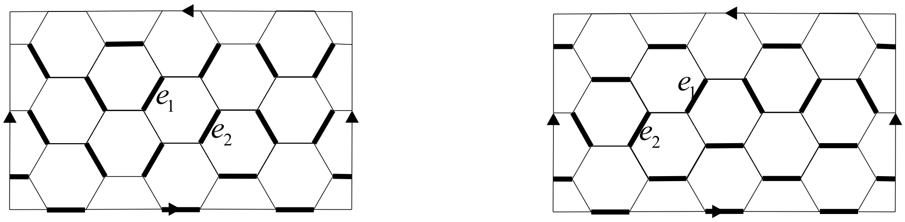

Fig. 16. Illustration for the proof of Lemma 4.8.

Lemma 4.8. *$N(p, q)$ is 2-extendable for $p \geq 4$ and $q \geq 5$.*

Proof. Let e_1 and e_2 be two independent edges in $N(p, q)$. Recall that we denote the y-coordinates of the end vertices of e_i by y_{i1} and y_{i2} with $y_{i1} \leq y_{i2}$, $i = 1, 2$. In the following, B'_y and B''_y can be chosen to be matchings containing both e_1 and e_2 according to the relative positions of e_1 and e_2.

Case 1. Both e_1 and e_2 are horizontal edges, i.e., $y_{11} = y_{12}$ and $y_{21} = y_{22}$. Then $\bigcup_{y \in Z_q} L_y$ is a perfect matching containing them.

Case 2. Exactly one of e_1 and e_2 is a horizontal edge, i.e., $y_{11} = y_{12}, y_{21} < y_{22}$ or $y_{11} < y_{12}, y_{21} = y_{22}$. By symmetry, we just consider the case $y_{11} = y_{12}, y_{21} < y_{22}$.

If $y_{21}, y_{22} \neq y_{11}$, then $(\bigcup_{y \in S} L_y) \cup B_{y_{22}}$ is a perfect matching containing e_1 and e_2, where $S = Z_q - \{y_{21}, y_{22}\}$. If $y_{11} = y_{12} = y_{21} < y_{22}$, then $(\bigcup_{y \in S} L_y) \cup B'_{y_{22}+1}$ is a perfect matching containing e_1 and e_2, where $S = Z_q - \{y_{22} - 1, y_{22}, y_{22} + 1\}$. If $y_{11} = y_{22} > y_{21}$, then $(\bigcup_{y \in S} L_y) \cup B'_{y_{11}}$ is a perfect matching containing e_1 and e_2, where $S = Z_q - \{y_{11} - 2, y_{11} - 1, y_{11}\}$.

Case 3. Both e_1 and e_2 are not horizontal edges, i.e., $y_{11} < y_{12}$ and $y_{21} < y_{22}$.

If $y_{21} \neq y_{11}, y_{12}$ and $y_{22} \neq y_{11}, y_{12}$, then $(\bigcup_{y \in S} L_y) \cup B_{y_{12}} \cup B_{y_{22}}$ is a perfect matching containing e_1 and e_2, where $S = Z_q - \{y_{11}, y_{12}, y_{21}, y_{22}\}$. If $y_{21} < y_{22} = y_{11} < y_{12}$, then $(\bigcup_{y \in S} L_y) \cup B''_{y_{12}+1}$, $S = Z_q - \{y_{21} - 1, y_{21}, y_{22}, y_{12}, y_{12} + 1\}$, is the desired perfect matching if e_1 and e_2 lie in a common hexagon (see Fig. 16(left)), and $(\bigcup_{y \in S} L_y) \cup B'_{y_{12}}$ otherwise, where $S = Z_q - \{y_{21}, y_{22}, y_{12}\}$ (see Fig. 16(right)). If $y_{11} < y_{12} = y_{21} < y_{22}$, then $(\bigcup_{y \in S} L_y) \cup B''_{y_{22}+1}$, where $S = Z_q - \{y_{11} - 1, y_{11}, y_{12}, y_{22}, y_{22} + 1\}$, is a perfect matching containing e_1 and e_2 if e_1 and e_2 lie in a common hexagon but $(\bigcup_{y \in S} L_y) \cup B'_{y_{22}}$ otherwise, where $S = Z_q - \{y_{11}, y_{12}, y_{12} + 1\}$. □

By the above arguments, the following theorem is obtained.

Theorem 4.9. *$N(p, q)$ is 2-extendable if and only if $p \geq 4$ and $q \geq 5$.*

5. *k*-resonance of $N(p, q)$

Since $N(p, 1)$ and $N(p, 2)$ contain a hexagonal face which is not bounded by a cycle, we restrict our consideration to $N(p, q)$ with $q \geq 3$. Let M be any matching of $N(p, q)$ and $h_{x,y}$ be any hexagon. By $M - h_{x,y}$, we mean the subset of M after deleting the edges of $E(h_{x,y}) \cap M$ from M. Note that the notations in this section have the same meaning as introduced before.

Theorem 5.1. *$N(p, q)$ is 1-resonant if and only if $q \geq 3$.*

Proof. Let h_{x_0,y_0} be any hexagon of $N(p, q)$. Then $(\bigcup_{y \in Z_q - Y} L_y) \cup B_{y_0} - h_{x_0,y_0}$ (the edges in bold in Fig. 17) is a perfect matching of $N(p, q) - h_{x_0,y_0}$, where $Y = \{y_0, y_0 - 1\}$. □

Theorem 5.2. *$N(p, q)$ is 2-resonant if and only if $q = 3$ or $(p, q) = (2, 4)$ or $q \geq 5$.*

Proof. For $q = 3$, by Theorem 5.1, $N(p, 3)$ is 1-resonant. Choose any two disjoint hexagons h_{x_1,y_1} and h_{x_2,y_2} in $N(p, 3)$ with $y_1 \geq y_2$. Then $0 \leq y_1 - y_2 \leq 2$. If $0 \leq y_1 - y_2 \leq 1$, $(\bigcup_{y \in Z_3 - Y} L_y) \cup B_{y_2} - h_{x_1,y_1} - h_{x_2,y_2}$ is a perfect matching of $N(p, 3) - h_{x_1,y_1} - h_{x_2,y_2}$ (see Fig. 18), where $Y = \{y_2, y_2 - 1\}$. Otherwise $B_{y_1} \cup L_{y_2} - h_{x_1,y_1} - h_{x_2,y_2}$ is a perfect matching of $N(p, 3) - h_{x_1,y_1} - h_{x_2,y_2}$ (see Fig. 19, in fact, $y_1 = 2$ and $y_2 = 0$).

For $q = 4$, $N(2, 4)$ is 2-resonant since it is 1-resonant and there are no two disjoint hexagons.

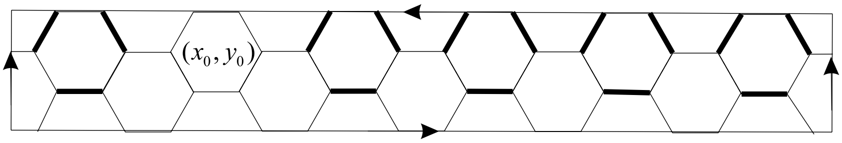

Fig. 17. Illustration for the proof of Theorem 5.1.

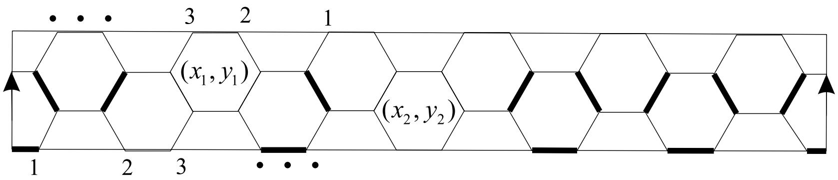

Fig. 18. $N(p, 3)$ and a perfect matching of $N(p, 3) - h_{x_1,y_1} - h_{x_2,y_2}$ when $y_1 - y_2 \leq 1$.

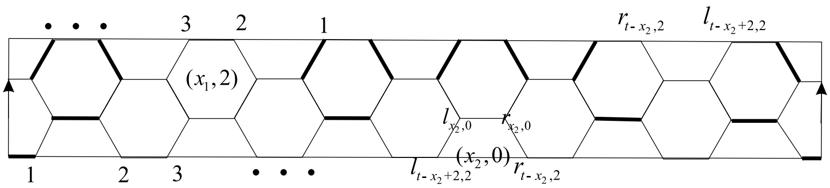

Fig. 19. $N(p, 3)$ and a perfect matching of $N(p, 3) - h_{x_1,y_1} - h_{x_2,y_2}$ when $y_1 - y_2 = 2$.

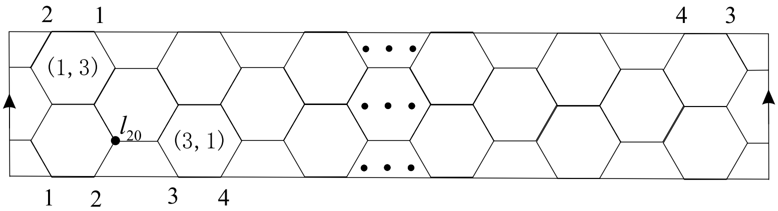

Fig. 20. $N(p, 4) - h_{1,3} - h_{3,1}$ has an isolated vertex $l_{2,0}$.

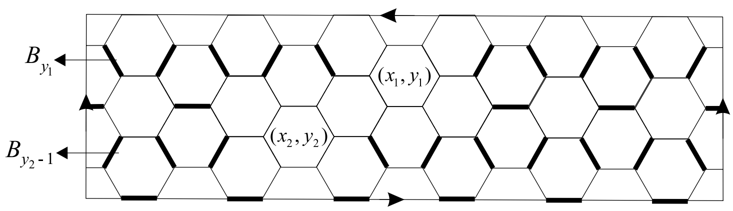

Fig. 21. Illustration for the proof of Case 3 in Theorem 5.2.

Now we consider $N(p, 4)$ with $p \geq 4$. Since all $N(p, q, t)$ are equivalent for $0 \leq t \leq p - 1$ with opposite parity with q, we may choose any t as we need. Then it can be seen that $N(p, 4)$ with $p \geq 4$ is not 2-resonant from Fig. 20, since $N(p, 4) - h_{1,3} - h_{3,1}$ has an isolated vertex $l_{2,0}$ if we choose $t = 1$.

Finally, for $N(p, q)$ with $p \geq 2$ and $q \geq 5$, let h_{x_1,y_1} and h_{x_2,y_2} ($y_1 \geq y_2$) be any two disjoint hexagons of $N(p, q)$. We distinguish four cases as follows (an example is provided for Case 3 in Fig. 21).

Case 1. $y_1 = y_2$. $(\bigcup_{y \in Z_q - Y} L_y) \cup B_{y_1} - h_{x_1,y_1} - h_{x_2,y_2}$ is a perfect matching of $N(p, q) - h_{x_1,y_1} - h_{x_2,y_2}$, where $Y = \{y_1, y_1 - 1\}$.

Case 2. $y_1 = y_2 + 1$. Similar to that of $N(p, 3)$, $(\bigcup_{y \in Z_q - Y} L_y) \cup B_{y_2} - h_{x_1,y_1} - h_{x_2,y_2}$ is a perfect matching of $N(p, q) - h_{x_1,y_1} - h_{x_2,y_2}$, where $Y = \{y_2, y_2 - 1\}$.

Case 3. $y_1 = y_2 + 2$. Since $q \geq 5$, B_{y_2-1} and B_{y_1} are disjoint. Then $(\bigcup_{y \in Z_q - Y} L_y) \cup B_{y_2-1} \cup B_{y_1} - h_{x_1,y_1} - h_{x_2,y_2}$ is a perfect matching of $N(p, q) - h_{x_1,y_1} - h_{x_2,y_2}$, where $Y = \{y_1 - 1, y_1, y_2 - 2, y_2 - 1\}$.

Case 4. $y_1 > y_2 + 2$. At this moment, $(\bigcup_{y \in Z_q - Y} L_y) \cup B_{y_2} \cup B_{y_1} - h_{x_1,y_1} - h_{x_2,y_2}$ is a perfect matching of $N(p, q) - h_{x_1,y_1} - h_{x_2,y_2}$, where $Y = \{y_1 - 1, y_1, y_2 - 1, y_2\}$. □

Lemma 5.3. *For $k \geq 3$, $N(p, q)$ is not k-resonant for $q \geq 7$, $p \geq 4$ or $q = 6$, $p \geq 4$ or $q = 5$, $p \geq 6$ or $q = 4$, $p \geq 4$.*

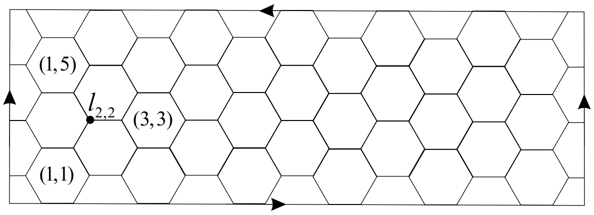

Fig. 22. Illustration for the proof of Lemma 5.3 for $N(p, q)$ with $q \geq 7$, $p \geq 4$.

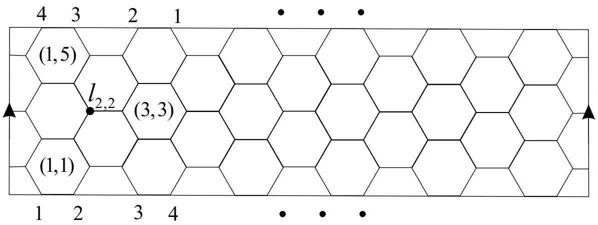

Fig. 23. Illustration for the proof of Lemma 5.3 for $N(p, 6)$ with $p \geq 4$.

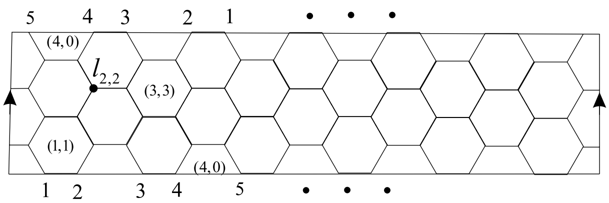

Fig. 24. Illustration for the proof of Lemma 5.3 for $N(p, 5)$ with $p \geq 6$.

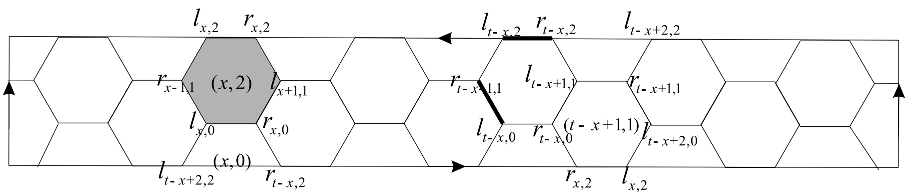

Fig. 25. A column of $h_{x,2}$, $h_{x,0}$ and $h_{t-x+1,1}$.

Proof. For $N(p, q)$ with $q \geq 7$, $p \geq 4$, it is not k-resonant, since $N(p, q) - h_{1,1} - h_{1,5} - h_{3,3}$ has an isolated vertex $l_{2,2}$ (see Fig. 22).

For $N(p, 6)$ with $p \geq 4$, $N(p, q) - h_{1,1} - h_{1,5} - h_{3,3}$ also has an isolated vertex $l_{2,2}$ if we choose $t = 3$ (see Fig. 23). Thus $N(p, 6)$ with $p \geq 4$ is not k-resonant. For $N(p, 5)$ with $p \geq 6$, $N(p, q) - h_{1,1} - h_{4,0} - h_{3,3}$ also has an isolated vertex $l_{2,2}$ if we choose $t = 4$ (Fig. 24). Hence it is not k-resonant. For $N(p, 4)$ with $p \geq 4$, it is not k-resonant for $k \geq 3$ since it is not 2-resonant by Theorem 5.2. □

Lemma 5.4. *For $k \geq 3$, $N(p, q)$ is k-resonant for $(p, q) = (4, 5)$ or $p = 2, q \geq 4$.*

Proof. If $(p, q) = (4, 5)$, then by Theorem 5.2, it is 2-resonant. Let $h_{x_1,y_1}, h_{x_2,y_2}, h_{x_3,y_3}$ be any three disjoint hexagons of $N(4, 5)$. Since $|V(N(4, 5))| = 20$, $|V(N(4, 5) - h_{x_1,y_1} - h_{x_2,y_2} - h_{x_3,y_3})| = 2$. If $N(p, q) - h_{x_1,y_1} - h_{x_2,y_2} - h_{x_3,y_3}$ has no perfect matchings, then it must contain exactly two isolated vertices. In order to isolate a vertex in $N(4, 5)$ by deleting three hexagons, two of the three hexagons must lie in the same column, i.e., with the same x_i in the labeling. But this kind of two hexagons must be adjacent in $N(4, 5)$. Thus $N(4, 5) - h_{x_1,y_1} - h_{x_2,y_2} - h_{x_3,y_3}$ has an isolated edge. Hence, $N(4, 5)$ is k-resonant ($k \geq 3$).

Finally, we consider the case of $p = 2, q \geq 4$. Let $h_{x_1,y_1}, h_{x_2,y_2}, \ldots, h_{x_t,y_t}$ be any t disjoint hexagons of $N(2, q)$. Since $h_{x_1,y_1}, h_{x_2,y_2}, \ldots, h_{x_t,y_t}$ are disjoint, $|y_i - y_j| > 2$ for $i \neq j$. Then $\bigcup_{y \in Z_q - Y} L_y$ is a perfect matching of $N(p, q) - h_{x_1,y_1} - \cdots - h_{x_t,y_t}$, where $Y = \{y_i \mid 1 \leq i \leq t\} \cup \{y_i - 1 \mid 1 \leq i \leq t\} \cup \{y_i - 2 \mid 1 \leq i \leq t\}$. Thus $N(2, q)$ is k-resonant for $q \geq 4$ and $k \geq 3$. □

Lemma 5.5. *For $k \geq 3$, $N(p, 3)$ is k-resonant.*

Proof. We just need to consider $N(p, 3, t)$ for any fixed t by Theorem 2.1. By the adjacency relation of $N(p, 3, t)$, $h_{x,2}$, $h_{x,0}$ and $h_{t-x+1,1}$ are mutually adjacent for any $0 \leq x \leq p - 1$ (see Fig. 25). We call the union of $h_{x,2}$, $h_{x,0}$ and $h_{t-x+1,1}$ a column.

Then one can see that the hexagons of $N(p, 3, t)$ can be decomposed into $\frac{p}{2}$ columns. In each column, if $h_{x,2}$ do not share the edge $r_{x,0}l_{x+1,1}$ or $r_{x-1,1}l_{x,0}$ with $h_{t-x+1,1}$, then all the edges not parallel to the x-axis form two cycles with length six. Otherwise they form one cycle with length six and the other three. The two cycles are called the boundary of this column. We can easily see that two adjacent columns share six common edges on their boundaries. Moreover, all these boundary cycles cover all the vertices of $N(p, 3, t)$.

Let F be an arbitrary set of disjoint hexagons of $N(p, 3, t)$. We will find a perfect matching covering all the vertices outside F by repeatedly covering the vertices in the six-length cycle and three-length cycle for each column.

We first establish a non-trivial automorphism ϕ_d of $N(p, q, t)$ for any fixed p, q and t. ϕ_d is defined as follows:

ϕ_d moves every vertex between two parallel edges in a hexagon downwards, but the x-coordinates may change. More precisely,

$$\phi_d(r_{xy}) = r_{x,y-2} \quad \text{and} \quad \phi_d(l_{xy}) = l_{x,y-2} \quad \text{for } 2 \le y \le q-1, \quad \text{and}$$
$$\phi_d(r_{x,1}) = l_{t-x+1,q-1}, \qquad \phi_d(l_{x,1}) = r_{t-x+1,q-1},$$
$$\phi_d(r_{x,0}) = l_{t-x+1,q-2}, \qquad \phi_d(l_{x,0}) = r_{t-x+1,q-2}.$$

By using the same method as in Theorem 2.1, we can also see that ϕ_d is a hexagon-preserving automorphism of $N(p, q, t)$. More precisely, $\phi_d(h_{x,2}) = h_{x,0}$ and $\phi_d(h_{x,0}) = h_{t-x+1,1}$.

It is obvious that each column contains at most one element of F. Choose any column such that one of the three hexagons belongs to F. Since the three hexagons are equivalent under the automorphism ϕ_d, let the column be the union of $h_{x,2}$, $h_{x,0}$ and $h_{t-x+1,1}$ with $h_{x,2} \in F$. Then $h_{x,0}, h_{t-x+1,1} \notin F$ and the boundary of the column is divided into two two-length paths, $P_1 = r_{t-x,2}l_{t-x+1,1}r_{t-x,0}$ and $P_2 = l_{t-x+2,2}r_{t-x+1,1}l_{t-x+2,0}$, after the deletion of $h_{x,2}$. (Note that if $h_{x,2}$ shares the edge $r_{x,0}l_{x+1,1}$ or $r_{x-1,1}l_{x,0}$ with $h_{t-x+1,1}$, there is exactly one two-length path left.)

We need only consider one of these two paths, symmetrically, say P_1. If $h_{t-x,2} \in F$, then P_1 is a subgraph of a hexagon in F. Then the column containing $h_{t-x,2}$ leaves a two-length path P' after the deletion of $h_{x,2}$ and $h_{t-x,2}$.

If $h_{t-x,2} \notin F$, then at least one of $h_{x+2,0}$ and $h_{t-x-1,1}$ does not belong to F, say $h_{x+2,0}$. Then we match $r_{t-x,2}$ with $l_{t-x,2}$ and $l_{t-x+1,1}$ with $r_{t-x,0}$ and further consider the column $h_{x+1,1} \cup h_{t-x,2} \cup h_{t-x,0}$. Consequently, the cycle $r_{x,2}l_{x+1,1}r_{x,0}r_{t-x,2}l_{t-x+1,1}r_{t-x,0}r_{x,2}$ of the boundary of $h_{x+1,1} \cup h_{t-x,2} \cup h_{t-x,0}$ also lies on the previous column $h_{x,2} \cup h_{x,0} \cup h_{t-x+1,1}$. Hence we need only consider the other cycle on its boundary. Let C be this cycle. Then C contains $l_{t-x,2}$. If $h_{t-x-1,1} \in F$, then C leaves two adjacent vertices unmatched. Thus just match the two vertices up. Otherwise, match $r_{t-x-1,1}$ with $l_{t-x,0}$. Since the vertices of $h_{t-x,2}$ have already been matched up, $h_{t-x,2}$ can be deleted. Then the boundary leaves one two-length path P' whose vertices have not yet been matched.

Then we can use the same method to extend the matching already obtained to cover the vertices of P' as P_1. Repeating the above argument, we finally obtain the desired matching. □

By the above arguments, the following theorem is obtained.

Theorem 5.6. *For $k \ge 3$, $N(p, q)$ is k-resonant if and only if $q = 3$ or $(p, q) = (4, 5)$ or $p = 2, q \ge 4$.*

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