# The Inertia of a Hermitian Matrix Having Prescribed Complementary Principal Submatrices

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ABSTRACT

For $i=1,2$ let $H_i$ be a given $n_i \times n_i$ Hermitian matrix. We characterize the set of inertias

$$\left\{ \mathrm{In}\left( \begin{bmatrix} H_1 & X \\ X^* & H_2 \end{bmatrix} \right) : X \text{ is } n_1 \times n_2 \right\}$$

in terms of $\mathrm{In}(H_1)$ and $\mathrm{In}(H_2)$.

## 1. INTRODUCTION

The inertia of an $n\times n$ complex matrix $A$ is the triple $\mathrm{In}(A)=(\pi,\nu,\delta)$, where $\pi$ (respectively $\nu,\delta$) is the number of eigenvalues $\lambda$ of $A$ with $\mathrm{Re}\,\lambda>0$ (respectively $\mathrm{Re}\,\lambda<0$, $\mathrm{Re}\,\lambda=0$). Since the multiplicities are counted fully, $\pi+\nu+\delta=n$. Hence, when the order of $A$ is known, $\mathrm{In}(A)$ can, and in the sequel often will, be specified by giving just $\pi$ and $\nu$ as follows: $\mathrm{In}(A)=(\pi,\nu,*)$.

The symbols $I_k$ and $0_k$ will denote the $k\times k$ identity matrix and the $k\times k$ zero matrix, respectively. The following notational conventions will also be

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used:

(1) The index $i$ takes the values 1 and 2,
(2) $H_i$ is an $n_i \times n_i$ Hermitian matrix with $\mathrm{In}(H_i)=(\pi_i, \nu_i, \delta_i)$,
(3) $X$ is an $n_1 \times n_2$ matrix, and
(4) $H=\begin{bmatrix} H_1 & X \\ X^* & H_2 \end{bmatrix}$ is $n\times n$.

It follows that $H$ is Hermitian, $n=n_1+n_2$, and $n_i=\pi_i+\nu_i+\delta_i$. Our main result is:

THEOREM. *The following are equivalent:*

(I) *Given $H_1$ and $H_2$ there exists an $X$ such that* $\mathrm{In}(H)=(\pi, \nu, *)$.
(II) *There exist $H_1$, $H_2$, and $X$ such that* $\mathrm{In}(H)=(\pi, \nu, *)$.
(III) *$\pi$ and $\nu$ are integers satisfying*

(1) $\pi+\nu \leqslant n$,
(2) $\max\{\pi_1, \pi_2\} \leqslant \pi \leqslant \min\{n_1+\pi_2, n_2+\pi_1\}$,
(3) $\max\{\nu_1, \nu_2\} \leqslant \nu \leqslant \min\{n_1+\nu_2, n_2+\nu_1\}$,
(4) $\pi-\nu \leqslant \pi_1+\pi_2$,
(5) $\nu-\pi \leqslant \nu_1+\nu_2$.

This theorem tells how much influence the pair $H_1$, $H_2$ of complementary principal submatrices has on the inertia of $H$. For example, that (II) and (III) are equivalent says that the inequalities (III)(1)–(5) describe exactly the set of inertias $H$ assumes as $H_1$, $H_2$, and $X$ vary subject to our conventions. That (I) and (III) are equivalent says that if instead $H_1$ and $H_2$ are fixed and only $X$ is varied, $H$ still assumes exactly the same set of inertias. Thus, when the goal is information about $\mathrm{In}(H)$, $X$ is arbitrary, and $\mathrm{In}(H_1), \mathrm{In}(H_2)$ are known, it is pointless to seek additional information about $H_1$ and $H_2$.

Some information on how the eigenvalues of the $H_i$ influence those of $H$ can be found in [8].

Some of the inequalities in (III) were established in [4]. However, this theorem tells much more about the problem it treats than the results of [4] do. Results on a similar problem can be found in [5]: there, some inequalities are obtained involving the inertias of $H$ and $H_1$ and the rank of $X$, in case $H_2=0$.

Our main technique has been widely used in connection with inertia theory (cf. for example [2]–[5], [9] and, in infinite dimensions, [1]), and it relies on the following

THEOREM 0. *Let*

$$H=\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$$

*be Hermitian, and suppose* $H_{11}$ *is nonsingular. Then*

(a) $H$ *is conjuctive with* $H_{11}\oplus K$, *where* $K=H_{22}-H_{12}^{*}H_{11}^{-1}H_{12}$;
(b) $\mathrm{In}(H)=\mathrm{In}(H_{11})+\mathrm{In}(K)$.

(*Convention*: *If* $H_{11}=H$, *then* $K$ *does not occur.*)

*Proof.* (a): $S^{*}HS=H_{11}\oplus K$ if

$$S=\begin{bmatrix} I & -H_{11}^{-1}H_{12} \\ 0 & I \end{bmatrix}.$$

(b): Apply (a) and Sylvester's theorem. ∎

We will follow [4] in referring to $K$ as the *Schur complement* of $H_{11}$. For a survey on Schur complements, we send the reader to [3]. It is worth noting that the method used here and an algorithm in [3, Sec. 7] are based on similar ideas.

## 2. PROOFS

(I)⇔(II)

That (I) implies (II) is trivial. For the converse assume that $\mathrm{In}(K)=(\pi,\nu,\delta)$ where

$$K=\begin{bmatrix} K_1 & Y \\ Y^{*} & K_2 \end{bmatrix},$$

and $K_i$ is an $n_i\times n_i$ Hermitian matrix with $\mathrm{In}(K_i)=(\pi_i,\nu_i,\delta_i)$, and $Y$ is $n_1\times n_2$. Assume also that the $H_i$ are given. There exists a nonsingular $n_i\times n_i$ matrix $S_i$ such that $S_i^{*}K_iS_i=H_i$, since $\mathrm{In}(H_i)=\mathrm{In}(K_i)$. Then $S_1^{*}YS_2$ will do for the required $X$ because $H=(S_1\oplus S_2)^{*}K(S_1\oplus S_2)$ has the same inertia as $K$, by Sylvester's theorem. Thus (II) implies (I).

(II)⇔(III)

Two symmetries will help shorten the proof: (1) The symbols subscripted with 1 ($H_1, \pi_1, \nu_1, \delta_1, n_1$) clearly play a role symmetrical to those subscripted with 2. (2) Multiplying the definition of $H$ by $-1$ interchanges the roles of $\pi, \pi_i$ with those of $\nu, \nu_i$, respectively, and this just interchanges (III)(2) with (III)(3) and (III)(4) with (III)(5).

We assume $n>0$ and observe that the theorem is clearly true when $n_1$ or $n_2$ is 0. In particular it is true if $n=1$, the first step of subsequent inductions.

LEMMA 2.1. *If* $\pi_1=\nu_1=\pi_2=\nu_2=0$, *then* (II) *is equivalent to* (III).

*Proof.* In this case (III)(1)–(5) reduce to

$$0\leqslant\pi=\nu\leqslant\min\{n_1,n_2\},$$

and $H_i=0$. Hence, the desired equivalence follows easily from a result of Wielandt (see [7, Lemma 1]). ∎

LEMMA 2.2. *If* $\pi_2=\nu_2=0$, *then* (II) *is equivalent to* (III).

*Proof.* In this case (III) can be expressed:

$$\pi+\nu\leqslant n, \tag{2.1}$$

$$\pi_1\leqslant\pi\leqslant n_1, \tag{2.2}$$

$$\nu_1\leqslant\nu\leqslant n_1, \tag{2.3}$$

$$\pi\leqslant n_2+\pi_1, \tag{2.4}$$

$$\nu\leqslant n_2+\nu_1, \tag{2.5}$$

$$\pi-\nu\leqslant\pi_1, \tag{2.6}$$

$$\nu-\pi\leqslant\nu_1. \tag{2.7}$$

By the preceding discussion we need only consider the case where $\pi_1+\nu_1>0$ and $n_2>0$. Furthermore the case $n=1$ of induction on $n=n_1+n_2$ is settled.

Assume (II) holds. We can assume without loss of generality that

$$H=\begin{bmatrix} \tilde{H}_1 & 0 & Y \\ 0 & 0_{\delta_1} & Z \\ Y^* & Z^* & 0_{n_2} \end{bmatrix},$$

where $\tilde{H}_1$ is $(\pi_1+\nu_1)\times(\pi_1+\nu_1)$ and nonsingular (if necessary we replace $H$ by a unitary similarity $U^*HU$ where $U=U_1\oplus I_{n_2}$). If $H'$ is the Schur complement of $\tilde{H}_1$, Theorem 0 gives

$$\mathrm{In}(H)=(\pi_1,\nu_1,0)+\mathrm{In}(H'), \tag{2.8}$$

where

$$H'=\begin{bmatrix} 0_{\delta_1} & Z \\ Z & H_1' \end{bmatrix} \quad \text{and} \quad H_1'=-Y^*\tilde{H}_1^{-1}Y.$$

Let $\mathrm{In}(H_1')=(\pi_1',\nu_1',\delta_1')$. Since $\mathrm{In}(-\tilde{H}_1^{-1})=(\nu_1,\pi_1,0)$, the Corollary to Theorem 1 of [6] shows that

$$0\leqslant \pi_1'\leqslant \nu_1, \tag{2.9}$$

$$0\leqslant \nu_1'\leqslant \pi_1, \tag{2.10}$$

$$\pi_1'+\nu_1'\leqslant n_2. \tag{2.11}$$

Set $n_1'=n_2$, $n_2'=\delta_1$, $n'=n_1'+n_2'$. Since $H'$ is $n'\times n'$ and $1\leqslant n'<n$, we can apply the induction hypothesis to $H'$. Letting $\mathrm{In}(H')=(\pi',\nu',\delta')$, we obtain

$$\begin{aligned} &\pi'+\nu'\leqslant n_2+\delta_1, && \pi'\leqslant \delta_1+\pi_1', \\ & && \nu'\leqslant \delta_1+\nu_1', \\ &\pi_1'\leqslant \pi'\leqslant n_2, && \pi'-\nu'\leqslant \pi_1', \\ &\nu_1'\leqslant \nu'\leqslant n_2, && \nu'-\pi'\leqslant \nu_1'. \end{aligned} \tag{2.12}$$

By (2.8) we have $\pi'=\pi-\pi_1$ and $\nu'=\nu-\nu_1$. So (2.12) becomes

$$\pi+\nu\leqslant n_1+n_2, \tag{2.13}$$

$$\pi_1'+\pi_1\leqslant\pi\leqslant n_2+\pi_1, \tag{2.14}$$

$$\nu_1'+\nu_1\leqslant\nu\leqslant n_2+\nu_1, \tag{2.15}$$

$$\pi\leqslant\pi_1+\delta_1+\pi_1', \tag{2.16}$$

$$\nu\leqslant\nu_1+\delta_1+\nu_1', \tag{2.17}$$

$$\pi-\nu\leqslant\pi_1-\nu_1+\pi_1', \tag{2.18}$$

$$\nu-\pi\leqslant\nu_1-\pi_1+\nu_1'. \tag{2.19}$$

It is easy to see that (2.13)–(2.19) and (2.9)–(2.10) imply (2.1)–(2.7). For example, (2.9) and (2.16) imply $\pi\leqslant\pi_1+\delta_1+\nu_1=n_1$; (2.9) and (2.18) imply (2.6). The rest either are obvious or follow by symmetry.

To prove the converse we suppose that $\pi$ and $\nu$ satisfy (2.1)–(2.7), and we set

$$n_1'=n_2,\qquad n_2'=n_1-\pi_1-\nu_1=\delta_1,\qquad n'=n_1'+n_2',$$

$$\pi'=\pi-\pi_1,\qquad \nu'=\nu-\nu_1,$$

$$\pi_1'=\max\{\pi+\nu_1-n_1,\pi-\pi_1-\nu+\nu_1,0\}, \tag{2.20}$$

$$\nu_1'=\max\{\nu+\pi_1-n_1,\nu-\nu_1-\pi+\pi_1,0\}. \tag{2.21}$$

We now prove that these primed integers satisfy conditions corresponding to (2.1)–(2.7). The omitted proofs are either easy or follow by symmetry:

$$\pi'+\nu'\leqslant n' \tag{2.1$'$}$$

(since $\pi'+\nu'=\pi+\nu-\pi_1-\nu_1\leqslant n_1+n_2-\pi_1-\nu_1=n_1'+n_2'$); since

$$\pi'=\pi-\pi_1\geqslant\begin{cases}\pi+\nu_1-n_1+\delta_1\geqslant\pi+\nu_1-n_1\\ \pi-\pi_1-\nu+\nu_1 & [\text{by (2.3)}]\\ 0 & [\text{by (2.2)}]\end{cases}$$

we have

$$\pi_1' \leqslant \pi' = \pi - \pi_1 \leqslant n_2 = n_1' \qquad [\text{by } (2.20), (2.4)], \tag{2.2'}$$

$$\nu_1' \leqslant \nu' \leqslant n_1' \qquad [\text{by symmetry with } (2.2')], \tag{2.3'}$$

$$\pi' = \pi - \pi_1 \leqslant \pi_1' + n_1 - \nu_1 - \pi_1 = \pi_1' + n_2' \qquad [\text{by } (2.20)], \tag{2.4'}$$

$$\nu' \leqslant \nu_1' + n_2', \tag{2.5'}$$

$$\pi' - \nu' = \pi - \pi_1 - \nu + \nu_1 \leqslant \pi_1' \qquad [\text{by } (2.20)], \tag{2.6'}$$

$$\nu' - \pi' \leqslant \nu_1'. \tag{2.7'}$$

Also, we can prove that

$$\pi_1' + \nu_1' \leqslant n_1'. \tag{2.22}$$

For we notice that, from the definitions (2.20) and (2.21), $\pi_1' + \nu_1' \leqslant n_1'$ splits into $9 = 3 \times 3$ inequalities without "max," that follow as easily from (2.20)–(2.21) and (2.1)–(2.7) as (2.1′)–(2.7′) did. Now, by $\pi_1' \geqslant 0$, $\nu_1' \geqslant 0$ and (2.22), $(\pi_1', \nu_1', n_1' - \pi_1' - \nu_1')$ is admissible as the inertia of an $n_1' \times n_1'$ matrix.

Since $\pi_1 + \nu_1 > 0$, we have $n' < n$; and so (2.1′)–(2.7′) and the induction hypothesis gives a Hermitian matrix (unitarily similar to)

$$H' = \begin{bmatrix} 0_{\delta_1} & Z \\ Z^* & H_1' \end{bmatrix},$$

such that $\operatorname{In}(H') = (\pi', \nu', n' - \pi' - \nu')$ and $\operatorname{In}(H_1') = (\pi_1', \nu_1', n_1' - \pi_1' - \nu_1')$. On the other hand, from (2.2), (2.6) and the definition of $\pi_1'$ we obtain $\pi_1' \leqslant \nu_1$. So, by symmetry,

$$\pi_1' \leqslant \nu_1 \quad \text{and} \quad \nu_1' \leqslant \pi_1.$$

Hence, since $\pi_1'$ and $\nu_1'$ also satisfy (2.22), by the Corollary to Theorem 1 of [6] there exists a $(\pi_1 + \nu_1) \times n_2$ matrix $Y$ such that

$$-Y^*(-I_{\nu_1} \oplus I_{\pi_1})Y = H_1'.$$

Thus, if

$$H=\begin{bmatrix} -I_{\nu_1}\oplus I_{\pi_1}\oplus 0_{\delta_1} & X \\ X^* & 0_{n_2} \end{bmatrix}$$

where

$$X=\begin{bmatrix} Y \\ Z \end{bmatrix}$$

is $n_1\times n_2$, then $H'$ is the Schur complement of $-I_{\nu_1}\oplus I_{\pi_1}$, and so by Theorem 0

$$\mathrm{In}(H)=(\pi_1,\nu_1,0)+(\pi',\nu',*)=(\pi,\nu,\delta).$$

This shows that (III)⇒(II); Lemma 2.2 is proven. ∎

LEMMA 2.3. *Let $\pi_2+\nu_2>0$. Then the integers $\pi$ and $\nu$ satisfy condition (II) if and only if the following inequalities hold:*

$$\begin{aligned} &\pi+\nu\leqslant n, \\ &\quad x\leqslant \pi\leqslant \min\{n_1+\pi_2,\, x+\delta_2\}, \\ &\quad y\leqslant \nu\leqslant \min\{n_1+\nu_2,\, y+\delta_2\}, \\ &\pi-\nu\leqslant x-\nu_2, \\ &\nu-\pi\leqslant y-\pi_2 \end{aligned} \tag{2.23}$$

*for some integers $x, y$ such that*

$$\begin{aligned} x+y&\leqslant n_1+\pi_2+\nu_2, \\ \max\{\pi_1,\pi_2\}&\leqslant x\leqslant \pi_1+\pi_2+\nu_2, \\ \max\{\nu_1,\nu_2\}&\leqslant y\leqslant \nu_1+\nu_2+\pi_2. \end{aligned} \tag{2.24}$$

*Proof.* If $H$ satisfies (II), it is unitarily similar to

$$\begin{bmatrix} H_1 & Y & Z \\ Y^* & 0_{\delta_2} & 0 \\ Z^* & 0 & \tilde{H}_2 \end{bmatrix},$$

where $\mathrm{In}(H_1)=(\pi_1,\nu_1,\delta_1)$ and $\mathrm{In}(\tilde{H}_2)=(\pi_2,\nu_2,0)$. By Theorem 0, $\mathrm{In}(H)=(\pi_2,\nu_2,0)+\mathrm{In}(K)$, where $K$ is the Schur complement of $\tilde{H}_2$. That means

$$K=\begin{bmatrix} L & Y \\ Y^* & 0_{\delta_2} \end{bmatrix}, \qquad \text{where} \quad L=H_1-Z^*\tilde{H}_2^{-1}Z.$$

Let $(P_1,N_1,*)=\mathrm{In}(L)$. By Theorem 5 of [6] [note: $\mathrm{In}(-\tilde{H}_2^{-1})=(\nu_2,\pi_2,0)$],

$$\begin{gathered} 0\leqslant P_1, \qquad 0\leqslant N_1, \qquad P_1+N_1\leqslant n_1, \\ \pi_1-\pi_2\leqslant P_1\leqslant \pi_1+\nu_2, \\ \nu_1-\nu_2\leqslant N_1\leqslant \nu_1+\pi_2. \end{gathered} \tag{2.25}$$

If $(P,N,*)=\mathrm{In}(K)$, then applying Lemma 2.2 to $K$ gives

$$\begin{gathered} P+N\leqslant n_1+\delta_2, \\ P_1\leqslant P\leqslant \min\{n_1,P_1+\delta_2\}, \\ N_1\leqslant N\leqslant \min\{n_1,N_1+\delta_2\}, \\ P-N\leqslant P_1, \\ N-P\leqslant N_1. \end{gathered} \tag{2.26}$$

Then $\mathrm{In}(H)=(\pi,\nu,\delta)=(\pi_2+P,\nu_2+N,\delta)$, and so introducing the notation $x=P_1+\pi_2$, $y=N_1+\nu_2$ converts (2.26) into (2.23) and (2.25) into (2.24).

Conversely, suppose $\pi,\nu$ satisfy (2.23) for some $x,y$ satisfying (2.24). Then $\pi$, $\nu$, $\delta=n-\pi-\nu$ are nonnegative. Also $P_1=x-\pi_2$, $N_1=y-\nu_2$ satisfy (2.25), and so, by Theorem 5 of [6], there exists an $n_1\times(\pi_2+\nu_2)$ matrix $Z$, and Hermitian matrices $H_1$, $\tilde{H}_2$ with $\mathrm{In}(H_1)=(\pi_1,\nu_1,\delta_1)$, $\mathrm{In}(\tilde{H}_2)=(\pi_2,\nu_2,0)$

such that $\mathrm{In}(L)=(P_1, N_1, *)$ where $L=H_1-Z^*\tilde{H}_2^{-1}Z$. Theorem 0 says that

$$H=\begin{bmatrix} H_1 & Y & Z \\ Y^* & 0_{\delta_2} & 0 \\ Z^* & 0 & \tilde{H}_2 \end{bmatrix}$$

will have the desired inertia $(\pi, \nu, \delta)$. provided that

$$K=\begin{bmatrix} L & Y \\ Y^* & 0_{\delta_2} \end{bmatrix},$$

the Schur complement of $\tilde{H}_2$, has inertia $(P, N, \delta)=(\pi-\pi_2, \nu-\nu_2, \delta)$. Furthermore, Lemma 2.2 and the equivalence of (I) and (II) tell us that there will exist an $n_1\times\delta_2$ matrix $Y$ such that $\mathrm{In}(K)=(P, N, \delta)$ if and only if $P$ and $N$ satisfy the counterparts, in the currently relevant notation, of (2.1)–(2.7). In other words, the proof will be finished when we have verified

$$P+N\leqslant n_1+\delta_2, \qquad P\leqslant\delta_2+P_1,$$

$$N\leqslant\delta_2+N_1,$$

$$P_1\leqslant P\leqslant n_1, \qquad P-N\leqslant P_1,$$

$$N_1\leqslant N\leqslant n_1, \qquad N-P\leqslant N_1.$$

But these inequalities are easy consequences of $P=\pi-\pi_2$, $N=\nu-\nu_2$, and (2.23). ■

The proof that (II) and (III) are equivalent is complete once Lemma 2.3 is combined with:

LEMMA 2.4. *There exist integers $x, y$ satisfying* (2.23)–(2.24) *if and only if the inequalities* (III)(1)–(5) *hold.*

*Proof.* The inequalities (2.23)–(2.24) can be rewritten as

$$\pi+\nu\leqslant n, \qquad \pi\leqslant n_1+\pi_2, \qquad \nu\leqslant n_1+\nu_2,$$

$$\begin{aligned} &\max\{\pi_1, \pi_2, \pi-\delta_2, \pi-\nu+\nu_2\}\leqslant x\leqslant\min\{\pi, \pi_1+\pi_2+\nu_2\}, \\ &\max\{\nu_1, \nu_2, \nu-\delta_2, \nu-\pi+\pi_2\}\leqslant y\leqslant\min\{\nu, \nu_1+\nu_2+\pi_2\}, \end{aligned} \tag{2.27}$$

$$x+y\leqslant n_1+\pi_2+\nu_2.$$

It is well known, and easy to prove, that a system of inequalities of the general form

$$a \leqslant x \leqslant A, \qquad b \leqslant y \leqslant B, \qquad x + y \leqslant C \tag{2.28}$$

has a solution $x, y$ iff $a \leqslant A$, $b \leqslant B$, and $a + b \leqslant C$. Moreover, if (2.28) is consistent, an integral solution exists whenever $a$, $b$, $A$, $B$, $C$ are integers.

Thus, there exist integers $x, y$ such that (2.27) holds iff

$$\pi + \nu \leqslant n, \qquad \pi \leqslant n_1 + \pi_2, \qquad \nu \leqslant n_1 + \nu_2,$$

$$\max\{\pi_1, \pi_2, \pi - \delta_2, \pi - \nu + \nu_2\} \leqslant \min\{\pi, \pi_1 + \pi_2 + \nu_2\}, \tag{2.29}$$

$$\max\{\nu_1, \nu_2, \nu - \delta_2, \nu - \pi + \pi_2\} \leqslant \min\{\nu, \nu_1 + \nu_2 + \pi_2\},$$

$$\max\{\pi_1, \pi_2, \pi - \delta_2, \pi - \nu + \nu_2\} + \max\{\nu_1, \nu_2, \nu - \delta_2, \nu - \pi + \pi_2\} \leqslant n_1 + \pi_2 + \nu_2.$$

We have to prove the equivalence of (2.29) with (III). For that, let us split (2.29) into a system of inequalities without "max" or "min." Among the 35 inequalities so obtained we find (III)(1)–(5); the remaining 24 inequalities are easy consequences of (III) and the nonnegativity of $\pi_i$, $\nu_i$, $\delta_i$. ∎

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