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Large N reduction on a twisted torus

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Abstract

We consider $SU(N)$ lattice gauge theory at infinite N defined on a torus with a CP invariant twist. Massless fermions are incorporated in an elegant way, while keeping them quenched. We present some numerical results which suggest that twisting can make numerical simulations of planar QCD more efficient.

1. Introduction

At an infinite number of colors, QCD on an Euclidean torus of size l^4 undergoes a staircase of transitions as l is reduced [1]. For $l > l_c$, where in ordinary QCD units $l_c \sim 1$ fermi, the system is in a phase where Wilson loop operators of arbitrary size have traces that are exactly l-independent. Thus, one can reduce the number of degrees of freedom from that of an infinite torus, without any loss of information at leading order in the $\frac{1}{N}$ expansion. This ought to be of help in getting at planar QCD using numerical simulation, as reduction holds on the lattice too. In practice this implies that increasing N reduces the finite size corrections, so that a balance between N and the size of the system can be reached which minimizes the computational effort required to get the planar limit of various physical QCD observables. It is likely that getting Monte Carlo numbers in the planar limit is cheaper than solving full QCD with the computer.

To obtain numbers appropriate for zero temperature infinite volume planar QCD one must make sure that all simulations are carried out at lattice spacings a and lattice sizes L which obey $La > l_c$. For sufficiently fine lattices, L turns out to be of order 10. In this Letter we aim to reduce this value even further by making use of an old idea due to González-Arroyo and Okawa [2]. We consider pure $SU(N)$ YM theory on a twisted torus. At infinite N the large volume phase should be independent of the boundary conditions in as much as it does not depend on the size of the box. When the

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volume is decreased, the theory enters a phase where some dependence on boundary conditions is restored. This phase must be distinct from the corresponding phase in the untwisted case, where it corresponds to finite temperature deconfined planar QCD. Therefore, it is conceivable that the critical size l_c^t for the twisted box is smaller than l_c and this follows from general arguments. It is also supported by the quantitative numerical work presented below.

2. Twisted torus—pure gauge

$SU(N)$ gauge fields are objects in $SU(N)/Z(N)$ and therefore the allowed bundles are those of $SU(N)/Z(N)$ over the torus [3]. Some of these bundles cannot be lifted to an $SU(N)$ bundle and when this happens we will say that we have a “twisted torus”. In this work we only interested in the CP invariant case where N is assumed to be even. We consider a nontrivial $SU(N)/Z(2)$ bundle over the torus. To ensure that the classical limit is as simple as possible, we further restrict N to be divisible by four. This ensures that there are flat connections in the bundle. It is easy to transfer this continuum gauge bundle to the lattice. If N were not divisible by four, only by two, the bundle would admit only half integral topological charges and the minimal action configuration would have nontrivial spacetime structure.

We use a single plaquette Wilson lattice action and the gauge group is $SU(N)$, where $N = 4M$ and M is chosen to be prime. Our choice of twist can be induced by choosing the sign of the lattice coupling β to be negative and taking a symmetrical hypercubic lattice of volume L^4 with L given by an odd integer. As is well known, the unusual sign of the coupling could be absorbed by a change of lattice gauge variables for even L, and there is no twist. The same change of variables is inconsistent at the boundaries when L is odd, where it becomes equivalent to the nontrivial boundary conditions one would use if one defined the twisted bundle in the continuum by starting from an open subhypercube of R^4. Another way to see that a negative lattice coupling amounts to twisting by -1 for odd L is to observe that the $Z(2)$ flux through any plane becomes $(-1)^{L^2}$.

The change of variables needed to bring the twisted action to a negative coupling Wilson action also affects observables. In particular, let C denote a closed finite curve C on an infinite lattice, mapped in the natural way to the torus. Associated with C there is a sign, $s(C) \equiv (-1)^{p(C)}$, where $p(C)$ is the number of plaquettes in a spanning surface with C as boundary on the infinite lattice. Let $W_L(C; b, N)$ denote a Wilson loop expectation value associated with the curve C in the presence of periodic boundary conditions for $SU(N)$ gauge theory at lattice coupling $\beta = 2bN^2$. Then, after the change of variables it transforms into $s(C)W_L(C; -b, N)$, where $W_L(C; -b, N)$ denotes the ordinary Wilson loop computed on a periodic lattice of volume L^4 with a negative value of the coupling constant. Inspection of the loop equations led Eguchi and Kawai [4] to conclude that in the large N limit $\frac{1}{N}\operatorname{tr} W_L(C; b, N)$ is L independent. Their proof can be extended [2] to twisted boundary conditions. This result can be also deduced from an analysis of the strong coupling expansion [5] directly. Thus, in the strong coupling region one has

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N}\operatorname{tr} W_L(C; b, N) \\
&\quad = \lim_{N\to\infty} \frac{1}{N}\operatorname{tr} W_\infty(C; b, N) \\
&\quad = \lim_{N\to\infty} \frac{s(C)}{N}\operatorname{tr} W_L(C; -b, N) \\
&\quad = \lim_{N\to\infty} \frac{s(C)}{N}\operatorname{tr} W_\infty(C; -b, N).
\end{aligned}$$

The above equation does not extend all the way to the continuum limit $|b| \to \infty$ [6], but previous work has produced evidence in favor of its validity beyond the radius of convergence of the strong coupling expansion. The basic idea of this Letter is to estimate $\lim_{N\to\infty} \frac{1}{N}\operatorname{tr} W_\infty(C; b, N)$ at some 't Hooft coupling $b > 0$ by numerically extrapolating $\frac{s(C)}{N}\operatorname{tr} W_L(C; -b, N)$ to infinite N at $-b$ and fixed L.

As explained in the introduction, for reduction to work at a given $b > 0$, one needs $L \geqslant L_c(b)$. Based upon past experience with twisting and on arguments to be given later we expect that $L_c(-b) < L_c(b)$. Our numerical findings indicate that this is true, opening the way to more efficient numerical work on planar QCD, employing a CP invariant twist.

The proof of reduction in perturbation theory [2,7,8] requires the vacuum structure to be relatively simple. For N given by $4M$, where M is prime, the minimal action configurations are made up of gauge orbits defined by the gauge configuration $U_\mu = \Gamma_\mu \otimes D_\mu$ where the Γ_μ are ordinary 4 by 4 Euclidean Dirac matrices, and the D_μ are diagonal matrices of size M. The moduli space is defined by the $4(M-1)$ angles associated with the matrices D_μ.

The system has a global $Z^4(N)$ symmetry, which is particularly important at finite volume. Any one of the vacua labeled by points in the moduli space preserves a $Z^4(2)$ subgroup of this symmetry. The remainder, $Z^4(M)$, would be restored by averaging over the moduli space of angles with flat measure. In other words, at infinite N one can say that the eigenvalues of Polyakov loops in all directions are uniformly distributed over the unit circle. Thus, uniform averaging would be a correct procedure if we knew that we are in a phase where the entire $Z^4(N)$ stays unbroken even at infinite N. This average over the moduli space at infinite N effects an extension of the discrete momentum sums associated with ordinary Feynman diagrams on a torus to continuous Feynman integrals on the smooth space of crystal momenta normally associated with an infinite lattice, with the angles filling in the momentum gaps typical of a finite spacetime torus. As we have seen above, effectively, twisting fractionalizes the Brillouin zone into 16 identical hypercubes, facilitating the gap-filling role assumed by the remaining angular parameters. All in all, twisting "helps" the system to maintain the global $Z^4(N)$ at $N=\infty$ and this is required for reduction to work.

When $|b|$ is increased too much, one expects the global $Z^4(N)$ to break spontaneously in the large N limit. We are not certain of the phase of the theory when $L < L_c(-b)$. The simplest guess would be a breaking of one of the $Z(N)$ factors down to $Z(2)$. Substantial numerical work would be needed to determine whether this is correct, or if another alternative takes over. This is an issue we postpone to the future. In this work, we shall carry out tests mainly at one particular b-coupling. From these results, we shall be able to also conclude that in the test cases the entire $Z^4(N)$ symmetry group was preserved and that reduction held.

3. Numerical tests—pure gauge

Our first goal is to check whether the twist indeed helps in reducing the lattice size L at which we can attain the low temperature symmetric phase. For that purpose, we take a lattice spacing a for which we know that reduction works on a periodic lattice only for L larger than a specific L_c. Then, we try to find out whether a simulation with twist on a smaller size $L^t < L_c$ torus is able to reproduce the value of various large N observables. Here we make use of the size independence of the results in the large N limit and in this phase.

In particular, we chose an inverse 't Hooft coupling, b set to $|b| = 0.36$; the corresponding lattice spacing is $a \approx (2.1\ \mathrm{GeV})^{-1}$, quite typical of current QCD simulations on what is considered a fine lattice. Using regular boundary conditions at this lattice spacing would require $L \geqslant 9$. We ran a series of tests which show that a twisted lattice of size $L^t = 5$ at the same value of $|b|$, is able to remain in the phase where full reduction holds, but staying away from the lattice strong coupling phase. (The latter phase extends from $|b| = 0$ to about $|b| = 0.36$, but at large N the tunneling rate into the strong coupling phase can be kept so low that one does not need to worry even about going slightly below $|b| = 0.36$.)

Our Wilson loops were built out of $\tilde{U}_\mu(x)$ matrices, rather than the original link matrices $U_\mu(x)$. The $\tilde{U}_\mu(x)$ matrices are defined in term of the $U_\mu(x)$ by an iterative "smearing" procedure [9]. Let $\Sigma_{U_\mu^{(n)}(x)}$ denote the "staple" associated with the link $U_\mu^{(n)}(x)$ in terms of the entire set of $U_\nu^{(n)}(y)$ matrices. One step in the iteration takes one from a set $U_\mu^{(n)}(x)$ to a set $U_\mu^{(n+1)}(x)$, by the following equation:

$$X_\mu^{(n+1)}(x) \equiv \alpha U_\mu^{(n)}(x) + \frac{1-\alpha}{6}\Sigma_{U_\mu^{(n)}(x)},$$
$$U_\mu^{(n+1)}(x) = X_\mu^{(n+1)}(x)\frac{1}{\sqrt{[X_\mu^{(n+1)}(x)]^\dagger X_\mu^{(n+1)}(x)}}.$$

We chose $\alpha = 0.45$ in the untwisted case and iterated twice:

$$\tilde{U}_\mu(x) = U_\mu^{(2)}(x).$$

Given the change of variables mentioned previously this changes to $\alpha = -0.45$ in the twisted case. Smearing has the effect of removing some of the ultraviolet fluctuations and produces more meaningful numbers for our comparison. Also, the test is made more stringent by including smearing because smearing is a relatively complicated operation involving longer loops and, although it should survive twisting theoretically, numerical effects might have marred the equivalence to the untwisted case.

In Table 1, we compare untwisted results obtained from the extrapolation to infinite N of a sequence of $b = 0.36$ 9^4 lattices with $N = 11, 17, 23, 29$ to results obtained from the extrapolation to infinite N of a sequence of $b = -0.36$ 5^4 lattices with $N = 12, 20, 28, 44, 52$. These were linear extrapolations in $\frac{1}{N^2}$ and are shown in Fig. 1. We compare the action density (raw plaquette average), and smeared n by n Wilson loops for $n = 1, 2, 3, 4, 5$. The agreement is within error bars, but the large N extrapolation works better in the untwisted case; nevertheless, the coefficients of the $\frac{1}{N^2}$ correction come out quite close in the twisted and untwisted cases, except in the twisted case of the 5×5 Wilson loop, where the linear fit in $\frac{1}{N^2}$ does not work well. Theoretically, one expects that the $\frac{1}{N^2}$ correction depend on the shape of the box, and since the twisted box is of size 5^4, this is a natural place to see some larger corrections. There is no question that the twisted simulations took less computer time, but a more quantitative comparison of efficiency needs to take into account the errors. The errors in Table 1 were estimated without taking cor-

Table 1
Action density s and smeared $n \times n$ Wilson loops $W(n)$ on twisted and untwisted lattices

Operator	Untwisted	Twisted
s	0.5581(1)	0.5580(1)
$W(1)$	0.90157(5)	0.90151(4)
$W(2)$	0.5343(2)	0.5345(2)
$W(3)$	0.2307(4)	0.2310(3)
$W(4)$	0.0844(4)	0.0849(3)
$W(5)$	0.0272(3)	0.0277(2)

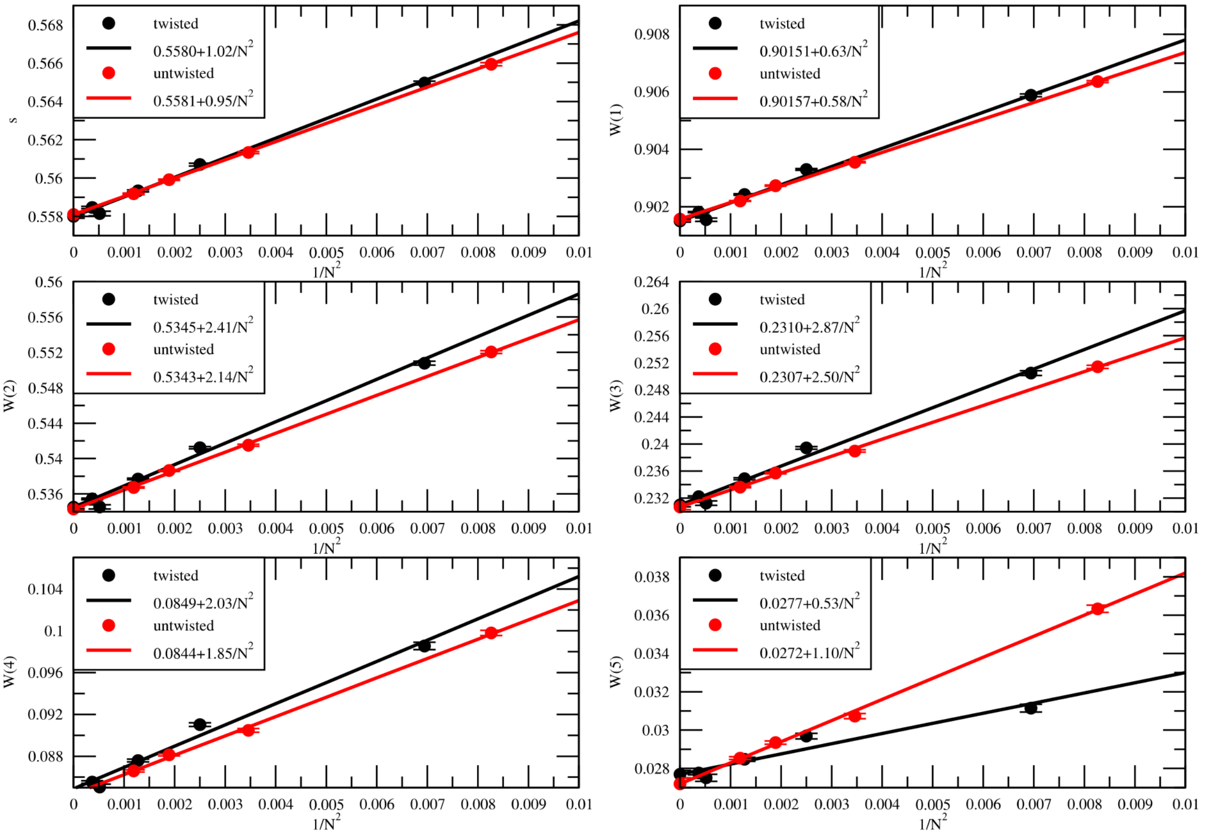

Fig. 1. Results for the untwisted case on a 9^4 lattice and for the twisted case on a 5^4 lattice at coupling $|b| \equiv \frac{|\beta|}{2N^2} = 0.36$.

relations into account because we did varied kinds or runs. The quoted errors should be viewed just as rough indicators. We do not have enough information for a quantitative efficiency estimate, but there is little doubt that it pays to twist.

4. Twisted torus—fermions

Perhaps the main numerical advantage of conventional planar QCD over real QCD is that fermions are quenched. This poses a problem on the twisted torus, as the fermions are in the fundamental of $SU(N)$ and no longer transform under just $SU(N)/Z(N)$. Another set of old tricks [10] can be brought to bear on this problem. First, enlarge the gauge group to $SU(N) \otimes SU(K)$. Next recall that with the choice of twists made here, only $SU(N)/Z(2)$ was exploited. Consider now the true group to be $[SU(N)/Z(2)] \otimes [SU(K)/Z(2)]$ and make the fermions transform under the latter as a bi-fundamental, canceling the $Z(2)$ twists between the two gauge groups. Now everything is in order and all we need to do is to take N and K divisible by 4. We choose $K = 4$ and $N = 4M$ with prime M, as before. We also wish to get rid of the gauge fields associated with the $SU(K)$ factor. We take its lattice $|b|$ coupling to infinity and freeze out those degrees of freedom. So long as we are considering fermion observables that are singlets under the full $SU(N) \otimes SU(K)$ group, the elementary fermions are still quenched in the large N limit. We expect there to be no difference between the twisted theory and the regular one at $N = \infty$, if we take four noninteracting flavors in the periodic case. For quenched fermions, the number of flavors is immaterial in the regular case, as the Dirac operator block decomposes over flavors. In the twisted case the flavors are coupled and this increases the cost of the fermion simulation relatively to the untwisted case. It is quite possible that this increase along with the need to go to larger N is outweighed by the smaller L one needs—only experimentation can determine the cost effectiveness of twisted torus for fermion simulations in the planar limit.

When the fermion twisting trick is taken to the lattice more checks are needed. First of all, we certainly want to preserve the hard won option of maintaining exact global chiral symmetry at finite lattice spacing [11]. In the untwisted case we know how to do this for each fermion flavor independently. Now we need to make sure that the coupling of the flavors in the specific way associated with twisting can be taken to the lattice. We cannot first put each flavor on the lattice and then couple them, because we would loose explicit lattice translational invariance. So, we must first couple the flavors and then carry out the overlap construction, as it is obvious that the twisting procedure meshes well with the sparse Wilson lattice fermion action.

In the untwisted case, the bilinear fermionic action for one flavor is described by the massless overlap Dirac operator D_o [11]:

$$D_o = \frac{1+V}{2},$$
$$V^{-1} = V^\dagger = \gamma_5 V \gamma_5 = \text{sign}\big(H_w(m)\big)\gamma_5.$$

$H_w(m)$ is the Wilson Dirac operator at mass m, which we shall choose as $m = -1.5$.

$$H_w(m) = \gamma_5\left[m + 4 - \sum_\mu \left(\frac{1-\gamma_\mu}{2} T_\mu + \frac{1+\gamma_\mu}{2} T_\mu^\dagger\right)\right].$$

The T_μ matrices are the lattice generators of parallel transport and depend parametrically and analytically on the lattice links $U_\mu(x)$ which are $SU(N)$ matrices at site x associated with the link connecting site x to site $x + \hat{\mu}$, where $\hat{\mu}$ is a unit vector in the positive μ-direction.

The internal fermion-line propagator, $\frac{2}{1+V}$ is not needed at infinite N, as the fermions are quenched at leading order in $\frac{1}{N}$. For fermion lines continuing external fermion sources we are allowed to use a slightly different quark propagator [12,13] defined by

$$\frac{1}{A} = \frac{1-V}{1+V},$$

$A = -A^\dagger$ and anticommutes with γ_5. The spectrum of A is unbounded, but is determined by the spectrum of V which is restricted to the unit circle. One should think of A as dimensionless, and of $|m|$ as providing the needed dimension. Up to a dimensional unit, A should be thought of as a lattice realization of the continuum massless Dirac operator, D:

$$2|m|A \leftrightarrow D = \gamma_\mu \partial_\mu + \cdots.$$

Twisting involves the T_μ operators, which would act now on two indices, color and flavor. There is also the notational inconvenience since we have to deal with three kinds of gamma matrices: the Lorentz γ_μ from above, the Γ_μ of color space that enter the classical vacua, and now, in addition, $\hat{\gamma}_\mu$ acting on flavor. To include the twist for fermions, the new T_μ operators are extended by

$$T_\mu \to T_\mu \otimes \hat{\gamma}_\mu.$$

As a result, the rank of D_o increases four fold. One still has global flavor singlet chiral symmetry, but no flavor-nonsinglet symmetry. Indeed, flavor is not a symmetry, as it implements the twist. It remains to check that in perturbation theory one effectively has four species of Dirac fermions.

To see this, we go to one of the vacua where all diagonal matrices D_μ are unity and can be suppressed. (We already know that when they are not unity they effectively induce some small amount of gap-filling momentum into the fermion lines.) Once the D_μ are ignored, the T_μ matrices get replaced by $\Gamma_\mu \otimes \hat{\gamma}_\mu$ for each μ. One needs now to diagonalize H_w. Specifically, the focus is on the gauge dependent Wilson mass term, defined as $4 - \sum_\mu (\pm)\Gamma_\mu \otimes \hat{\gamma}_\mu$. There, with all momenta zero and any Dirac index, one finds sixteen states with eigenvalues 8, 6, 4, 2, 0 and respective degeneracies of 1, 4, 6, 4, 1. As indicated by the $\pm$ signs, one also needs to take into account all other 15 momenta where some subset of zero momentum components get replaced by π. The set of eigenvalues and degeneracies stays the same for each one of the sixteen momenta. With our choice of the parameter m, only the sixteen states of zero eigenvalue will produce a massless Dirac fermion. So, in total we obtain $16M$ massless Dirac fermions, including all flavors and colors. Since the number of colors is $N = 4M$ we have $4N$ fermions, exactly as expected in the continuum: four flavors of colored fundamental multiplets.

The way this worked out is quite remarkable. Similarly to staggered fermions, the split of the Brillouin zone into sixteen components contributes one species for each compartment. However, unlike in the case of staggered fermions, ordinary Dirac indices are not being mixed in and the global chiral symmetry is the ordinary continuous group we know from continuum. Only flavor is scrambled up, but it had to be, because if the fermionic action fully factorized, we would have concluded that one can define a single fermion flavor on our twisted bundle, something that is geometrically impossible in the continuum. However, in the planar limit there is an equivalence of the flavor and gauge singlet Green functions of the twisted theory to an untwisted theory in which the four flavors are decoupled. This does not really mean that the flavors are decoupled in the twisted theory; rather, so long as one is restricted to only considering flavor singlets, the fermion flavors act as if they were decoupled.

Given the rather intricate nature of this mechanism, a numerical check is highly desirable; as we shall see, it works very well. Before turning over to numerical results, we wish to point out that had we been interested in dynamical fermions, the case of fermions in the double indexed antisymmetric representation of $SU(N)$ is an excellent candidate theoretically, as it is unaffected by our $Z(2)$ twist. In this case, no extra flavors are needed. Thus, for projects trying to get at the planar limit of supersymmetric QCD [14], our $Z(2)$ twist poses no fermionic problems.

5. Numerical tests—fermions

Previous work employing an untwisted action at infinite N in the physical phase, showed how the use of random matrix theory [15] to calculate the fermion condensate and establish spontaneous chiral symmetry breaking on the lattice. In the twisted case, at infinite N, we can do the same. We wish to check that after proper rescaling we find a condensate identical to the condensate we found using the regular method, just multiplied by four on account of the four flavors [16].

In the untwisted case we gathered a large amount of data at $b = 0.35$, which is a coarser lattice. To facilitate comparison of the bare quantities directly we now focus on the twisted case with at $b = -0.35$ on a 5^4 lattice at $N = 44$. We first establish that the two lowest eigenvalues of $\sqrt{-A^2}$ indeed have a ratio r distributed by the parameter free prediction $p(r)$ of the Shuryak–Verbaarschot model. This is shown in the right panel of Fig. 2. Next, consistent estimates for the condensate can be extracted using the two smallest eigenvalues independently: after scaling each by a fitted number their distributions are predicted to be given by universal functions, $p_i(z_i)$, $i = 1, 2$ where z_i is the rescaled value of the ith eigenvalue. The two fits are consis-

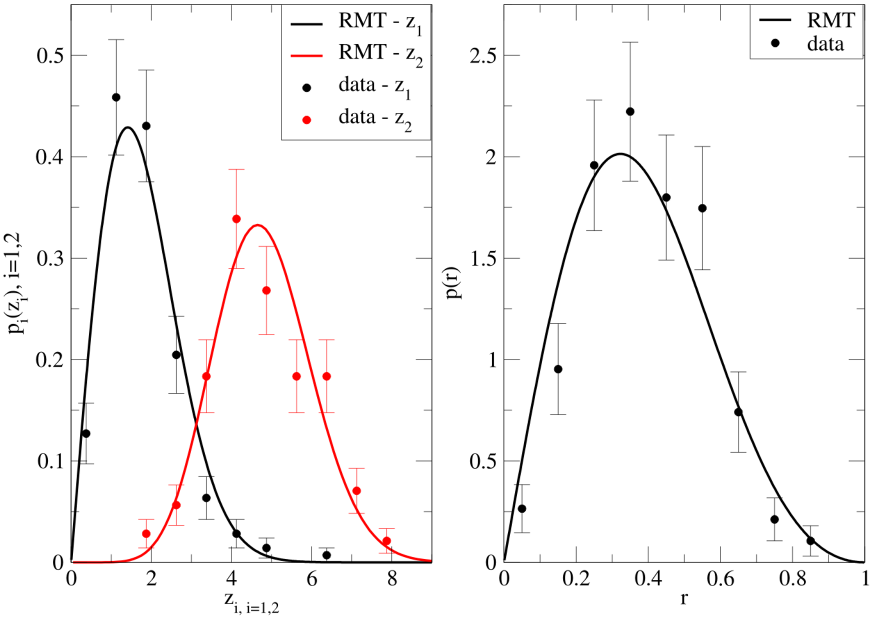

Fig. 2. Distributions associated with overlap Dirac operator eigenvalues for the twisted case on a 5^4 lattice at coupling $|b| \equiv \frac{|\beta|}{2N^2} = 0.35$.

tent with each other and are compared to the data in the left panel of Fig. 2. We have generated 189 gauge configurations; from the lowest eigenvalue we obtain $\frac{1}{N}\langle\bar{\psi}\psi\rangle = (0.140(2))^3$ and from the second lowest we get $\frac{1}{N}\langle\bar{\psi}\psi\rangle = (0.143(1))^3$. In the untwisted case [16] the result at $b = 0.35$ was $\frac{1}{N}\langle\bar{\psi}\psi\rangle = (0.142(6))^3$.

We have also gone to the finer lattice spacing corresponding to $b = -0.36$ in the twisted case keeping $N = 44$ and $L = 5$. Here we accumulated 480 configurations, as the decrease in lattice spacing increases the numerical values of the bare eigenvalues, speeding up the algorithm that calculates them. Now we see a small but clear deviation of the ratio distribution away from the universal curve, showing preference for higher ratios, as typical in these cases, where eigenvalue repulsion has not yet become fully active. Therefore, one needs to increase the N or L in order to reach agreement with random matrix theory. This does not rule out that the lowest eigenvalue is already correctly distributed and indeed the match to theory with a fitted condensate looks fine. This gives us $\frac{1}{N}\langle\bar{\psi}\psi\rangle = (0.106(1))^3$. We do not have untwisted data at $b = 0.36$, but extrapolating from the data at the smaller b couplings we do have, we can roughly estimate that $\frac{1}{N}\langle\bar{\psi}\psi\rangle = (0.106(4))^3$ would be the result at $b = 0.36$. This looks good in comparison, but we must keep in mind that the second eigenvalue in the twisted case also appears to obey its universal prediction after rescaling, but now we get $\frac{1}{N}\langle\bar{\psi}\psi\rangle = (0.1102(4))^3$. Therefore, the second eigenvalue has not yet converged to its random matrix distribution, and we cannot be sure that the first already has, although the indication is that it did.

6. Three dimensions

Very similar constructions hold in there dimensions, with the obvious difference that one has no topological charge to worry about and one can take $N = 2M$ with M prime. Brief tests we have carried out in three dimensions also indicate that twisting allows one to deal with fine lattices at smaller lattice volumes than in the regular approach.

7. Summary and discussion

The outstanding question is to clearly characterize the phase that the twisted system decays into when l decreases to just below l_c^t and numerically check that this phase survives the continuum limit. Until this is done, one cannot claim to have an understanding of twisted simulations at the same level as of untwisted ones. Twisted simulations hold the promise of substantial savings in computer time due to the ability to work at even smaller volumes than when using conventional periodic boundary conditions on the torus.

For our twisted simulations we used our regular untwisted code, and simply used a negative b and a negative α, leaving everything else the same. Our regular code was tuned for the untwisted case, but seemed to perform reasonably also on the twisted case. Further work is needed to tune a code specifically for the twisted case, specifically for weaker couplings than the ones used in untwisted numerical simulations.

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