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On some methods of construction of invariant normalizations of lightlike hypersurfaces

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Abstract: The authors study the geometry of lightlike hypersurfaces on pseudo-Riemannian manifolds (M, g) of Lorentzian signature. Such hypersurfaces are of interest in general relativity since they can be models of different types of physical horizons. For a lightlike hypersurface $V \subset (M, g)$ of general type and for some special lightlike hypersurfaces (namely, for totally geodesic, umbilical, and belonging to a manifold (M, g) of constant curvature), in a third-order neighborhood of a point $x \in V$, the authors construct invariant normalizations intrinsically connected with the geometry of V and investigate affine connections induced by these normalizations. For this construction, they used relative and absolute invariants defined by the first and second fundamental forms of V. The authors show that if $\dim M = 4$, their methods allow to construct three invariant normalizations and affine connections intrinsically connected with the geometry of V. Such a construction is given in the present paper for the first time. The authors also consider the fibration of isotropic geodesics of V and investigate their singular points and singular submanifolds.

Keywords: Pseudo-Riemannian manifold, Lorentzian signature, lightlike hypersurface, invariant normalization, affine connection, isotropic geodesics, singular point, isotropic sectional curvature.

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0. Introduction

The lightlike hypersurfaces V of a pseudo-Riemannian manifold (M, g) of Lorentzian signature produce models of horizons of different types in general relativity. This is the reason that they were studied intensively by geometers and physicists (see the books [16, 23, 19, 20] as well as many papers quoted in these books).

In the study of lightlike hypersurfaces, the problem of construction of their normalizations and finding affine connections on such hypersurfaces arises naturally. This problem does not arise for the spacelike and timelike hypersurfaces since on them a family of normals is defined intrinsically in a first-order neighborhood: their normals are polar-conjugate of tangent

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hyperplanes $T_x(V)$, $x \in V$, with respect to the isotropic cones C_x of the manifold (M, g). For a lightlike hypersurface, a hyperplane $T_x(V)$ is tangent to the cone C_x. Hence a straight line orthogonal to $T_x(V)$ belongs to $T_x(V)$, and the family of these straight lines does not determine a normalization of a lightlike hypersurface V and consequently an affine connection on V.

For a normalization of a lightlike hypersurface $V \subset (M, g)$ some authors (see [11, 14, 17, 21, 27]) assign a field N of isotropic directions not belonging to the tangent hyperplanes $T_x(V)$. Other authors (see, for example, the papers [9, 10] and the book [16]) assign a screen distribution S on V which belongs to the tangent bundle $T(V)$. Since an isotropic direction N_x at a point $x \in V$ can be chosen being conjugate to a screen subspace S_x with respect to the isotropic cone C_x, these two methods of normalization of a lightlike hypersurface $V \subset (M, g)$ are equivalent.

The important problem is to construct on a lightlike hypersurface $V \subset (M, g)$ a field N of isotropic directions or a screen distribution S intrinsically connected with the geometry of V. Such a problem was open until now.

In this paper we present a few methods of construction of an invariant normalization on a lightlike hypersurface V of a pseudo-Riemannian manifold (M, g) of Lorentzian signature which is intrinsically connected with the geometry of V. In these constructions we use relative and absolute invariants defined by the first and second fundamental forms of V. The normalizations we have constructed are defined in a third-order neighborhood of a point x of a lightlike hypersurface V. Each of the constructed normalizations induces an affine connection whose curvature tensor is expressed in terms of quantities connected with a fourth-order neighborhood of a point $x \in V$.

We describe briefly the contents of the paper. In Sections 1–3 we give the basic equations of the manifold (M, g) of Lorentzian signature and construct on (M, g) an isotropic frame bundle. In Sections 4–5 we consider lightlike hypersurfaces V on a manifold (M, g), construct an isotropic frame bundle on them, and present the existence theorem for lightlike hypersurfaces. In Section 6 we study the fibration of isotropic geodesics on a lightlike hypersurface V, singular points, and singular submanifolds of V. In Section 7 we find conditions defining invariant normalizations and affine connections on V.

Using the first and second fundamental forms of V, in Section 8 we construct on V a series of relative and absolute invariants connected with a second-order neighborhood of a point $x \in V$. In Section 9 we consider the isotropic sectional curvature defined by Harris in [18]; see also [8]).

Sections 10–11 are devoted to the construction of invariant normalizations intrinsically connected with the geometry of a lightlike hypersurface V. As we have indicated earlier, these normalizations are constructed by means of the invariants that were found in Section 8, and they are defined in a third-order neighborhood of a point $x \in V$.

In the following two sections we address the problem of construction of an invariant normalization and an affine connection on lightlike hypersurfaces of some special classes: totally geodesic, totally umbilical, and belonging to a pseudo-Riemannian manifold of constant curvature. In these sections we clarify the role of the isotropic sectional curvature in the geometry of such hypersurfaces.

Note that in the papers [9, 10] and the book [16, Chapter 4] for a lightlike hypersurface of a pseudo-Riemanninan manifold (M, g) (in particular, in a semi-Euclidean space $\mathbf{R}^n_q$), a rigging (it is called a canonical screen distribution for $\mathbf{R}^n_q$) and an induced affine connection have been constructed. However, the authors did not give the proof of independence of the constructed distribution and connection relative to a choice of a coordinate system in (M, g) (in $\mathbf{R}^n_1$), that is, they did not prove that these distribution and connection are intrinsically connected with the geometry of V.

Finally, in Section 14, we consider a construction of an intrinsic normalization and an intrinsic affine connection on lightlike hypersurfaces V of a four-dimensional manifold (M, g) of Lorentzian signature. We prove that in general, one can construct three normalizations and affine connections intrinsically connected with the geometry of V. Since a four-dimensional manifold (M, g) of Lorentzian signature is directly connected with general relativity, the invariant normalizations we have constructed can have a physical meaning. In order to clarify the physical meaning, an assistance from physicists is needed.

In our study of lightlike hypersurfaces $V \subset (M, g)$ we use the method of moving frames and exterior differential forms of É. Cartan (see, for example, [12, 15, 1]). This allows us to shorten computations and clarify a geometric meaning of constructed objects which is much more difficult in other methods.

The contents of this paper is directly connected with our papers [3, 4, 5, 6, 7] where we studied lightlike hypersurfaces in a pseudoconformal space, the de Sitter space and on a manifold endowed with a conformal structure.

1. Pseudo-Riemannian manifolds of Lorentzian signature

Consider an n-dimensional pseudo-Riemannian manifold (M, g) of Lorentzian signature, where M is a differentiable manifold of dimension n, $\dim M = n$, and g is a metric differential quadratic form of signature $(n-1, 1)$, $\operatorname{sign} g = (n-1, 1)$ (for definition see [25]).

A local frame associated with (M, g) consists of a point $x \in M$ and n vectors $e_i \in T_x(M)$, $i = 1, \ldots, n$, where $T_x(M)$ is a pseudo-Euclidean space tangent to the manifold M at a point x.

For any two vectors $\xi, \eta \subset T_x(M)$, $\xi = \xi^i e_i$, $\eta = \eta^i e_i$, the quadratic form g defines the scalar product

$$(\xi, \eta) = g(\xi, \eta) = g_{ij}\xi^i \eta^j, \tag{1}$$

where $g_{ij} = (e_i, e_j)$.

The equation

$$g(\xi, \xi) = 0 \tag{2}$$

determines an isotropic cone $C_x \subset T_x(M)$ at $x \in M$. The cone C_x is real, and it bears rectilinear generators.

The equations of infinitesimal displacement of this frame have the form

$$dx = \omega^i e_i, \quad de_i = \omega_i^j e_j, \tag{3}$$

where ω^i are basis forms of this manifold, and ω_j^i are the forms of the Levi-Civita connection.

From (3) it follows that for a vector $\xi = \xi^i e_i$ we have

$$d\xi = \big(d\xi^i + \xi^j \omega_j^i\big) e_i.$$

The quantities

$$\nabla \xi^i = d\xi^i + \xi^j \omega_j^i$$

are covariant differentials of the coordinates of the vector ξ in the Levi-Civita connection. The conditions of parallel displacement of the vector ξ have the form $\nabla \xi^i = 0$. Since the scalar product remains unchanged under parallel displacement, we have $d(\xi, \eta) = 0$. It follows that in the Levi-Civita connection, the metric tensor g_{ij} satisfy the following differential equations:

$$\nabla g_{ij} = dg_{ij} - g_{ik}\omega_j^k - g_{kj}\omega_i^k = 0. \tag{4}$$

Equations (4) mean that the metric tensor is covariantly constant with respect to the Levi-Civita connection.

Note that the components g_{ij} and the 1-forms ω^i are defined in a first-order differential neighborhood of a point $x \in (M, g)$, and the 1-forms ω_j^i are defined in its second-order neighborhood.

2. The structure equations

The forms ω^i and ω_j^i are the forms of the Levi-Civita connection. They satisfy the following structure equations:

$$d\omega^i = \omega^j \wedge \omega_j^i, \quad d\omega_j^i = \omega_j^k \wedge \omega_k^i + R^i_{jkl}\omega^k \wedge \omega^l, \tag{5}$$

where $i, j, k, l = 1, \dots, n$, and R^i_{jkl} is the curvature tensor of the manifold (M, g). The curvature tensor is defined in a third-order differential neighborhood of a point $x \in (M, g)$.

Consider the tensor

$$R_{ijkl} = g_{im} R^m_{jkl}. \tag{6}$$

This tensor satisfies the following equations:

$$\begin{cases} R_{ijkl} = -R_{jikl} = -R_{ijlk}, \\ R_{ijkl} = R_{klij}, \\ R_{ijkl} + R_{iklj} + R_{iljk} = 0. \end{cases} \tag{7}$$

If the curvature tensor vanishes, $R^i_{jkl} = 0$, then (M, g) is a pseudo-Euclidean space $\mathbf{R}_1^n$ of signature $(n-1, 1)$ (for $n = 4$, it is a Minkowski space), and equations (3) are completely integrable for such a space.

If the curvature tensor does not vanish, $R^i_{jkl} \neq 0$, then equations (3) are integrable along a curve $x = x(t) \subset M$. A solution of these equations defines a *development* of this line and the frame bundle along the curve onto the tangent pseudo-Euclidean space $(\mathbf{R}_1^n)_x$ at the point $x \in M$.

3. An isotropic frame bundle on (M, g)

Let C_x be an isotropic cone, let η be an isotropic hyperplane, and let e_1 be an isotropic vector along which the hyperplane η is tangent to the cone C_x. Let further the vectors $e_a \in \eta$, $a = 2, \ldots, n-1$, be spacelike vectors, and let e_n be an isotropic (normalizing) vector not belonging to η and conjugate to the vector e_a. Suppose that ζ is a hyperplane tangent to C_x along e_n. Then the $(n-2)$-dimensional subspace $S_x = \eta \cap \zeta = e_2 \wedge \ldots \wedge e_{n-1}$ is called a *screen subspace*.

In the isotropic frame described above the matrix of the metric tensor g has the form

$$(g_{ij}) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & g_{ab} & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad a, b = 2, \ldots, n-1. \tag{8}$$

Here $a, b = 2, \ldots, n-1, g_{1n} = (e_1, e_n) = -1$ is a normalizing condition, $\det(g_{ab}) \neq 0$, rank $(g_{ab}) = n-2$, and $g_{ab}\xi^a\xi^b > 0$.

It follows from equations (1), (4), and (8) that

$$g = g_{ab}\xi^a\xi^b - 2\xi^1\xi^n, \tag{9}$$

$$\begin{cases} \omega_1^n = \omega_n^1 = 0, & \omega_1^1 + \omega_n^n = 0, \\ \omega_a^n = g_{ab}\omega_1^b, & \omega_a^1 = g_{ab}\omega_n^b, \\ dg_{ab} - g_{ac}\omega_b^c - g_{cb}\omega_a^c = 0. \end{cases} \tag{10}$$

4. Lightlike hypersurfaces

Suppose that $V \subset (M, g), \dim V = n-1$, is a lightlike hypersurface on the manifold (M, g), and $x \in V$ is a point of V. Then the tangent hyperplane $\eta = T_x(V)$ is isotropic, i.e., it is tangent to the cone C_x. Let e_1 be an isotropic vector in η which together with vectors e_a, $a = 2, \ldots, n-1$, form a basis of the subspace η. Finally suppose that $e_n \notin \eta$ is also an isotropic vector (see Section 3).

Then the equation of V is

$$\omega^n = 0. \tag{11}$$

On the hypersurface V we have

$$g = g_{ab}\xi^a\xi^b, \quad \text{rank } g = n-2. \tag{12}$$

This form is called the *first fundamental form* of V, and the equations $\omega^a = 0$ define *isotropic lines* on V.

Consider a first-order frame bundle associated with a lightlike hypersurface $V \subset (M, g)$. Since by (3) and (11) we have

$$dx = \omega^1 e_1 + \omega^a e_a, \tag{13}$$

the forms ω^1 and ω^a are basis forms on the hypersurface V. If we fix a point $x \in V$, we obtain that $\omega^1 = \omega^a = 0$. As a result, equations (3) take the form

$$\begin{cases} \delta e_1 = \pi_1^1 e_1, \\ \delta e_a = \pi_a^1 e_1 + \pi_a^b e_b, \\ \delta e_n = \qquad\quad + \pi_n^a e_a - \pi_1^1 e_n, \end{cases} \tag{14}$$

where $\delta = d|_{\omega^1=\omega^a=0}$ is the symbol of differentiation with respect to fiber parameters and $\pi_\eta^\xi = \omega_\eta^\xi(\delta) = \omega_\eta^\xi|_{\omega^1=\omega^a=0}$.

By (10), we find that

$$\pi_n^a = g^{ab}\pi_b^1. \tag{15}$$

Thus the forms π_1^1, π_b^a, and π_a^1 are independent fiber forms. These forms are invariant forms of the group of admissible transformations of first-order frames whose dimension is $1+(n-2)+(n-2)^2 = n-1+(n-2)^2$.

Among the fiber forms the forms π_1^a play a special role. They define a displacement of a screen distribution S_x in the tangent hyperplane $T_x(V)$ of a lightlike hypersurface V. By (15) there is a bijective correspondence between the screen subspaces S_x and the normalizing isotropic straight lines $xe_n = N_x$.

Taking exterior derivatives of equation (11), we arrive at the exterior quadratic equation

$$\omega^a \wedge \omega_a^n = 0. \tag{16}$$

Applying Cartan's lemma to this equation, we find that

$$\omega_a^n = \lambda_{ab}\omega^a, \quad \lambda_{ab} = \lambda_{ba}. \tag{17}$$

The tensor λ_{ab} is the *second fundamental tensor* of the hypersurface V, and the *second fundamental form* of V is

$$\varphi = \lambda_{ab}\omega^a\omega^b. \tag{18}$$

Equations (10) and (17) imply that

$$\omega_1^a = \lambda_b^a\omega^b, \tag{19}$$

where $\lambda_b^a = g^{ac}\lambda_{cb}$ is the *Burali–Forti affinor* of V (see [13]). Note that the authors of [16] called λ_b^a the shape operator (see [16, pp. 85, 154, and 160]).

Equations (3) and (10) imply that

$$de_1 = \omega_1^1 e_1 + \omega_1^a e_a. \tag{20}$$

The point x and the vector e_1 define an isotropic direction xe_1 on the hypersurface V. By (19), the system of equations $\omega^a = 0$ defines an isotropic fibration $\mathcal{F}$ on V and $V = M^{n-2} \times l$, where l is a straight line whose image is an isotropic geodesic xe_1 on the manifold (M, g), $f(l) = xe_1$ (see [7]).

5. 5. The existence theorem

Applying the Cartan test (see [12]) to the system of equations (11), (16), and (17) in the same way as in [7], we arrive at the following theorem.

Theorem 1. *Lightlike hypersurfaces on a manifold (M, g) exist, and the solution of a system defining such hypersurfaces depends on one function of $n-2$ variables.*

Proof. The proof of Theorem 1 coincides with the proof of the existence theorem for lightlike hypersurfaces V on a manifold (M, c) endowed with a conformal structure of Lorentzian signature given in [7]. $\square$

6. Isotropic geodesics on $V \subset (M, g)$

It follows from (12) and (18) that integral curves γ of the vector field e_1 defined by the equations $\omega^a = 0$ are isotropic and asymptotic on V. These curves form a foliation $\mathcal{F}$ on V.

Theorem 2. *Isotropic lines γ of a lightlike hypersurface V are geodesic lines of the manifold (M, g).*

Proof. In fact, the equations of geodesic lines on a Riemannian manifold have the form

$$d\omega^i + \omega^j \omega_j^i = \alpha\omega^i, \tag{21}$$

where α is an 1-form. For $i = a$, these equations become

$$d\omega^a + \omega^1\omega_1^a + \omega^b\omega_b^a = \alpha\omega^a.$$

It follows from (19) that for $\omega^a = 0$, equations (21) are satisfied identically. $\square$

Note that the isotropic geodesics on pseudo-Riemannian manifolds were considered in [3] (see also [2]), where, in particular, their invariance under conformal transformations of a pseudo-Riemannian metric has been proved.

Theorem 2 implies that the *foliation $\mathcal{F}$ is also a geodesic foliation on V.*

Under the development of the manifold (M, g) onto the tangent pseudo-Euclidean space $(\mathbf{R}_1^n)_x = T_x(M)$, to the isotropic geodesic xe_1 there corresponds the straight line l. Consider a point $y = x + se_1$ on the straight line l. From equations (20) it follows that

$$dy = \big(ds + s\omega_1^1 + \omega^1\big)e_1 + \big(\omega^a + s\omega_1^a\big)e_a.$$

But by (19), we have

$$\omega^a + s\omega_1^a = \big(\delta_b^a + s\lambda_b^a\big)\omega^b.$$

This allows us to write the equation for dy in the form

$$dy = \big(ds + s\omega_1^1 + \omega^1\big)e_1 + \big(\delta_b^a + s\lambda_b^a\big)\omega^b e_a. \tag{22}$$

The matrix $(J^a_b) = (\lambda^a_b + s\delta^a_b)$ is the Jacobi matrix of the mapping $f : M^{n-2} \times l \to V \subset (M, g)$, and its determinant,

$$J = \det\left(\lambda^a_b + s\delta^a_b\right)$$

is the Jacobian of this mapping.

Since the affinor $\lambda^a_b = g^{ac}\lambda_{cb}$ is symmetric, its characteristic equation

$$\det\left(\lambda^a_b - \lambda\delta^a_b\right) = 0 \tag{23}$$

has $n-2$ real roots λ_a if each of them is counted as many times as its multiplicity. This implies the following theorem.

Theorem 3. *Any isotropic geodesic l of a lightlike hypersurface V of a manifold (M, g) bears $n-2$ real singular points if each of them is counted as many times as its multiplicity.*

Proof. Consider the development $\widetilde{V}$ of the hypersurface V onto the tangent space $(\mathbf{R}^n_1)_x = T_x(M)$. The tangent subspace $T_y(\widetilde{V})$ to $\widetilde{V}$ at a point y is a subspace of the space $T_x(M)$. By (22), this subspace is determined by the point y and the vectors e_1 and $f_b = (\lambda^a_b + s\delta^a_b)e_a$. If the Jacobian J is different from 0, then these vectors are linearly independent and determine the $(n-1)$-dimensional tangent subspace $T_{\underset{\sim}{y}}(V)$. In this case the point y is a regular point of the hypersurface $\widetilde{V}$, and to such a point, on $\widetilde{V}$ there corresponds a regular point of $V \subset (M, g)$. If at a point $y \in xe_1$ the Jacobian J is equal to 0, then at this point $\dim T_y(\widetilde{V}) < n-1$, and this point is a singular point of $\widetilde{V}$. To such a point, on $\widetilde{V}$ there corresponds a singular point of the hypersurface $V \subset (M, g)$.

Singular points are defined by the equation

$$\det\left(\lambda^a_b + s\delta^a_b\right) = 0. \tag{24}$$

Comparing equations (23) and (24), we find the coordinates s_a of these singular points: $s_a = -1/\lambda_a$. Thus the singular points of the straight line l are

$$F_a = x - \frac{1}{\lambda_a} e_1. \quad \square \tag{25}$$

Note that if $\lambda_a = 0$, then F_a is the point at infinity. It is obvious that the point x is a regular point of the straight line l.

To an eigenvalue λ_a of the affinor (λ^a_b) there corresponds an invariant two-dimensional eigenplane passing through the vector e_1. The eigenplanes corresponding to distinct eigenvalues λ_a and $\lambda_b \neq \lambda_a$ are orthogonal with respect to the scalar product $(\xi, \eta) = g_{ab}\xi^a\eta^b$.

If λ_a is a simple root of equation (23), then the focus F_a describes a *lightlike focal submanifold* (F_a), $\dim(F_a) = n-2$, bearing an $(n-3)$-parameter family of isotropic lines. The eigenplane corresponding to such a root λ_a is the osculating plane for these lines.

In the paper [4], for a lightlike hypersurface of a pseudo-Riemannian de Sitter space we investigated the structure of such singular points, and the structure of V itself taking into account multiplicities of singular points. Many of the results of [4] are still valid for a lightlike hypersurface $V \subset (M, g)$.

7. An affine connection on $V \subset (M, g)$

From equations (5) it follows that the basis forms ω^1 and ω^a of the hypersurface V satisfy the following structure equations:

$$\begin{cases} d\omega^1 = \omega^1 \wedge \omega_1^1 + \omega^a \wedge \omega_a^1, \\ d\omega^a = \omega^1 \wedge \omega_1^a + \omega^b \wedge \omega_b^a. \end{cases} \tag{26}$$

Thus the 1-form

$$\omega = \begin{pmatrix} \omega_1^1 & \omega_a^1 \\ \omega_1^a & \omega_b^a \end{pmatrix}$$

defines an affine structure on V. To define an affine connection, the form ω must satisfy the structure equation

$$d\omega + \omega \wedge \omega = \Omega, \tag{27}$$

where Ω is the curvature 2-form of this connection which is a linear combination of exterior products of the basis forms ω^1 and ω^a (see, for example, [22, Ch. III]).

Taking the exterior derivative of the form ω componentwise and applying equations (5), (10), and (11), we find that

$$\begin{cases} d\omega_1^1 + \omega_a^1 \wedge \omega_1^a = R_{1kl}^1 \omega^k \wedge \omega^l, \\ d\omega_a^1 + \omega_1^1 \wedge \omega_a^1 + \omega_b^1 \wedge \omega_a^b = R_{akl}^1 \omega^k \wedge \omega^l, \\ d\omega_1^a + \omega_1^a \wedge \omega_1^1 + \omega_b^a \wedge \omega_1^b = R_{1kl}^a \omega^k \wedge \omega^l, \\ d\omega_b^a + \omega_1^a \wedge \omega_b^1 + \omega_c^a \wedge \omega_b^c = \omega_b^n \wedge \omega_n^a + R_{bkl}^a \omega^k \wedge \omega^l. \end{cases} \tag{28}$$

Equations (28) and (17) show that conditions (27) are satisfied if and only if the 1-form ω_a^1, and by (10) the form ω_n^a as well, are expressed in terms of the basis forms of the hypersurface V:

$$\omega_a^1 = \nu_a \omega^1 + \nu_{ab} \omega^b, \qquad \omega_n^a = g^{ab} \omega_b^1. \tag{29}$$

It follows from (3) that the vectors e_a and e_n satisfy the differential equations

$$de_a = \omega_a^1 e_1 + \omega_a^b e_b + \omega_a^n e_n, \qquad de_n = \omega_n^a e_a - \omega_1^1 e_n. \tag{30}$$

For $\omega^1 = \omega^a = 0$, equations (30) take the form

$$de_a = \omega_a^b e_b, \qquad de_n = -\omega_1^1 e_n. \tag{31}$$

This means that conditions (29) are satisfied if and only if the screen distribution $S = \bigcup_{x \in V} S_x$, or equivalently the field of normalizing isotropic straight lines $N = \bigcup_{x \in V} x e_1$, are defined invariantly. Note in these two expressions, $x \in V$ are the regular points of V.

Hence *an affine connection on V is defined if and only if there is given an invariant screen distribution S* (*or a field of normalizing isotropic straight lines N*) *on V*. This result is well-known and was discussed in many papers. Note that Bonnor [11], Cagnac [14], Galstyan [17], Katsuno [21], Lemmer [24] (see also [27]) constructed a field of isotropic normalizing vectors while Duggal and Bejancu in their book [16] considered a screen distribution.

However, in all papers on this subject known to the authors, the problem of construction of a screen distribution S or a field of normalizing isotropic straight lines N that are *intrinsically connected* with the geometry of a lightlike hypersurface $V \in (M, g)$ was not considered. In what follows we present a few solutions of this problem.

8. Invariants of a lightlike hypersurface

A lightlike hypersurface $V \subset (M, g)$ in an isotropic first-order frame is determined by equation (11) whose prolongation gives equation (17).

Exterior differentiation of equations (17) by means of structure equations (5) and equations (10) leads to the following exterior quadratic equations:

$$\left[\nabla\lambda_{ab} - \lambda_{ab}\omega_1^1 + (\lambda_{ac}g^{ce}\lambda_{eb} + 2R^n_{ab1})\omega^1 + R^n_{abc}\omega^c\right] \wedge \omega^b = 0,$$

where $\nabla\lambda_{ab} = d\lambda_{ab} - \lambda_{ac}\omega_b^c - \lambda_{cb}\omega_a^c$. Applying Cartan's lemma to the last equation, we find that

$$\nabla\lambda_{ab} - \lambda_{ab}\omega_1^1 + (\lambda_{ac}g^{ce}\lambda_{eb} + 2R^n_{ab1})\omega^1 + R^n_{abc}\omega^c = \mu_{abc}\omega^c. \tag{32}$$

Here the quantities μ_{abc} are symmetric with respect to all indices.

The quantities R^n_{ab1} are symmetric with respect to the indices a and b since by (6) and (7) we have

$$R^n_{ab1} = -R_{1ab1} = -R_{b11a} = -R_{1ba1} = R^n_{ba1}.$$

Now if we alternate equations (32) with respect to the indices a and b, then we find that $R^n_{[ab]c} = 0$. This implies $R^n_{abc} = R^n_{bac}$. But since by (7) we have $R^n_{abc} = -R^n_{acb}$, we find that

$$R^n_{abc} = -R^n_{acb} = -R^n_{cab} = R^n_{cba} = R^n_{bca} = -R^n_{bac} = -R^n_{abc}.$$

It follows that

$$R^n_{abc} = 0. \tag{33}$$

Hence on a lightlike hypersurface $V \subset (M, g)$ conditions (33) are satisfied. As a result, equations (32) take the form

$$\nabla\lambda_{ab} - \lambda_{ab}\omega_1^1 + \left(\lambda_{ac}g^{ce}\lambda_{eb} + 2R^n_{ab1}\right)\omega^1 = \mu_{abc}\omega^c. \tag{34}$$

For a fixed point $x \in V$ (i.e., for $\omega^1 = \omega^a = 0$), we find from (34) that

$$\nabla_\delta\lambda_{ab} = \lambda_{ab}\pi_1^1, \tag{35}$$

where

$$\nabla_\delta\lambda_{ab} = \delta\lambda_{ab} - \lambda_{ac}\pi_b^c - \lambda_{cb}\pi_a^c.$$

Equations (35) prove that the quantities λ_{ab} form a relative $(0,2)$-tensor of weight 1. This tensor is the *second fundamental tensor* of the hypersurface V. It is defined in a second-order neighborhood of a point $x \in V$.

It follows from (10) and (35) that for a fixed point $x \in V$ the affinor λ_b^a satisfies the equations

$$\nabla_\delta \lambda_b^a = \lambda_b^a \pi_1^1. \tag{36}$$

Hence it is also of weight 1.

Consider characteristic equation (23) of the affinor λ_b^a. We write it in the expanded form

$$\lambda^{n-2} - I_1 \lambda^{n-3} + \cdots + (-1)^{n-2} I_{n-2} = 0. \tag{37}$$

The coefficients of this equation are relative invariants of weights equal to their labels. These invariants are the sums of the diagonal minors of corresponding orders of the matrix (λ_b^a):

$$I_1 = \lambda_a^a, \qquad I_2 = \lambda_{[a}^b \lambda_{b]}^a, \qquad \ldots, \qquad I_{n-2} = \det(\lambda_b^a). \tag{38}$$

These coefficients form a complete system of relative invariants of the affinor λ_b^a. We can get another complete system of relative invariants of the affinor λ_b^a if we consider the following contractions:

$$\widetilde{I}_1 = I_1 = \lambda_a^a, \qquad \widetilde{I}_2 = \lambda_a^b \lambda_b^a, \qquad \ldots, \qquad \widetilde{I}_{n-2} = \lambda_{a_1}^{a_{n-2}} \lambda_{a_2}^{a_1} \ldots \lambda_{a_{n-2}}^{a_{n-3}}. \tag{39}$$

Moreover, the roots λ_a, $a = 2, \ldots, n-1$, of characteristic equation (37) also form a complete system of invariants of weights 1 of the affinor λ_b^a.

We can find invariants of weights 1 from nonvanishing invariants (38) and (39) if we take from them the root of degree equal to their labels: the quantities $|I_p|^{1/p}$ and $|\widetilde{I}_p|^{1/p}$ are invariants of weight 1.

Equations (36) imply that for a fixed point $x \in V$, each relative invariant I of weight 1 satisfies the differential equation

$$\delta I = I \pi_1^1. \tag{40}$$

Any nonvanishing relative invariant I of weight 1 allows us to normalize the isotropic vector e_1 by setting $\widetilde{e}_1 = (1/I)e_1$, and the new vector $\widetilde{e}_1$ is invariant. In fact, it follows from (3) and (10) that for a fixed point $x \in V$ we have

$$\delta e_1 = \pi_1^1 e_1.$$

This and equation (40) imply that $\delta \widetilde{e}_1 = 0$, and thus the vector $\widetilde{e}_1$ does not depend on a choice of normalizing parameter on an isotropic geodesic xe_1.

Absolute invariants of a hypersurface V can be constructed by taking ratios of two nonvanishing relative invariants of the same weight. For a fixed point $x \in V$, an absolute invariant J satisfies the equation

$$\delta J = 0. \tag{41}$$

Since the affinor λ_b^a is defined in a second-order neighborhood of a point $x \in V$, it follows that all absolute and relative invariants of a hypersurface V constructed by means of λ_b^a are defined also in a second-order neighborhood of $x \in V$,

9. Isotropic sectional curvature of a lightlike hypersurface

Harris introduced the notion of isotropic sectional curvature of an isotropic 2-plane σ of a pseudo-Riemannian manifold (M, g) (see [18]; see also the book [8, Appendix A, p. 571]). If N is an isotropic nonzero element of a one-dimensional space of isotropic vectors belonging to σ, and P is an arbitrary (nonzero) nonisotropic vector from σ, then the isotropic sectional curvature $K_N(\sigma)$ is defined as

$$K_N(\sigma) = \frac{(R(P, N)N, P)}{(P, P)}. \tag{42}$$

This expression does not depend on a vector $P \subset \sigma$ but depends quadratically on an isotropic vector N.

Denote by n^i coordinates of an isotropic vector N and by p^i coordinates of a vector P. Then for the standard coordinate representation of the curvature tensor (see (5) and (6)) the nominator of (42) can be written as

$$\big(R(P, N)N, P\big) = R_{ijkl} n^i p^j p^k n^l,$$

and its denominator is $(P, P) = g_{ij} p^i p^j$.

Let V be a lightlike hypersurface of a pseudo-Riemannian manifold (M, g) of Lorentzian signature, and let $T_x(V)$ be its tangent hyperplane. In the isotropic frame considered in Section 4, the vector e_1 is isotropic, and this vector and a vector $P = p^1 e_1 + p^a e_a$ determine an isotropic 2-plane $\sigma = e_1 \wedge P$. For this 2-plane the isotropic sectional curvature has the following expression:

$$K_N(\sigma) = \frac{R_{1ab1} p^a p^b}{g_{ab} p^a p^b}. \tag{43}$$

A lightlike hypersurface $V \subset (M, g)$ is called a *hypersurface of null isotropic sectional curvature* if for all its tangent two-dimensional isotropic planes σ, their isotropic sectional curvatures vanish.

Consider equation (32) for the second fundamental tensor λ_{ab} of a lightlike hypersurface $V \subset (M, g)$. This equation contains the components R^n_{ab1} of the curvature tensor of the manifold (M, g). But by (7) we have

$$R_{1ab1} = -R^n_{ab1}. \tag{44}$$

Now we prove the following theorem.

Theorem 4. *The isotropic sectional curvature of a lightlike hypersurface $V \subset (M, g)$ vanishes if and only if the derivative of the second fundamental tensor of V along the field of isotropic directions on V is expressed in terms of objects of a second-order neighborhood.*

Proof. The field of isotropic directions on V is defined by the equations $\omega^a = 0$. It follows from equation (34) that the derivative of the tensor λ_{ab} along an isotropic direction on V is determined by the formula

$$\big(\nabla\lambda_{ab} - \lambda_{ab}\omega^1_1\big)_{,1} = -\lambda_{ac} g^{ce} \lambda_{eb} - 2R^n_{ab1}. \tag{45}$$

In the right-hand side of this equation the first term is defined in a second-order neighborhood of a point $x \in V$, and the second term in its third-order neighborhood. By (43) and (44), the second term vanishes if and only if a hypersurface V has its isotropic sectional curvature equal to 0. □

It follows from Theorem 4 that the derivatives of all the invariants of a lightlike hypersurface with the vanishing isotropic sectional curvature taking along a field of isotropic directions of V are also defined in terms of second-order objects.

10. Construction of a screen distribution by means of absolute invariants

We prove the following theorem.

Theorem 5. *If $J = J(x)$ is an absolute invariant defined on a lightlike hypersurface $V \subset (M, g)$, and the level $(n-2)$-dimensional submanifolds of $J(x)$ are transversal to isotropic geodesics of V, then the distribution S tangent to these level submanifolds is an invariant screen distribution. If the invariant $J(x)$ is connected with the hypersurface V intrinsically, then the same is true for a screen distribution S generated by J. If the order of an invariant $J(x)$ is equal to p, then the normalization is defined in a neighborhood of a point $x \in V$ of order $p+1$, and the curvature tensor is defined in a neighborhood of a point $x \in V$ of order $p+2$.*

Proof. By (41), the differential of the invariant J has the form

$$dJ = K\omega^1 + \widetilde{K}_a\omega^a, \tag{46}$$

where $K \neq 0$. On a level submanifold, $dJ = 0$. It follows that

$$\omega^1 = K_a\omega^a, \tag{47}$$

where $K_a = -\widetilde{K}_a/K$. Thus on a level surface we have

$$dx = \omega^a\widetilde{e}_a,$$

where $\widetilde{e}_a = e_a + K_a e_1$. At a point $x \in V$, the vectors $\widetilde{e}_a$ determine an invariant screen subspace $S_x = \widetilde{e}_2 \wedge \widetilde{e}_3 \wedge \ldots \wedge \widetilde{e}_{n-1}$. The distribution $S = \bigcup_{x\in V} S_x$ is an invariant screen distribution generated by the invariant $J = J(x)$. If this invariant is intrinsically connected with the hypersurface V, then the same is true for the screen distribution S generated by J.

Let us make a reduction in the isotropic first-order frame bundle by superposing the vectors e_a with the vectors $\widetilde{e}_a$. Then we have $K_a = 0$, and equation (47) takes the form

$$\omega^1 = 0.$$

Since this equation determines a family of level submanifolds of the invariant J, it must be completely integrable. Hence

$$d\omega^1 \wedge \omega^1 = 0.$$

By (5), the last equation can be written as

$$\omega^1 \wedge \omega^a \wedge \omega^1_a = 0.$$

This implies that

$$\omega_a^1 = \nu_a \omega^1 + \nu_{ab} \omega^a, \tag{48}$$

where $\nu_{ab} = \nu_{ba}$. Equation (48) coincides with the first equation of equations (29). However the condition $\nu_{ab} = \nu_{ba}$ shows that an affine connection generated by an absolute invariant J is a connection of special type. If an absolute invariant $J = J(x)$ is constructed by means of the affinor λ_b^a, then it is defined in a second-order neighborhood of a point $x \in V$, the quantities K_a and $\widetilde{K}_a$ defining the screen distribution are defined in a third-order neighborhood, and finally, the quantities ν and ν_a from equations (48) are defined in a fourth-order neighborhood. Thus the curvature tensor of the affine connection generated by the absolute invariant J is also defined in a fourth-order neighborhood of a point $x \in V$. $\square$

11. Construction of a screen distribution by means of relative invariants

In a first-order frame bundle of a lightlike hypersurface V constructed in Section 4, we define a screen subspace S_x by vectors c_a:

$$c_a = e_a + z_a e_1, \quad a = 2, \dots, n-1.$$

This subspace is invariant if and only if

$$\delta c_a = \sigma_a^b c_b, \tag{49}$$

where as earlier, δ is the symbol of differentiation with respect to fiber parameters, and σ_a^b are some 1-forms.

Applying equations (3), (10), (11) and (17), we find that

$$\delta c_a = \left(\nabla_\delta z_a + z_a \pi_1^1 + \pi_a^1\right) e_1 + \pi_a^b c_b. \tag{50}$$

Comparing equations (50) and (49), we see that the screen subspace $S_x = [x, c_2, \dots, c_{n-1}]$ is invariant if and only if the following conditions hold:

$$\nabla_\delta z_a + z_a \pi_1^1 + \pi_a^1 = 0. \tag{51}$$

The coordinates of a *normalizing object* z_a defining an invariant screen subspace S_x must satisfy this equation.

Consider a nonvanishing relative invariant $I = I(x)$ of weight 1 defined in a second-order neighborhood of a point $x \in V$. Equation (40) which this invariant satisfies can be written as

$$\delta \ln |I| = \pi_1^1.$$

The last equation is equivalent to the equation

$$d \ln |I| - \omega_1^1 = -K\omega^1 - K_a \omega^a. \tag{52}$$

The coefficients K and K_a in (52) are defined in a third-order neighborhood of a point $x \in V$.

We prove the following theorem.

Theorem 6. *If the coefficient K in equation* (52) *is not a root of characteristic equation* (37), *then the coefficients K_a in equation* (52) *allow one to construct an object defining an invariant normalization of a lightlike hypersurface $V \subset (M, g)$. This normalization is intrinsically connected with the geometry of V and defined in a third-order neighborhood of a point $x \in V$.*

Proof. Taking exterior derivatives of equation (52), we find that

$$\begin{aligned}\big(dK - K\omega_1^1\big) \wedge \omega^1 + \big(\nabla K_a + (\lambda_a^b - K\delta_a^b)\omega_b^1\big) \wedge \omega^a + K_b\lambda_a^b\omega^1 \wedge \omega^a \\ - R_{1kl}^1\omega^k \wedge \omega^l = 0,\end{aligned} \tag{53}$$

where $\nabla K_a = dK_a - K_b\omega_a^b$. It follows from equation (53) that

$$\begin{cases} dK - K\omega_1^1 = M\omega^1 + M_a\omega^a, \\ \nabla K_a + \big(\lambda_a^b - K\delta_a^b\big)\omega_b^1 = \widetilde{M}_a\omega^1 + M_{ab}\omega^b. \end{cases} \tag{54}$$

The coefficients M, M_a, $\widetilde{M}_a$, and M_{ab} are defined in a fourth-order neighborhood of a point $x \in V$ and satisfy the relations

$$M_a - \widetilde{M}_a = K_b\lambda_a^b - 2R_{11a}^1, \qquad M_{ab} = -R_{1ab}^1, \tag{55}$$

which are obtained if we substitute expansions (54) into equations (53).

For a fixed point $x \in V$, equations (54) become

$$\delta K = K\pi_1^1, \tag{56}$$

and

$$\nabla_\delta K_a + \big(\lambda_a^b - K\delta_a^b\big)\pi_b^1 = 0. \tag{57}$$

Equation (56) shows that the quantity K is a relative invariant of weight 1.

Since by theorem hypothesis, the quantity K is not a root of characteristic equation (37), the affinor

$$\Lambda_a^b = \lambda_a^b - K\delta_a^b \tag{58}$$

is nondegenerate. As the affinor λ_a^b, the affinor Λ_a^b is of weight 1. Thus the inverse affinor $\widetilde{\Lambda}_b^a$ of the affinor λ_b^a is of weight -1, i.e., this inverse affinor satisfies the equations

$$\nabla_\delta \widetilde{\Lambda}_a^b = -\widetilde{\Lambda}_a^b\pi_1^1. \tag{59}$$

Further consider the quantities

$$L_a = \widetilde{\Lambda}_a^b K_b. \tag{60}$$

Differentiating equations (60) with respect to fiber parameters and taking into account conditions (59) and (57), we find that

$$\nabla_\delta L_a + L_a\pi_1^1 + \pi_a^1 = 0. \tag{61}$$

Comparing equations (61) and (51), we see the quantities L_a form a *normalizing object* of a hypersurface $V \subset (M, g)$ intrinsically defined by the geometry of V in its third-order neighborhood.

Moreover, the vectors

$$\widetilde{e}_a = e_a + L_a e_1$$

define an invariant screen subspace S_x and, along with it, an invariant screen distribution $S = \bigcup_{x \in V} S_x$ that is intrinsically connected with a lightlike hypersurface $V \subset (M, g)$. $\square$

We make a reduction in the frame bundle associated with a hypersurface V by superposing the vectors e_a and $\widetilde{e}_a$. Then we obtain $L_a = 0$, $K_a = 0$, and as a result, the second group of equations (54) takes the form

$$\Lambda_a^b \omega_b^1 = \widetilde{M}_a \omega^1 + M_{ab} \omega^b.$$

Since we assume that the tensor Λ_a^b is nondegenerate, we can solve the last equations with respect to the 1-forms ω_a^1. As a result, we obtain equations (29) where

$$\nu_a = \widetilde{\Lambda}_a^b \widetilde{M}_b, \qquad \nu_{ab} = \widetilde{\Lambda}_a^c M_{cb}. \tag{62}$$

These quantities are defined in a fourth-order neighborhood of a point $x \in V$. This and (28) imply that the curvature tensor of the affine connection Γ induced by the screen distribution S we have constructed is defined in a fourth-order neighborhood of a point $x \in V$.

In the same way as in Section 10, one can prove that the screen distribution S is integrable if and only if $\nu_{ab} = \nu_{ba}$.

Note that in the papers [9, 10] as well as in the book [16], the authors consider canonical screen distributions on a lightlike hypersurface M of a pseudo-Euclidean space $\mathbf{R}_q^n$ or a pseudo-Riemannian space $(\widetilde{M}, \widetilde{g})$ (here we used their notations). However, this distribution and affine connections induced by them are not intrinsically connected with the geometry of a lightlike hypersurface M since they are defined by means of a vector field V connected with a coordinate system of the ambient space $\mathbf{R}_q^n$ or $(\widetilde{M}, \widetilde{g})$. In fact, for example, in $\mathbf{R}_q^n$ this vector field V is defined by formula (6.8) (see [16, p. 115]) which in the case $q = 1$ take the form $V = -D^0 \partial / \partial x^0$, i.e., the vector field V is a field of tangents vectors to the lines x^0 of the curvilinear coordinate system of $\mathbf{R}_1^n$. Thus the vector field V as well as the vector field N (see (6.10) in [16]) and the screen distribution S (see p. 116 in [16]) constructed by means of V are neither invariant nor intrinsically connected with the geometry of M.

Note also that a canonical screen distribution constructed in [9, 10] and [16] is defined by elements of a first-order differential neighborhood of a hypersurface M. As we showed in Sections 10 and 11, screen distributions intrinsically connected with the geometry of a lightlike hypersurface M can be constructed only in a third-order differential neighborhood of M.

Finally note that a screen distribution similar to that in [9, 10] and [16] was constructed by Bonnor in 1972 (see [11]) who gave a physical justification for such a distribution.

12. An affine connection on totally geodesic and totally umbilical lightlike hypersurfaces

We prove the following theorem.

Theorem 7. *The second fundamental tensor of the pseudo-Riemannian space (M, g) vanishes on a totally geodesic lightlike hypersurface $V \subset (M, g)$. For any choice of isotropic normal-*

ization of a totally geodesic lightlike hypersurface V, an affine connection is induced on V, and the curvature tensor of this connection is completely determined by the curvature tensor of the manifold (M, g).

Proof. The equations of geodesic lines on a pseudo-Riemannian manifold (M, g) have the form (21). Since in a first-order frame a hypersurface V is defined by equation (11), V will be totally geodesic if equations (21) are identically satisfied on it.

For $i = n$, equations (21) give

$$\omega^i \omega_i^n = 0, \quad i = 1, \ldots, n-1.$$

But by (10), we have $\omega_1^n = 0$, and as a result, the above equation becomes

$$\omega^a \omega_a^n = 0.$$

Substituting the values of ω_a^n from (17) into the last equation, we find that

$$\lambda_{ab} = 0. \tag{63}$$

From equation (34) it follows that

$$R^n_{ab1} = 0, \quad \mu_{abc} = 0.$$

The first of these equations shows that a totally geodesic lightlike hypersurface V has the vanishing isotropic sectional curvature, $K_N(\sigma) = 0$. Since the second fundamental tensor of such a V also vanishes, it is impossible to find an invariant normalization of V intrinsically connected with the geometry of V by means of this tensor.

However, an affine connection on totally geodesic lightlike hypersurfaces can be defined uniquely. In fact, equations (63) are equivalent to the equations $\omega_a^n = 0$. It follows from these equations that in structure equations (28) of the affine connection induced on V, the term $\omega_b^n \wedge \omega_n^a$ in the right-hand side of the last equation vanishes. This proves Theorem 7. □

Corollary 8. *If the curvature tensor of the manifold (M, g) vanishes (i.e., this manifold is a Minkowski space $\mathbf{R}^n_1$), then totally geodesic lightlike hypersurfaces are isotropic hyperplanes of $\mathbf{R}^n_1$.*

Next we consider totally umbilical lightlike hypersurfaces $V \subset (M, g)$. They are defined by the equations

$$\lambda_{ab} = \lambda g_{ab}, \tag{64}$$

where $\lambda \neq 0$. It follows from equations (64) and (25) that the isotropic geodesic xe_1 of the hypersurface V bears a single singular point

$$F = x - \frac{1}{\lambda} e_1. \tag{65}$$

Differentiating equation (65) and applying equations (3) and (20), we find that

$$dF = \frac{1}{\lambda^2}\big(d\lambda - \lambda\omega_1^1 + \lambda^2\omega^1\big)e_1. \tag{66}$$

Substituting expressions (64) into equations (34), we obtain that

$$g_{ab}\big(d\lambda - \lambda\omega_1^1 + \lambda^2\omega^1\big) + 2R^n_{ab1}\omega^1 = \mu_{abc}\omega^c. \tag{67}$$

This implies that

$$d\lambda - \lambda\omega_1^1 + \lambda^2\omega^1 = \mu\omega^1 + \mu_a\omega^a. \tag{68}$$

If we substitute this expression into equations (67), we find that

$$g_{ab}\big(\mu\omega^1 + \mu_a\omega^a\big) + 2R^n_{ab1}\omega^1 = \mu_{abc}\omega^c.$$

Equating coefficients in linearly independent 1-forms ω^1 and ω^a, we obtain

$$R^n_{ab1} = -\tfrac{1}{2}g_{ab}\mu \tag{69}$$

and

$$g_{ab}\mu_c = \mu_{abc}. \tag{70}$$

Since the quantities μ_{abc} are symmetric with respect to all indices, it follows from (70) that

$$g_{ab}\mu_c = g_{ac}\mu_b.$$

Contracting these equations with g^{ab}, we find that

$$(n-3)\mu_c = 0. \tag{71}$$

It follows that if $n \geqslant 4$, then $\mu_c = 0$. Note that the case $n = 3$ is not interesting since for $n = 3$, a lightlike hypersurface becomes an isotropic curve.

Now equations (68) take the form

$$d\lambda - \lambda\omega_1^1 + \lambda^2\omega^1 = \mu\omega^1. \tag{72}$$

Taking the exterior derivative of equation (72), we find that

$$\big(d\mu - 2\mu\omega_1^1\big) \wedge \omega^1 - \mu\omega_a^1 \wedge \omega^a + \lambda R^1_{1kl}\omega^k \wedge \omega^l = 0. \tag{73}$$

If $\lambda \neq 0$, then for $\mu = 0$ equation (73) implies that

$$R^1_{1kl} = 0. \tag{74}$$

If $\mu \neq 0$, then it follows from (73) that

$$\frac{d\mu}{\mu} - 2\omega_1^1 = \nu\omega^1 + \nu_a\omega^a, \qquad -\omega_a^1 = \widetilde{\nu}_a\omega^l + \nu_{ab}\omega^b. \tag{75}$$

Substituting these decompositions into equation (73), we find that

$$\nu_a - \widetilde{\nu}_a = \frac{2\lambda}{\mu}R^1_{11a}, \qquad \nu_{[ab]} = \frac{\lambda}{\mu}R^1_{1ab}. \tag{76}$$

The quantities ν, ν_a, and $\widetilde{\nu}_a$ are defined in a fourth-order differential neighborhood of a point $x \in (M, g)$

Using equations of this section, we prove further three theorems.

Theorem 9. *The isotropic sectional curvature of a totally umbilical lightlike hypersurface $V \subset (M, g)$ depends on its point $x \in V$ and does not depend on an isotropic 2-plane $\sigma = e_1 \wedge P$, where $P \in T_x(V)$.*

Proof. In fact, it follows from (69) that $R_{1ab1} = \frac{1}{2} g_{ab}\mu$, and this and formula (42) give $K_N(\sigma) = \frac{1}{2}\mu$. □

Theorem 10. *If for $n \geqslant 4$, the isotropic sectional curvature of a totally umbilical hypersurface $V \subset (M, g)$ vanishes, then the hypersurface V is an isotropic cone of the manifold (M, g). On such a hypersurface V, it is impossible to construct an invariant normalization and an invariant affine connection intrinsically connected with the geometry of V. The components of the curvature tensor of the manifold (M, g) satisfy the equations*

$$R^n_{ab1} = 0, \qquad R^1_{1kl} = 0. \tag{77}$$

Proof. The proof of the main part of this theorem follows from equations (72) and (66). Since for $\mu = 0$ differentiation of equation (72) gives only equations (74) that does not contain the 1-forms ω^1_a defining a screen distribution S, an intrinsic normalization and an intrinsic affine connection on such a hypersurface V cannot be found. Relations (77) follow from (69) and (74). □

Theorem 11. *If the isotropic sectional curvature of a totally umbilical manifold (M, g) does not vanish, then a singular point F of its isotropic geodesic xe_1 describes an isotropic line γ. On V one can define an invariant screen distribution S intrinsically connected with the geometry of V. This distribution is integrable if and only if $R^1_{1ab} = 0$.*

Proof. In fact, by (66) and (72), we have

$$dF = \frac{\mu}{\lambda^2}\,\omega^1 e_1. \tag{78}$$

This means that the point F describes a line γ tangent to the vector e_1, i. e., an isotropic curve. The equation $\omega^1 = 0$ defines on V a screen distribution S intrinsically connected with the geometry of V. If a point x moves along integral lines of the distribution S, then by (78), the point F is fixed. It follows from the second equation of (76) that the screen distribution S is integrable if and only if the components R^1_{1ab} of the curvature tensor of the manifold (M, g) vanish on V, $R^1_{1ab} = 0$. In this case the fibration of isotropic geodesics decomposes into a one-parameter family of cones. □

13. Lightlike hypersurfaces on a pseudo-Riemannian manifold (M, g) of Lorentzian signature and constant curvature

The tensor of Riemannian curvature of a Riemannian or pseudo-Riemannian manifold (M, g) of constant curvature has the form

$$R_{ijkl} = K\big(g_{ik}g_{jl} - g_{il}g_{jk}\big), \tag{79}$$

where K is the curvature of the manifold. By Schur's theorem (see [28] or [22, Section 5.3]), for $n \geqslant 3$, the curvature K does not depend on a point $x \in (M, g)$, i.e, K is constant on the manifold (M, g).

For $K = 0$, the manifold (M, g) of Lorentzian signature and constant curvature is the Minkowski space $\mathbf{R}_1^n$; for $K > 0$, it is the de Sitter space $\mathbf{S}_1^n$ of first kind whose projective model was considered in detail in [4] and [6]; and for $K < 0$, it is the de Sitter space $\mathbf{H}_1^n$ of second kind (see [8, pp. 115–117]).

Harris in [18] proved the following theorem.

Theorem 12. *A pseudo-Riemannian manifold (M, g) of Lorentzian signature has a constant curvature if and only if its isotropic sectional curvature $K_N(\sigma)$ vanishes.*

Proof. It is not so difficult to prove the necessity of this theorem. In fact, consider an isotropic frame bundle on a manifold (M, g). In this frame bundle the metric tensor g_{ij} has the form (8). This and equations (79) imply that

$$R_{1ab1} = 0. \tag{80}$$

But since e_1 is an arbitrary isotropic vector, by (43), condition (80) means that $K_N(\sigma) = 0$ on the manifold (M, g).

The proof of sufficiency is more complicated (see [18]). $\square$

By conditions (80), equations (34) on a lightlike hypersurface of a manifold (M, g) of constant curvature take the form

$$\nabla\lambda_{ab} - \lambda_{ab}\omega_1^1 + \lambda_{ac}g^{ce}\lambda_{eb}\omega^1 = \mu_{abc}\omega^c. \tag{81}$$

As a result, the covariant derivative of the tensor λ_{ab} in the direction of the vector e_1 has the following expression:

$$(\nabla\lambda_{ab} - \lambda_{ab}\omega_1^1)_{,1} = -\lambda_{ac}g^{ce}\lambda_{eb}.$$

It is expressed only in terms of quantities defined in a second-order differential neighborhood of a point $x \in (M, g)$.

A construction of an invariant normalization and an invariant affine connection for a lightlike hypersurface $V \subset (M, g)$ of constant curvature can be done in the same way as in the general case following the scheme indicated in Sections 10 and 11 with the only difference that in formulas (46) and (52) the quantity K is defined now in a second-order differential neighborhood of a point $x \in (M, g)$ (not the third-order as this was in the general case).

Consider a totally umbilical lightlike hypersurface V on a manifold (M, g) of Lorentzian signature and constant curvature. By Theorem 12, on such a hypersurface the isotropic sectional curvature $K_N(\sigma)$ vanishes. This and Theorem 10 imply the following result.

Theorem 13. *Totally umbilical lightlike hypersurface V on a manifold (M, g) of Lorentzian signature and constant curvature are the light cones of (M, g).*

Note that any Riemannian or pseudo-Euclidean manifold (M, g) of constant curvature is conformally flat (see, for example, [26, §122]). Hence Theorem 13 follows from [6, Theorem 7, part b].

14. An intrinsic normalization of a lightlike hypersurface V on a four-dimensional manifold (M, g) of Lorentzian signature

Consider a lightlike hypersurface on a manifold (M, g), dim $M = 4$, sign $g = (3, 1)$. All formulas of Sections 4-8 hold on such a hypersurface, and the range of the indices a, b, c is 2, 3: $a, b, c = 2, 3$. We reduce simultaneously the first and the second fundamental tensors of the hypersurface V to diagonal forms

$$(g_{ab}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (\lambda_{ab}) = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{pmatrix} \tag{82}$$

and assume that $\lambda_2/\lambda_3 \neq \text{const}$, and $\lambda_2 \neq 0$, $\lambda_3 \neq 0$.

From the last equation of (10) and the first relation of (82) it follows that on V we have

$$\omega_2^2 = \omega_3^3 = 0, \qquad \omega_2^3 + \omega_3^2 = 0, \tag{83}$$

and equations (34) take the form

$$\begin{cases} d\lambda_2 - \lambda_2\omega_1^1 + \big((\lambda_2)^2 + R^4_{221}\big)\omega^1 = \mu_{22c}\omega^c, \\ d\lambda_3 - \lambda_3\omega_1^1 + \big((\lambda_3)^2 + R^4_{331}\big)\omega^1 = \mu_{32c}\omega^c, \\ (\lambda_2 - \lambda_3)\omega_2^3 + R^4_{231}\omega^1 = \mu_{23c}\omega^c. \end{cases} \tag{84}$$

Since $\lambda_2 \neq \lambda_3$, the last equation implies that

$$\omega_2^3 = \frac{1}{\lambda_2 - \lambda_3}\big(R_{1231}\omega^1 + \mu_{232}\omega^2 + \mu_{233}\omega^3\big). \tag{85}$$

The first two equations of (84) can be written as

$$\begin{cases} d\lambda_2 - \lambda_2\omega_1^1 = \big(R_{121} - (\lambda_2)^2\big)\omega^1 + \mu_{222}\omega^2 + \mu_{223}\omega_3, \\ d\lambda_3 - \lambda_3\omega_1^1 = \big(R_{1331} - (\lambda_3)^2\big)\omega^1 + \mu_{332}\omega^2 + \mu_{333}\omega^3. \end{cases} \tag{86}$$

The quantities λ_2 and λ_3 are relative invariants of weight 1. The equations to which these invariants satisfy can be written in the form (52), where

$$\begin{aligned} K_2 &= \lambda_2 - \frac{R_{1221}}{\lambda_2}, \qquad K_{22} = -\frac{\mu_{222}}{\lambda_{23}}, \qquad K_{23} = -\frac{\mu_{223}}{\lambda_2}, \\ K_3 &= \lambda_3 - \frac{R_{1331}}{\lambda_3}, \qquad K_{32} = -\frac{\mu_{332}}{\lambda_3}, \qquad K_{33} = -\frac{\mu_{333}}{\lambda_3}. \end{aligned} \tag{87}$$

The first index in the left-hand sides of these equations is the index of the relative invariant λ_a.

By Theorem 6, if the coefficients K_a are not roots of the characteristic equation of the affinor (λ_b^a), then by means of the coefficients K_{ab} we can construct the normalizing objects L_{ab}.

These normalizing objects determine two invariant normalizations intrinsically connected with the geometry of the hypersurface V.

The ratio λ_2/λ_3 of the eigenvalues of the affinor (λ^a_b) is an absolute invariant. It follows from equations (86) that this absolute invariant satisfies the equation

$$\ln\left|\frac{\lambda_2}{\lambda_3}\right| = \left(\frac{K_2}{\lambda_2} - \frac{K_3}{\lambda_3}\right)\omega^1 + \left(\frac{K_{22}}{\lambda_2} - \frac{K_{32}}{\lambda_3}\right)\omega^2 + \left(\frac{K_{23}}{\lambda_2} - \frac{K_{33}}{\lambda_3}\right)\omega^3. \tag{88}$$

By Theorem 5, if the coefficient in ω^1 in equation (88) is different from 0 (i.e., if the quantities K_2 and K_3 are not proportional to the eigenvalues λ_2 and λ_3 of the affinor (λ^a_b)), then the absolute invariant λ_2/λ_3 allows us to construct one more invariant normalization intrinsically connected with the geometry of the hypersurface V. The screen distribution defining this normalization is tangent to level submanifolds of the invariant λ_2/λ_3.

Thus we have proved the following result.

Theorem 14. *If the eigenvalues λ_2 and λ_3 of the affinor (λ^a_b) of a lightlike hypersurface $V \subset (M, g)$, $\dim M = 4$, are different from 0, the absolute invariant $\lambda_2/\lambda_3 \neq \mathrm{const}$, and the coefficients K_2 and K_3 defined by formulas (87) do not coincide with any of the eigenvalues λ_2 and λ_3 and are not proportional to them, then on such a hypersurface we can construct three invariant normalizations intrinsically connected with the geometry of V, and the screen distribution of one of these normalizations is integrable.*

Note also that the eigenvectors e_2 and e_3 corresponding to the eigenvalues λ_2 and λ_3 of the affinor (λ^a_b) generate two orthogonal vector fields on screen distributions of normalizations we have constructed. These vector fields with the field of isotropic vectors e_1 determine the coordinate net on the hypersurface V. In general, this net is not holonomic. This means that in general, the two-dimensional distributions defined by the eigenvectors of the affinor (λ^a_b) and the vectors e_1 are not integrable.

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