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On a subclass of n-p-valent prestarlike functions

M.K. Aouf, A.O. Mostafa*

Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

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Abstract

The object of this paper is to study the subclass $R_p[\alpha, \beta, n]$ of n-p-valent α-prestarlike functions of order β with negative coefficients. Extreme points, integral operators and distortion theorems of this class are obtained. We also obtain some results for the modified Hadamard products of functions belonging to this class.

Keywords: p-valent; Prestarlike functions; Distortion theorems; Modified Hadamard products

1. Introduction

Let $A(p)$ denote the class of functions of the from:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in N = \{1, 2, \ldots\}), \tag{1.1}$$

which are analytic and p-valent in the unit disk $U = \{z : |z| < 1\}$. A function $f(z)$ in the class $A(p)$ is called p-valent starlike of order β $(0 \le \beta < p)$ if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta, \tag{1.2}$$

and

$$\int_0^{2\pi} \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \mathrm{d}\theta = 2\pi p \quad (z \in U). \tag{1.3}$$

We denote the class of p-valent starlike functions of order β by $S^*(p, \beta)$. This class was introduced by Patil and Thakare [1].

The function

$$s_\alpha^p(z) = z^p (1 - z)^{-2(p-\alpha)} \quad (0 \le \alpha < p;\ p \in N), \tag{1.4}$$

* Corresponding author.
E-mail addresses: mkaouf127@yahoo.com (M.K. Aouf), adelaeg254@yahoo.com (A.O. Mostafa).

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is the familiar extremal function for the class $S^*(p,\alpha)$. Setting

$$G^p(\alpha,k)=\frac{\prod_{m=2}^{k}[2(p-\alpha)+m-2]}{(k-1)!}\quad (k\geqslant 2), \tag{1.5}$$

then $s_\alpha^p(z)$ can be written in the form:

$$s_\alpha^p(z)=z^p+\sum_{k=1}^{\infty}G^p(\alpha,k+1)z^{p+k}. \tag{1.6}$$

Clearly, $G^p(\alpha,k)$ is a decreasing function in $\alpha(0\le\dot{\alpha}\le\frac{2p-1}{2};\ p\in N)$ and

$$\lim_{k\to\infty}G^p(\alpha,k+1)=\begin{cases}\infty, & \alpha<\dfrac{2p-1}{2},\\ 1, & \alpha=\dfrac{2p-1}{2},\\ 0, & \alpha>\dfrac{2p-1}{2}.\end{cases}$$

Let $(f*g)(z)$ denote the Hadamard product (or convolution) of the functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z)=z^p+\sum_{k=1}^{\infty}b_{p+k}z^{p+k}\quad (p\in N), \tag{1.7}$$

then

$$(f*g)(z)=z^p+\sum_{k=1}^{\infty}a_{p+k}b_{p+k}z^{p+k}. \tag{1.8}$$

For a function $f(z)$ in $A(p)$, we define

$$D^0f(z)=f(z),$$

$$D^1f(z)=Df(z)=zf'(z)=pz^p+\sum_{k=1}^{\infty}(p+k)a_{p+k}z^{p+k},$$

$$D^2f(z)=D(Df(z))=p^2z^p+\sum_{k=1}^{\infty}(p+k)^2a_{p+k}z^{p+k},$$

$$D^nf(z)=D(D^{n-1}f(z))=p^nz^p+\sum_{k=1}^{\infty}(p+k)^na_{p+k}z^{p+k}. \tag{1.9}$$

For $p=1$, the differential operator D^n was introduced by Salagean [2].

A function $f(z)$ in $A(p)$ is said to be n-p-valent starlike of order $\beta(n\in N_0=N\cup\{0\}, 0\leqslant\beta<p$ and $p\in N)$ if and only if

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^nf(z)}\right\}>\beta\quad z\in U. \tag{1.10}$$

We denote by $S_n^*(p,\beta)$ the class of n-p-valent starlike functions of order β. We note that $S_0^*(p,\beta)=S^*(p,\beta)$.

Define the class $R_p(\alpha,\beta,n)$ by

$$R_p(\alpha,\beta,n)=\{f\in A(p):(D^nf*s_\alpha^p)(z)\in S^*(p,\beta)\}, \tag{1.11}$$

where $0\le\alpha<p, 0\le\beta<p, p\in N$ and $n\in N_0$.

The class $R_1(\alpha, \beta, 0)$ is called the class of α-prestarlike functions of order $\beta (0 \leq \alpha < 1; 0 \leq \beta < 1)$ and was introduced by Sheil-Small et al. [3].

Let $T(p)$ be the subclass of $A(p)$ consisting of functions of the form:

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (a_{k+p} \geq 0; \ p \in N). \tag{1.12}$$

We denote by $R_p[\alpha, \beta, n]$ the class obtained by taking the intersection of the class $R_p(\alpha, \beta, n)$ with the class $T(p)$. Thus, we have

$R_p[\alpha, \beta, n] = R_p(\alpha, \beta, n) \cap T(p)$, $p \in N, n \in N_0$.

We note that $R_p[\alpha, \beta, 0] = R_p[\alpha, \beta]$ (Aouf and Silverman [4]).

By specializing the parameters α, β, n and p we obtain the following classes studied by various authors:

(i) $R_1[\alpha, \alpha, 0] = R[\alpha]$ (Silverman and Silvia [5]);

(ii) $R_1[\alpha, \beta, 0] = R[\alpha, \beta]$ (Aouf and Salagean [6], Sheil-Small et al. [3] and Uralegaddi and Sarangi [7]);

(iii) $R_1[\alpha, \beta, 1] = R[\alpha, \beta, 1]$ (Aouf and Salagean [6] and Owa and Uralegaddi [8]);

(iv) $R_1[\alpha, \beta, n] = R[\alpha, \beta, n]$ (Aouf and Salagean [9]);

(v) $R_1[\frac{1}{2}, \beta, n] = R[\frac{1}{2}, \beta, n]$ (Salagean [10,11]).

2. Coefficient estimates

Theorem 1. *Let the function $f(z)$ be defined by (1.12). Then $f(z)$ is in the class $R_p[\alpha, \beta, n]$ if and only if*

$$\sum_{k=1}^{\infty} \left(\frac{k+p}{p}\right)^n (k+p-\beta) G^p(\alpha, k+1) a_{k+p} \leq (p-\beta), \tag{2.1}$$

where $0 \leq \beta < p$, $0 \leq \alpha < p$, $p \in N$ and $n \in N_0$.

Proof. Assume that the inequality (2.1) holds true. Then we find that

$$\begin{aligned}
\left|\frac{z((D^n f * s_\alpha^p)(z))'}{(D^n f * s_\alpha^p)(z)} - p\right| &= \left|\frac{\left[pz^p - \sum_{k=1}^{\infty} \left(\frac{k+p}{p}\right)^n (k+p) G^p(\alpha, k+1) a_{k+p} z^{k+p}\right]}{z^p - \sum_{k=1}^{\infty} \left(\frac{k+p}{p}\right)^n G^p(\alpha, k+1) a_{k+p} z^{k+p}} - p\right| \\
&\leq \frac{\sum_{k=1}^{\infty} k \left(\frac{k+p}{p}\right)^n G^p(\alpha, k+1) a_{k+p} |z|^k}{1 - \sum_{k=1}^{\infty} (\frac{k+p}{p})^n G^p(\alpha, k+1) a_{k+p} |z|^k} \\
&\leq \frac{\sum_{k=1}^{\infty} k \left(\frac{k+p}{p}\right)^n G^p(\alpha, k+1) a_{k+p}}{1 - \sum_{k=1}^{\infty} \left(\frac{k+p}{p}\right)^n G^p(\alpha, k+1) a_{k+p}} \leq p - \beta.
\end{aligned}$$

This shows that the values of the function

$$\Phi(z) = \frac{z((D^n f * s_\alpha^p)(z))'}{(D^n f * s_\alpha^p)(z)} \tag{2.2}$$

lie in a circle which is centred at $w = p$ and whose radius is $p - \beta$. Hence $f(z)$ satisfies the condition $\mathrm{Re}\{\Phi(z)\} > \beta$, $0 \leq \beta < p$, that is $f(z) \in R_p[\alpha, \beta, n]$.

Conversely, assume that the function $f(z)$ defined by (1.12) is in the class $R_p[\alpha, \beta, n]$. Then we have

$$\mathrm{Re}\left\{\frac{z((D^n f * s_\alpha^p)(z))'}{(D^n f * s_\alpha^p)(z)}\right\} = \mathrm{Re}\left\{\frac{p - \sum\limits_{k=1}^{\infty}(k+p)(\frac{k+p}{p})^n G^p(\alpha, k+1) a_{k+p} z^k}{1 - \sum\limits_{k=1}^{\infty}(\frac{k+p}{p})^n G^p(\alpha, k+1) a_{k+p} z^k}\right\} > \beta, \tag{2.3}$$

for $\beta(0 \leq \beta < p), \alpha(0 \leq \alpha < p)$, $p \in N$, $n \in N_0$, and $z \in U$. Choose values of z on the real axis so that $\Phi(z)$ given by (2.2) is real. Upon clearing the denominator in (2.3) and letting $z \to 1^-$ through real values, we can see that

$$p - \sum_{k=1}^{\infty}(k+p)\left(\frac{k+p}{p}\right)^n G^p(\alpha, k+1) a_{k+p} \geq \beta\left(1 - \sum_{k=1}^{\infty}\left(\frac{k+p}{p}\right)^n G^p(\alpha, k+1) a_{k+p}\right).$$

Thus we have the inequality (2.1).

Corollary 1. *Let the function $f(z)$ defined by (1.12) be in the class $R_p[\alpha, \beta, n]$. Then*

$$a_{k+p} \leq \frac{(p-\beta)}{\left(\frac{k+p}{p}\right)^n (k+p-\beta) G^p(\alpha, k+1)} \quad (k, p \in N; n \in N_0). \tag{2.4}$$

The equality in (2.4) is attained for the function $f(z)$ given by

$$f(z) = z^p - \frac{(p-\beta)}{\left(\frac{k+p}{p}\right)^n (k+p-\beta) G^p(\alpha, k+1)} z^{k+p} \quad (k, p \in N; n \in N_0). \tag{2.5}$$

3. Extreme points of the class $R_p[\alpha, \beta, n]$

From Theorem 1, we see that $R_p[\alpha, \beta, n]$ is closed under convex linear combination, which enables us to determine the extreme points of this class.

Theorem 2. *Let $f_p(z) = z^p$ and*

$$f_{p+k}(z) = z^p - \frac{(p-\beta)}{\left(\frac{k+p}{p}\right)^n (k+p-\beta) G^p(\alpha, k+1)} z^{k+p} \quad (k, p \in N). \tag{3.1}$$

Then $f(z) \in R_p[\alpha, \beta, n]$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=0}^{\infty} \lambda_{p+k} f_{p+k}(z), \tag{3.2}$$

where $\lambda_{p+k} \geqslant 0$ and $\sum_{k=0}^{\infty} \lambda_{p+k} = 1$.

Proof. Suppose that

$$f(z) = \sum_{k=0}^{\infty} \lambda_{p+k} f_{p+k}(z) = z^p - \sum_{k=1}^{\infty} \frac{(p-\beta)}{\left(\frac{k+p}{p}\right)^n (k+p-\beta) G^p(\alpha, k+1)} \lambda_{p+k} z^{k+p}. \tag{3.3}$$

Then

$$\sum_{k=1}^{\infty} \frac{\left(\frac{k+p}{p}\right)^n (k+p-\beta) G^p(\alpha, k+1)}{(p-\beta)} \cdot \frac{(p-\beta)}{\left(\frac{k+p}{p}\right)^n (k+p-\beta) G^p(\alpha, k+1)} \lambda_{p+k} = \sum_{k=1}^{\infty} \lambda_{p+k} = 1 - \lambda_p \leq 1.$$

Therefore, by Theorem 1, $f(z) \in R_p[\alpha, \beta, n]$.

Conversely, let the function $f(z)$ defined by (1.12) be in the class $R_p[\alpha, \beta, n]$.
Setting

$$\lambda_{p+k} = \frac{\left(\frac{k+p}{p}\right)^n (k+p-\beta)G^p(\alpha, k+1)}{(p-\beta)} a_{k+p},$$

where a_{k+p} is given by (2.4) and

$$\lambda_p = 1 - \sum_{k=1}^{\infty} \lambda_{p+k}.$$

Then

$$\sum_{k=0}^{\infty} \lambda_{k+p} f_{k+p}(z) = z^p - \sum_{k=1}^{\infty} a_{k+p} z^{k+p} = f(z).$$

This completes the proof of Theorem 2.

Corollary 2. *The extreme points of the class $R_p[\alpha, \beta, n]$ are the functions $f_p(z) = z^p$ and $f_{k+p}(z)$ given by (3.1).*

4. Distortion theorems

As a consequence of Theorem 2, we will obtain distortion theorems for the class $R_p[\alpha, \beta, n]$. We shall need the following lemma given by Aouf and Silverman [4].

Lemma 1 ([4]). *For $0 \le \alpha \le \frac{2p-1}{p}$, $0 \le \beta < p$, $p \in N$, then $(k+p-\beta)G^p(\alpha, k+1)$ is an increasing function of k, where $G^p(\alpha, k+1)$ is defined by (1.4).*

In the remainder of this paper, we assume that $f(z)$ is defined by (1.12), $0 \le \alpha \le \frac{2p-1}{2}$, $0 \le \beta < p$, $p \in N$ and $n \in N_0$.

Theorem 3. *Let the function $f(z)$ be in the class $R_p[\alpha, \beta, n]$, then*

$$|z|^p - \frac{(p-\beta)}{2\left(\frac{1+p}{p}\right)^n (1+p-\beta)(p-\alpha)} |z|^{p+1} \le |f(z)|$$
$$\le |z|^p + \frac{(p-\beta)}{2\left(\frac{1+p}{p}\right)^n (1+p-\beta)(p-\alpha)} |z|^{p+1} \quad (z \in U). \tag{4.1}$$

Equality holds for the function

$$f(z) = z^p - \frac{(p-\beta)}{2\left(\frac{1+p}{p}\right)^n (1+p-\beta)(p-\alpha)} z^{p+1} \quad (z \in U). \tag{4.2}$$

Proof. From Theorem 2, we have

$$|z|^p - \max_{k \in N} \frac{(p-\beta)}{\left(\frac{p+k}{p}\right)^n (k+p-\beta)G^p(\alpha, k+1)} |z|^{p+k} \le |f(z)|$$
$$\le |z|^p + \max_{k \in N} \frac{(p-\beta)}{\left(\frac{p+k}{p}\right)^n (k+p-\beta)G^p(\alpha, k+1)} |z|^{p+k}. \tag{4.3}$$

By virtue of Lemma 1, we see that the max in (4.3) occurs when $k = 1$. This completes the proof of Theorem 3.

Corollary 3. *If the function $f(z)$ is in the class $R_p[\alpha,\beta,n]$. Then $f(z)$ is included in a disc with centre at the origin and radius r, where*

$$r = 1 + \frac{(p-\beta)}{2\left(\frac{1+p}{p}\right)^n (1+p-\beta)(p-\alpha)}. \tag{4.4}$$

Theorem 4. *Let the function $f(z)$ be in the class $R_p[\alpha,\beta,n]$, then*

$$p\,|z|^{p-1} - \frac{(p+1)(p-\beta)}{2\left(\frac{1+p}{p}\right)^n (1+p-\beta)(p-\alpha)}|z|^p \le \left|f'(z)\right|$$
$$\le p\,|z|^{p-1} + \frac{(p+1)(p-\beta)}{2\left(\frac{1+p}{p}\right)^n (1+p-\beta)(p-\alpha)}|z|^p \quad (z\in U). \tag{4.5}$$

Equality holds for the function $f(z)$ given by (4.2).

Proof. We know that

$$p\,|z|^{p-1} - \max_{k\in N}\frac{(p+k)(p-\beta)}{\left(\frac{k+p}{p}\right)^n (k+p-\beta)G^p(\alpha,k+1)}|z|^p \le \left|f'(z)\right|$$
$$\le p\,|z|^{p-1} + \max_{k\in N}\frac{(p+k)(p-\beta)}{\left(\frac{k+p}{p}\right)^n (k+p-\beta)G^p(\alpha,k+1)}|z|^p, \quad z\in U. \tag{4.6}$$

From Lemma 1, we see that the max in (4.6) accurs when $k=1$. This completes the proof of Theorem 4.

5. Integral operators

Theorem 5. *Let the function $f(z)$ defined by (1.12) be in the class $R_p[\alpha,\beta,n]$ and let c be a real number such that $c>-p$. Then the function $F(z)$ defined by*

$$F(z) = \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)\mathrm{d}t \tag{5.1}$$

also belongs to the class $R_p[\alpha,\beta,n]$.

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z^p - \sum_{k=1}^{\infty} b_{p+k} z^{k+p}, \tag{5.2}$$

where

$$b_{p+k} = \left(\frac{c+p}{c+p+k}\right) a_{p+k}.$$

Therefore

$$\sum_{k=1}^{\infty}\left(\frac{k+p}{p}\right)^n (k+p-\beta)G^p(\alpha,k+1)b_{p+k} = \sum_{k=1}^{\infty}\left(\frac{k+p}{p}\right)^n (k+p-\beta)G^p(\alpha,k+1)\left(\frac{c+p}{k+c+p}\right)a_{k+p}$$
$$\le \sum_{k=1}^{\infty}\left(\frac{k+p}{p}\right)^n (k+p-\beta)G^p(\alpha,k+1)a_{k+p} \le p-\beta,$$

since $f(z)\in R_p[\alpha,\beta,n]$. Hence, by Theorem 1, $F(z)\in R_p[\alpha,\beta,n]$.

Theorem 6. *Let c be a real number such that $c > -p$. If $F(z) \in R_p[\alpha, \beta, n]$, then the function $f(z)$ defined by (5.1) is p-valent in $|z| < R_p$, where*

$$R_p = \inf_k \left\{ \frac{\left(\frac{k+p}{p}\right)^{n-1} \left(\frac{k+p-\beta}{p-\beta}\right) G^p(\alpha, k+1)}{\left(\frac{k+p+c}{p+c}\right)} \right\}^{\frac{1}{k}} \quad (k \geqslant 1). \tag{5.3}$$

The result is sharp.

Proof. Let

$$F(z) = z^p - \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (a_{k+p} \geqslant 0).$$

It follow from (5.1) that

$$f(z) = \frac{z^{1-c}(z^c F(z))'}{(c+p)} \quad (c > -p).$$

To prove the result, it suffices to show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p \quad \text{for } |z| < R_p.$$

Now

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| = \left| \sum_{k=1}^{\infty} (k+p) \left(\frac{k+p+c}{p+c} \right) a_{k+p} z^k \right| \leq \sum_{k=1}^{\infty} (k+p) \left(\frac{k+p+c}{p+c} \right) a_{k+p} |z|^k .$$

Thus

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p \quad \text{if } \sum_{k=1}^{\infty} \left(\frac{k+p}{p} \right) \left(\frac{k+p+c}{p+c} \right) a_{k+p} |z|^k \leq 1. \tag{5.4}$$

But Theorem 1 confirms that

$$\sum_{k=1}^{\infty} \left(\frac{k+p}{p} \right)^n \left(\frac{k+p-\beta}{p-\beta} \right) G^p(\alpha, k+1) a_{k+p} \leq 1.$$

Thus (5.4) will be satisfied if

$$\left(\frac{k+p}{p} \right) \left(\frac{k+p+c}{p+c} \right) |z|^k \leq \left(\frac{k+p}{p} \right)^n \left(\frac{k+p-\beta}{p-\beta} \right) G^p(\alpha, k+1) \quad (k \geqslant 1).$$

That is, if

$$|z| \leq \left\{ \frac{\left(\frac{k+p}{p}\right)^{n-1} \left(\frac{k+p-\beta}{p-\beta}\right) G^p(\alpha, k+1)}{\left(\frac{k+p+c}{p+c}\right)} \right\}^{\frac{1}{k}} \quad (k \geqslant 1).$$

The result is sharp for the function

$$f(z) = z^p - \frac{(p-\beta)(k+p+c)}{\left(\frac{k+p}{p}\right)^n (k+p-\beta)(p+c) G^p(\alpha, k+1)} z^{k+p} \quad (k, p \in N; n \in N_0). \tag{5.5}$$

6. Modified Hadamard products

Let the functions $f_\nu(z)$ $(\nu = 1, 2)$ defined by

$$f_\nu(z) = z^p - \sum_{k=1}^{\infty} a_{k+p,\nu} z^{k+p} \quad (a_{k+p,\nu} \geq 0). \tag{6.1}$$

The modified Hadamard product of the functions $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^p - \sum_{k=1}^{\infty} a_{k+p,1} a_{k+p,2} z^{k+p}. \tag{6.2}$$

Theorem 7. *Let the functions* $f_\nu(z)(\nu = 1, 2)$ *defined by* (6.1) *be in the class* $R_p[\alpha, \beta, n]$. *Then we have* $(f_{1*} f_2)(z) \in R_p[\alpha, \gamma, n]$, *where*

$$\gamma = \gamma(\alpha, \beta, p) = p - \frac{(p-\beta)^2}{2(p-\alpha)(1+p-\beta)^2 \left(\frac{1+p}{p}\right)^n - (p-\beta)^2}. \tag{6.3}$$

The result is sharp for the functions $f_\nu(z)$ *given by*

$$f_\nu(z) = z^p - \frac{p-\beta}{2(p-\alpha)(1+p-\beta)^2 \left(\frac{1+p}{p}\right)^n} z^{p+1} \quad (\nu = 1, 2;\ p \in N;\ n \in N_0). \tag{6.4}$$

Proof. Employing the technique used earlier by Schild and Silverman [12], we need to find the largest $\gamma = \gamma(\alpha, \beta, p)$ such that

$$\sum_{k=1}^{\infty} \left(\frac{k+p}{p}\right)^n \left(\frac{k+p-\gamma}{p-\gamma}\right) G^p(\alpha, k+1) a_{k+p,1} a_{k+p,2} \leq 1. \tag{6.5}$$

Since the functions $f_\nu(z)(\nu = 1, 2)$ belong to the class $R_p[\alpha, \beta, n]$, then

$$\sum_{k=1}^{\infty} \left(\frac{k+p}{p}\right)^n \left(\frac{k+p-\beta}{p-\beta}\right) G^p(\alpha, k+1) a_{k+p,\nu} \leq 1. \tag{6.6}$$

By the Cauchy–Schwarz inequality, we have

$$\sum_{k=1}^{\infty} \left(\frac{k+p}{p}\right)^n \left(\frac{k+p-\beta}{p-\beta}\right) G^p(\alpha, k+1) \sqrt{a_{k+p,1} a_{k+p,2}} \leq 1. \tag{6.7}$$

Thus, it is sufficient to show that

$$\left(\frac{k+p}{p}\right)^n \left(\frac{k+p-\gamma}{p-\gamma}\right) G^p(\alpha, k+1) a_{k+p,1} a_{k+p,2} \leq \left(\frac{k+p}{p}\right)^n \left(\frac{k+p-\beta}{p-\beta}\right) G^p(\alpha, k+1) \sqrt{a_{k+p,1} a_{k+p,2}},$$

that is, that

$$\sqrt{a_{k+p,1} a_{k+p,2}} \leq \frac{(p-\gamma)(k+p-\beta)}{(p-\beta)(k+p-\gamma)} \quad (k \geqslant 1). \tag{6.8}$$

But from (6.7) we have

$$\sqrt{a_{k+p,1} a_{k+p,2}} \leq \frac{(p-\beta)}{\left(\frac{k+p}{p}\right)^n (k+p-\beta) G^p(\alpha, k+1)} \quad (k \geqslant 1). \tag{6.9}$$

Consequently, we need only to prove that

$$\frac{(p-\beta)(k+p-\gamma)}{(p-\gamma)(k+p-\beta)} \leq \left(\frac{k+p}{p}\right)^n \left(\frac{k+p-\beta}{p-\beta}\right) G^p(\alpha, k+1),$$

or, equivalently, that

$$\gamma \leq p - \frac{k(p-\beta)^2}{(k+p-\beta)^2 \left(\frac{k+p}{p}\right)^n G^p(\alpha, k+1) - (p-\beta)^2} = A(\alpha, \beta, n, k) \quad (k \geqslant 1). \tag{6.10}$$

But

$$A(\alpha, \beta, n, k+1) - A(\alpha, \beta, n, k) = \frac{(p-\beta)^2}{B(\alpha, \beta, n, k)} \left\{ \frac{G^p(\alpha, k+1)}{k+1} H(\alpha, \beta, n, k) + (p-\beta)^2 \right\},$$

where

$$B(\alpha, \beta, n, k) = \left[(k+p-\beta)^2 \left(\frac{k+p}{p}\right)^n G^p(\alpha, k+1) - (p-\beta)^2\right] \cdot \left[(k+p+1-\beta)^2 \left(\frac{k+p+1}{p}\right)^n G^p(\alpha, k+2) - (p-\beta)^2\right] > 0,$$

and

$$H(\alpha, \beta, n, k) = \left[k \left(\frac{k+1+p}{p}\right)^n (k+1+p-\beta)^2 (k+2p-2\alpha) - (k+1)^2 \left(\frac{k+p}{p}\right)^n (k+p-\beta)^2\right].$$

For $0 \leq \alpha \leq \frac{2p-1}{2}$, $0 \leq \beta < p$, $n \in N_0$ and $k \geqslant 1$, we have

$$H(\alpha, \beta, n, k) \geqslant k(k+1) \left(\frac{k+1+p}{p}\right)^n (k+1+p-\beta)^2 - (k+1)^2 \left(\frac{k+p}{p}\right)^n (k+p-\beta)^2 > 0,$$

that is, that $A(\alpha, \beta, n, k)$ is an increasing function of k, $k \geqslant 1$. Hence, we have

$$\gamma \leq A(\alpha, \beta, n, 1) = p - \frac{(p-\beta)^2}{2(p-\alpha)(1+p-\beta)^2 \left(\frac{1+p}{p}\right)^n - (p-\beta)^2}.$$

This completes the proof of Theorem 7.

Remark 1. Putting $n = 0$ in Theorem 7 we obtain the result obtained by Aouf and Silverman [4, Theorem 11].

Remark 2. Putting $n = 1$ in Theorem 7 we obtain the result obtained by Aouf and Salagean [9, Theorem 5].

Theorem 8. *Let the functions $f_\nu(z)$ $(\nu = 1, 2, \ldots, m)$ defined by (6.1) be in the class $R_p[\alpha, \beta, n]$. Then the function*

$$h(z) = z^p - \sum_{k=1}^{\infty} \left(\sum_{\nu=1}^{m} a_{k+p,\nu}^2\right) z^{k+p}, \tag{6.11}$$

is in the class $R_p[\alpha, \eta, n]$, where

$$\eta = \eta(\alpha, \beta, p) = p - \frac{m(p-\beta)^2}{2(p-\alpha)(1+p-\beta)^2 \left(\frac{1+p}{p}\right)^n - m(p-\beta)^2}. \tag{6.12}$$

The result is sharp for the functions $f_\nu(z)$ given by (6.4), $(\nu = 1, 2, \ldots, m)$.

Proof. By virtue of Theorem 1 we have

$$\sum_{k=1}^{\infty}\left\{\left(\frac{k+p}{p}\right)^{n}\left(\frac{k+p-\beta}{p-\beta}\right)G^{p}(\alpha,k+1)\right\}^{2}a_{k+p,\nu}^{2} \leq \left\{\sum_{k=1}^{\infty}\left(\frac{k+p}{p}\right)^{n}\left(\frac{k+p-\beta}{p-\beta}\right)G^{p}(\alpha,k+1)a_{k+p,\nu}\right\}^{2} \leq 1.$$

Then it follows that for $\nu = 1, 2, \ldots, m$,

$$\frac{1}{m}\sum_{k=1}^{\infty}\left\{\left(\frac{k+p}{p}\right)^{n}\left(\frac{k+p-\beta}{p-\beta}\right)G^{p}(\alpha,k+1)\right\}^{2}\left(\sum_{\nu=1}^{m}a_{k+p,\nu}^{2}\right) \leq 1. \tag{6.13}$$

Therefore,we need to find the largest $\eta = \eta(\alpha, \beta, p)$ such that

$$\sum_{k=1}^{\infty}\left(\frac{k+p}{p}\right)^{n}\left(\frac{k+p-\eta}{p-\eta}\right)G^{p}(\alpha,k+1)\left(\sum_{\nu=1}^{m}a_{k+p,\nu}^{2}\right) \leq 1.$$

This implies that

$$\eta \leq p - \frac{mk(p-\beta)^{2}}{\left(\frac{k+p}{p}\right)^{n}(k+p-\beta)^{2}G^{p}(\alpha,k+1) - m(p-\beta)^{2}} = A_{1}(\alpha,\beta,n,k) \quad (k \geqslant 1). \tag{6.14}$$

But

$$A_{1}(\alpha,\beta,n,k+1) - A_{1}(\alpha,\beta,n,k) = \frac{m(p-\beta)^{2}}{B_{1}(\alpha,\beta,n,k)}\left\{\frac{G^{p}(\alpha,k+1)}{k+1}H_{1}(\alpha,\beta,n,k) + m(p-\beta)^{2}\right\},$$

where

$$B_{1}(\alpha,\beta,n,k) = \left[\left(\frac{k+p}{p}\right)^{n}(k+p-\beta)^{2}G^{p}(\alpha,k+1) - m(p-\beta)^{2}\right] \cdot \left[\left(\frac{k+1+p}{p}\right)^{n}(k+1+p-\beta)^{2}G^{p}(\alpha,k+2) - m(p-\beta)^{2}\right] \geqslant 0,$$

and

$$H_{1}(\alpha,\beta,n,k) = k\left(\frac{k+1+p}{p}\right)^{n}(k+1+p-\beta)^{2}(k+2p-2\alpha) - (k+1)^{2}\left(\frac{k+p}{p}\right)^{n}(k+p-\beta)^{2}.$$

For $0 \leq \alpha \leq \frac{2p-1}{2}$, $0 \leq \beta < p$, $n \in N_0$ and $k \geqslant 1$, we have

$$H_{1}(\alpha,\beta,n,k) \geqslant k(k+1)\left(\frac{k+1+p}{p}\right)^{n}(k+1+p-\beta)^{2} - (k+1)^{2}\left(\frac{k+p}{p}\right)^{n}(k+p-\beta)^{2} > 0.$$

Hence, $A_1(\alpha, \beta, n, k)$ is an increasing function of k, $k \geqslant 1$. Setting $k = 1$ in (6.14) we have

$$\eta \leq A_{1}(\alpha,\beta,n,1) = p - \frac{m(p-\beta)^{2}}{2(p-\alpha)(1+p-\beta)^{2}\left(\frac{1+p}{p}\right)^{n} - m(p-\beta)^{2}}.$$

This completes the proof of Theorem 8.

Remark 3. Putting in Theorem 8 (i) $n = 0$, (ii) $p = 1$, we obtain the following results.

Corollary 4. *If the functions $f_\nu(z)(\nu = 1, 2, \ldots, m)$ defined by (6.1) be in the class $R_p[\alpha, \beta, 0] = R_p[\alpha, \beta]$, then the function $h(z)$ defined by (6.11) is in the class $R_p[\alpha, \delta]$, where*

$$\delta = p - \frac{m(p-\beta)^{2}}{2(p-\alpha)(1+p-\beta)^{2} - m(p-\beta)^{2}}. \tag{6.15}$$

The result is sharp.

Corollary 5. *If the functions* $f_\nu(z)$ $(\nu = 1, 2, \ldots, m)$ *defined by* (6.1) *be in the class* $R_1[\alpha, \beta, n] = R[\alpha, \beta, n]$, *then the function* $h(z)$ *defined by* (6.11) *is in the class* $R_1[\alpha, \sigma, n]$, *where*

$$\sigma = 1 - \frac{m(1-\beta)^2}{2^{n+1}(1-\alpha)(2-\beta)^2 - m(1-\beta)^2}. \tag{6.16}$$

The result is sharp.

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