Multipliers with Natural Local Spectra on Commutative Banach Algebras

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Several notions from the abstract spectral theory of bounded linear operators on Banach spaces are investigated and characterized in the context of multipliers on a semi-simple commutative Banach algebra. Particular emphasis is given to the determination of the local spectra of such multipliers in connection with Dunford's property (C), Bishop's property (β), and decomposability in the sense of Foiaş. The strongest results are obtained for regular Tauberian Banach algebras with approximate units and for multipliers whose Gelfand transforms on the spectrum of the Banach algebra vanish at infinity. The general theory is then applied to convolution operators induced by measures on a locally compact abelian group G. Our results give new insight into the spectral theory of convolution operators on the group algebra $L_1(G)$ and on the measure algebra $M(G)$. In particular, we identify large classes of measures for which the corresponding convolution operators have excellent spectral properties. We also obtain a number of negative results such as examples of convolution operators on $L_1(G)$ without natural local spectra, but with natural spectrum in the sense of Zafran.

1. Introduction

In this article, we shall investigate certain decomposition and spectral mapping properties of multipliers on a semi-simple commutative Banach algebra. It will be shown that various weak and strong versions of the basic

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decomposability concept due to Foiaş [14] tend to coincide for this class of operators. Moreover, the spectral decomposition properties of a multiplier will be related to the structure of its spectrum and also to certain continuity and growth conditions for its Gelfand transform on the spectrum of the typically non-regular multiplier algebra. In this connection, the hull-kernel topology will play an important role.

The relevant preliminaries from local spectral theory and the theory of decomposable operators will be collected in Section 2. The central theme of Section 3 will be the discussion of those multipliers for which the local spectra can be calculated most efficiently in terms of the Gelfand transform. Under fairly mild assumptions on the Banach algebra, it turns out that this class of multipliers is characterized by Dunford's property (C) and also by the notion of quasi-decomposability. In Section 4, particularly definitive results will be obtained for the more manageable class of multipliers whose Gelfand transforms on the spectrum of the underlying Banach algebra vanish at infinity. For such multipliers, it will be shown that basically all our conditions from local spectral theory coincide and that they impose a severe restriction on the Gelfand transform.

Most of these results are new even for convolution operators given by measures on a locally compact abelian group \( G \). In Section 5, we shall obtain a variety of new examples and characterizations of decomposable convolution operators on the group algebra \( L_1(G) \) and also considerable new insight into the role of Bishop's property (\( \beta \)) and Dunford's property (C) in the context of convolution operators. Moreover, the question of which measures on \( G \) induce convolution operators on \( L_1(G) \) with natural local spectra will be connected with some interesting problems in harmonic analysis and spectral theory. We shall also obtain examples of convolution operators on \( L_1(G) \) with natural spectrum in the sense of Zafran [32], but without natural local spectra.

Finally, in Section 6, the local spectral properties of a multiplier will be related to those of the corresponding multiplication operator on the multiplier algebra. In this context, bounded approximate identities will be an important tool. In particular, we shall exhibit examples of multiplication operators on measure algebras without natural local spectra. The present note is very much in the spirit of our recent papers [20–22], where the interested reader will find further information and references.

2. Local Spectral Theory

We first recall some basic notions from the axiomatic spectral theory of linear operators on Banach spaces; further information may be obtained from the monographs [8] and [31]. Given a complex Banach space \( X \) and
the Banach algebra \( \mathcal{L}(X) \) of all bounded linear operators on \( X \), an operator \( T \in \mathcal{L}(X) \) is called decomposable if, for every open covering \( \{ U_1, U_2 \} \) of the complex plane \( \mathbb{C} \), there exists a pair of \( T \)-invariant closed linear subspaces \( Y_1 \) and \( Y_2 \) of \( X \) such that \( Y_1 + Y_2 = X \) and \( \sigma(T) Y_k \subseteq U_k \) for \( k = 1, 2 \) where \( \sigma \) denotes the spectrum, cf. [3, 31]. If it is only required that the sum \( Y_1 + Y_2 \) be dense in \( X \), one obtains the definition of the weak 2-spectral decomposition property (weak 2-SDP), cf. [11, 12]. Although it is known from [2] that, in general, this condition in strictly weaker than decomposability, we shall see that for certain classes of multipliers the weak 2-SDP implies decomposability.

We shall also need the following closely related notions. An operator \( T \in \mathcal{L}(X) \) is said to have Bishop’s property (\( \beta \)) if, for every open subset \( U \) of \( \mathbb{C} \) and for every sequence of analytic functions \( f_n : U \to X \) for which \((T - \lambda) f_n(\lambda) \) converges uniformly to zero on each compact subset of \( U \), it follows that \( f_n(\lambda) \to 0 \) as \( n \to \infty \), uniformly on each compact subset of \( U \), cf. [6]. Obviously, property (\( \beta \)) implies that \( T \) has the single valued extension property, which means that, for every open \( U \subseteq \mathbb{C} \), the only analytic solution \( f : U \to X \) of the equation \((T - \lambda) f(\lambda) = 0 \) for all \( \lambda \in U \) is the constant \( f \equiv 0 \), cf. [8]. Finally, an operator \( T \in \mathcal{L}(X) \) is said to have the decomposition property (\( \delta \)) if, given an arbitrary open covering \( \{ U_1, U_2 \} \) of \( \mathbb{C} \), every \( x \in X \) admits a decomposition \( x = u_1 + u_2 \) where the vectors \( u_1, u_2 \in X \) satisfy \( u_k = (T - \lambda) f_k(\lambda) \) for all \( \lambda \in \mathbb{C} \setminus U_k^- \) and some analytic function \( f_k : \mathbb{C} \setminus U_k^- \to X \) for \( k = 1, 2 \); cf. [5]. Note that the unilateral left shift on the Hilbert space \( l^2 \) has property (\( \delta \)), but not the weak 2-SDP, cf. [23].

It is easily seen that an operator \( T \in \mathcal{L}(X) \) is decomposable if and only if it has both properties (\( \beta \)) and (\( \delta \)). Moreover, it has recently been shown in [5] that the properties (\( \beta \)) and (\( \delta \)) are dual to each other: an operator \( T \in \mathcal{L}(X) \) satisfies (\( \beta \)) if and only if the adjoint operator \( T^* \) on the dual space \( X^* \) satisfies (\( \delta \)), and the corresponding statement remains valid if both properties are interchanged. It has also been shown in [5] that an operator \( T \in \mathcal{L}(X) \) has property (\( \beta \)) if and only if \( T \) is similar to the restriction of a decomposable operator to one of its closed invariant subspaces and that \( T \) has property (\( \delta \)) if and only if \( T \) is similar to a quotient of a decomposable operator. These characterizations have been very useful in recent work in the spirit of Scott Brown on the invariant subspace problem for operators on Banach spaces; see for instance [13]. Here we shall investigate properties (\( \beta \)) and (\( \delta \)) in the context of multipliers.

Given an arbitrary operator \( T \in \mathcal{L}(X) \), let \( \sigma_T(x) \subseteq \mathbb{C} \) denote the local spectrum of \( T \) at the point \( x \in X \), i.e. the complement of the set \( p_T(x) \) of all \( \lambda \in \mathbb{C} \) for which there exist an open neighborhood \( U \) of \( \lambda \) in \( \mathbb{C} \) and an analytic function \( f : U \to X \) such that \((T - \mu)f(\mu) = x \) holds for all \( \mu \in U \), cf. [3, 31]. For every closed subset \( F \) of \( \mathbb{C} \), let \( X_T(F) := \{ x \in X : \sigma_T(x) \subseteq F \} \) denote the corresponding analytic spectral subspace of \( T \). If \( T \) has the single
valued extension property, then property \((\delta)\) for \(T\) means precisely that 
\[
X = X_T(U^-) + X_T(V^-)
\]
for every open covering \(\{U, V\}\) of \(C\).

Finally, recall that an operator \(T \in \mathcal{L}(X)\) is said to have Dunford’s property \((C)\) if \(X_T(F)\) is closed for every closed \(F \subseteq \mathbb{C}\). This condition plays an important role in the theory of spectral operators, cf. [10]. It is known that Bishop’s property \((\beta)\) implies Dunford’s property \((C)\) and that property \((C)\) implies the single valued extension property, cf. Proposition 1.2 of [23]. It is, however, an intriguing open problem whether \((\beta)\) and \((C)\) are equivalent in general. Note that it follows from Proposition 1.3.8 of [8] that an operator is decomposable if and only if it has both properties \((C)\) and \((\delta)\).

3. Multipliers on Semi-simple Commutative Banach Algebras

Throughout this section, let \(A\) denote a semi-simple commutative complex Banach algebra with or without identity, and let \(\mathcal{M}(A)\) stand for the spectrum of \(A\), i.e. the set of all non-trivial multiplicative linear functionals on \(A\). For each \(a \in A\), let \(a^\wedge: \mathcal{M}(A) \to \mathbb{C}\) denote the corresponding Gelfand transform given by \(a^\wedge(\varphi) := \varphi(a)\) for all \(\varphi \in \mathcal{M}(A)\). On \(\mathcal{M}(A)\) we shall consider both the usual Gelfand topology and the hull-kernel topology, which is determined by the Kuratowski closure operation \(\text{cl}(E) := \text{hul}(\ker(E))\), i.e.,

\[
\text{cl}(E) = \{\psi \in \mathcal{M}(A): \psi(u) = 0 \text{ for all } u \in A \text{ with } \varphi(u) = 0 \text{ for each } \varphi \in E\}
\]

for all \(E \subseteq \mathcal{M}(A)\), cf. [7] and [28]. The hull-kernel topology is always coarser than the Gelfand topology, and they coincide if and only if the algebra \(A\) is regular. Therefore, for some \(a \in A\) the Gelfand transform \(a^\wedge\) will not be hull-kernel continuous on \(\mathcal{M}(A)\) whenever the algebra \(A\) is non-regular. Moreover, if \(A\) has an identity, then the hull-kernel topology on \(\mathcal{M}(A)\) is Hausdorff if and only if \(A\) is regular.

We shall also need some basic results from the theory of multipliers on Banach algebras as presented, for instance, in [16]. Recall that a mapping \(T: A \to A\) is called a multiplier on \(A\) if \(T(u) v = u T(v)\) holds for all \(u, v \in A\). By semi-simplicity, every multiplier is a continuous linear operator which satisfies \(T(uv) = T(u) v\) for all \(u, v \in A\) and has the single valued extension property, cf. Proposition 6.2.3 of [8]. Moreover, the set \(\mathcal{M}(A)\) of all multipliers on \(A\) is a semi-simple commutative closed subalgebra of \(\mathcal{L}(A)\) which contains the identity operator \(I\) and the algebra \(A\), where each \(a \in A\) is, of course, identified with the corresponding multiplication operator \(L_a: A \to A\) given by \(L_a(u) := au\) for all \(u \in A\). Note that the norm of \(A\) may be strictly greater than the operator norm inherited from \(\mathcal{M}(A)\) and that \(A\) need not be closed in \(\mathcal{M}(A)\).
By the theory developed in Section 1.4 of [16], the spectrum $\Delta(M(A))$ of the multiplier algebra $M(A)$ may be represented as the disjoint union of $\Delta(A)$ and $\text{hul}(A)$, where $\Delta(A)$ is canonically embedded in $\Delta(M(A))$ and $\text{hul}(A)$ denotes the hull of $A$ in $\Delta(M(A))$. When $\Delta(A)$ is regarded as a subset of $\Delta(M(A))$, the hull-kernel topology of $\Delta(A)$ coincides with the relative hull-kernel topology induced by $\Delta(M(A))$, and the same result holds with respect to the Gelfand topology. Obviously, $\Delta(A)$ is hull-kernel and hence Gelfand open in $\Delta(M(A))$. Moreover, by semi-simplicity, $\Delta(A)$ is always hull-kernel dense in $\Delta(M(A))$, but this is certainly not true for the Gelfand topology in general. Next let:

$$M_{\partial}(A) := \{ T \in M(A) : T^\wedge | \Delta(A) \text{ vanishes at infinity in the Gelfand topology of } \Delta(A) \}$$

$$M_{\partial\partial}(A) := \{ T \in M(A) : T^\wedge \equiv 0 \text{ on } \text{hul}(A) \} = \text{ker}(\text{hul}(A))$$

Clearly, $M_{\partial}(A)$ and $M_{\partial\partial}(A)$ are closed ideals in $M(A)$ with $A \subseteq M_{\partial\partial}(A) \subseteq M_{\partial}(A)$. It follows from elementary Gelfand theory that the spectrum of each multiplier $T \in M(A)$ is given by $\sigma(T) = T^\wedge (\Delta(M(A)))$. Borrowing a term from harmonic analysis [32], we shall say that a multiplier $T \in M(A)$ has natural spectrum if $\sigma(T) = T^\wedge (\Delta(M(A)))^-$. This condition is certainly fulfilled if $T^\wedge$ is hull-kernel continuous on $\Delta(M(A))$. Also, every multiplier in $M_{\partial\partial}(A)$ has natural spectrum, but, as we shall see shortly, it imposes a severe restriction on a multiplier to have natural spectrum; see also [1, 21, 22, 25] for some related results.

Of particular interest is the case of the group algebra $A = L_1(G)$ for an arbitrary locally compact abelian group $G$. In this case, the spectrum $\Delta(A)$ is given by the dual group $G^\wedge$, and the multiplier algebra $M(A)$ may be canonically identified with the measure algebra $M(G)$ via convolution, cf. [16]. The systematic investigation of convolution operators with natural spectrum dates back to Zafran [32]. In particular, Zafran observed that, except for the trivial case of a discrete group, there are always multipliers in $M_{\partial}(L_1(G))$ with non-natural spectrum; see also Proposition 12 below. This fact is, of course, related to the inversion problem for measures in $M(G)$ from classical Fourier analysis, cf. Chapter 8 of [15]. In the following, we shall obtain further information about multipliers with natural spectrum.

**Proposition 1.** Assume that the multiplier $T \in M(A)$ has the weak 2-SDP. Then $T|_{\Delta(A)}$ is hull-kernel continuous on $\Delta(A)$ and $\sigma(T) = T^\wedge (\Delta(A))^-$.  

**Proof.** Suppose that the restriction $T^\wedge | \Delta(A)$ is not hull-kernel continuous on $\Delta(A)$. Then there exists a closed subset $F$ of $C$ such that
observer that functional A obtained in [22]. Note that, even in the case of the group algebra AT of non-decomposable multipliers which do have natural spectrum. This result, we collect some elementary facts on natural local spectra. The multiplier T in [23]. But it is trivial that A has natural spectrum. Let \( T \) be an arbitrary A, and A T has natural local spectra if and only if \( T \vdash (\text{supp}_x) \vdash \sigma_T(x) \) holds for all \( x \in A \), where \( \text{supp}_x \vdash \vdash \) denotes the closure of \( \{ \varphi \in A(A) : \varphi \vdash \varphi \neq 0 \} \) in the Gelfand topology of A(A). In the following result, we collect some elementary facts on natural local spectra.

**Proposition 2.** Let \( T \in M(A) \). Then \( T \vdash (\text{supp}_x) \vdash \vdash \sigma_T(x) \) for all \( x \in A \) and \( A_T(F) \vdash Z_T(F) \) for all closed \( F \subseteq C \) where \( Z_T(F) := \{ x \in A : T \vdash (\text{supp}_x) \vdash \vdash F \} \). Moreover, \( T \) has natural local spectra if and only if \( A_T(F) \vdash Z_T(F) \) for all closed \( F \subseteq C \). Finally, every multiplier with natural local spectra has natural spectrum.
Proof. Let \( \lambda \in \mathbb{C} \setminus \sigma_f(x) \) and assume that \( \lambda = T^\wedge(\varphi) \) for some \( \varphi \in \mathcal{A}(A) \). Since \( (T - \lambda) u = x \) for some \( u \in A \), we obtain \( 0 = (T^\wedge - \lambda) u^\wedge(\varphi) = x^\wedge(\varphi) \). This shows that \( T^\wedge(\{ \varphi \in \mathcal{A}(A) : x^\wedge(\varphi) \neq 0 \}) \subseteq \sigma_f(x) \) and hence \( T^\wedge(\text{supp } x^\wedge) \subseteq \sigma_f(x) \) by the Gelfand continuity of \( T^\wedge \) on \( \mathcal{A}(A) \). In particular, it follows that the inclusion \( A_f(F) \subseteq Z_f(F) \) holds in general. If \( T \) has natural local spectra, then obviously \( A_f(F) = Z_f(F) \) for all closed \( F \subseteq \mathbb{C} \). The converse follows by considering closed sets of the form \( F = T^\wedge(\text{supp } x^\wedge)^- \). Finally recall that all multipliers on semi-simple Banach algebras have the single valued extension property, cf. \( \cite{8} \). Hence Theorem 1.9 of \( \cite{11} \) shows that \( \sigma(T) \) is the union of the local spectra \( \sigma_f(x) \) over all \( x \in A \), from which the last assertion is immediate.

Given a multiplier \( T \in M(A) \) and a closed subset \( F \subseteq \mathbb{C} \), it is clear that \( Z_f(F) \) is always a closed ideal in \( A \). Because of \( \sigma_f(uT) \subseteq \sigma_f(u) \cap \sigma_f(T) \) for all \( u, v \in A \), the space \( A_f(F) \) is also an ideal in \( A \), but not necessarily closed. Standard examples of multipliers without natural spectrum in the group algebra setting \( \cite{32} \) show that \( A_f(F) \) and \( Z_f(F) \) need not coincide. However, under rather mild conditions on \( A \), we shall show that these ideals are always very near in the sense that \( A_f(F) \) is dense in \( Z_f(F) \).

**Proposition 3.** Assume that \( A \) is regular. Then \( \sigma_f(x) = T^\wedge(\text{supp } x^\wedge) \) holds for all multipliers \( T \in M(A) \) and all \( x \in A \) for which \( \text{supp } x^\wedge \) is compact in \( \mathcal{A}(A) \). Moreover, \( \text{hul}(A_f(F)) = \text{hul}(Z_f(F)) = (T^\wedge \mathbb{C} \setminus F)^- \) for all closed \( F \subseteq \mathbb{C} \).

**Proof.** Let \( x \in A \) such that \( \text{supp } x^\wedge \) is compact in \( \mathcal{A}(A) \). By the regularity of \( A \), there exists an \( e \in A \) so that \( e^\wedge \equiv 1 \) on \( \text{supp } x^\wedge \). Clearly \( J := \{ u \in A : \text{supp } u^\wedge \subseteq \text{supp } x^\wedge \} \) is a closed ideal in \( A \) which is invariant under all multipliers in \( M(A) \) and hence an ideal in \( M(A) \). Moreover, the multiplier \( S := T(J \in \vee J) \) satisfies \( \sigma_f(x) \subseteq \sigma_s(x) \subseteq \sigma(S) \) and \( S = L_e(\infty) \) with the choice \( e := T^\wedge(1) \in A \). Since \( A \) is regular, we conclude from Theorems 3 and 4 of \( \cite{25} \) that \( S \) is decomposable on \( J \) as the restriction of a decomposable multiplication operator on \( A \). By Proposition 1, this implies that \( \sigma(S) = S^\wedge(\mathcal{A}(A))^- \). But from the definition of \( J \) it is easily seen that \( \mathcal{A}(J) \subseteq \text{supp } x^\wedge \), where \( \mathcal{A}(J) \) is canonically embedded in \( \mathcal{A}(A) \) in the sense of Theorem 2.66 of \( \cite{28} \). An obvious combination of these results yields \( \sigma_f(x) \subseteq \sigma(S) \subseteq a^\wedge(\text{supp } x^\wedge) = T^\wedge e^\wedge(\text{supp } x^\wedge) = T^\wedge(\text{supp } x^\wedge) \), since \( e^\wedge \equiv 1 \) on \( \text{supp } x^\wedge \). By the first part of Proposition 2, we conclude that \( \sigma_f(x) = T^\wedge(\text{supp } x^\wedge) \). Finally, given an arbitrary closed \( F \subseteq \mathbb{C} \), we observe that the inclusions \( \text{hul}(A_f(F)) \supseteq \text{hul}(Z_f(F)) \supseteq (T^\wedge \mathbb{C} \setminus F)^- \) follow immediately from the definition of a hull. Conversely, let \( \varphi \in \mathcal{A}(A) \setminus (T^\wedge \mathbb{C} \setminus F)^- \). Then, by the regularity of \( A \), there exists some \( x \in A \) such that \( x^\wedge(\varphi) = 1 \) and that \( \text{supp } x^\wedge \) is compact and disjoint from \( T^\wedge(\mathbb{C} \setminus F)^- \). By the first
part of the proof, we obtain $\sigma_T(x) = T^\wedge (\text{supp } x^\wedge) \subseteq F$ and therefore $x \in A_T(F)$. We conclude that $\varphi \notin \text{hul}(A_T(F))$, which completes the proof.

For regular $A$, it follows that $A_T(F)$ is dense in $Z_T(F)$ whenever $(T^\wedge)^{-1}(C \setminus F)^-$ is a set of spectral synthesis in $A(A)$. This criterion is appropriate for the class of $N$-algebras [28], but not for group algebras because of the problem of spectral synthesis: it is known that the algebra $A = L_1(G)$ for a locally compact abelian group $G$ is an $N$-algebra if and only if $G$ is compact, cf. [30]. On the other hand, we shall see that, for certain algebras, it suffices to know that the empty set is of spectral synthesis. This means exactly that $A$ is Tauberian in the sense that the set of all $x \in A$ with compact $\text{supp } x^\wedge$ is norm dense in $A$. Also, recall that $A$ is said to have approximate units if for each $x \in A$ and $\varepsilon > 0$ there exists some $u \in A$ such that $\|x - ux\| < \varepsilon$. This is the weakest notion in the hierarchy of reasonable approximate identity conditions. Note that the group algebra $A = L_1(G)$ for an arbitrary locally compact abelian group $G$ satisfies all the assumptions on $A$ in the following result, cf. [30].

**Proposition 4.** Assume that $A$ is regular and Tauberian and has approximate units. Then, for each $T \in M(A)$, we have

$$A_T(F) \subseteq E_T(F) \subseteq \bigcap_{\lambda \in C \setminus F} (T - \lambda) A \subseteq Z_T(F) = A_T(F)^-$$

for all closed $F \subseteq C$, where $E_T(F)$ denotes the largest linear subspace $Y$ of $A$ with the property that $(T - \lambda) Y = Y$ for all $\lambda \in C \setminus F$.

**Proof.** It suffices to show that $Z_T(F)$ is contained in $A_T(F)^-$, since all the other inclusions can be easily verified even without any additional assumption on $A$. Given an arbitrary $x \in Z_T(F)$ and $\varepsilon > 0$, we choose some $u \in A$ so that $\|x - ux\| < \varepsilon$. Next, since $A$ is Tauberian, there exists a $v \in A$ such that $\text{supp } v^\wedge$ is compact and $\|v - e\| < \varepsilon/(1 + |x|)$. Obviously, $\|x - vx\| < 2 \varepsilon$ and $vx \in Z_T(F)$. Moreover, $\text{supp } (vx)^\wedge$ is compact which, by Proposition 3, implies that $\sigma_T(vx) = T^\wedge(\text{supp } (vx)^\wedge) \subseteq F$ and therefore $vx \in A_T(F)$. We conclude that $x \in A_T(F)^-$, as desired.

An operator $T \in \mathfrak{L}(X)$ on a complex Banach space $X$ is called quasi-decomposable if $X_T(F)$ is closed for each closed $F \subseteq C$ and if, for every finite open covering $\{U_1, \ldots, U_n\}$ of $C$, the sum $X_T(U_1) + \cdots + X_T(U_n)$ is dense in $X$. Since Dunford’s property $(C)$ implies the single valued extension property [23], it follows from Proposition 1.3.8 of [8] that the preceding simple definition is equivalent to the one given in [11]. Also, it is clear that all quasi-decomposable operators have the weak 2-SDP, but they need not be decomposable as shown by the example given in [2]. For multipliers we obtain the following characterization.
Assume that $A$ is regular and Tauberian, then a multiplier on $A$ is quasi-decomposable if and only if it has Dunford’s property (C).

Proof. To show that any $T \in M(A)$ with property (C) is quasi-decomposable, let $\{U_1, \ldots, U_n\}$ be an arbitrary open covering of $\mathbb{C}$, and consider an $x \in A$ such that $\operatorname{supp} x^\wedge$ is compact in $A(A)$. By regularity, there exist $e_1, \ldots, e_n \in A$ such that $e_1^\wedge + \cdots + e_n^\wedge \equiv 1$ on $\operatorname{supp} x^\wedge$ and $\operatorname{supp} e_k^\wedge \subseteq T^{\sim -1}(U_k)$ for $k = 1, \ldots, n$. Since $\operatorname{supp} x^\wedge$ and hence $\operatorname{supp} e_k^\wedge x^\wedge$ is compact, we conclude from Proposition 3 that $\sigma_T(e_k x) = T^{\sim -1}(U_k)$ and therefore $e_k x \in A_T(U_k^\wedge)$ for $k = 1, \ldots, n$. Moreover, by semi-simplicity, it follows that $x = e_1 x + \cdots + e_n x$ and hence $x \in A_T(U_1^\wedge) + \cdots + A_T(U_n^\wedge)$. Since $A$ is Tauberian, we conclude that $A_T(U_1^\wedge) + \cdots + A_T(U_n^\wedge)$ is dense in $A$, which proves that $T$ is quasi-decomposable.

Finally, an operator $T \in \mathfrak{L}(X)$ on a Banach space $X$ is said to be admissible if, for each closed $F \subseteq \mathbb{C}$, the algebraic spectral subspace $E_T(F)$ is closed. Such operators arise naturally in the automatic continuity theory for intertwining linear transformations, cf. [17, 20, 24]. It follows from Proposition 1.5 of [19] that all super-decomposable operators without divisible subspaces are admissible; recall that the operator $T \in \mathfrak{L}(X)$ is said to be super-decomposable if, for every open covering $\{U, V\}$ of $\mathbb{C}$, there exists some operator $R \in \mathfrak{L}(X)$ such that $RT = TR$, $\sigma(R \cap \mathcal{R}(X)^\circ \cap V)$, and $\sigma(T)(I - R \cap \mathcal{R}(X)^\circ \cap V) \subseteq V$. On the other hand, by Proposition 3.4 of [17], every admissible $T \in \mathfrak{L}(X)$ satisfies $X_T(F) = E_T(F)$ for all closed $F \subseteq \mathbb{C}$, which shows that all admissible operators have Dunford’s property (C).

The following theorem is the main result of this section.

Theorem 6. Assume that $A$ is regular and Tauberian and has approximate units. Then, for each $T \in M(A)$, the following assertions are equivalent:

1. $T$ has natural local spectra.
2. $T$ has Dunford’s property (C).
3. $T$ is quasi-decomposable.
4. $T$ is admissible.

The proof of this result follows from an obvious combination of Propositions 2, 4, 5 and Proposition 3.4 of [17]. If $A$ is regular and Tauberian and has approximate units, then it is clear from Theorem 6 and Proposition 1 that the following chain of implications

property (β) ⇒ natural local spectra ⇒ weak 2-SDP ⇒ natural spectrum
holds for an arbitrary multiplier $T \in M(A)$. Under certain additional assumptions on $A$ and $T$, we shall prove that all these conditions are equivalent, but it is an interesting open question which of these implications (if any) can be reversed in general. Again, the case of group algebras should be of particular interest in this context. We shall see in Section 5 that, in this case, there exist examples of multipliers with natural spectrum, but without natural local spectra. Another intriguing open problem is whether all decomposable multipliers on an arbitrary semi-simple commutative Banach algebra have natural local spectra. If a multiplier $T \in M(A)$ satisfies the slightly stronger assumption that left multiplication by $T$ on $M(A)$ is decomposable, then it follows from Theorems 2.3 and 2.4 of [21] that $T$ has indeed natural local spectra, but we do not know the answer in general. The following result provides a positive solution for algebras with approximate units. Some related results will be obtained in Section 6.

**Proposition 7.** Assume that $A$ has approximate units and that the multiplier $T \in M(A)$ has the weak 2-SDP. Then $T$ has natural local spectra if and only if $T$ has Dunford’s property (C). In particular, every quasi-decomposable $T \in M(A)$ has natural local spectra.

**Proof.** It remains to be shown that a multiplier $T \in M(A)$ with the weak 2-SDP and property (C) has natural local spectra. Given an arbitrary closed $F \subseteq \mathbb{C}$ and an open neighborhood $U$ of $F$, we choose an open subset $V$ of $C$ such $U \cap V = \varnothing$ and $F \cap \overline{V} = \varnothing$. Now, let $x \in Z_T(F)$ and $\varepsilon > 0$. Since $A$ has approximate units, there is an $e \in A$ so that $\|x - ex\| < \varepsilon$. Next, from the weak 2-SDP we obtain some $e_U \in A_T(U^-)$ and some $e_V \in A_T(V^-)$ such that $\|e - e_U - e_V\| < \varepsilon((1 + \|x\|)$. Because of $F \cap \overline{V} = \varnothing$, we have $e_U x \in Z_T(F) \cap A_T(V^-) \subseteq Z_T(F) \cap Z_T(V^-) = \{0\}$ and therefore

$$\|x - e_U x\| = \|x - e_U x - e_V x\| \leq \|x - ex\| + \|e - e_U - e_V\| < 2\varepsilon.$$ 

Since $e_U x \in A_T(U^-)$ and $A_T(U^-)$ is closed, we conclude that $x \in A_T(U^-)$. Intersection over all open neighborhoods $U$ of $F$ leads to $x \in A_T(F)$, hence $A_T(F) = Z_T(F)$ for all closed $F \subseteq \mathbb{C}$. By Proposition 2, it follows that $T$ has natural local spectra.

It is possible to describe the condition of having natural local spectra solely in terms of the closed ideals $Z_T(F)$. For instance, it follows easily from Proposition 2 and Proposition 1.3.8 of [8] that a multiplier $T \in M(A)$ has natural local spectra if and only if $\sigma(T|Z_T(F)) \subseteq F$ for all closed $F \subseteq \mathbb{C}$. Moreover, if the multiplier $T \in M(A)$ is super-decomposable [19], then a slight modification of the proof of Proposition 7 shows that $T$ has natural local spectra if and only if, for each closed $F \subseteq \mathbb{C}$, the space $Z_T(F)$ is hyper-invariant for $T$. 
For the sake of illustration, we close this section with an application to isometric multipliers. The following result can also be obtained without the specific tools from local spectral theory; a different and somewhat more elementary proof has recently been communicated to us by T. J. Ransford. Note that this result ceases to be true for non-regular algebras, as can be easily seen by shift operators acting on convolution algebras on the half line $\mathbb{R}_+$, cf. [9].

**Proposition 8.** Assume that $A$ is regular and Tauberian. Then every isometric multiplier $T \in \mathbb{M}(A)$ is surjective and hence decomposable.

**Proof.** By Proposition 1.19 of [18], every isometry has property (B) and therefore property (C). By Propositions 1 and 5, it follows that $T$ has natural spectrum and hence satisfies $T^\sim(A(A)) = \sigma(T) \subseteq \mathbb{D}$ where $\mathbb{D}$ denotes the closed unit disc. Suppose that $|T^\sim(\varphi)| < 1$ for some $\varphi \in A(A)$ and choose an $r$ and a compact neighborhood $U$ of $\varphi$ in $\mathcal{A}(A)$ so that $|T^\sim| < r < 1$ on $U$. By regularity, there exists some $a \in A$ such that $a^\sim(\varphi) = 1$ and $\text{supp } a^\sim \subseteq U$. Since $\text{supp } a^\sim$ is compact, we obtain from Proposition 3 that $\sigma_T(a) = T^\sim(\text{supp } a^\sim) \subseteq T^\sim(U)$ and hence $a \in A_T(F)$ where $F := \{z \in \mathbb{C} : |z| < r\}$. On the other hand, since $A_T(F)$ is closed, Proposition 1.3.8 of [8] yields $\sigma(T|A_T(F)) \subseteq F$. Since $r < 1$ and $T|A_T(F)$ is an isometry, this implies that $A_T(F) = \{0\}$ and therefore $a = 0$, which is impossible because of $a^\sim(\varphi) = 1$. This contradiction shows that $\sigma(T)$ is contained in the unit circle and hence that $T$ is invertible. Finally, it is well-known that all invertible isometries are generalized scalar and therefore decomposable; see for instance Corollary 4.6 of [19] or Corollary 4.2 of [24].

4. **The Case of Multipliers in $\mathbb{M}_0(A)$**

Again, let $A$ be a semi-simple commutative Banach algebra over $\mathbb{C}$ with or without identity. In this section, we shall discuss the spectral properties of multipliers whose Gelfand transforms on $\mathcal{A}(A)$ vanish at infinity. Under certain mild assumptions on $A$, a sharp dichotomy will be established for this class of multipliers: inside $\mathbb{M}_0(A)$, we shall obtain very nice spectral properties, whereas multipliers in $\mathbb{M}_0(A) \setminus \mathbb{M}_0(A)$ will not satisfy any of our spectral conditions.

**Theorem 9.** For each $T \in \mathbb{M}_0(A)$, the following assertions are equivalent:

(a) $T$ has natural local spectra and $T^\sim | A(A)$ is hull-kernel continuous on $\mathcal{A}(A)$.
(b) \( T \in \mathcal{M}_{\Omega}(A) \) and \( T^* \mid \mathcal{A}(A) \) is hull-kernel continuous on \( \mathcal{A}(A) \).
(c) \( T^* \) is hull-kernel continuous on \( \mathcal{A}(\mathcal{M}(A)) \).
(d) \( L_z : \mathcal{M}(A) \to \mathcal{M}(A) \) is decomposable.
(e) \( T : A \to A \) is super-decomposable.
(f) \( T : A \to A \) has property \((\delta)\).
(g) \( T : A \to A \) is decomposable.

Proof. The equivalence of the conditions \((b)-(g)\) follows immediately from a combination of Theorem 3.2 of [19], Theorem 2.3 of [21], and Theorem 8 of [22]. Moreover, the implication \((c) \Rightarrow (a)\) is a consequence of Theorem 2.4 of [21] and the standard fact that the hull-kernel topology on \( \mathcal{A}(A) \) coincides with the relative hull-kernel topology of \( \mathcal{A}(\mathcal{M}(A)) \), see Theorem I.43 of [16]. It remains to be shown that \((a)\) implies \((g)\). As noted earlier, by Proposition 2 and Proposition 1.3.8 of [8], the assumption of natural local spectra implies that \( \mathcal{A}_T(F) = Z_T(F) \) and \( \sigma(T) \) coincides with \( Z_T(F) \) for every closed \( F \subseteq \mathbb{C} \). Therefore, it suffices to prove the decomposition \( A = Z_T(U^-) + Z_T(V^-) \) for an arbitrary open covering \( \{ U, V \} \) of \( \mathbb{C} \). Without loss of generality, we may assume that \( 0 \in V \). Then there exists an \( \varepsilon > 0 \) such that \( z \in V \) whenever \( |z| < \varepsilon \). Since \( T^* \mid \mathcal{A}(A) \) vanishes at infinity, we conclude that \( T^* \mid \mathcal{A}(\mathcal{M}(A)) \) is Gelfand compact in \( \mathcal{A}(A) \). Moreover, since \( T^* \mid \mathcal{A}(A) \) is hull-kernel continuous on \( \mathcal{A}(A) \), the sets \( T^{-1}(C \setminus U) \) and \( T^{-1}(C \setminus V) \) are disjoint hulls in \( \mathcal{A}(A) \). Hence, by Corollary 3.6.10 of [28], there exists some \( e \in A \) such that \( e^* \equiv 1 \) on \( T^{-1}(C \setminus V) \) and \( e^* \equiv 0 \) on \( T^{-1}(C \setminus U) \). This implies that \( \text{supp } e^* \subseteq T^{-1}(U^-) \) and \( \text{supp}(1 - e^*) \subseteq T^{-1}(V^-) \). Consequently, for arbitrary \( x \in A \), we obtain the decomposition \( x = xe + (x - xe) \) with \( xe \in Z_T(U^-) \) and \( x - xe \in Z_T(V^-) \), which proves that \( T \) is decomposable on \( A \).

The preceding theorem shows that decomposable multipliers in \( \mathcal{M}_{\Omega}(A) \) always have natural local spectra. Moreover, if \( A \) is regular, then the converse holds. Indeed, in this case, the hull-kernel topology coincides with the Gelfand topology on \( \mathcal{A}(A) \) so that, for each \( T \in \mathcal{M}(A) \), the restriction \( T^* \mid \mathcal{A}(A) \) is automatically hull-kernel continuous on \( \mathcal{A}(A) \). Hence, for regular \( A \), it follows from Theorem 9 that a multiplier \( T \in \mathcal{M}_{\Omega}(A) \) has natural local spectra if and only if \( T \) is decomposable on \( A \) and that this happens precisely when \( T \) belongs to \( \mathcal{M}_{\Omega}(A) \). Under suitable additional assumptions, this list of equivalences can be extended as follows.

Corollary 10. Assume that \( A \) is regular and Tauberian and has approximate units. Then, for each \( T \in \mathcal{M}_{\Omega}(A) \), the following conditions are equivalent: decomposability, super-decomposability, quasi-decomposability, property \((\delta)\), Dunford’s property \((C)\), Bishop’s property \((\beta)\), natural local spectra.
spectra, admissibility, and hull-kernel continuity of $T^*$ on $\mathcal{A}(M(A))$. Moreover, a multiplier $T \in M_{\delta}(A)$ satisfies these equivalent conditions exactly when it belongs to $M_{oo}(A)$.

Since all decomposable operators have Bishop’s property ($\beta$) and since ($\beta$) implies Dunford’s property ($C$), the proof of Corollary 10 follows immediately from Theorems 6 and 9. Note that regularity of $A$ is an appropriate assumption for the equivalences of the preceding result to hold. For instance, if $A$ is the disc algebra, i.e. the algebra of all continuous functions on the closed unit disc $\mathbb{D}$ which are analytic in the interior of $\mathbb{D}$, then all multiplies on $A$ have property ($\beta$), whereas only those multipliers, which are multiples of the identity operator, have property ($\delta$), cf. [22].

Also, it will follow from Theorem 19 below that, on measure algebras, there exist multiplication operators without natural local spectra.

Further characterizations of multipliers in $M_{\delta}(A)$ with natural local spectra can be obtained when topological assumptions are imposed on the spectrum $\mathcal{A}(A)$. Recall that a locally compact Hausdorff space $\Omega$ is said to be scattered (or dispersed) if every non-empty compact subset of $\Omega$ contains an isolated point, cf. [27] and [29]. Clearly, every discrete space is scattered and every scattered space is totally disconnected. Moreover, the results of [27] and [29], when applied to the one-point compactification of $\Omega$, lead to the following characterization: the space $\Omega$ is scattered if and only if every continuous complex-valued function on $\Omega$ that vanishes at infinity has countable range. Also, recall that an operator $T \in M(X)$ on a complex Banach space $X$ is said to be a Riesz operator if, for each $\lambda \in \mathbb{C}\setminus\{0\}$, the dimension of the kernel $\ker(T - \lambda)$ and the codimension of the range $(T - \lambda)X$ in $X$ are both finite. The following result extends Theorem 3.1 of [21].

**Corollary 11.** Assume that $\mathcal{A}(A)$ is scattered in the Gelfand topology. Then, for each $T \in M_{\delta}(A)$, the following statements are equivalent:

(a) $T$ is decomposable.
(b) $T$ is quasi-decomposable.
(c) $T$ has the weak 2-SDP.
(d) $T$ has property ($\delta$).
(e) $T$ has natural local spectra.
(f) $T$ has natural spectrum.
(g) $T$ has countable spectrum.
(h) $T \in M_{oo}(A)$.

Moreover, if $\mathcal{A}(A)$ is discrete in the Gelfand topology, then a multiplier $T \in M_{\delta}(A)$ satisfies these equivalent conditions if and only if $T: A \to A$ is a Riesz operator.
Proof. As already noted in [21], the Shilov idempotent theorem [7] implies that a semi-simple commutative Banach algebra with totally disconnected maximal ideal space has to be regular. Since \( A(A) \) is scattered and hence totally disconnected, we conclude that \( A \) is regular. Hence Theorem 9 shows that \( (a) \Leftrightarrow (d) \Leftrightarrow (c) \Leftrightarrow (h) \). Moreover, the implications \( (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (f) \) is contained in Proposition 1. Next observe that \( T^\wedge(A(A))^{-} \cong T^\wedge(M(A)) \cup \{0\} \) because of \( T \in M_{\text{ot}}(A) \). Since \( M(A) \) is scattered, we obtain \( (f) \Rightarrow (g) \). Finally, \( (g) \Rightarrow (a) \) is clear, since all operators with totally disconnected spectrum are known to be decomposable; see for instance Example 3.1.20 of [8]. The last assertion follows from Theorem 3.2 of [21].

5. Applications to Convolution Operators

The preceding results apply, in particular, to the group algebra \( A = L_{1}(G) \) for an arbitrary locally compact abelian group \( G \). This algebra is a semi-simple commutative Banach algebra which has a bounded approximate identity and is regular and Tauberian, cf. [30]. Moreover, its multiplier algebra \( M(A) \) may be identified, via convolution, with the measure algebra \( M(G) \) of all regular complex Borel measures on \( G \), cf. [16]. With this identification, \( M_{\text{ot}}(A) \) becomes the subalgebra \( M_{\text{ot}}(G) \) of all measures \( \mu \) on \( G \) whose Fourier–Stieltjes transforms \( \mu^\wedge \) on the dual group \( G^\wedge \) vanish at infinity, and \( M_{\text{ot}}(A) \) becomes the subalgebra \( M_{\text{ot}}(G) \) of all measures \( \mu \) on \( G \) whose Fourier–Stieltjes transforms on \( M(G) \) vanish outside \( G^\wedge \). Note that \( M_{\text{ot}}(G) \) is also known as \( \text{Rad}(L_{1}(G)) \). As usual, the algebras of all absolutely continuous, discrete, and continuous measures on \( G \) will be denoted by \( M_{\text{ac}}(G) \), \( M_{\text{d}}(G) \), and \( M_{\text{c}}(G) \), respectively. Finally, for each \( \mu \in M(G) \), let \( T_{\mu} : L_{1}(G) \rightarrow L_{1}(G) \) denote the corresponding convolution operator given by \( T_{\mu}(f) := \mu * f \) for all \( f \in L_{1}(G) \).

We obtain the following results. First, Proposition 2 shows that the operator \( T_{\mu} \) has natural local spectra exactly when

\[
A_{T_{\mu}}(F) = \{ f \in L_{1}(G) : \mu^\wedge(\text{supp } f^\wedge) \subseteq F \} \quad \text{for all closed } F \subseteq \mathbb{C},
\]

where the Fourier and Fourier-Stieltjes transforms are taken on the dual group \( G^\wedge \). Note that, by Proposition 4, the analytic spectral subspace on the left hand side is always dense in the ideal on the right hand side. Theorem 6 and Corollary 10 characterize those measures \( \mu \) in \( M(G) \), resp. \( M_{\text{ot}}(G) \), for which \( T_{\mu} \) has natural local spectra.

In particular, it follows that, for each \( \mu \in M_{\text{ot}}(G) \setminus M_{\text{ot}}(G) \), the operator \( T_{\mu} \) does not have Bishop's property \((\beta)\). It seems that these are the first known examples of convolution operators without property \((\beta)\). In this

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connection, note that $M_{cl}(G)$ is strictly larger than $M_{ocl}(G)$ whenever $G$ is non-discrete. Also, it has been an open question if all convolution operators on $L_1(G)$ are admissible, cf. Example 2 of [20]. Again, it follows from Corollary 10 that $M_{cl}(G)\setminus M_{ocl}(G)$ provides a rich supply of counter-examples for any non-discrete locally compact abelian group $G$.

Further note that Corollary 11 applies to all measures in $M_{ocl}(G)$ for a compact abelian group $G$. In general, it is clear from Theorem 6 or Proposition 7 that $T_\mu$ has natural local spectra whenever $T_\mu$ is decomposable. A variety of characterizations and examples of decomposable convolution operators has been given in Section 4 of [21]. In particular, it follows that $T_\mu$ is decomposable and hence has natural local spectra for every measure $\mu \in M_{ocl}(G) + M_1(G) + L_1(H)$. Here $H$ is an arbitrary closed subgroup of $G$ and $L_1(H)$ is canonically identified with the space of all measures on $G$ which are concentrated on $H$ and absolutely continuous with respect to Haar measure on $H$.

We now derive some further results on convolution operators with natural spectrum from the theory of measures on thin sets. We shall use the standard notions and results from Chapter 6 of [15] and Chapter 5 of [30]. Recall that a measure $\mu \in M(G)$ is said to be Hermitian if $\mu = \mu^*$ where $\mu^*(E)$ denotes the complex conjugate of $\mu(-E)$ for each Borel set $E \subseteq G$.

**Proposition 12.** If $\mu \in M(G)$ is a Hermitian probability measure with independent powers, then $T_\mu$ does not have natural spectrum and hence is not decomposable.

Since Theorem 6.1.1 of [15] shows that the spectrum of such a measure is the closed unit disc, the result follows immediately from Proposition 1 and from the fact that $\mu^*$ is real-valued on $G^*$ if $\mu$ is Hermitian. Note that, by Theorem 6.8.1 of [15], $M_{ocl}(G)$ contains even a tame Hermitian probability measure with independent convolution powers whenever $G$ is non-discrete.

**Theorem 13.** Assume that $K$ is a compact independent subset of a non-discrete locally compact abelian group $G$, and consider a measure $\mu \in M(G)$ which is concentrated on $K$. Then $T_\mu$ is decomposable if and only if $\mu$ is discrete. In particular, if $\mu \in M_1(G)$ is a non-zero continuous measure which is concentrated on a compact Kronecker set, then $T_\mu$ has natural spectrum, but is not decomposable.

**Proof.** Since Kronecker sets are independent, the second assertion follows from the first and from Theorem 5.5.2 of [30]. To prove the first statement, we observe that all discrete measures induce decomposable convolution operators on $L_1(G)$ and that decomposability on $L_1(G)$ is
preserved under addition and absolute continuity, cf. Theorems 2.5, 4.2 and 4.4 of [21]. Since \( \mu \) may be replaced by the total variation of its continuous part if necessary, it remains to be shown that \( T_\mu \) is not decomposable for every probability measure \( \mu \in M_1(G) \) which is concentrated on a compact independent set \( K \). Suppose that \( T_\mu \) is decomposable and consider the probability measures \( v := \mu^- \) and \( \lambda := (\mu + v)/2 \). Since \( T_\mu \) and \( T_\nu \) are similar, it is clear that \( T_\nu \) is also decomposable. But then Theorem 2.5 of [21] shows that \( T_\nu \) is decomposable and hence has natural spectrum by Proposition 1. On the other hand, since \( \lambda \) is continuous and concentrated on \( K \cup (-K) \), it follows from Theorem 5.3.2 of [30] that \( \lambda \) has independent convolution powers. Since \( \lambda \) is Hermitian, we obtain a contradiction to Proposition 12. The assertion follows.

To put Theorem 13 into perspective, we note that every Cantor set in \( G \) supports a continuous probability measure and that every \( I \)-group contains a Cantor set which is a Kronecker set, cf. [30]. Hence, for every \( I \)-group \( G \), there exists a measure on \( G \) for which the corresponding convolution operator on \( L_1(G) \) has natural spectrum, but is not decomposable, see also [4]. Since compact Kronecker sets are Helson sets, Theorem 5.6.10 of [30] shows that the measures obtained by this method do not belong to \( M_\partial(G) \).

It should be noted that it is also possible to construct non-decomposable convolution operators with natural spectrum from certain measures in \( M_\partial(G) \). By Corollary 10, this means, in other words, that one can find measures in \( M_\partial(G) \setminus M_{oo}(G) \) with natural spectrum. Of course, Corollary 11 says that this cannot happen for compact groups, but for certain non-compact groups, elementary examples of measures in \( M_\partial(G) \setminus M_{oo}(G) \) with natural spectrum have recently been communicated to us by C. C. Graham. We do not know if such examples can be found for every non-compact locally compact abelian group.

The following observation shows that the obstacle is the issue of additivity of the set of multipliers with natural spectrum. In \( M(G) \) this issue has been settled: Zafran [32] gives examples of measures \( \mu, \nu \in M(G) \) for which the operators \( T_\mu \) and \( T_\nu \) both have natural spectrum, but \( T_{\mu+\nu} \), does not have natural spectrum. Note that the following characterization for measures in \( M_\partial(G) \) contains the compact case, because if \( G \) is compact and \( \mu \in M_\partial(G) \) has natural spectrum, then this spectrum is countable, and hence the additivity condition holds.

**Proposition 14.** Let \( NS_\partial(G) := \{ \mu \in M_\partial(G) : T_\mu \) has natural spectrum\}. Then \( NS_\partial(G) = M_{oo}(G) \) if and only if \( NS_\partial(G) \) is closed under addition.

**Proof.** Since \( M_{oo}(G) \) is always contained in \( NS_\partial(G) \), it remains to be seen that \( NS_\partial(G) \subseteq M_{oo}(G) \) when \( NS_\partial(G) \) is closed under addition. Let
\(\mu \in NS_\delta(G)\) and \(\varepsilon > 0\). Then \(T_{\mu,\varepsilon}\) has natural spectrum for every \(\nu \in M_\delta(G)\). By the Stone–Weierstrass theorem, the range of the Fourier transform on \(L_1(G)\) is dense in the continuous functions on \(G^\wedge\) which vanish at infinity. Since \(\mu \in M_\delta(G)\), there exists some \(\nu \in M_\delta(G)\), such that \(|\mu^\wedge + \nu^\wedge| \leq \varepsilon\) on \(G^\wedge\). The natural spectrum assumption implies that \(|\mu^\wedge + \nu^\wedge| \leq \varepsilon\) on \(\mathcal{A}(M(G))\). Since to \(\nu \in M_\delta(G)\), we conclude that \(|\mu^\wedge| \leq \varepsilon\) on \(\mathcal{A}(M(G))\)\(\cap G^\wedge\) and therefore \(\mu \in M_\delta(G)\).

In view of Corollary 10, all measures in \(M_\delta(G)\setminus M_{\delta\Omega}(G)\) with natural spectrum provide examples of multipliers with natural spectrum, but without natural local spectra. It is an open problem whether all quasi-decomposable convolution operators on \(L_1(G)\) are decomposable. By Corollary 10, we know that this is indeed the case for convolution by measures in \(M_\delta(G)\). Since the construction of quasi-decomposable operators which are not decomposable is quite involved [2], it would be interesting to know if such examples can be found among convolution operators. On the other hand, if all quasi-decomposable convolution operators were decomposable, then Theorem 13 would provide another class of examples of multipliers which have natural spectrum, but not natural local spectra.

Another outstanding problem in this area, the question of Katznelson on measures with real spectra [15], has recently been solved by Parreau [26]: for every non-discrete locally compact abelian group \(G\), Parreau has found some \(\mu \in M(G)\) for which \(T_\mu\) has real spectrum, but not natural spectrum. This leaves the interesting problem of finding suitable additional assumptions on a measure with real spectrum which will imply natural spectrum. In Theorem 5.1 of [26], it is shown that a certain growth condition on the measure is sufficient to guarantee natural spectrum. The following proposition is a slight generalization of this result and puts it into the context of decomposable operators.

**Proposition 15.** Assume that the measure \(\mu \in M(G)\) satisfies

\[
\sum_{n=1}^{\infty} \frac{\log a_n}{n^2} < \infty \quad \text{where} \quad a_n := \max\{\|e^{in\rho}\|, \|e^{-in\rho}\|\} \quad \text{for all} \quad n \in \mathbb{N}.
\]

Then \(T_\mu\) has real spectrum. Moreover, \(T_\mu\) is decomposable, hence has natural spectrum.

**Proof.** Without loss of generality, we may assume that \(\|\mu\| = 1\). If \(S\) denotes the convolution operator corresponding to the measure \(e^{in\rho} \in M(G)\), then the assumption on \(\mu\) implies that \(|n|^{-1} \log \|S^n\| \to 0\) as \(|n| \to \infty\). Hence both \(S\) and \(S^{-1}\) have spectral radius 1, which shows that \(\sigma(S)\) is
contained in the unit circle. Moreover, it follows that \( \| S^n \| \geq 1 \) for all integers \( n \in \mathbb{Z} \) and that \( S \) satisfies Beurling's condition

\[
\sum_{n \in \mathbb{Z}} \log \frac{\| S^n \|}{1 + n^2} < \infty.
\]

Hence Theorem 5.3.2 of [8] implies that \( S \) is decomposable. Because of \( S = e^{i\mu} \) and \( \| T_\mu \| = 1 \), we conclude from the spectral mapping theorem and from Proposition 2.1.12 of [8] that \( T_\mu \) has real spectrum and is decomposable. The last assertion is then clear from Proposition 1.

6. Multipliers as Multiplication Operators on \( M(A) \)

In this final section, we shall compare certain local spectral properties of a multiplier \( T \in M(A) \) with those of the corresponding multiplication operator \( L_T \) on the multiplier algebra \( M(A) \). Except in Proposition 16, we shall continue with our assumption that \( A \) be a semi-simple commutative Banach algebra over \( \mathbb{C} \). One of the motivations for these investigations comes again from harmonic analysis: for a measure \( \mu \) on a locally compact abelian group \( G \), it is interesting to relate the spectral properties of convolution by \( \mu \) on the group algebra \( L_1(G) \) to those of convolution by \( \mu \) on the measure algebra \( M(G) \).

The basic tool is the following proposition, stated and proved with minimal assumptions on the Banach algebra \( A \). Neither commutativity nor semi-simplicity is needed here; we only require that \( A \) be without order which means that only the zero element annihilates the whole algebra by multiplication. This guarantees that, with the same definition as in Section 3, the set \( M(A) \) of all multipliers on \( A \) is a closed commutative subalgebra of \( L(A) \), cf. Chapter 1 of [16]. Additionally, at this level of generality, we must make the single valued extension property an explicit assumption.

**Proposition 16.** Assume that \( A \) is an arbitrary Banach algebra without order, and consider a multiplier \( T \in M(A) \) with the single valued extension property. Then the corresponding left multiplication operator \( L_T \) on \( M(A) \) has the single valued extension property. Moreover, its local spectra are given by

\[
\sigma_{L_T}(S) = \left[ \bigcup_{a \in A} \sigma_T(Sa) \right]^{-} \quad \text{for all} \quad S \in M(A),
\]

and \( M(A)_{L_T}(F) = \{ S \in M(A); S(A) \subseteq A_T(F) \} \) for each closed \( F \subseteq \mathbb{C} \).
The first assertion is obvious and the last follows immediately from the claimed description of the local spectra of \( L_T \). Also, given an arbitrary \( S \in M(A) \), it is easily seen that the local spectrum of \( L_T \) at \( S \) contains \( \bigcup_{a \in A} \sigma_T(Sa) \) and hence its closure. Conversely, assume that \( \lambda \) is an interior point of \( \sigma_T(S\lambda) \). Thus, for some open neighborhood \( N(\lambda) \) of \( \lambda \), there is, for every \( a \in A \), an analytic function \( f_a : N(\lambda) \to A \) which satisfies \( (T - \mu) f_a(\mu) = Sa \) for all \( \mu \in N(\lambda) \). For fixed \( \mu \in N(\lambda) \), we define the map \( W^a_{\mu} : A \to A \) by \( W^a_{\mu}(a) := f_a(\mu) \) for all \( a \in A \). Evidently, \( (L_T - \mu) W^a_{\mu} = S \).

Therefore it remains to be shown that \( W^a_{\mu} \) is a multiplier on \( A \) and that \( W^a_{\mu} \) depends analytically on \( \mu \in N(\lambda) \). To prove that \( W^a_{\mu} \) belongs to \( M(A) \), let \( a, b \in A \) be arbitrarily given and consider \( c := aW^a_{\mu}(b) - W^a_{\mu}(a) b \). Then it is clear that \((T - \mu)(c) = aS(b) - S(a) b = 0 \) and hence \( c \in \ker(T - \mu) \subseteq A_T(\mu) \).

On the other hand, we know from general theory that \( \sigma_T(W^a_{\mu}(a)) = \sigma_T(f_a(\mu)) = \sigma_f(S\mu) \) and \( \sigma_T(W^a_{\mu}(b)) = \sigma_T(f_a(\mu)) = \sigma_f(Sb) \), which implies that \( \mu \in A_T(\sigma_f(Sa) \cup \sigma_f(Sb)) \). Since \( \mu = \sigma_f(Sa) \cup \sigma_f(Sb) \) and \( T \) has the single valued extension property, we conclude that \( c = 0 \) and therefore \( W^a_{\mu} \in M(A) \). Finally, to show that \( W^a_{\mu} \) depends analytically on \( \mu \in N(\lambda) \), fix \( \mu \in N(\lambda) \) and choose \( r > 0 \) so that the closed disc \( D \) with center \( \mu \) and radius \( 2r \) is contained in \( N(\lambda) \). Then, for each \( a \in A \), a standard application of Cauchy’s formula to the analytic function \( f_a : N(\lambda) \to A \) leads to the estimate \( \| (W^\zeta_{\mu}(a) - W^\zeta_{\mu}(a))/\zeta - \mu \| \leqslant M_\mu/r \) for every \( \zeta \in C \) with \( 0 < |\zeta - \mu| \leqslant r \), where \( M_\mu := \sup \{ \| f_a(z) \| : z \in D \} \). By the uniform boundedness principle, we obtain a constant \( M > 0 \) such that \( \| W^\zeta_{\mu} - W^\zeta_{\mu} \| \leqslant M \) and therefore \( \| W^\zeta_{\mu} \| \leqslant M \| \zeta - \mu \| \) for all \( \zeta \in C \) for which \( |\zeta - \mu| \leqslant r \). This shows that the function \( \zeta \to W^\zeta_{\mu} \) is norm continuous at \( \mu \). It is now a straightforward application of Morera’s theorem to see that \( \mu \to W^\mu_{\mu} \) is an analytic function on \( N(\lambda) \). Thus \( N(\lambda) \) is contained in \( \rho_{L_T}(S) \), which completes the proof.

The above applies to a much larger class of algebras than those primarily considered here. For instance, let \( A \) be a commutative Banach algebra which is semi-prime in the sense that \( A \) contains no non-zero nilpotent elements. Then \( A \) is obviously without order, and it is not difficult to see that any multiplier on \( A \) has the single valued extension property. Thus the above proposition gives, for instance, a description of the analytic spectral subspaces of multipliers on the radical weighted convolution algebras \( L \), \( \omega, \omega \), cf. [9]. We shall not pursue this here. Instead, for the rest of this section, we assume \( A \) to be a semi-simple commutative Banach algebra.

**Proposition 17.** Assume that \( T \in M(A) \) has natural local spectra. Then the multiplication operator \( L_T \) on \( M(A) \) has natural local spectra and satisfies

\[
M(A)_{L_T}(F) = \{ S \in M(A) : \text{supp } S^\ast \cap A(A) \subseteq T \ast^{-1}(F) \}
\]

for all closed \( F \subseteq C \).
Proof. Since $T$ has natural local spectra, we obtain, for every $S \in \mathcal{M}(A)$ and $a \in A$, that $\sigma_T(Sa) = T^*(\text{supp}(Sa)^-) \subseteq T^*(\text{supp} S^\wedge \cap \mathcal{M}(A))^-$ $\subseteq T^*(\text{supp} S^\wedge)$, where $\text{supp} S^\wedge$ denotes, of course, the support of the Gelfand transform $S^\wedge$ on $\mathcal{M}(A)$. With $L := L_T$ we conclude from Propositions 2 and 16 that $T^*(\text{supp} S^\wedge) \subseteq \sigma_L(S) \subseteq T^*(\text{supp} S^\wedge \cap \mathcal{M}(A))^-$ $\subseteq T^*(\text{supp} S^\wedge)$ for all $S \in \mathcal{M}(A)$. This shows that $L_T$ has natural local spectra and implies also the desired formula for the analytic spectral subspaces.

There are other circumstances under which $L_T$ can be seen to have natural local spectra. The next result may be viewed as a supplement to Proposition 7.

**Proposition 18.** If the multiplier $T \in \mathcal{M}(A)$ has the weak 2-SDP and Dunford's property (C), then $L_T$ has natural local spectra. In particular, if $T \in \mathcal{M}(A)$ is quasi-decomposable, then $L_T$ has natural local spectra.

**Proof.** By Proposition 2, it remains to be seen that the operator $L := L_T$ satisfies $\mathcal{M}(A)_f(F) \cong Z_L(F)$ for each closed $F \subseteq \mathbb{C}$. We shall show a bit more, namely that the spectral subspace $\mathcal{M}(A)_f(F)$ contains and hence equals the ideal $Y_L(F) := \{S \in \mathcal{M}(A) : \text{supp} S^\wedge \cap \mathcal{M}(A) \subseteq T^\wedge \cap \{F\}\}$ which appeared in the previous result. Because of the description of $\mathcal{M}(A)_f(F)$ from Proposition 16, it suffices to show that, for every $S \in Y_L(F)$ and $a \in A$, we have $Sa \in A_T(F)$. The argument for this is reminiscent of that of Proposition 7. Obviously, it is enough to show that $Sa \in A_T(U^-)$ for an arbitrary open neighborhood $U$ of $F$. Choose an open set $V$ such that $U \cup V = \mathbb{C}$ and $F \cap V^- = \emptyset$, and let $\varepsilon > 0$. Then the weak 2-SDP for $T$ gives us elements $a_U \in A_T(U^-)$ and $a_V \in A_T(V^-)$ so that $\|a - a_U - a_V\| < \varepsilon(1 + |S|)$. Since $\text{supp} S^\wedge \cap \mathcal{M}(A) \cap \text{supp} a_V^* = \emptyset$, we have $Sa_V = 0$ and therefore $\|Sa - Sa_U\| < \varepsilon$. Since $A_T(U^-)$ is closed and contains $Sa_U$, we conclude that $Sa \in A_T(U^-)$. The assertion follows.

We have a converse to Proposition 17 for algebras with suitable approximate identities. Note that the following theorem applies, in particular, to the group algebra $A = L_1(G)$ for an arbitrary locally compact abelian group $G$. It follows that, for every measure on $G$, the corresponding convolution operator on $L_1(G)$ has natural local spectra if and only if this condition holds for convolution on $M(G)$. Hence Proposition 12 shows that, for any Hermitian probability measure on $G$ with independent powers, the corresponding convolution operator on $M(G)$ does not have natural local spectra. But clearly all convolution operators on $M(G)$ have natural spectrum with respect to $M(G)$.
**Theorem 19.** Let $T \in M(A)$ be an arbitrary multiplier. If $L_T$ has natural local spectra, then $Z_T(F) = \{ x \in A : xA \subseteq A_T(F) \}$ for all closed $F \subseteq \mathbb{C}$. Hence, if $A$ has approximate units, then $T$ has natural local spectra if and only if $T$ has Dunford's property (C) and $L_T$ has natural local spectra. Moreover, if $A$ has a bounded approximate identity, then $T$ has natural local spectra if and only if $L_T$ has natural local spectra.

**Proof.** In the following argument, we tacitly identify the elements $a \in A$ with the corresponding multiplication operators $L_a \in M(A)$. Let $L := L_T$ and fix an arbitrary closed subset $F$ of $\mathbb{C}$. Because of $A \subseteq M_{DG}(A)$, it is easily seen that $Z_T(F) = Z_L(F) \cap A$ holds without any assumptions on $A$ or $T$. Now suppose that $L$ has natural local spectra. Then the last identity, combined with the results of Propositions 2 and 16, leads to $Z_T(F) = \{ x \in A : xA \subseteq A_T(F) \}$. If we assume, in addition, that $A$ has approximate units and that $A_T(F)$ is closed, then for each $x \in Z_T(F)$ we obtain $x \in (x \cdot A) \subseteq A_T(F)$. Therefore $A_T(F) = Z_T(F)$ which shows that $T$ has natural local spectra. Similarly, if we assume that $A$ has a bounded approximate identity, then the module version of the Cohen factorization theorem [7] implies that $Z_T(F) = Z_T(F) A \subseteq A_T(F)$ and hence that $T$ has natural local spectra. The converse is clear from Propositions 2 and 17.

**References**