Mathematical analysis of radionuclides displacement in porous media with nonlinear adsorption

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Abstract

This paper is devoted to the mathematical analysis of the miscible displacement of a set of radionuclides in a flow occurring in a heterogeneous porous medium. The flow is governed by Darcy’s law, and the motion of the chemical species is given by a nonclassical advection–diffusion–reaction equations system. The novelty of the model lies in the adsorption phenomenon that leads to a time derivative of a nonlinear term in these equations. A semi-discretization method is used to establish the existence of weak solutions to this system. Uniform $L^\infty$-estimates on the solutions are specified.

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1. Introduction

This paper deals with the mathematical analysis of the miscible displacement of radioactive elements in a heterogeneous porous medium.

Physically, we consider a water-saturated area of the ground that is polluted by radionuclides coming from outside or from a leak of their storage site. This problem associates two phenomena: the flow, governed by Darcy’s filtration velocity, and the displacement of the chemical species...
into this flow. The miscible motion of a free substratum of concentration $c$ in a porous medium is usually modeled by a parabolic partial differential equation including transport and diffusion–dispersion effects. Geological observations show the motion of the substratum is always delayed by fixation in the solid rock of the porous matrix, known as the adsorption phenomenon. The adsorbed phase of the substratum is assumed to be a mapping on the variable $c$, and several studies (see for, e.g., [4,9,10]) deal with the linear adsorption $F(x, c) = \gamma(x)c$. In this paper we are interested about what happens when $F$ is not necessarily linear. For example, usual adsorptions (see [7]) are

\[
\text{Langmuir’s isotherm: } F(x, c) = \frac{\gamma_1(x)c}{1 + \gamma_2(x)c},
\]

\[
\text{Freundlich’s isotherm: } F(x, c) = \gamma_1(x)c^{\frac{1}{n}},
\]

where $n \in \mathbb{N}^*$, $\gamma_1$ and $\gamma_2$ are nonnegative bounded functions. Note that Freundlich’s isotherm is not locally Lipschitz continuous with respect to $c$.

The adsorption phenomenon induces a not necessarily linear term in the time derivative. This kind of nonlinearity and the dependence of isotherm function on space are not usual, and constitutes the main work behind this article. Additionally radioactive decay is modeled by a nonlinear reaction term.

The multiple displacement of radio-elements is also considered and leads to a nonlinear advection–diffusion–reaction system. In this case, the radioactive filiation phenomenon is taken into account and constitutes another difficulty of the problem.

A discretization method inspired by [1,14] is used to prove the existence of a weak solution for single and multiple displacements of radionuclides. Darcy’s velocity is not uniformly bounded in general and requires to obtain $L^\infty$-estimates in the concentrations.

The paper is organized as follows. In Section 2, we formulate the mathematical problem and the concept of weak solution is defined for this model. The main results are the existence of at least one weak solution and the regularity and boundedness properties verified by the concentrations. Sections 3–6 are devoted to the proof of these results. Note that the discretization method for the existence of a weak solution is detailed through the Section 4.

2. Model and main results

This section summarizes the statements of the problem. The cases of the displacement of single and multiple radionuclides are separated for the sake of clearness. Let be $T > 0$, and let $\Omega$ be an open bounded subset of $\mathbb{R}^d$, with a Lipschitz continuous boundary, $\Gamma = \partial \Omega$, such that $\Omega$ lies locally on one side of $\Gamma$. The domain $\Omega$ represents a porous medium and $d$ is space dimension. We define $Q_T = (0, T) \times \Omega$, $\Sigma_T = (0, T) \times \Gamma$.

2.1. Displacement of a single radionuclide

We consider the miscible displacement of a single radionuclide in the ground water. The unknowns of the problem are the concentration $c$, the pressure $p$, and the filtration velocity $V$, defined in $Q_T$. The equations that describe such a flow (see [3,4,7]) are

\[
\begin{align*}
\partial_t G(x, c(t, x)) + \text{div}(c(t, x)V(t, x)) - \text{div}(D(x, c(t, x), V(t, x))\nabla c(t, x)) \\
+ \lambda G(x, c(t, x)) - f(t, x) = 0, \quad (t, x) \in Q_T,
\end{align*}
\]
\[
\text{div } V(t, x) = 0, \quad (t, x) \in Q_T,
\]
\[
V(t, x) = -\frac{1}{\mu(c(t, x))} K(x) \left( \nabla p(t, x) - \rho(c(t, x)) \vec{g}(x) \right), \quad (t, x) \in Q_T,
\]

with the boundary conditions
\[
D\left( \sigma, c(t, \sigma), V(t, \sigma) \right) \nabla c(t, \sigma) \cdot \vec{v}(\sigma) + \left( c(t, \sigma) - g(t, \sigma) \right) \left( V(t, \sigma) \cdot \vec{v}(\sigma) \right)^- = 0,
\]
\[
V(t, \sigma) \cdot \vec{v}(\sigma) = V(t, \sigma)
\]
for \((t, \sigma) \in \Sigma_T\), where \(\vec{v}(\sigma)\) is the outward unit normal to \(\Gamma\) at location \(\sigma\). The initial condition for the concentration is
\[
c(0, x) = c_0(x) \quad \text{for a.e. } x \in \Omega
\]
with
\[
c_0 \in L^\infty(\Omega), \quad c_0(x) \geq 0 \quad \text{for a.e. } x \in \Omega.
\]

For any real number \(u\), we define
\[
u^+ = \frac{|u| + u}{2}, \quad u^- = \frac{|u| - u}{2},
\]
so that \(u^+ \geq 0, u^- \geq 0, \) and \(u = u^+ - u^-\). Equation (1) represents the mass conservation of the radionuclide, Eq. (2) is the incompressibility equation of the fluid, and Eq. (3) is usual Darcy’s law. The flow is assumed to only occur in the saturated area, but nevertheless, exchanges of fluid with the exterior are modeled by a normal velocity (5) at the boundary. The region where \(V(t, \sigma) < 0\) represents an injection of possibly contaminated fluid with a concentration \(g\). Then, the total flux through \(\Gamma\) in this region is modeled by a convective flux polluted by \(g\). The region where \(V(t, \sigma) > 0\) stands for an ejection. In this case, the total flux equals a convective flux containing the concentration of the flow \(c\). The region where \(V(t, \sigma)\) vanishes is an impervious boundary. It is assumed that
\[
V \in L^2((0, T); L^2(\Gamma)).
\]

According to (2), \(V\) has to verify the compatibility condition
\[
\int_{\Gamma} V(t, \sigma) d\sigma = 0 \quad \text{for a.e. } t \in (0, T).
\]

We also assume
\[
g \in L^\infty(\Sigma_T), \quad g(t, \sigma) \geq 0 \quad \text{a.e. in } \Sigma_T.
\]

The quantity \(f\) denotes a gain of radionuclide coming from a leak in a nuclear waste repository located into the aquifer. We suppose that
\[
f \in L^1((0, T); L^\infty(\Omega)) \cap L^2((0, T); (H^1(\Omega))'), \quad f(t, x) \geq 0 \text{ a.e. in } Q_T.
\]
The function $G$ is defined from $\Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ by

$$G(x,u) = \phi(x)u + (1 - \phi(x))\rho_s(x)F(x,u), \quad \text{for a.e. } x \in \Omega, \forall u \in \mathbb{R}_+.$$  \hspace{1cm} (12)

In the case where $G(x,u)$ does not depend on $x$, taking $v = G(u)$ as unknown instead of $u$, Eq. (1) is reduced to a weakly degenerate parabolic equation. The kind of degeneracy can be found in Alt and Luckhaus [1] for general quasilinear elliptic–parabolic differential equations. Incompressible and compressible two phases flows in porous media lead to degenerate parabolic equations; lots of works deal on such problems [8,9,11–13,15].

The porosity $\phi$ and the density of the solid matrix $\rho_s$ are two measurable functions in $\Omega$ and there exist two positive constants $\underline{\phi}$ and $\bar{\rho}_s$ such that

$$0 < \underline{\phi} \leq \phi(x) \leq 1, \quad 0 \leq \rho_s(x) \leq \bar{\rho}_s \quad \text{for a.e. } x \in \Omega.$$  \hspace{1cm} (13)

According to the examples in the introduction, the adsorption function $F$, belonging to the class of Carathéodory functions, is defined from $\Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ and verifies

$$F(x,u) \text{ is measurable with respect to } x,$$  \hspace{1cm} (14)

$$F(.,u) \in L^\infty(\Omega) \quad \text{for all } u \in \mathbb{R}_+,$$  \hspace{1cm} (15)

$$F(x,0) = 0 \quad \text{for a.e. } x \in \Omega,$$  \hspace{1cm} (16)

$$u \mapsto F(x,u) \text{ is continuous and monotone nondecreasing in } \mathbb{R}_+.$$  \hspace{1cm} (17)

The term $\lambda G$ represents the radioactive decay for $c$, $\lambda$ being its radioactive decay factor. One assumes

$$\lambda \geq 0.$$  \hspace{1cm} (18)

The diffusion–dispersion tensor $D$ defined from $\Omega \times \mathbb{R} \times \mathbb{R}^d$ to $\mathcal{M}_d(\mathbb{R})$ belongs to the class of Carathéodory functions and satisfies

$$D(.,u,\eta) \text{ is measurable} \quad \forall u \in \mathbb{R}, \forall \eta \in \mathbb{R}^d,$$  \hspace{1cm} (19)

$$D(x,.,.) \in C^0(\mathbb{R} \times \mathbb{R}^d; \mathcal{M}_d(\mathbb{R})) \quad \text{for a.e. } x \in \Omega.$$  \hspace{1cm} (20)

Furthermore, we assume the existence of $\overline{D} \in \mathbb{R}_+^*$ such that

$$\forall (u,\eta) \in \mathbb{R} \times \mathbb{R}^d, \text{ for a.e. } x \in \Omega \quad \|D(x,u,\eta)\|_{\mathcal{M}_d(\mathbb{R})} \leq \overline{D}$$  \hspace{1cm} (21)

and $\alpha \in \mathbb{R}_+^*$ such that for all $(u,\eta) \in \mathbb{R} \times \mathbb{R}^d$, for a.e. $x \in \Omega$,

$$\forall \xi \in \mathbb{R}^d, \quad D(x,u,\eta)\xi \cdot \xi \geq \alpha|\xi|^2.$$  \hspace{1cm} (22)

Finally, let us give the hypotheses for Eq. (3).

The permeability $K: \Omega \to \mathcal{M}_d(\mathbb{R})$ verifies

$$K_{i,j} \in L^\infty(\Omega) \quad \forall i, j = 1, \ldots, d,$$  \hspace{1cm} (23)

\hspace{1cm}
and there exists $\beta \in \mathbb{R}_+^*$ such that for a.e. $x \in \Omega$,
\[
\forall \xi \in \mathbb{R}^d, \quad K(x)\xi \cdot \xi \geq \beta |\xi|^2.
\] (24)

The density $\rho$ and the viscosity $\mu$ of the fluid belong to $C^0(\mathbb{R}_+; \mathbb{R}_+)$ and
\[
\exists \gamma > 0 \text{ such that } \forall u \in \mathbb{R}^+, \quad \mu(u) \geq \gamma.
\] (25)

The gravity vector $\vec{g}$ is defined from $\Omega$ to $\mathbb{R}^d$ and satisfies
\[
\vec{g} \in (L^\infty(\Omega))^d.
\] (26)

We now introduce the definition of a weak solution of (1)–(6).

**Definition 1.** Let (7)–(26) hold. A pair $(c, p)$ is a weak solution of (1)–(6) if it satisfies

\[
c \in L^2((0, T); H^1(\Omega)) \cap L^\infty(Q_T), \quad c(t, x) \geq 0 \quad \text{a.e. in } Q_T,
\]
\[
t \mapsto G(., c(., .)) \in C^0([0, T]; (H^1(\Omega))^'),
\]
\[
t \mapsto \partial_t G(., c(., .)) \in L^2((0, T); (H^1(\Omega))^'),
\]
\[
G(., c(0, .)) = G(., c_0(., .)) \quad \text{a.e. in } \Omega,
\]
\[
p \in L^2((0, T); H^1(\Omega)/\mathbb{R}), \quad V \in L^2((0, T); (L^2(\Omega))^d),
\]
\[
\int_0^T \int_{Q_T} \{\partial_t G(., c(., .)), \varphi(., .)\}_{(H^1(\Omega))^', H^1(\Omega)} dt - \int_{Q_T} c(t, x)V(t, x) \cdot \nabla \varphi(t, x) dx dt
\]
\[
+ \int_{Q_T} D(x, c(t, x), V(t, x))\nabla c(t, x) \cdot \nabla \varphi(t, x) dx dt
\]
\[
+ \int_{\Sigma_T} (\gamma_T^+ c(t, \sigma)\nabla^+(t, \sigma) - g(t, \sigma)\nabla^-(t, \sigma))\gamma_T \varphi(t, \sigma) d\sigma dt
\]
\[
+ \lambda \int_{Q_T} G(x, c(t, x))\varphi(t, x) dx dt - \int_{Q_T} f(t, x)\varphi(t, x) dx dt = 0
\]
\[
\text{for all } \varphi \in L^2((0, T); H^1(\Omega)),
\] (27)
\[
\int_{Q_T} \frac{1}{\mu(c(t, x))} K(x)\left(\nabla p(t, x) - \rho(c(t, x))\vec{g}(x)\right) \cdot \nabla \psi(t, x) dx dt
\]
\[
+ \int_{\Sigma_T} V(t, \sigma)\gamma_T \psi(t, \sigma) d\sigma dt = 0 \quad \text{for all } \psi \in L^2((0, T); H^1(\Omega)),
\] (28)
\[
V(t, x) = -\frac{1}{\mu(c(t, x))} K(x)\left(\nabla p(t, x) - \rho(c(t, x))\vec{g}(x)\right) \quad \text{a.e. in } Q_T.
\] (29)
In this definition, $\gamma\Gamma$ denotes the trace operator.

**Remark 2.** For any weak solution, since the space $L^\infty(\Omega)$ is continuously embedded into $(H^1(\Omega))^\prime$, and $t \mapsto G(., c(t,.))$ belongs to $L^\infty((0, T); L^\infty(\Omega)) \cap C^0([0, T]; (H^1(\Omega))^\prime)$, this application is also weakly continuous from $[0, T]$ to $L^\infty(\Omega)$. Thus $G(., c(t,.))$ belongs to $L^\infty(\Omega)$ everywhere in $[0, T]$, and $c(t, .)$ can be viewed everywhere in $[0, T]$ as an element of $L^\infty(\Omega)$.

We state the main results.

**Theorem 3.** There exists a weak solution $(c, p)$ of (1)–(6) in the sense of Definition 1. Furthermore, the function $c$ satisfies

$$0 \leq c(t, x) \leq \bar{c} + \int_0^t \left\| \frac{f(s, .)}{\phi} \right\|_{L^\infty(\Omega)} ds$$

for a.e. $(t, x) \in QT$, with

$$\bar{c} = \max(\|c_0\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Sigma T)}).$$

Because the adsorption $F(x,c)$ is usually a concave mapping with respect to $c$ in geological models, we precise the long-time behaviour of the concentration with respect to time when the source term $g$ vanishes.

**Proposition 4.** Assume the mapping $c \mapsto F(x,c)$ is concave, for almost every $x \in \Omega$. Assume there exists $t_0 \in [0, T)$ such that $g(t, \sigma) = 0$ for a.e. $(t, \sigma) \in [t_0, T] \times \Gamma$. For a.e. $(t, x) \in QT$, the concentration $c$ in Theorem 3 verifies

$$c(t, x) \leq e^{-\lambda(t-t_0)} \left( \bar{c} + \int_0^{t_0} \left\| \frac{f(s, .)}{\phi} \right\|_{L^\infty(\Omega)} ds \right) + \int_{t_0}^t e^{-\lambda(t-s)} \left\| \frac{f(s, .)}{\phi} \right\|_{L^\infty(\Omega)} ds.$$

Note that if $f$ belongs to some $L^p(\mathbb{R}_+; L^\infty(\Omega))$, this result indicates a uniform $L^\infty(\Omega)$ bound for the solution independently of $T$ and an exponential decay to 0 if the support of $f$ is bounded in $\mathbb{R}_+ \times \Omega$ and $T$ is large enough.

### 2.2. Displacement of multiple radionuclides

We consider a miscible flow transporting $m$ radionuclides in the ground water, $m \geq 1$. The model includes several phenomena as adsorption, radioactive decay and filiation. The evolution of the concentrations $c = (c_1, \ldots, c_m)^T$, the pressure $p$ and the filtration velocity $V$ are governed by

$$\frac{\partial}{\partial t} G_k(x, c_k(t,x)) + \text{div}(c_k(t,x)V(t,x))$$

$$- \text{div}(D_k(x, c_k(t,x), V(t,x))\nabla c_k(t,x)) + \lambda_k G_k(x, c_k(t,x))$$

$$- \sum_{l<k} \lambda_l R_{k,l} G_l(x, c_l(t,x)) - f_k(t,x) = 0 \quad \text{in } QT, \; k = 1, \ldots, m, \quad (30)$$

$$\text{div } V(t,x) = 0 \quad \text{in } QT, \quad (31)$$

$$V(t,x) = -\frac{1}{\mu(c(t,x))} K(x)(\nabla p(t,x) - \rho(c(t,x))\vec{g}(x)) \quad \text{in } QT, \quad (32)$$
with boundary and initial conditions given for $k = 1, \ldots, m$ by

$$D_k(\sigma, c_k(t, \sigma), V(t, \sigma)) \nabla c_k(t, \sigma) \cdot \vec{v}(\sigma) + (c_k(t, \sigma) - g_k(t, \sigma))(V(t, \sigma) \cdot \vec{v}(\sigma))^- = 0 \quad \text{on } \Sigma_T,$$

$$V(t, \sigma) \cdot \vec{v}(\sigma) = \mathcal{V}(t, \sigma) \quad \text{on } \Sigma_T, \quad c_k(0, x) = c_{k,0}(x) \quad \text{in } \Omega. \quad (33)$$

Initial conditions are supposed essentially bounded and nonnegative:

$$c_{k,0} \in L^\infty(\Omega), \quad c_{k,0}(x) \geq 0 \quad \text{for a.e. } x \in \Omega, \quad k = 1, \ldots, m. \quad (35)$$

The function $\mathcal{V}$ is assumed to fulfill hypotheses (8) and (9). We suppose that

$$g_k \in L^\infty(\Sigma_T), \quad g_k(t, \sigma) \geq 0 \quad \text{a.e. in } \Sigma_T, \quad k = 1, \ldots, m,$$

$$f_k \in L^1((0, T); L^\infty(\Omega)) \cap L^2((0, T); (H^1(\Omega))'), \quad f_k(t, x) \geq 0 \quad \text{a.e. in } Q_T, \quad k = 1, \ldots, m. \quad (36)$$

The functions $G_k$ go from $\Omega \times \mathbb{R}_+$ to $\mathbb{R}_+$ and are defined by

$$G_k(x, u) = \phi(x)u + (1 - \phi(x))\rho_s(x)F_k(x, u) \quad \text{for a.e. } x \in \Omega, \quad u \in \mathbb{R}_+, \quad (38)$$

$F_k$ being the adsorption for $c_k$. The functions $F_k, k = 1, \ldots, m,$ verify

$$F_k(x, u) \text{ is measurable with respect to } x, \quad (39)$$

$$F_k(., u) \in L^\infty(\Omega) \quad \text{for all } u \in \mathbb{R}_+, \quad (40)$$

$$F_k(x, 0) = 0 \quad \text{for a.e. } x \in \Omega, \quad (41)$$

$$u \mapsto F_k(x, u) \text{ is continuous and monotone nondecreasing in } \mathbb{R}_+. \quad (42)$$

The nonnegative number $\lambda_k$ is the radioactive decay factor for the $k$th species. The coefficients $R_{k,l}$ are defined in the filiation term for $l, k = 1, \ldots, m$ by $R_{k,l} = \alpha_{k,l}m_l/m_k$, where $m_k$ stands for the molar mass of $c_k$, and $\alpha_{k,l}$ is the production rate of $c_k$ by the disintegration of $c_l$. Thus, $R_{k,l} = 0$ if $c_l$ do not produce $c_k$ by disintegration. Nuclear reactions are assumed irreversible, thus radionuclides are supposed ordered such that $(R_{k,l})_{k,l}$ is a lower triangular matrix, so the filiation term $\sum_{l<k} \lambda_l R_{k,l}G_l(., c_l)$ in (30) has a “triangular” form and is missing for $k = 1$. One assumes

$$\lambda_k \geq 0, \quad R_{k,l} \geq 0 \quad \text{for } l, k = 1, \ldots, m. \quad (43)$$

For all $k = 1, \ldots, m$, we assume there exist $\bar{D}_k \in \mathbb{R}_+^d$ and $\alpha_k \in \mathbb{R}_+^*$ such that the tensor $D_k$ verifies for all $u \in \mathbb{R}$, $\eta \in \mathbb{R}^d$, for a.e. $x \in \Omega$,

$$D_k(., u, \eta) \text{ is measurable}, \quad D_k(x, ., \eta) \in C^0(\mathbb{R} \times \mathbb{R}^d; \mathcal{M}_d(\mathbb{R})), \quad (44)$$

$$\left\|D_k(x, u, \eta)\right\|_{\mathcal{M}_d(\mathbb{R})} \leq \bar{D}_k, \quad \forall \xi \in \mathbb{R}^d, \quad D_k(x, u, \eta)\xi \cdot \xi \geq \alpha_k||\xi||^2. \quad (45)$$

The functions $\rho$ and $\mu$ belong to $C(\mathbb{R}_+^m; \mathbb{R}_+)$. Data $\phi$, $\rho_s$, $K$, $\rho$, $\mu$ and $\vec{g}$ still satisfy (13), (23), (24), (25) and (26), respectively.
Definition 5. Let (13), (23)–(26), (35)–(45) hold. A pair \((c, p)\) is a weak solution of (30)–(34) if for all \(k = 1, \ldots, m\),
\[
c_k \in L^2((0, T); H^1(\Omega)) \cap L^\infty(Q_T), \quad c_k(t, x) \geq 0 \quad \text{a.e. in } Q_T,
\]
t \mapsto G_k(., c_k(t, .)) \in C^0([0, T]; (H^1(\Omega))^\prime), \quad t \mapsto \partial_t G_k(., c_k(t, .)) \in L^2((0, T); (H^1(\Omega))^\prime),
\]
\[
G_k(., c_k(0, .)) = G_k(., c_k(0, .)) \quad \text{a.e. in } \Omega,
\]
p \in L^2((0, T); H^1(\Omega)/\mathbb{R}), \quad V \in L^2((0, T); (L^2(\Omega))^d),
\]
\[
\int_0^T \langle \partial_t G_k(., c_k(t, .), \varphi(t, .)) |_{(H^1(\Omega))^\prime}, H^1(\Omega) \rangle dt - \int_{Q_T} c_k(t, x) V(t, x) \cdot \nabla \varphi(t, x) dx dt
\]
\[
+ \int_{Q_T} D_k(x, c_k(t, x), V(t, x)) \nabla c_k(t, x) \cdot \nabla \varphi(t, x) dx dt
\]
\[
+ \int_{\Sigma_T} \left( \gamma_{\Gamma} c_k(t, \sigma) V^+(t, \sigma) - g_k(t, \sigma) V^-(t, \sigma) \right) \gamma_{\Gamma} \varphi(t, \sigma) d\sigma dt
\]
\[
+ \lambda_k \int_{Q_T} G_k(x, c_k(t, x)) \varphi(t, x) dx dt - \sum_{l < k} \lambda_l R_{k,l} \int_{Q_T} G_l(x, c_l(t, x)) \varphi(t, x) dx dt
\]
\[
- \int_{Q_T} f_k(t, x) \varphi(t, x) dx dt = 0 \quad \text{for all } \varphi \in L^2((0, T); H^1(\Omega)),
\]
\[
\int_{Q_T} \frac{1}{\mu(c(t, x))} K(x) \left( \nabla p(t, x) - \rho(c(t, x)) \vec{g}(x) \right) \cdot \nabla \psi(t, x) dx dt
\]
\[
+ \int_{\Sigma_T} \mathcal{V}(t, \sigma) \gamma_{\Gamma} \psi(t, \sigma) d\sigma dt = 0 \quad \text{for all } \psi \in L^2((0, T); H^1(\Omega)),
\]
\[
V(t, x) = - \frac{1}{\mu(c(t, x))} K(x) \left( \nabla p(t, x) - \rho(c(t, x)) \vec{g}(x) \right) \quad \text{a.e. in } Q_T.
\]

We retrieve the existence result for multiple radionuclides.

Theorem 6. There exists a weak solution of (30)–(34) in the sense of Definition 5.

3. Analysis of an auxiliary nonlinear elliptic equation

Let \(D, F, G, \lambda\) be functions verifying (12)–(22). Along the lines of Alt and Luckhaus (see [1, Section 1.1]), we define a mapping \(B\) by
\[
B(x, u) = \frac{1}{\mu(c(t, x))} K(x) \left( \nabla p(t, x) - \rho(c(t, x)) \vec{g}(x) \right) \quad \text{for a.e. } x \in \Omega, \text{ for every } u \in \mathbb{R}.
\]
Since $G(x, u)$ is increasing in $u$, $B$ is nonnegative. In [1], a convexity argument establishes: for almost every $x \in \Omega$, for all $z_1, z_2 \in \mathbb{R}$,

$$\left(G(x, z_1) - G(x, z_2)\right)z_2 \leq B(x, z_1) - B(x, z_2) \leq \left(G(x, z_1) - G(x, z_2)\right)z_1.$$  \hspace{1cm} (46)

We shall now be concerned with a nonlinear elliptic equation. Let $f \in L^\infty(\Omega)$, $g \in L^\infty(\Gamma)$ be two nonnegative functions; let be $U \in (L^2(\Omega))^d$ and $\mathcal{U} \in L^2(\Gamma)$ such that $\operatorname{div} U = 0$ and $\gamma^\nu_U = \mathcal{U}$, where $\gamma^\nu$ denotes the application $\eta \mapsto \eta \cdot \nu$ on the set of functions in $(L^2(\Omega))^d$ having a divergence in $L^2(\Omega)$. We show the following proposition.

**Proposition 7.** Let be $h \in \mathbb{R}^*_+, u \in L^\infty(\Omega)$, and $\bar{u} \in \mathbb{R}_+$ such that $0 \leq u(x) \leq \bar{u}$ and $0 \leq g(\sigma) \leq \bar{u}$ for a.e. $x \in \Omega$, $\sigma \in \Gamma$.

There exists $c \in H^1(\Omega)$ such that

$$0 \leq c(x) \leq \bar{u} + h \| \frac{f}{\overline{\phi}} \|_{L^\infty(\Omega)} \quad \text{for a.e. } x \in \Omega,$$

$$\frac{1}{h} \int_\Omega \left(G(x, c) - G(x, u)\right)\varphi \, dx - \int_\Omega c U \cdot \nabla \varphi \, dx + \int_\Omega D(x, c, U) \nabla c \cdot \nabla \varphi \, dx$$

$$+ \int_\Gamma \left(\gamma^\Gamma U^+ - g \mathcal{U}^-\right) \gamma^\Gamma \varphi \, d\sigma + \lambda \int_\Omega G(x, c) \varphi \, dx - \int_\Omega f \varphi \, dx = 0 \quad \text{for all } \varphi \in H^1(\Omega).$$  \hspace{1cm} (48)

Furthermore,

$$\frac{1}{h} \int_\Omega \left(B(x, c) - B(x, u)\right) \, dx + \alpha \int_\Omega |\nabla c|^2 \, dx + \lambda \int_\Omega G(x, c) \, dx$$

$$+ \int_\Gamma |U||\gamma^\Gamma c|^2 \, d\sigma \leq \int_\Omega c \, dx + \int_\Gamma U^- g \gamma^\Gamma c \, d\sigma.$$  \hspace{1cm} (49)

**Proof.** Setting $M = \bar{u} + h \| \frac{\overline{f}}{\overline{\phi}} \|_{L^\infty(\Omega)}$, we define a function $H$, and extensions of $F$ and $G$ to $\mathbb{R}$ for a.e. $x \in \Omega$ by

$$H(u) = \begin{cases} 0 & \text{if } u \leq 0, \\ u & \text{if } 0 \leq u \leq M, \\ M & \text{if } u \geq M, \end{cases} \quad \tilde{F}(x, u) = \begin{cases} 0 & \text{if } u \leq 0, \\ F(x, u) & \text{if } 0 \leq u \leq M, \\ F(x, M) & \text{if } u \geq M, \end{cases}$$

$$\tilde{G}(x, u) = \phi(x) u + (1 - \phi(x)) \rho_s(x) \tilde{F}(x, u).$$

We show there exists $c \in H^1(\Omega)$ verifying: $\forall \varphi \in H^1(\Omega),$

$$\frac{1}{h} \int_\Omega \left(\tilde{G}(x, c) - \tilde{G}(x, u)\right)\varphi \, dx - \int_\Omega H(c) U \cdot \nabla \varphi \, dx + \int_\Omega D(x, c, U) \nabla c \cdot \nabla \varphi \, dx$$

$$+ \int_\Gamma \left(\gamma^\Gamma c \mathcal{U}^+ - g \mathcal{U}^-\right) \gamma^\Gamma \varphi \, d\sigma + \lambda \int_\Omega \tilde{G}(x, c) \varphi \, dx - \int_\Omega f \varphi \, dx = 0.$$  \hspace{1cm} (50)
The main tool for proving the existence of $c$ is the Schauder fixed point theorem. A straightforward application of the Lax–Milgram theorem using (22) allows to define the application $\Theta$ as

$$\Theta: \ L^2(\Omega) \longrightarrow L^2(\Omega), \ w \longmapsto v,$$

where $v$ is the unique function of $H^1(\Omega)$ such that for all $\varphi \in H^1(\Omega),$

$$\left(\lambda + \frac{1}{h}\right) \int_\Omega \phi \varphi \, dx + \int \frac{\rho_s}{h} \bar{F}(x, w) \varphi \, dx + \int \gamma^+ \varphi \, dx = -\lambda \int_\Omega (\bar{G}(x, u) \varphi \, dx + \int H(w) U \cdot \nabla \varphi \, dx + \int g \varphi \, d\sigma. \quad (51)$$

Every fixed point of $\Theta$ is a solution of (50). Let us show first $\Theta(\mathbb{L}^2(\Omega))$ is relatively compact in $\mathbb{L}^2(\Omega)$. Taking $\varphi = v$ in (51),

$$\left(\lambda + \frac{1}{h}\right) \int_\Omega \phi |v|^2 \, dx + \int D(x, w, U) \nabla v \cdot \nabla v \, dx + \int \gamma^+ |\gamma^v\varphi|^2 \, d\sigma = -\lambda \int_\Omega (\bar{G}(x, u) v \, dx + \int H(w) U \cdot \nabla v \, dx + \int g \varphi \, d\sigma + \int |\varphi| \, dx. \quad (52)$$

Consider three positive real numbers $a_1, a_2, a_3$ and use the coerciveness of $D$ (22), the boundedness (13), the Young inequality to obtain

$$\frac{\phi}{h} \|v\|^2_{L^2(\Omega)} + \alpha \|\nabla v\|^2_{L^2(\Omega)} \leq a_1 \|v\|^2_{L^2(\Omega)} + \frac{1}{4a_1} \left(\lambda + \frac{1}{h}\right)^2 \|\bar{F}(., w)\|^2_{L^2(\Omega)} + \frac{1}{h^2} \|\bar{G}(., u)\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)}$$

$$+ a_2 \|\nabla v\|^2_{L^2(\Omega)} + \frac{\|H(w)\|^2_{L^\infty(\Omega)} \|U\|^2_{L^2(\Omega)}}{4a_2}$$

$$+ a_3 \|\gamma^v\varphi\|^2_{L^2(\Gamma)} + \frac{\|g\|^2_{L^\infty(\Gamma)} \|U^-\|^2_{L^2(\Gamma)}}{4a_3}.$$

Then, select $a_1 = \frac{\phi}{4h}, a_2 = \frac{\alpha}{4}, a_3 = \frac{1}{A^2} \min(\frac{\varphi}{4}, \frac{\phi}{4h}),$ where $A^2$ is a constant such that $\forall \varphi \in H^1(\Omega), \|\gamma^v\varphi\|^2_{L^2(\Gamma)} \leq A^2 \|\varphi\|^2_{H^1(\Omega)}.$ Since $\bar{F}$ is uniformly bounded in $L^\infty(\Omega)$, the above estimate is reduced to

$$\frac{\phi}{2h} \|v\|^2_{L^2(\Omega)} + \frac{\alpha}{2} \|\nabla v\|^2_{L^2(\Omega)} \leq A. \quad (52)$$
where $A$ is a constant independent of $w$. Thus $\Theta(L^2(\Omega))$ is bounded in $H^1(\Omega)$, so $\Theta(L^2(\Omega))$ is relatively compact in $L^2(\Omega)$.

Let us show $\Theta$ is a continuous mapping. Let $(w_n)_n$ be a sequence converging towards a function $w$ in $L^2(\Omega)$. Setting $v_n = \Theta(w_n)$, the objective is to show $v_n$ converges to $\Theta(w)$ in $L^2(\Omega)$.

Extract first from $(w_n)_n$ a sequence $(w_{nj})_j$ converging almost everywhere in $\Omega$ to $w$. Since $\tilde{F}(x,u), H(u), D(x,u,\eta)$ are bounded, and since they are continuous in the variable $u$, the dominated convergence theorem claims that

$$\tilde{F}(., w_{nj}) \to \tilde{F}(., w), \quad H(w_{nj}) \to H(w) \quad \text{strongly in } L^2(\Omega)$$

$$D(., w_{nj}, U) \to D(., w, U) \quad \text{strongly in } (L^2(\Omega))^{d^2}, \quad \text{as } j \to +\infty.$$ 

From (52), $(v_{nj})_j$ is bounded in $H^1(\Omega)$ which is relatively compact in $H^{1-\varepsilon}(\Omega)$, $0 < \varepsilon \leq 1$, thus there exists a subsequence $(v_{njq})_q$ and a function $v \in H^1(\Omega)$ such that as $q \to +\infty$,

$$v_{njq} \rightharpoonup v \quad \text{weakly in } H^1(\Omega),$$

$$v_{njq} \to v \quad \text{strongly in } L^2(\Omega) \text{ and a.e. in } \Omega,$$

$$\gamma_G v_{njq} \to \gamma_G v \quad \text{strongly in } L^2(\Gamma).$$

Passing to the limit as $q \to +\infty$ in (51) where $v$ and $w$ are replaced by $v_{njq}$ and $w_{njq}$ respectively yields

$$v = \Theta(w).$$

The subsequence $v_{njq}$ converges to $\Theta(w)$ as $q \to +\infty$, and the same arguments also show that every subsequence of $(v_n)_n$ converging in $L^2(\Omega)$ has for limit $\Theta(w)$. Hence the sequence $(v_n)_n$ has a unique accumulation point, and since it is included in a relatively compact subset of $L^2(\Omega)$, the whole sequence $(v_n)_n$ converges to $\Theta(w)$ in $L^2(\Omega)$, which proves $\Theta$ is continuous.

The Schauder fixed point theorem allows to conclude on the existence of a fixed point $c \in H^1(\Omega)$ for $\Theta$. This completes (50).

The maximum principle (47) is obtained in the same way as in [8–10].

Assuming $0 \leq u \leq \bar{u}$, let us show that $c \geq 0$ a.e. in $\Omega$. From a result of Stampacchia (see [16, p. 54]) $c^-$ belongs to $H^1(\Omega)$. Taking $\varphi = -c^-$ in (50), one has

$$-\left(\frac{1}{h} + \lambda\right) \int_\Omega \tilde{G}(x, c^-) c^- \, dx + \frac{1}{h} \int_\Omega \tilde{G}(x, u) c^- \, dx$$

$$+ \int_\Omega H(c) U \cdot \nabla c^- \, dx + \int_\Omega D(x, c, U) \nabla c^- \cdot \nabla c^- \, dx + \int_\Gamma \mathcal{U}^+ |\gamma_G c^-|^2 \, d\sigma$$

$$+ \int_\Gamma g(\sigma) \mathcal{U}^- \gamma_G c^- \, d\sigma + \int_\Omega f(x) c^- \, dx = 0. \quad (53)$$
All the terms except the first one of the left-hand side of this equation are nonnegative, so
\[
\left(\frac{1}{h} + \lambda\right) \int_{\Omega} -\tilde{G}(x, c)c^- \, dx \leq 0,
\]
and then, since \(-\tilde{G}(x, c)c^- = \phi(x)|c^-|^2 \geq 0\), \(c^-\) finally vanishes almost everywhere.

Next, recalling \(M = \bar{u} + h\|f/\phi\|_{L^{\infty}(\Omega)}\), let us show \((c - M)^+ = 0\) for a.e. \(x \in \Omega\). For this, taking \(\varphi = (c - M)^+\) in (50), we get
\[
\frac{1}{h} \int_{\Omega} (\tilde{G}(x, c) - \tilde{G}(x, u))(c - M)^+ \, dx - \int_{\Omega} H(c) U \cdot \nabla (c - M)^+ \, dx
\]
\[
+ \int_{\Gamma} (\gamma_\Gamma \partial U^+ - \gamma_\Gamma^-) \gamma_\Gamma (c - M)^+ \, d\sigma - \int_{\Omega} f(c - M)^+ \, dx
\]
\[
+ \lambda \int_{\Omega} \tilde{G}(x, c)(c - M)^+ \, dx + \int_{\Omega} D(x, c, U) \nabla (c - M)^+ \cdot \nabla (c - M)^+ \, dx = 0. \quad (54)
\]
Using one more time (22), and since \(\tilde{G}(x, c(x))\) is nonnegative if \(c(x)\) is nonnegative, the two last terms of the left-hand side of the above equation are nonnegative. Let us prove the inequality
\[
I := -\int_{\Omega} H(c) U \cdot \nabla (c - M)^+ \, dx + \int_{\Gamma} (\gamma_\Gamma \partial U^+ - \gamma_\Gamma^-) \gamma_\Gamma (c - M)^+ \, d\sigma \geq 0. \quad (55)
\]
From the definition of \(H\) and according to Stampacchia (see [16]) one has \(H(c) U \cdot \nabla (c - M)^+ = MU \cdot \nabla (c - M)^+\) almost everywhere in \(\Omega\); so, by using Stokes’ formula, \(\text{div} \, U = 0\), and \(\gamma_\partial U = U\), we obtain
\[
I = -\int_{\Gamma} MU_\gamma_\Gamma (c - M)^+ \, d\sigma + \int_{\Gamma} (\gamma_\Gamma \partial U^+ - \gamma_\Gamma^-) \gamma_\Gamma (c - M)^+ \, d\sigma
\]
\[
= \int_{\Gamma} U^+ (\gamma_\Gamma (c - M)^+ \gamma_\Gamma (c - M)^+ \, d\sigma + \int_{\Gamma} U^- (M - g) \gamma_\Gamma (c - M)^+ \, d\sigma.
\]
Since \(\gamma_\Gamma (c - M)^+ = (\gamma_\Gamma (c - M))^+\), the first term of the above equation is nonnegative, and since \(0 \leq g \leq \bar{u}\), the second term is also nonnegative, which proves (55).

From (54), we deduce that
\[
\frac{1}{h} \int_{\Omega} \phi \left( c - u - h \frac{f}{\phi} \right)(c - M)^+ \, dx + \frac{1}{h} \int_{\Omega} (1 - \phi) \rho_\delta (\tilde{F}(x, c) - \tilde{F}(x, u))(c - M)^+ \, dx \leq 0.
\]
Because \(u \leq \bar{u} \leq M\) a.e. and \(v \mapsto \tilde{F}(x, v)\) is a monotone nondecreasing mapping, the second term of the left-hand side of the above inequality is nonnegative. Moreover, since \(u + h \frac{1}{\phi} \leq M\) a.e.,
which proves \((c - M)^+ = 0\) for a.e. \(x \in \Omega\).

To prove (49) select \(\varphi = c\) as test function in (48). Then remark

\[
\int_\Omega c U \cdot \nabla c \, dx = \frac{1}{2} \int_\Gamma |\gamma \Gamma| c^2 \, d\sigma,
\]
so that

\[
\frac{1}{h} \int_\Omega (G(x, c) - G(x, u)) \, c \, dx + \int_\Omega D(x, c, U) \nabla c \cdot \nabla c \, dx
+ \lambda \int_\Omega G(x, c)c \, dx + \frac{1}{2} \int_\Gamma |U| |\gamma \Gamma| c^2 \, d\sigma = \int_\Omega fc \, dx + \int_\Gamma U^{-g} \gamma \Gamma c \, d\sigma.
\]

Estimate (49) is then a consequence of (22) and (46). \(\square\)

4. Proof of Theorem 3

The proof is based on a discretization method in Banach spaces. It is inspired by [1,14]. The semi-discretization method consists on approaching \(\partial_t G(x, c(t, x))\) in (27) by

\[
\frac{G(x, c(t + h, x)) - G(x, c(t, x))}{h}.
\]

This approach is particularly adapted to the model, because \(G\) is generally nonlinear and is not assumed locally Lipschitz continuous.

4.1. Construction of an approximating solution

Similarly to [14], we start with defining two interpolation operators. Let \(\mathcal{E}\) be a Banach space. Let be \(T > 0, N \in \mathbb{N}^*, h = \frac{T}{N}\).

For all \(u = (u_0, u_1, \ldots, u_N) \in \mathcal{E}^{N+1}\), the constant interpolation operator is defined by \(\Pi^0_N u : [0; T] \rightarrow \mathcal{E}\),

\[
\begin{cases}
\Pi^0_N u(0) = u_0, \\
\Pi^0_N u(t) = \sum_{n=0}^{N-1} u_{n+1} \chi_{[nh, (n+1)h]}(t) & \text{if } 0 < t \leq T,
\end{cases}
\]

\(\chi_{[nh, (n+1)h]}\) being the characteristic function in \([nh, (n+1)h]\). We let \(\tilde{\Pi}^0_N u\) denote the extension of \(\Pi^0_N u\) in \([-h; T]\), with \(u_0\) value on \([-h, 0[\].

The linear interpolation operator is defined by \(\Pi^1_N u : [0; T] \rightarrow \mathcal{E}\),

\[
\Pi^1_N u(t) = \sum_{n=0}^{N-1} \left[ \left( 1 + \frac{t - \frac{n}{h}}{h} \right) u_n + \left( \frac{t}{h} - n \right) u_{n+1} \right] \chi_{[nh, (n+1)h]}(t).
\]
The function $\Pi_N^1 u$ is continuous and its derivative is given for all $t \neq nh$ by

$$
\frac{d}{dt} \left( \Pi_N^1 u(t) \right) = \sum_{n=0}^{N-1} \frac{1}{h} \left( u_{n+1} - u_n \right) \chi_{[nh, (n+1)h)}(t).
$$

Note that

$$
\left\| \Pi_N^0 u \right\|_{L^p((0,T);\mathcal{E})} = \left( h \sum_{n=1}^{N} \|u_n\|_{\mathcal{E}}^p \right)^{1/p} \quad \text{if } 1 \leq p < \infty,
$$

$$
\Pi_N^0 u \xrightarrow{L^\infty((0,T);\mathcal{E})} \max_{n=1,\ldots,N} (\|u_n\|_{\mathcal{E}}).
$$

Next, for all function $\zeta$ in $L^1((0,T);\mathcal{E})$, we define the averaging operator $\Lambda_N$ by $\Lambda_N \zeta = ((\Lambda_N \zeta)_0, \ldots, (\Lambda_N \zeta)_N) \in \mathcal{E}^{N+1}$, with

$$
(\Lambda_N \zeta)_0 = 0, \quad (\Lambda_N \zeta)_n = \frac{1}{h} \int_{(n-1)h}^{nh} \zeta(t) \, dt \quad \text{for } 0 < n \leq N.
$$

If $\zeta \in L^p((0,T);\mathcal{E})$, there holds

$$
\left\| \Pi_N^0 \Lambda_N \zeta \right\|_{L^p((0,T);\mathcal{E})} \leq \|\zeta\|_{L^p((0,T);\mathcal{E})} \quad \text{if } 1 \leq p \leq \infty,
$$

$$
\Pi_N^0 \Lambda_N \zeta \xrightarrow{N \to +\infty} \zeta \quad \text{strongly in } L^p((0,T);\mathcal{E}) \quad \text{if } 1 \leq p < \infty.
$$

Finally, we define the $t'$-translated of $\zeta$ in $(0, T - t')$ by

$$
(\tau_{-t'} \zeta)(t) = \zeta(t + t').
$$

Setting $f^N = \Lambda_N f$, $g^N = \Lambda_N g$, $\nu^N = \Lambda_N \nu$, we define $c^N = (c^N_0, c^N_1, \ldots, c^N_N)$, $p^N = (p^N_0, p^N_1, \ldots, p^N_N)$ and $V^N = (V^N_0, V^N_1, \ldots, V^N_N)$ by

$$
c^N_0 = c_0, \quad p^N_0 = 0, \quad V^N_0 = 0.
$$

For $n = 0, \ldots, N$, if $c^N_n \in L^2(\Omega)$ is known, $p^N_{n+1}$ is defined as the unique solution of

$$
\begin{cases}
p^N_{n+1} \in H^1(\Omega), & \int_\Omega p^N_{n+1} \, dx = 0, \\
\forall \psi \in H^1(\Omega), & \int_\Omega \frac{1}{\mu(c^N_n)} K(\nabla p^N_{n+1} - \rho(c^N_n) g) \cdot \nabla \psi \, dx + \int_\Gamma \nu^N_{n+1} \gamma_\Gamma \psi \, d\sigma = 0.
\end{cases}
$$

Then, setting

$$
V^N_{n+1} = -\frac{1}{\mu(c^N_n)} K(\nabla p^N_{n+1} - \rho(c^N_n) g),
$$

$c^N_{n+1} \in H^1(\Omega)$ is defined as a solution of

$$
0 \leq c^N_{n+1}(x) \leq \max\left(\|c^N_n\|_{L^\infty(\Omega)}, \|g^N_{n+1}\|_{L^\infty(\Gamma)}\right) + h \left\| \frac{f^N_{n+1}}{\phi} \right\|_{L^\infty(\Omega)} \quad \text{a.e. in } \Omega,
$$

$$
f^N_{n+1}(x) \leq \max\left(\|c^N_n\|_{L^\infty(\Omega)}, \|g^N_{n+1}\|_{L^\infty(\Gamma)}\right) + h \left\| \frac{f^N_{n+1}}{\phi} \right\|_{L^\infty(\Omega)} \quad \text{a.e. in } \Omega.
$$
\[
\frac{1}{h} \int_{\Omega} (G(x, c_{n+1}^N) - G(x, c_n^N)) \varphi \, dx - \int_{\Omega} c_{n+1}^N \nabla \varphi \, dx \\
+ \int_{\Omega} D(x, c_{n+1}^N, V_{n+1}^N) \nabla c_{n+1}^N \cdot \nabla \varphi \, dx + \int_{\Gamma} (\gamma_{\Gamma} c_{n+1}^N (V_{n+1}^N)^+ - g_{n+1}^N (V_{n+1}^N)^-) \gamma_{\Gamma} \varphi \, d\sigma \\
+ \lambda \int_{\Omega} G(x, c_{n+1}^N) \varphi \, dx - \int_{\Omega} f_{n+1}^N \varphi \, dx = 0 \quad \text{for all } \varphi \in H^1(\Omega). \tag{62}
\]

The existence of a unique \( p_{n+1}^N \) satisfying (59) with \( \psi \in H^1(\Omega) \) such that \( \int_{\Omega} \psi \, dx = 0 \) is a straightforward consequence of the Lax–Milgram theorem, and then, since (9) holds, existence of \( p_{n+1}^N \) satisfying (59) for all \( \psi \in H^1(\Omega) \) is obtained using \( \psi - \int_{\Omega} \psi \, dx \) as test function combined with the Poincaré–Wirtinger inequality, and uniqueness follows (see [8,10]). Moreover, \( \text{div} \ V_{n+1} = 0 \) a.e. in \( \Omega \), \( \gamma_{\Gamma} V_{n+1} = V_{n+1} \), and therefore the existence of \( c_{n+1}^N \) verifying (61), (62) comes from Proposition 7.

In the next two paragraphs, uniform a priori estimates on the interpolations of \( c^N, p^N \) and \( V^N \) with respect to \( N \) are obtained, and letting \( N \) tend to \(+\infty\), we conclude on the existence of a weak solution of (1)–(6).

### 4.2. Estimates

From (56) and (57), we deduce the uniform bounds

\[
\left\| \Pi_N^0 f^N \right\|_{L^1((0,T);L^1(\Omega))} \leq \| f \|_{L^1((0,T);L^1(\Omega))}, \tag{63}
\]

\[
\left\| \Pi_N^0 f^N \right\|_{L^2((0,T);(H^1(\Omega))')} \leq \| f \|_{L^2((0,T);(H^1(\Omega))')}, \tag{64}
\]

\[
\left\| \Pi_N^0 V^N \right\|_{L^p((0,T);L^p(\Gamma))} \leq \| V \|_{L^p((0,T);L^p(\Gamma))}, \quad p = 1,2, \tag{65}
\]

\[
\left\| g_{n}^N \right\|_{L^\infty(\Gamma)} \leq \| g \|_{L^\infty(\Sigma_T)}. \tag{66}
\]

This section begins with a uniform \( L^\infty \) estimate for \( \Pi_N^0 c^N \). It is the key point of the proof.

**Proposition 8.** For all \( N \), for all \( n = 1, \ldots, N \), for a.e. \( x \in \Omega \),

\[
0 \leq c_n^N(x) \leq \tilde{c} + \frac{nh}{\phi} \left\| f(s,.) \right\|_{L^\infty(\Omega)} \quad ds \leq M_0, \tag{67}
\]

with \( M_0 = \tilde{c} + \| f/\phi \|_{L^1((0,T);L^\infty(\Omega))} \).

**Proof.** A recurrence process in (61) combined with (66) supplies \( 0 \leq c_n^N \leq \tilde{c} + h \sum_{i=1}^{n} \| f_i^N /\phi \|_{L^\infty(\Omega)} \) a.e. in \( \Omega \) for \( n \geq 1 \). The proof is achieved since

\[
\sum_{i=1}^{n} h \left\| f_i^N /\phi \right\|_{L^\infty(\Omega)} \leq \sum_{i=1}^{n} \int_{(i-1)h}^{ih} \left\| f(s,.) /\phi \right\|_{L^\infty(\Omega)} \quad ds = \int_{0}^{nh} \left\| f(s,.) /\phi \right\|_{L^\infty(\Omega)} \quad ds. \quad \square
\]
It allows to obtain an $L^2((0, T); (L^2(\Omega))^d)$-estimate for $\Pi_N^0 V^N$ in the next proposition.

**Proposition 9.** There exists a constant $A$ independent of $N$ such that

$$\left\| \Pi_N^0 V^N \right\|_{L^2((0, T); (L^2(\Omega))^d)} \leq A. \tag{68}$$

**Proof.** The proof is classical (see [8,10] and the references contained therein). Its basic steps are given for the sake of completeness. From the continuity of $\rho$ and $\mu$ and Proposition 8, the functions $\rho(c^N_n)$ and $\mu(c^N_n)$ are uniformly bounded in $L^\infty(Q_T)$ independently of $N$ and $n$. Then, taking $\psi = p^N_n$ in (59), using assumptions (23)–(26), and the Cauchy–Schwarz inequality, it follows easily that there exists a constant $A'$ independent of $N$ and $n$ such that

$$\left\| \nabla p^N_{n+1} \right\|_{(L^2(\Omega))^d}^2 \leq A' \left( \left\| \nabla p^N_{n+1} \right\|_{(L^2(\Omega))^d} + \left\| \gamma^N p^N_{n+1} \right\|_{L^2(\Gamma)} \right),$$

and with the continuity from $H^1(\Omega)$ to $L^2(\Gamma)$ of the trace application, the Poincaré–Wirtinger inequality and Young’s inequality, there exists a constant still denoted by $A'$ such that

$$\left\| \nabla p^N_{n+1} \right\|_{(L^2(\Omega))^d}^2 \leq A'(1 + \left\| \gamma^N_{n+1} \right\|_{L^2(\Gamma)}^2).$$

Multiplying by $h$ and summing on from $n = 0$ to $n = N - 1$,

$$\left\| \nabla \Pi_N^0 p^N \right\|_{L^2((0, T); (L^2(\Omega))^d)}^2 \leq A'T + A' \left\| \Pi_N^0 V^N \right\|_{L^2((0, T); (L^2(\Gamma)))}^2,$$

and therefore (65) leads to a uniform $L^2((0, T); (L^2(\Omega))^d)$-estimate with respect to $N$ for $\nabla \Pi_N^0 p^N$. Looking, finally, at (60), estimate (68) is a direct consequence of

$$\Pi_N^0 V^N = -\frac{1}{\mu(\tau h \Pi_N^0 c^N)} K \left( \nabla \Pi_N^0 p^N - \rho(\tau h \Pi_N^0 c^N) g \right). \tag{69}$$

Now we are concerned with some estimates which lead to a compactness result on the concentrations. Define $G(\cdot, c^N) \in (L^2(\Omega))^{N+1}$ by

$$G(\cdot, c^N) = (G(\cdot, c^0(\cdot)), G(\cdot, c^1(\cdot)), \ldots, G(\cdot, c^N(\cdot))),$$

and remark that for all $t \in [0, T]$, for a.e. $x \in \Omega$,

$$(\Pi_N^0 G(\cdot, c^N))(t, x) = G(x, \Pi_N^0 c^N(t, x)).$$

This commutativity property for $\Pi_N^0$ does not occur for $\Pi_N^1$. The following proposition holds.
Proposition 10. There exists a constant $A$ independent of $N$ such that

$$
\| \Pi^0_N c^N \|_{L^2((0,T);H^1(\Omega))} \leq A,
$$
(70)

$$
\forall h' > 0, \quad \| \tau_{-h'} \Pi^0_N c^N - \Pi^0_N c^N \|_{L^2((0,T-h');L^2(\Omega))}^2 \leq Ah',
$$
(71)

$$
\| \partial_t (\Pi^1_N G(\cdot,c^N)) \|_{L^2((0,T);(H^1(\Omega)))'} \leq A,
$$
(72)

$$
\| \Pi^1_N G(\cdot,c^N) - \Pi^0_N c^N \|_{L^2((0,T);(H^1(\Omega)))'} \leq Ah.
$$
(73)

Remark 11. The estimate (71) deals with time translates of approximate solutions, and estimate (70) involves an analogous estimate on space translates, then the sequence $(\Pi^0_N c^N)_N$ is relatively compact in $L^2((0,T);L^2(\Omega))$, which is an immediate consequence of the Kolmogorov theorem [2,5,6].

In the other hand, estimate (71) means $\Pi^0_N c^N$ is bounded in the Nikolskii space $N^1_{1/2}((0,T);L^2(\Omega))$ (see [18]). This uniform bound together with (70) allows to use the compactness criterion of Simon (see [17]), the result of which is the strong convergence of the sequence $\Pi^0_N c^N$ in $L^2((0,T);L^2(\Omega))$.

Proof of Proposition 10. Let us prove estimate (70). Inequality (49) in Proposition 7 applied to (62), together with (66) and (67) ensures: $\forall n = 0, \ldots, N - 1$

$$
\frac{1}{h} \left( \int_\Omega B(x,c^N_{n+1}) d x - \int_\Omega B(x,c^N_n) d x \right) + \alpha \| \nabla c^N_{n+1} \|^2_{(L^2(\Omega))d} \leq M_0 \| f^N_{n+1} \|_{L^1(\Omega)} + \tilde{c} M_0 \| \gamma^N_{n+1} \|_{L^1(\Gamma)},
$$
where $M_0$ is defined in Proposition 8. Multiplying the above inequality by $h$ and summing from $n = 0$ to $n = N - 1$,

$$
\int_\Omega B(x,c^N_n) d x - \int_\Omega B(x,c^N_0) d x + \alpha \sum_{n=0}^{N-1} h \| \nabla c^N_{n+1} \|^2_{(L^2(\Omega))d} \leq \sum_{n=0}^{N-1} h \left( M_0 \| f^N_{n+1} \|_{L^1(\Omega)} + \tilde{c} M_0 \| \gamma^N_{n+1} \|_{L^1(\Gamma)} \right),
$$
and then, using (63) and (65),

$$
\int_\Omega B(x,c^N_n) d x + \alpha \| \nabla \Pi^0_N c^N \|^2_{L^2((0,T);(L^2(\Omega)))d} \leq \int_\Omega B(x,c^N_0(x)) d x + M_0 \| f \|_{L^1(Q_T)} + \tilde{c} M_0 \| \gamma \|_{L^1(\Sigma_T)},
$$

which establishes (70) since $B(x,c^N_N(x))$ is a.e. nonnegative.
Now let us prove (71), which is crucial to obtaining compactness property for the sequence $\Pi_{N}^{0}c^{N}$ in $L^2((0, T); L^2(\Omega))$. Such a result on time translates estimation of approximate solutions is classical in the analysis of the convergence of implicit finite volume schemes for nonlinear parabolic equations (see [5, Chapter 4] and [6, for example]). The proof of estimate (71) is inspired by [5] and especially from the section devoted to the analysis of convergence in the nonlinear case.

Let be $h' > 0$, one has

$$I := \left\| \tau_{-h'}^{T-h'} \Pi_{N}^{0}c^{N} - \Pi_{N}^{0}c^{N} \right\|^2_{L^2((0, T-h'), L^2(\Omega))}$$

$$= \int_0^{T-h'} \int_{\Omega} \left( \Pi_{N}^{0}c^{N}(t + h', x) - \Pi_{N}^{0}c^{N}(t, x) \right)^2 dx dt = \int_0^{T-h'} A(t) dt (74)$$

for almost every $t \in (0, T - h')$, $A(t) = \int_{\Omega} \left( \Pi_{N}^{0}c^{N}(t + h', x) - \Pi_{N}^{0}c^{N}(t, x) \right)^2 dx = \int_{\Omega} \left( c^{N}_{\lfloor (t+h')/h \rfloor}(x) - c^{N}_{\lfloor t/h \rfloor}(x) \right)^2 dx$ (denoting by $\lfloor x \rfloor$ the integer part of a real $x$) which also reads

$$A(t) = \int_{\Omega} \sum_{n=\lceil \frac{t}{h} \rceil}^{\lfloor \frac{t+h'}{h} \rfloor - 1} (c^{N}_{n+1}(x) - c^{N}_{n}(x))(c^{N}_{\lfloor (t+h')/h \rfloor}(x) - c^{N}_{\lfloor t/h \rfloor}(x)) dx.$$

Denote by $n_0(t) = \lfloor \frac{t}{h} \rfloor$ and $n_1(t) = \lfloor \frac{t+h'}{h} \rfloor$. Summing Eq. (62) from $n = n_0$ to $n_1 - 1$ and choosing $\varphi = c^{N}_{n_1} - c^{N}_{n_0}$, we have

$$\frac{1}{h} \sum_{n=n_0}^{n_1-1} \int_{\Omega} \left( G(x, c^{N}_{n+1}) - G(x, c^{N}_{n}) \right)(c^{N}_{n+1} - c^{N}_{n}) dx$$

$$\leq \sum_{n=n_0}^{n_1-1} \left( \int_{\Omega} c^{N}_{n+1} \nabla c^{N}_{n+1} \cdot \nabla (c^{N}_{n+1} - c^{N}_{n}) dx - \int_{\Omega} D(x, c^{N}_{n+1}, V^{N}_{n+1}) \nabla c^{N}_{n+1} \cdot \nabla (c^{N}_{n+1} - c^{N}_{n}) dx \right.$$

$$- \int_{\partial\Omega} (\gamma_{\Gamma} c^{N}_{n+1} V^{N}_{n+1} - \gamma_{n+1} V^{N}_{n+1}) \gamma_{\Gamma} (c^{N}_{n+1} - c^{N}_{n}) d\sigma - \lambda \int_{\Omega} G(x, c^{N}_{n+1}) (c^{N}_{n+1} - c^{N}_{n}) dx$$

$$+ \int_{\Omega} f^{N}_{n+1} (c^{N}_{n+1} - c^{N}_{n}) dx \right). \quad (75)$$

One defines $v_i := \int_{\Omega} |\nabla c^{N}_{i+1}|^2 dx$, and

$$u_j := \int_{\Omega} M_0^2 |V^{N}_{j}|^2 + |\nabla c^{N}_{j}|^2 + 2M_0 |\lambda G(x, c^{N}_{j}) + f^{N}_{j}| dx$$

$$+ 2M_0 \int_{\partial\Omega} M_0 |V^{N}_{j}| + |g^{N}_{j}| |V^{N}_{j}| d\sigma.$$
Since $G(x, z_1) - G(x, z_2) \geq \phi(z_1 - z_2)$, from (74) and (75), the use of Young's inequality and estimate (67), implies that

$$I \leq \frac{h}{\phi}(I_1 + I_2 + I_3), \quad (76)$$

where

$$I_1 = \int_0^{T-h'} \sum_{n=n_0(t)+1}^{n_1(t)} u_n \, dt, \quad I_2 = \int_0^{T-h'} \sum_{n=n_0(t)+1}^{n_1(t)} v_{n_0(t)} \, dt, \quad I_3 = \int_0^{T-h'} \sum_{n=n_0(t)+1}^{n_1(t)} v_{n_1(t)} \, dt.$$

Next, we follow the proof given in [5, p. 106] to estimate each term in the right-hand side of (76). So that

$$I_1 \leq h' \sum_{i=1}^{N} u_i.$$

For more clarity, we reproduce this proof. Define $\chi_n(t, t + h') = 1$, if $nh \in (t, t + h')$, and $\chi_n(t, t + h') = 0$, if $nh \notin (t, t + h')$. Thus, $I_1$ may be rewritten such as

$$I_1 = \int_0^{T-h'} \sum_{n=1}^{N} u_n \chi_n(t, t + h') \, dt = \sum_{n=1}^{N} u_n \int_0^{T-h'} \chi_n(t, t + h') \, dt \leq h' \sum_{n=1}^{N} u_n,$$

since $\int_0^{T-h'} \chi_n(t, t + h') \, dt \leq h'$.

To estimate $I_2$ and $I_3$, we reproduce again the proof proposed in [5, p. 108] to get

$$I_2 \leq h' \sum_{n=1}^{N} v_n \quad \text{and} \quad I_3 \leq h' \sum_{n=1}^{N} v_n.$$

Finally, we deduce that $I \leq \frac{1}{\phi} h' \sum_{i=1}^{N} h(u_i + 2v_i)$, and returning to the definition of $v_i$, $u_i$, inequalities (63), (65), (66), and the previous a priori estimates (67), (68) and (70) allow to write

$$I \leq Ah', \quad (77)$$

where $A$ is a constant independent of $N$. This establishes (71).

To prove estimates (72) and (73), remark first that

$$\left\| \partial_t (P_N^t G(\cdot, c^N)) \right\|_{L^2((0,T);(H^1(\Omega))')}^2 = \frac{1}{h} \sum_{n=0}^{N-1} \left\| G(\cdot, c^N_{n+1}) - G(\cdot, c^N_n) \right\|_{(H^1(\Omega))'}^2.$$

Let us show

$$\frac{1}{h} \sum_{n=0}^{N-1} \left\| G(\cdot, c^N_{n+1}) - G(\cdot, c^N_n) \right\|_{(H^1(\Omega))'}^2 \leq A. \quad (78)$$
From (62), since for \( \varphi \in H^1(\Omega) \), the Cauchy–Schwarz inequality and property (21) lead to

\[
\frac{1}{h} |G(., c_{n+1}^N) - G(., c_n^N), \varphi|_{(H^1(\Omega))', H^1(\Omega)}
\]

\[
= \frac{1}{h} \int_\Omega (G(x, c_{n+1}^N) - G(x, c_n^N)) \varphi \, dx
\]

\[
\leq (\| c_{n+1}^N \|_{L^\infty(\Omega)} \| V_{n+1}^N \|_{(L^2(\Omega))^d})^d + \mathcal{D} \| \nabla c_{n+1}^N \|_{(L^2(\Omega))^d} \| \nabla \varphi \|_{(L^2(\Omega))^d}
\]

\[
+ (\| \gamma_T c_{n+1}^N \|_{L^\infty(\Gamma)} \| V_{n+1}^N \|_{L^2(\Gamma)} + \| s_{n+1}^N \|_{L^\infty(\Gamma)} \| V_{n+1}^N \|_{L^2(\Gamma)} \| \gamma_T \varphi \|_{L^2(\Gamma)}
\]

\[
+ \lambda \| G(x, c_{n+1}^N) \|_{L^2(\Omega)} \varphi \|_{L^2(\Omega)} + \| f_{n+1}^N \|_{(H^1(\Omega))'} \| \varphi \|_{H^1(\Omega)},
\]

so, using (66) and (67), there exists a positive constant \( A \) not depending on \( N \) such that for all \( n \),

\[
\frac{1}{h^2} \| G(., c_{n+1}^N) - G(., c_n^N) \|_{(H^1(\Omega))'}^2
\]

\[
\leq A(1 + \mathcal{D}^2 \| c_{n+1}^N \|_{H^1(\Omega)}^2 + \| V_{n+1}^N \|_{(L^2(\Omega))^d}^2 + \| V_{n+1}^N \|_{L^2(\Gamma)}^2 + \| f_{n+1}^N \|_{(H^1(\Omega))'}^2).
\]

Multiplying by \( h \), summing from \( n = 0 \) to \( n = N - 1 \), and using estimates (64), (65), (68) and (70), we, finally, obtain (78), and (72). Estimate (73) is a consequence of (78) since

\[
\| \Pi_N^1 G(., c^N) - \Pi_N^0 G(., c^N) \|^2_{L^2((0,T);(H^1(\Omega))')}
\]

\[
= \sum_{n=0}^{N-1} \int_{\frac{n}{h}}^{\frac{n+1}{h}} \left(1 + \frac{t}{h}\right)^2 (G(x, c_n^N) - G(x, c_{n+1}^N)) \|_{(H^1(\Omega))'}^2 \, dt
\]

\[
= \frac{h}{3} \sum_{n=0}^{N-1} \| G(x, c_n^N) - G(x, c_{n+1}^N) \|_{(H^1(\Omega))'}^2 \leq \frac{A h^2}{3}. \quad \square
\]

4.3. Passing to the limit

**Proposition 12.** There exists a subsequence of \( (c^N)_{N \in \mathbb{N}^*}, \) still denoted by \( (c^N)_{N \in \mathbb{N}^*}, \) and a function \( c \) in \( L^\infty(Q_T) \cap L^2((0,T);H^1(\Omega)) \) such that \( t \mapsto G(., c(t,.)) \in C^0((0,T);(H^1(\Omega))') \) and

\[
\Pi_N^0 c^N \rightharpoonup c \quad \text{weakly-* in } L^\infty(Q_T),
\]

\[
\Pi_N^0 c^N \rightharpoonup c \quad \text{weakly in } L^2((0,T);H^1(\Omega)),
\]

\[
\Pi_N^0 c^N \rightharpoonup c \quad \text{strongly in } L^2(Q_T) \text{ and a.e. in } Q_T,
\]

\[
\gamma_T \Pi_N^0 c^N \rightharpoonup \gamma_T c \quad \text{strongly in } L^2(\Sigma_T),
\]

\[
\Pi_N^0 G(., c^N) \rightharpoonup G(., c) \quad \text{strongly in } L^2(Q_T) \text{ and a.e. in } Q_T,
\]

\[
\Pi_N^1 G(., c^N) \rightharpoonup G(., c) \quad \text{weakly-* in } L^\infty(Q_T),
\]
\[ \partial_t \left( \Pi_N^1 G(.,c^N) \right) \rightarrow \partial_t (G(.,c)) \text{ weakly in } L^2((0,T);(H^1(\Omega))'), \quad (85) \]

\[ \Pi_N^1 G(.,c^N) \rightarrow G(.,c) \text{ strongly in } C^0([0,T];(H^1(\Omega))') \quad (86) \]

as \( N \rightarrow +\infty \). Moreover, \( c \) verifies:

\[ G(x,c(0,x)) = G(x,c_0(x)) \quad \text{a.e.}, \quad (87) \]

\[ 0 \leq c(t,x) \leq \bar{c} + \int_0^t \left\| \frac{f(s,.)}{\phi} \right\|_{L^\infty(\Omega)} ds \quad \text{for a.e. } (t,x) \in Q_T. \quad (88) \]

**Proof.** These convergences are straightforward applications of the estimates of Proposition 10. We proceed step-by-step by extracting of a new subsequence of \((c^N)_{N \in \mathbb{N}^*}\) still denoted by \((c^N)_{N \in \mathbb{N}^*}\). First, assertions (79) and (80) are directly deduced from estimates (67) and (70). Since \( H^1(\Omega) \) is compactly embedded in \( H^{1-\varepsilon}(\Omega) \) for \( 0 < \varepsilon \leq 1 \), a compactness criterion of Simon [17] used with estimates (70) and (71) ensures the sequence \( \Pi_0^N c_N \) belongs to a relatively compact subset of \( L^2((0,T);(H^1(\Omega))') \). This yields (81), (82), and since \( G(x,u) \) is continuous in the variable \( u \) and \( \Pi_0^N c_N \) is uniformly bounded, convergence (83) follows from the Lebesgue dominated convergence theorem.

Next, estimates (67) and (72) permit to prove there exist \( w \) and a subsequence \( c^N \) such that as \( N \rightarrow +\infty \),

\[ \Pi_N^1 G(.,c^N) \rightharpoonup w \text{ weakly-\( \star \) in } L^\infty(Q_T), \]

\[ \partial_t \left( \Pi_N^1 G(.,c^N) \right) \rightharpoonup \partial_t w \text{ weakly in } L^2((0,T);(H^1(\Omega))'). \]

The space \( L^\infty(\Omega) \) being compactly embedded into \( (H^1(\Omega))' \), one deduces (see [17])

\[ \Pi_N^1 G(.,c^N) \rightarrow w \text{ strongly in } C^0([0,T];(H^1(\Omega))') \quad \text{as } N \rightarrow +\infty. \]

Furthermore, estimate (73) in invoked to obtain

\[ \left\| \Pi_N^1 G(.,c^N) - \Pi_N^0 G(.,c^N) \right\|_{L^2((0,T);(H^1(\Omega))')} \xrightarrow{N \rightarrow +\infty} 0. \]

This, combined with the strong convergence of \( \Pi_N^0 G(.,c^N) \) towards \( G(.,c) \) in \( L^2(Q_T) \) shows \( \Pi_N^1 G(.,c^N) \) tends to \( G(.,c) \) strongly in \( L^2((0,T);(H^1(\Omega))') \), and therefore \( w = G(.,c) \), which establishes (84)-(86).

Next, from (86), \( \Pi_N^1 G(.,c^N)(0) \rightarrow G(.,c(0,.)) \) strongly in \( (H^1(\Omega))' \), and since for all \( N \), \( \Pi_N^1 G(.,c^N)(0) = G(.,c_0), G(.,c(0,.)) = G(.,c_0(.)), \) that is (87).

Finally, to prove (88), one sets \( \tilde{n} = \left[ \frac{T}{N} \right] + 1 \), for all \( t \in (0,T) \), where \( [\cdot] \) denotes the integer part function, and one remarks first that \( \Pi_N^0 c_N(t,x) = c_N^\tilde{n}(x) \), so from (67), one has for a.e. \( (t,x) \in Q_T \),

\[ 0 \leq \Pi_N^0 c_N(t,x) \leq \bar{c} + \int_0^T \left\| \frac{f(s,.)}{\phi} \right\|_{L^\infty(\Omega)} \chi_{[0,\tilde{n}h]}(s) ds. \]
Observing
\[ \tilde{h} = \left( \left( \frac{N}{T} \right) \right) t + 1 \]
the almost everywhere convergence of \( \Pi_N^0 c^N \) towards \( c \) and the dominated convergence theorem enable to end the proof. \( \square \)

Now a strong convergence result for the sequence \( \Pi_N^0 V^N \) is shown.
First, there exists a unique \( p \in L^\infty((0, T); H^1(\Omega)) \) such that for all \( \psi \in L^2((0, T); H^1(\Omega)) \),
\[
\int_{Q_T} \frac{1}{\mu(c)} K \left( \nabla p - \rho(c) \bar{g} \right) \cdot \nabla \psi \, dx + \int_{\Sigma_T} \gamma_{\Gamma} \psi \, d\sigma \, dt = 0,
\]
\[
\int_{\Omega} p(t, x) \, dx = 0 \quad \text{for a.e.} \, t \in (0, T),
\]
where \( c \) is the limit obtained at Proposition 12. This result is classical (see, e.g., [8–10]). Next, define \( V \), which belongs to \( L^2((0, T); (L^2(\Omega))^d) \) and verifies \( \text{div} \, V = 0 \), by
\[ V = -\frac{1}{\mu(c)} K \left( \nabla p - \rho(c) \bar{g} \right). \]
We prove
\[ \Pi_N^0 V^N \rightarrow V \quad \text{strongly in} \quad (L^2(Q_T))^d \quad \text{as} \quad N \rightarrow +\infty. \]
Equations (59) can be rewritten as: \( \forall \psi \in L^2((0, T); H^1(\Omega)) \),
\[
\int_{Q_T} \frac{1}{\mu(\tau_h \Pi_N^0 c^N)} K \left( \nabla \Pi_N^0 p^N - \rho(\tau_h \Pi_N^0 c^N) \bar{g} \right) \cdot \nabla \psi \, dx \, dt + \int_{\Sigma_T} \Pi_N^0 \nu^N \gamma_{\Gamma} \psi \, d\sigma \, dt = 0.
\]
Subtract this equation with (90) to get: for all \( \psi \in L^2((0, T); H^1(\Omega)) \),
\[
\int_{Q_T} \frac{1}{\mu(\tau_h \Pi_N^0 c^N)} K \left( \nabla \Pi_N^0 p^N - \nabla p \right) \cdot \nabla \psi \, dx \, dt
\]
\[ = -\int_{Q_T} \left( \frac{1}{\mu(\tau_h \Pi_N^0 c^N)} - \frac{1}{\mu(c)} \right) K \nabla p \cdot \nabla \psi \, dx \, dt
\]
\[ + \int_{Q_T} \left( \frac{\rho(\tau_h \Pi_N^0 c^N)}{\mu(\tau_h \Pi_N^0 c^N)} - \frac{\rho(c)}{\mu(c)} \right) K \bar{g} \cdot \nabla \psi \, dx \, dt - \int_{\Sigma_T} \left( \Pi_N^0 \nu^N - \nu \right) \gamma_{\Gamma} \psi \, d\sigma \, dt. \]
The end of the proof follows the same lines as [8–10]. Roughly speaking, taking \( \psi = \Pi_0 p^N - p \) as test function in (92), using (24), the continuity of the trace application and Young’s inequality,

\[
\frac{B}{4\bar{\mu}} \left\| \nabla \Pi_0^N p^N - \nabla p \right\|^2_{(L^2(Q_T))}\,d
\]

\[
\leq \frac{\bar{\mu}}{\beta} \left( \int_{Q_T} \left( \frac{1}{\mu(\tau_h \Pi_0^N c^N)} - \frac{1}{\mu(c)} \right) K \cdot \nabla p \right)^2 \, dx \, dt
\]

\[
+ \int_{Q_T} \left( \frac{\rho(\tau_h \Pi_0^N c^N)}{\mu(\tau_h \Pi_0^N c^N)} - \frac{\rho(c)}{\mu(c)} \right) K \bar{g} \right)^2 \, dx \, dt + A_T^2 \int_{\Sigma_T} \left| \Pi_0^N V^N - \mathcal{V} \right|^2 \, d\sigma \, dt
\]

(93)

where \( \bar{\mu} \) is a constant independent of \( N \) such that \( \mu(\tau_h \Pi_0^N c^N) \leq \bar{\mu} \). Convergence (58) applied to \( V \) drives to

\[
\Pi_0^N V^N \rightarrow \mathcal{V} \quad \text{in} \quad L^2((0, T); L^2(Q)).
\]

(94)

This, combined with the Lebesgue dominated convergence theorem on the continuous and bounded functions \( \rho \) and \( \mu \) allows the right-hand side of (93) tend to 0 as \( N \rightarrow +\infty \), and consequently \( \nabla \Pi_0^N p^N \) converges to \( \nabla p \) strongly in \( (L^2(Q_T))\,d \) as \( N \rightarrow +\infty \). The strong convergence of \( \Pi_0^N V^N \) towards \( V \) in \( (L^2(Q_T))\,d \) follows from (69).

To prove the pair \( (c, p) \) is a weak solution of (1)–(6) in the sense of Definition 5, it remains to check whether the variational equality (27) holds. Equations (62) yield: for all \( \varphi \in L^2((0, T); H^1(\Omega)) \),

\[
\int_0^T \left\langle \partial_t \Pi_N^N G(x, c^N, \varphi)_{(H^1(\Omega))'} , H^1(\Omega) \right\rangle \, dt - \int_{Q_T} \Pi_N^N \nabla \Pi_N^N \cdot \nabla \varphi \, dx \, dt
\]

\[
+ \int_{Q_T} D(x, \Pi_N^N c^N, \Pi_N^N V^N) \nabla \Pi_N^N c^N \cdot \nabla \varphi \, dx \, dt
\]

\[
+ \int_{\Sigma_T} \left( \gamma_{\Gamma} \Pi_N^N \left( \Pi_N^N V^N \right)^+ - \Pi_N^N g^N \left( \Pi_N^N V^N \right)^- \right) \gamma_{\Gamma} \varphi \, d\sigma \, dt
\]

\[
+ \lambda \int_{Q_T} G(x, \Pi_N^N c^N) \varphi \, dx \, dt - \int_0^T \left\langle \Pi_N^N f^N, \varphi \right\rangle_{(H^1(\Omega))'} \, dt = 0.
\]

(95)

Up to a subsequence, one can assume \( \Pi_N^N V^N \rightarrow V \) a.e. in \( Q_T \), as \( N \rightarrow +\infty \), so

\[
D(x, \Pi_N^N c^N, \Pi_N^N V^N) \xrightarrow[N \rightarrow +\infty]{} D(x, c, V) \quad \text{a.e. in} \quad Q_T.
\]

(96)

Furthermore, (58) ensures

\[
\Pi_N^N g^N \rightarrow g \quad \text{strongly in} \quad L^2(\Sigma_T), \quad \Pi_N^N f^N \rightarrow f \quad \text{strongly in} \quad L^2((0, T); (H^1(\Omega))').
\]
so that

\[ \int_0^T \langle \Pi_0^N f_N, \varphi \rangle_{(H^1(\Omega))', H^1(\Omega)} \, dt \to \int_0^T \langle f, \varphi \rangle_{(H^1(\Omega))', H^1(\Omega)} \, dt = \int_{Q_T} f \varphi \, dx \, dt. \]

The convergence result of Proposition 12 and (91), (94), (96) allow to pass to the limit in the other terms of formulation (95) and finally prove (27). This completes the proof of Theorem 3.

5. Proof of Proposition 4

The adsorption functions of the geology literature are usually concave with respect to the concentration (see Marsily [7] and the examples of the introduction). It permits to say more about the long-time behaviour of the concentration when there is no longer injection of polluted fluid in the domain.

Assume the mapping \( u \mapsto F(x, u) \) is concave, and there exists \( t_0 \in [0, T) \) such that for a.e. \((t, \sigma) \in [t_0, T] \times \Gamma, g(t, \sigma) = 0 \). Let \( c^N, p^N, V^N \) be the approximating sequences defined at subsection 4.1. Let be \( t > t_0 \). One sets

\[ \tilde{c}_n = \left[ \frac{t_0 + n}{T} \right] + 1 \quad \text{and} \quad \tilde{n} = \left[ \frac{N}{T} \right] + 1, \quad \text{so that} \quad \Pi_0^N c^N(t) = c_{\tilde{n}}^N. \]

We state the following lemma.

Lemma 13. Let be \( n \geq \tilde{c}_n, \tilde{u} > 0 \). If \( c^N_n \leq \tilde{u} \), then \( c^N_{n+1} \leq \frac{\tilde{u} + h \| f^N_{n+1}/\phi \|_{L^\infty(\Omega)}}{1 + \lambda h} \).

Proof. As \( g = 0 \) in \([nh, (n + 1)h] \times \Gamma\), Eq. (62) for \( c^N_{n+1} \) reads

\[
\frac{1}{h} \int_\Omega \left( G(x, c^N_{n+1}) - G(x, c^N_n) \right) \varphi \, dx - \int_\Omega c^N_{n+1} V^N_{n+1} \cdot \nabla \varphi \, dx \\
+ \int_\Omega D(x, c^N_{n+1}, V^N_{n+1}) \nabla c^N_{n+1} \cdot \nabla \varphi \, dx \quad + \int_\Gamma \gamma \Gamma c^N_{n+1} (V^N_{n+1})^+ \gamma \Gamma \varphi \, d\sigma \\
+ \lambda \int_\Omega G(x, c^N_{n+1}) \varphi \, dx \quad - \int_\Omega f^N_{n+1} \varphi \, dx = 0. \quad (97)
\]

The idea consists on not neglecting the nonnegative term \( \lambda G(x, c^N_{n+1}) \) in the maximum principle. Setting \( M = \tilde{u} + h \| f^N_{n+1}/\phi \|_{L^\infty(\Omega)} \), and taking \( \varphi = ((1 + \lambda h) c^N_{n+1} - M)^+ \) in (97), previous arguments drive to

\[
\int_\Omega \phi \left( (1 + \lambda h) c^N_{n+1} - c^N_n - h \frac{f^N_{n+1}}{\phi} \right) ((1 + \lambda h) c^N_{n+1} - M)^+ \, dx \\
+ \int_\Omega (1 - \phi) \rho_s \left( (1 + \lambda h) F(x, c^N_{n+1}) - F(x, c^N_n) \right) ((1 + \lambda h) c^N_{n+1} - M)^+ \, dx \leq 0. \quad (98)
\]
Since $c^N_{n+1}$ is nonnegative and $F(x, 0) = 0$, the concavity of $F$ ensures $(1 + \lambda h) F(x, c^N_{n+1}) \geq F(x, (1 + \lambda h) c^N_{n+1})$, and therefore, since $u \mapsto F(x, u)$ is monotone nondecreasing,

$$((1 + \lambda h) F(x, c^N_{n+1}) - F(x, c^N_n))(1 + \lambda h) c^N_{n+1} - M)^+ \geq 0,$$

and inequality (98) gives $$((1 + \lambda h) c^N_{n+1} - M)^+ \leq 0 \text{ a.e. in } \Omega.$$

Assume $N$ is large enough so that $\tilde{n} \geq \tilde{n}_0 + 1$. From estimate (67) in Proposition 8, $c^N_{\tilde{n}_0} \leq \int_{0}^{\tilde{n}_0 h} \| f(s, .) / \phi \|_{L^\infty(\Omega)} ds$, and then a recursive argument from $n = \tilde{n}_0$ to $n = \tilde{n}$ in (62) with the previous lemma shows that

$$\Pi_N^0 c^N(t, x) = c^N_{\tilde{n}}(x)$$

$$\leq \frac{1}{(1 + \lambda h) \tilde{n}_0 h} \left( \tilde{c} + \int_{0}^{\tilde{n}_0 h} \| f(s, .) / \phi \|_{L^\infty(\Omega)} ds + \sum_{i=\tilde{n}_0}^{\tilde{n}-1} \frac{h}{(1 + \lambda h) (\tilde{n} - i)} \left( \int_{ih}^{(i+1)h} \| f(s, .) / \phi \|_{L^\infty(\Omega)} ds \right) \right).$$

(99)

It remains to pass to the limit in (99) as $N \to +\infty$. In the same manner as (89), one has

$$\int_{0}^{\tilde{n}_0 h} \| f(s, .) / \phi \|_{L^\infty(\Omega)} ds \xrightarrow{N \to +\infty} \int_{0}^{\tilde{n}_0} \| f(s, .) / \phi \|_{L^\infty(\Omega)} ds. \quad (100)$$

Next, since $(1 + \lambda h)^{i+1} \leq e^{(i+1)\lambda h}$, there holds

$$\sum_{i=\tilde{n}_0}^{\tilde{n}-1} \frac{h}{(1 + \lambda h) (\tilde{n} - i)} \left( \int_{ih}^{(i+1)h} \| f(s, .) / \phi \|_{L^\infty(\Omega)} ds \right) \leq \frac{1}{(1 + \lambda h) \tilde{n}_0 h} \sum_{i=\tilde{n}_0}^{\tilde{n}-1} \left( \int_{ih}^{(i+1)h} \| f(s, .) / \phi \|_{L^\infty(\Omega)} ds \right) e^{(i+1)\lambda h}.$$

(101)

Denote by $e^\lambda$ the vector $(e^0, e^\lambda h, \ldots, e^{N\lambda h})$. The interpolation $\Pi_N^0 e^\lambda$ has the constant value $e^{(i+1)\lambda h}$ on each interval $[ih, (i+1)h]$, and $(\Pi_N^0 e^\lambda)(s) \to e^{\lambda s}$ for $s \in \mathbb{R}$. Thus

$$\sum_{i=\tilde{n}_0}^{\tilde{n}-1} \int_{ih}^{(i+1)h} \| f(s, .) / \phi \|_{L^\infty(\Omega)} e^{(i+1)\lambda h} ds = \int_{\tilde{n}_0 h}^{\tilde{n} h} \| f(s, .) / \phi \|_{L^\infty(\Omega)} \left( \Pi_N^0 e^\lambda \right)(s) ds$$

$$\xrightarrow{N \to +\infty} \int_{\tilde{n}_0}^{t} \| f(s, .) / \phi \|_{L^\infty(\Omega)} e^{\lambda s} ds. \quad (102)$$
Finally, remark that
\[
\frac{1}{(1 + \lambda h)^{(\bar{n} - \bar{n}_0)}} = \exp \left( - (\bar{n} - \bar{n}_0) \ln (1 + \lambda h) \right) \rightarrow e^{-\lambda (t - t_0)}, \tag{103}
\]
\[
\frac{1}{(1 + \lambda h)^{\bar{n} + 1}} \rightarrow e^{-\lambda t} \tag{104}
\]
as \(N \rightarrow +\infty\). Proposition 4 is then established from (99), using (100)–(104) and the almost everywhere convergence of \(\Pi_N^0 c^N\) towards \(c\).

6. Proof of Theorem 6

Now, consider a general displacement of \(m\) substratums, \(m \geq 1\). This corresponds to Eqs. (30)–(34). The additional difficulty in comparison to the previous section is the presence of the filiation term \(\sum_{l<k} \lambda_{l,Rk,l} G_l(., c^N_l)\). The inherent \(L^\infty\)-estimates for the concentrations are obtained in this case from the triangular form of this filiation term.

The proof of Theorem 6 follows the same steps than the one of Theorem 3.

Let be \(T > 0, N \in \mathbb{N}^*,\) and define \(h = \frac{T}{N}\). For all \(n = 0, \ldots, N\), set
\[
f^N_k = \Lambda^N f_k, \quad g^N_k = \Lambda^N g_k \quad \text{and} \quad \mathcal{V}_N = \Lambda^N \mathcal{V}, \ k = 1, \ldots, m.
\]
Also set \(\zeta^N_{k,n} = (\tilde{c}^N_k)_n, \xi_k = f_k, g_k\). In the light of Section 4.1, we introduce an algorithm approximating (30)–(34).

Define \(c^N_{k,0} = c_{k,0}, p^N_0 = 0, V^N_0 = 0\), and consider \(c^N_n = (c^N_{k,n})_{k=1,\ldots,m} \in (L^\infty(\Omega))^m\) known.

Then, \(p^N_{n+1}\) is defined by (59), and \(V^N_{n+1}\) by (60). Next, for \(k = 1, \ldots, m\), the concentrations \(c^N_{k,n+1}\) are solutions of
\[
\forall \varphi \in H^1(\Omega), \quad \frac{1}{h} \int_\Omega (G(x, c^N_{k,n+1}) - G(x, c^N_{k,n})) \varphi \, dx
\]
\[
- \int_\Omega c^N_{k,n+1} V^N_{n+1} \cdot \nabla \varphi \, dx + \int_\Omega D(x, c^N_{k,n+1}, V^N_{n+1}) \nabla c^N_{k,n+1} \cdot \nabla \varphi \, dx
\]
\[
+ \int_\Gamma (\gamma \tau c^N_{k,n+1} (\gamma^N_{n+1})^+ - g^N_{k,n+1} (\gamma^N_{n+1})^-) \gamma \tau \varphi \, d\sigma \, dx
\]
\[
+ \lambda \int_\Omega G(x, c^N_{k,n+1}) \varphi - \sum_{l<k} \lambda_{l,Rk,l} \int_\Omega G_l(x, c^N_l) \varphi \, dx - \int_\Omega f^N_{k,n+1} \varphi \, dx = 0. \tag{105}
\]

The existence of \(c^N_{k,n+1} \in H^1(\Omega) \cap L^\infty(\Omega)\) is ensured for all \(k\) by Proposition 7 using \(f = f^N_{k,n+1} + \sum_{l<k} \lambda_{l,Rk,l} G_l(., c^N_l)\). Estimate (47) becomes
\[
0 \leq c^N_{k,n+1}
\]
\[
\leq \max \left( \|c^N_{k,n}\|_{L^\infty(\Omega)}, \|g^N_{k,n+1}\|_{L^\infty(\Gamma)} \right) + h \left\| \frac{1}{\phi} \left( \sum_{l<k} \lambda_{l,Rk,l} G_l(., c^N_l) + f^N_{k,n+1} \right) \right\|_{L^\infty(\Omega)} . \tag{106}
\]
If uniform $L^\infty$-estimates with respect to $N$ and $n$ are found for $c_{k,n}^N$, the end of the proof is achieved exactly in the same manner as Section 4 because they drive to uniform estimates for $\Pi_0^N V^N$ (Proposition 9), and for the concentrations (Proposition 10). These $L^\infty$-estimates are found in the following proposition.

**Proposition 14.** There exists $M > 0$ such that for all $N$, $n = 1, \ldots, N$,

$$0 \leq c_{k,n}^N(x) \leq M \quad \text{a.e. in } \Omega, \ k = 1, \ldots, m.$$  

**Proof.** The proof is obtained with recurrence on $k$ using the bound (106) and the triangular form of the filiation term. Recall the filiation term is missing in (105) for the first substratum ($k = 1$), and only depends on the $l$th radionuclides, $l < k$, for $k \geq 2$. First, for all $k = 1, \ldots, m$, we define $m$ nonnegative constants $\tilde{c}_k$ by

$$\tilde{c}_k = \max \left( \| c_{k,0} \|_{L^\infty(\Omega)}, \| g_k \|_{L^\infty(\Sigma_T)} \right).$$

For $k = 1$, Eq. (105) is similar to (62) and allows to use Proposition 8 to get

$$0 \leq c_{1,n}^N(x) \leq M_1 \quad \text{for a.e. } x \in \Omega, \text{ for all } N, n,$$

where $M_1 = \tilde{c}_1 + \| f_1/\phi \|_{L^1((0,T);L^\infty(\Omega))}$. For $k \geq 2$, assume that for all $l < k$, there exists $M_l > 0$ such that

$$0 \leq c_{l,n}^N(x) \leq M_l \quad \text{for a.e. } x \in \Omega, \text{ for all } N, n.$$

Then from (106), for all $N, n$,

$$0 \leq c_{k,n+1}^N \leq \max \left( \| c_{k,n}^N \|_{L^\infty(\Omega)}, \| g_k \|_{L^\infty(\Sigma_T)} \right) + h r_k + h \left\| \frac{f_{k,n+1}}{\phi} \right\|_{L^\infty(\Omega)}$$

where

$$r_k = \left\| \sum_{l=1}^{k-1} \lambda_l R_{k,l} G_l(.,M_l) \frac{\phi}{\phi} \right\|_{L^\infty(\Omega)}.$$

Hence a recurrence argument for $n \geq 1$ yields: for all $N, n$,

$$0 \leq c_{k,n}^N(x) \leq \tilde{c}_k + n h r_k + \sum_{i=1}^{n} h \left\| \frac{f_{k,n}}{\phi} \right\|_{L^\infty(\Omega)}$$

$$\leq \tilde{c}_k + Tr_k + \left\| \frac{f_k}{\phi} \right\|_{L^1((0,T);L^\infty(\Omega))} := M_k,$$

which completes the proof. □

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