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Self-Adjoint Subspace Extensions of Nondensely Defined Symmetric Operators

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The self-adjoint subspace extensions of a possibly nondensely defined symmetric operator in a Hilbert space are characterized in terms of "generalized boundary conditions."

1. INTRODUCTION

Suppose S is a densely defined symmetric operator in a Hilbert space $\mathfrak{H}$, and let $\mathfrak{E}(\pm i) = \{h \in \mathfrak{D}(S^*) \mid S^*h = \pm ih\}$, where S^* is the adjoint of S and $\mathfrak{D}(S^*)$ is the domain of S^*. It was shown by von Neumann that S has a self-adjoint extension H in $\mathfrak{H}$ if and only if $\mathfrak{E}(+i)$ and $\mathfrak{E}(-i)$ have the same dimension. Let $\dim \mathfrak{E}(+i) = \dim \mathfrak{E}(-i) = \omega < \infty$, and let H be any self-adjoint extension of S in $\mathfrak{H}$. It satisfies $S \subset H = H^* \subset S^*$, and $\mathfrak{D}(H)$ may be characterized in terms of certain abstract boundary conditions in the following way. For $f, g \in \mathfrak{D}(S^*)$, let $\langle fg \rangle = (S^*f, g) - (f, S^*g)$. There exist $\delta_1, \ldots, \delta_\omega$ in $\mathfrak{D}(S^*)$, linearly independent mod $\mathfrak{D}(S)$ and satisfying $\langle \delta_j \delta_k \rangle = 0$, $j, k = 1, \ldots, \omega$, such that $\mathfrak{D}(H)$ is the set of all $f \in \mathfrak{D}(S^*)$ for which $\langle f\delta_j \rangle = 0$, $j = 1, \ldots, \omega$ (see [2, Theorem 3]). This characterization of $\mathfrak{D}(H)$ is especially appropriate in describing the self-adjoint extensions of a symmetric ordinary differential operator.

Now suppose that S is a symmetric operator in $\mathfrak{H}$, whose domain $\mathfrak{D}(S)$ is not dense in $\mathfrak{H}$. Its adjoint is not a well-defined operator. However, the set of all pairs $\{h, k\} \in \mathfrak{H}^2 = \mathfrak{H} \oplus \mathfrak{H}$ such that $(Sf, h) = (f, k)$ for all $f \in \mathfrak{D}(S)$ is a closed linear manifold (subspace) in $\mathfrak{H}^2$ which can be thought of as the adjoint subspace S^* to the graph of S (which we can identify with S) in $\mathfrak{H}^2$. More generally, we can consider symmetric subspaces S in $\mathfrak{H}^2$, which are not necessarily the graphs of operators in $\mathfrak{H}$;

these satisfy $(g, f') = (f, g')$ for all $\{f, g\}$, $\{f', g'\} \in S$. The adjoint subspace S^* is then the set of all $\{h, k\} \in \mathfrak{H}^2$ for which $(g, h) = (f, k)$ for all $\{f, g\} \in S$. A self-adjoint subspace H in $\mathfrak{H}^2$ is one such that $H = H^*$. An analog of the von Neumann result is valid for symmetric subspaces S in $\mathfrak{H}^2$ ([3, Theorem 15] and Theorem B below). It is the purpose of this paper to show how to apply this result to obtain a characterization (Theorem 3 below) of the self-adjoint subspace extensions H of a symmetric subspace S in $\mathfrak{H}^2$ in terms of "generalized boundary conditions", for that case when S is the graph of a symmetric operator ($\mathfrak{D}(S)$ not necessarily dense in $\mathfrak{H}$) satisfying $\dim(S^* \ominus S) < \infty$ and $\dim(\mathfrak{H} \ominus \mathfrak{D}(S)) < \infty$. Applications to ordinary differential operators will be considered in a subsequent paper. Announcements of these results appeared in [4] and [5].

2. Symmetric and Self-Adjoint Subspaces

In this section we collect together the definitions and results from [3] which we require. Let $\mathfrak{H}$ be a Hilbert space over the complex field $\mathbf{C}$, and let $\mathfrak{H}^2 = \mathfrak{H} \oplus \mathfrak{H}$ be the Hilbert space of all pairs $\{f, g\}$, where $f, g \in \mathfrak{H}$, with the inner product $(\{f, g\}, \{h, k\}) = (f, h) + (g, k)$. A *subspace* T in $\mathfrak{H}^2$ is a closed linear manifold in $\mathfrak{H}^2$, which we view as a linear relation whose *domain* $\mathfrak{D}(T)$ and *range* $\mathfrak{R}(T)$ are given by

$$\mathfrak{D}(T) = \{f \in \mathfrak{H} \mid \{f, g\} \in T \text{ for some } g \in \mathfrak{H}\},$$

$$\mathfrak{R}(T) = \{g \in \mathfrak{H} \mid \{f, g\} \in T \text{ for some } f \in \mathfrak{H}\}.$$

For subspaces T, S we define αT ($\alpha \in \mathbf{C}$), ST, $T + S$, T^{-1} as follows:

$$\begin{aligned} \alpha T &= \{\{f, \alpha g\} \mid \{f, g\} \in T\}, \\ ST &= \{\{f, k\} \mid \{f, g\} \in T, \{g, k\} \in S \text{ for some } g \in \mathfrak{H}\}, \\ T + S &= \{\{f, g + k\} \mid \{f, g\} \in T, \{f, k\} \in S \text{ for some } f \in \mathfrak{H}\}, \\ T^{-1} &= \{\{g, f\} \mid \{f, g\} \in T\}. \end{aligned}$$

For $f \in \mathfrak{D}(T)$, we let $T(f) = \{g \in \mathfrak{H} \mid \{f, g\} \in T\}$. A subspace T is the graph of a linear function if $T(0) = \{0\}$, and in this case we say T is an *operator* in $\mathfrak{H}$ and denote $T(f)$ by the more usual Tf. The *null space* (or *kernel*) of T is the set

$$\nu(T) = \{f \in \mathfrak{H} \mid \{f, 0\} \in T\} = T^{-1}(0).$$

There are two other sums naturally associated with any two subspaces T, S in $\mathfrak{H}^2$, the *algebraic sum* $T \dot{+} S$ of the two linear manifolds,

$$T \dot{+} S = \{\{f + h, g + k\} \mid \{f, g\} \in T, \{h, k\} \in S\},$$

and the *orthogonal sum* $T \oplus S$, which is $T \dot{+} S$ when T and S are orthogonal in $\mathfrak{H}^2$. The *orthogonal complement* of a linear manifold M in a subspace N is denoted by $N \ominus M$, and if N is all of the Hilbert space under consideration, we denote this by $M^\perp$.

The *adjoint* T^* of a subspace T in $\mathfrak{H}^2$ is defined by

$$T^* = \{\{h, k\} \in \mathfrak{H}^2 \mid (g, h) = (f, k) \text{ for all } \{f, g\} \in T\}.$$

It is a subspace, and its properties can be easily analyzed by noting that $T^* = \mathfrak{H}^2 \ominus JT = (JT)^\perp$, where J is the unitary operator on $\mathfrak{H}^2$ defined by $J\{f, g\} = \{g, -f\}$.

For any subspace T in $\mathfrak{H}^2$, let T_∞ be the set of all elements of the form $\{0, g\}$ in T, and let $T_s = T \ominus T_\infty$. Then T_s is a closed operator in $\mathfrak{H}$, called the *operator part* of T, and we have the orthogonal decomposition $T = T_s \oplus T_\infty$, with $\mathfrak{D}(T_s) = \mathfrak{D}(T)$ dense in $(T^*(0))^\perp$ and $\mathfrak{R}(T_s) \subset (T(0))^\perp$ The subspace T_∞ may be viewed as the *purely multivalued* part of T.

A *symmetric subspace* S in $\mathfrak{H}^2$ is one satisfying $S \subset S^*$, and a *self-adjoint subspace* H is one for which $H = H^*$. If $H = H_s \oplus H_\infty$ is a self-adjoint subspace in $\mathfrak{H}^2$, then H_s, considered as an operator in the Hilbert space $(H(0))^\perp$, is a densely defined self-adjoint operator (R. Arens [1, Theorem 5.3]). This allows a spectral analysis of H once its operator part H_s and purely multivalued part H_∞ have been identified.

We are interested in the *self-adjoint extensions* H of a given symmetric subspace S in $\mathfrak{H}^2$, that is, those self-adjoint H satisfying $S \subset H$. All such H, if they exist, satisfy $S \subset H \subset S^*$. In [3] (Theorems 12 and 15), we gave two characterizations of these self-adjoint extensions, which we now state as Theorems A and B.

THEOREM A. *A subspace H in $\mathfrak{H}^2$ is a self-adjoint extension of a symmetric subspace S in $\mathfrak{H}^2$ if and only if $H = S \oplus M_1$, where M_1 is a subspace of $M = S^* \ominus S$ satisfying $JM_1 = M \ominus M_1$. Hence such H can also be characterized by $H = S^* \ominus JM_1$, where $JM_1 = M \ominus M_1$.*

The subspace M can be written as $M = M^+ \oplus M^-$ where

$$M^\pm = \{\{h, k\} \in S^* \mid k = \pm ih\},$$

and in these terms Theorem B can be phrased as follows.

THEOREM B. *A subspace H in $\mathfrak{H}^2$ is a self-adjoint extension of a symmetric subspace S in $\mathfrak{H}^2$ if and only if there exists an isometry V of M^+ onto M^- such that $H = S \oplus (I - V)M^+$, where I is the identity operator. Thus S has a self-adjoint extension in $\mathfrak{H}^2$ if and only if* $\dim M^+ = \dim M^-$.

3. THE ADJOINT OF A NONDENSELY DEFINED SYMMETRIC OPERATOR

Let T_0 be a closed operator in the Hilbert space $\mathfrak{H}$ whose domain $\mathfrak{D}(T_0)$ is dense in $\mathfrak{H}$, and let $\mathfrak{H}_0$ be a subspace of $\mathfrak{H}$. We define the operator T by the requirements

$$\mathfrak{D}(T) = \mathfrak{D}(T_0) \cap \mathfrak{H}_0^{\perp}, \qquad T \subset T_0 . \tag{3.1}$$

When viewed in $\mathfrak{H}^2$, we have $T = T_0 \cap (\mathfrak{H}_0^{\perp} \oplus \mathfrak{H})$, and hence T is a subspace (that is, T is a closed operator) with $\mathfrak{D}(T)$ not dense in $\mathfrak{H}$ if $\mathfrak{H}_0 \neq \{0\}$. Thus T^* is in general a subspace which is not an operator. Under certain assumptions (which will be verified in the symmetric case), T^* can be computed in terms of $T_0{}^*$ and $\mathfrak{H}_0$.

THEOREM 1. *Let T_0 be a densely defined closed operator in $\mathfrak{H}$, and let the closed operator T be defined by* (3.1). *Suppose that* (a) $\dim \mathfrak{H}_0 = p < \infty$, (b) $\mathfrak{R}(T_0)$ *is closed*, (c) $\mathfrak{R}(T_0{}^*) = \mathfrak{R}(T^*)$. *Then*

(i) $T^*(0) = \mathfrak{H}_0 , (T^*)_\infty = \{0\} \oplus \mathfrak{H}_0$,

(ii) $\nu(T^*) = \{v \in \mathfrak{D}(T_0{}^*) \mid T_0{}^*v \in \mathfrak{H}_0\} = (T_0{}^*)^{-1}(\mathfrak{H}_0)$,

(iii) $\dim \nu(T^*) = \dim \nu(T_0{}^*) + \dim \mathfrak{H}_0$,

(iv) $T^* = T_0{}^* \dotplus (T^*)_\infty = \{\{h, T_0{}^*h + \varphi\} \mid h \in \mathfrak{D}(T_0{}^*), \varphi \in \mathfrak{H}_0\}$.

Proof. The proof of (i) makes use of (a) only. Since $\mathfrak{D}(T_0)$ is dense in $\mathfrak{H}$ and $\dim \mathfrak{H}_0 = p < \infty$, $\mathfrak{D}(T)$ is dense in $\mathfrak{H}_0{}^{\perp}$. We sketch the simple argument due to Gohberg and Krein ([6, Lemma 2.1]). Let $\varphi_1 ,..., \varphi_p$ be an orthonomal basis for $\mathfrak{H}_0$. For any $f \in \mathfrak{H}_0{}^{\perp}$, there is a sequence $f^{(k)} \in \mathfrak{D}(T_0)$ such that $f^{(k)} \to f$, $k \to \infty$. Since $\det((\varphi_j , \varphi_r)) = 1$, we can choose $\psi_1 ,..., \psi_p \in \mathfrak{D}(T_0)$ so close to $\varphi_1 ,..., \varphi_p$, respectively, that $\det((\psi_j , \varphi_r)) \neq 0$. Then let

$$g^{(k)} = \alpha_1^{(k)}\psi_1 + \cdots + \alpha_p^{(k)}\psi_p + f^{(k)},$$

where the $\alpha_j^{(k)}$ are chosen so that $g^{(k)} \in \mathfrak{H}_0^\perp$. Thus the $\alpha_j^{(k)}$ are the unique solutions of the equations $(g^{(k)}, \varphi_r) = 0$, $r = 1,..., p$, or

$$\sum_{j=1}^{p} \alpha_j^{(k)}(\psi_j, \varphi_r) = -(f^{(k)}, \varphi_r), \qquad r = 1,..., p.$$

Now $f^{(k)} \to f$ and $(f, \varphi_r) = 0$ imply that $\alpha_j^{(k)} \to 0$, $k \to \infty$, and hence $g^{(k)} \to f$, with $g^{(k)} \in \mathfrak{D}(T)$. Now $T^*(0) = (\mathfrak{D}(T))^\perp = \mathfrak{H}_0$, and consequently $(T^*)_\infty = \{0\} \oplus \mathfrak{H}_0$.

Since $\mathfrak{R}(T^*) \supset \mathfrak{H}_0$, we have from (c) that $\mathfrak{R}(T_0^*) \supset \mathfrak{H}_0$. Let $Z = (T_0^*)^{-1}(\mathfrak{H}_0)$; we show that Z is closed and that

$$\dim Z = \dim \nu(T_0^*) + \dim \mathfrak{H}_0 .$$

It is clear that $\nu(T_0^*) \subset Z$. As above, let $\varphi_1 ,..., \varphi_p$ be an orthonormal basis for $\mathfrak{H}_0$, and let $w_j \in \mathfrak{D}(T_0^*)$ satisfy the conditions $T_0^*w_j = \varphi_j$ and $w_j \in (\nu(T_0^*))^\perp$. It is easy to see that w_j exists and is unique. Indeed, for each φ_j there exists a $v_j \in \mathfrak{D}(T_0^*)$ such that $T_0^*v_j = \varphi_j$, and we can verify that $w_j = v_j - \pi_0 v_j$, where π_0 is the orthogonal projection of $\mathfrak{H}$ onto the subspace $\nu(T_0^*)$ of $\mathfrak{H}$. If W is the span of $w_1 ,..., w_p$, then clearly $\nu(T_0^*) \oplus W \subset Z$. The opposite inclusion is also valid. For let $v \in Z$ and $T_0^*v = \varphi \in \mathfrak{H}_0$. Then we can write $v = \alpha + w$, where

$$\alpha = v - \sum_{j=1}^{p} (\varphi, \varphi_j)\, w_j , \qquad w = \sum_{j=1}^{p} (\varphi, \varphi_j)\, w_j ,$$

with $w \in W$ and $T_0^*\alpha = 0$, and thus $v \in \nu(T_0^*) \oplus W$. We now have $Z = \nu(T_0^*) \oplus W$, which shows that Z is closed. The w_j are linearly independent, for if $\sum a_j w_j = 0$, then

$$T_0^* \left(\sum a_j w_j\right) = \sum a_j T_0^* w_j = \sum a_j \varphi_j = 0,$$

which implies that all $a_j = 0$, since the φ_j are a basis for $\mathfrak{H}_0$. Thus $\dim W = \dim \mathfrak{H}_0$, and we have

$$\dim Z = \dim \nu(T_0^*) + \dim W = \dim \nu(T_0^*) + \dim \mathfrak{H}_0 . \tag{3.2}$$

Now we prove that $Z = \nu(T^*)$. Let $v \in Z$ with $T_0^*v = \varphi \in \mathfrak{H}_0$. Then for all $f \in \mathfrak{D}(T) = \mathfrak{D}(T_0) \cap \mathfrak{H}_0^\perp$, we have $(Tf, v) = (T_0 f, v) = (f, T_0^*v) = (f, \varphi) = 0$. Thus $v \in (\mathfrak{R}(T))^\perp = \nu(T^*)$, and so $Z \subset \nu(T^*)$. In order to prove that $\nu(T^*) \subset Z$, we make use of the assumption (b). We shall show that $Z^\perp \subset \mathfrak{R}(T)$, which implies $\nu(T^*) = (\mathfrak{R}(T))^\perp \subset Z$. Since $\nu(T_0^*) \subset Z$, we have $(\nu(T_0^*))^\perp = (\Re(T_0))^c = \Re(T_0) \supset Z^\perp$, where $(\Re(T_0))^c$ represents the closure of $\Re(T_0)$. Thus given any $f^* \in Z^\perp$, there exists an $f \in \mathfrak{D}(T_0)$ such that $T_0 f = f^*$. Let $\varphi \in \mathfrak{H}_0$ and $v \in Z$ such that $T_0^* v = \varphi$. Then

$$0 = (f^*, v) = (T_0 f, v) = (f, T_0^* v) = (f, \varphi),$$

which shows that $f \in \mathfrak{H}_0^\perp$ and hence that $f \in \mathfrak{D}(T)$. Therefore $Tf = T_0 f = f^*$ and $Z^\perp \subset \Re(T)$, completing the proof that $Z = \nu(T^*)$, which is assertion (ii). Combining this with (3.2) we have (iii).

Finally, let us prove (iv). Since $T_0^* \subset T^*$ and $(T^*)_\infty \subset T^*$, we know that $T_0^* \dotplus (T^*)_\infty \subset T^*$, and so we just have to verify that $T^* \subset T_0^* \dotplus (T^*)_\infty$. Let $\{h, k\} \in T^*$. From (c) there exists an $h_0 \in \mathfrak{D}(T_0^*)$ such that $k = T_0^* h_0$, and so $\{h_0, k\} \in T_0^* \subset T^*$. Thus $\{h - h_0, 0\} \in T^*$ or $v = h - h_0 \in \nu(T^*) = Z$. So $v \in \mathfrak{D}(T_0^*)$ and $T_0^* v = \varphi \in \mathfrak{H}_0$. Hence $h = h_0 + v \in \mathfrak{D}(T_0^*)$, and $T_0^* h = k + \varphi$, or

$$\{h, k\} = \{h, T_0^* h - \varphi\} = \{h, T_0^* h\} + \{0, -\varphi\} \in T_0^* \dotplus (T^*)_\infty .$$

This completes the proof of Theorem 1.

Remarks (1). If $\nu(T_0) = \{0\}$, then conditions (b) and (c) of Theorem 1 are automatically true, since $T \subset T_0$ implies that $T_0^* \subset T^*$ and hence $\mathfrak{H} = \Re(T_0^*) = \Re(T^*)$.

(2) The conclusions (i) and (iv) in Theorem 1 imply that

$$(T^*)_s h = Q_0 T_0^* h, \qquad h \in \mathfrak{D}(T_0^*) = \mathfrak{D}(T^*),$$

where Q_0 is the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{H}_0^\perp$. Indeed, $(T^*)_s = T^* \ominus (T^*)_\infty$, and so the element $\{h, k\} = \{h, T_0^* h + \varphi\}$ in T^* will be in $(T^*)_s$ if and only if $0 = (T_0^* h + \varphi, \psi) = (P_0 T_0^* h + \varphi, \psi)$ for all $\psi \in \mathfrak{H}_0$, where P_0 is the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{H}_0$. Thus $\varphi = -P_0 T_0^* h$ and $(T^*)_s h = k = (I - P_0) T_0^* h = Q_0 T_0^* h$.

Now suppose that S_0 is a densely defined (closed) symmetric operator in $\mathfrak{H}_0$ and that $\mathfrak{H}_0$ is a finite-dimensional subspace of $\mathfrak{H}$. We define S to be the operator given by

$$\mathfrak{D}(S) = \mathfrak{D}(S_0) \cap \mathfrak{H}_0^\perp, \qquad S \subset S_0, \tag{3.3}$$

that is, $S = S_0 \cap (\mathfrak{H}_0^\perp \oplus \mathfrak{H})$. Then S is a closed symmetric operator in $\mathfrak{H}$. If $M = S^* \ominus S$ and $M_0 = S_0^* \ominus S_0$, we have

$$M = M^+ \oplus M^-, \qquad M_0 = (M_0)^+ \oplus (M_0)^-,$$

where

$$M^{\pm} = \{\{h, k\} \in S^* \mid k = \pm ih\},$$
$$(M_0)^{\pm} = \{\{h, S_0{}^*h\} \in S_0{}^* \mid S_0{}^*h = \pm ih\}.$$

We let

$$\mathfrak{E}(\pm i) = \nu(S^* \mp iI), \qquad \mathfrak{E}_0(\pm i) = \nu(S_0{}^* \mp iI),$$

and then we see that

$$\mathfrak{D}(M^{\pm}) = \mathfrak{E}(\pm i), \qquad \mathfrak{D}((M_0)^{\pm}) = \mathfrak{E}_0(\pm i).$$

In order to determine S^*, we require the following lemma concerning $S_0 \pm iI$ and $S_0{}^* \pm iI$.

LEMMA. *Suppose S_0 is a densely defined (closed) symmetric operator in $\mathfrak{H}$. Then*

(i) $\mathfrak{R}(S_0 \pm iI) = (\mathfrak{E}_0(\pm i))^{\perp}$,

(ii) $\mathfrak{R}(S_0{}^* \pm iI) = \mathfrak{H}$.

Proof. Since $(\mathfrak{R}(S_0 \pm iI))^{\perp} = \nu(S_0{}^* \pm iI) = \mathfrak{E}_0(\pm i)$, we just have to verify that $\mathfrak{R}(S_0 \pm iI)$ is closed in order to prove (i). This follows from the fact that S_0 is closed and the equality

$$\|(S_0 \pm iI)f\|^2 = \| S_0 f\|^2 + \|f\|^2, \qquad f \in \mathfrak{D}(S_0).$$

Turning to the proof of (ii), we note that $S_0{}^* = S_0 \oplus (M_0)^+ \oplus (M_0)^-$ implies that

$$\mathfrak{D}(S_0{}^*) = \mathfrak{D}(S_0) \dotplus \mathfrak{E}_0(+i) \dotplus \mathfrak{E}_0(-i),$$

a direct sum. From (i) it follows that any $k \in \mathfrak{H}$ may be written uniquely as

$$k = (S_0 + iI)f + \varphi^+, \qquad f \in \mathfrak{D}(S_0), \qquad \varphi^+ \in \mathfrak{E}_0(+i),$$

and, if $h = f + (1/2i)\varphi^+ \in \mathfrak{D}(S_0) \dotplus \mathfrak{E}_0(+i) \subset \mathfrak{D}(S_0{}^*)$, then

$$(S_0{}^* + iI)h = (S_0 + iI)f + \varphi^+ = k.$$

Thus $\mathfrak{R}(S_0{}^* + iI) = \mathfrak{H}$, and similarly $\mathfrak{R}(S_0{}^* - iI) = \mathfrak{H}$.

THEOREM 2. *Let S be defined by* (3.3) *where* $\dim \mathfrak{H}_0 < \infty$. *Then S is a (closed) symmetric operator such that*

(i) $\mathfrak{D}(S)$ *is dense in* $\mathfrak{H}_0^\perp$, $S^*(0) = \mathfrak{H}_0$, $(S^*)_\infty = \{0\} \oplus \mathfrak{H}_0$,

(ii) $S^* = S_0^* \dotplus (S^*)_\infty = \{\{h, S_0^*h + \varphi\} \mid h \in \mathfrak{D}(S_0^*), \varphi \in \mathfrak{H}_0\}$,

(iii) $\dim M^\pm = \dim(M_0)^\pm + \dim \mathfrak{H}_0$.

Proof. The three statements in (i) are equivalent, and their validity is a consequence of Theorem 1(i) (applied to $T_0 = S_0$, $T = S$) since the proof of that part of Theorem 1 only made use of the condition $\dim \mathfrak{H}_0 < \infty$.

For the remainder of the proof, we need the observations that for any subspace A in $\mathfrak{H}^2$, $\alpha \in \mathbf{C}$,

$$(A + \alpha I)^* = A^* + \bar{\alpha} I, \qquad (A + \alpha I)_\infty = A_\infty . \tag{3.4}$$

If we now let $T_0 = S_0 + iI$, $T = S + iI$, we see that $\mathfrak{D}(T_0) = \mathfrak{D}(S_0)$, $\mathfrak{D}(T) = \mathfrak{D}(S)$, and (3.3) then shows that (3.1) is valid, i.e., $\mathfrak{D}(T) = \mathfrak{D}(T_0) \cap \mathfrak{H}_0^\perp$, $T \subset T_0$. We shall show that the hypotheses (a)–(c) of Theorem 1 are true for T_0, T. Clearly (a) is true by assumption, and $\mathfrak{R}(T_0) = \mathfrak{R}(S_0 + iI)$ is closed by the assertion (i) of the Lemma. Finally, by (3.4) we have $T_0^* = S_0^* - iI$, $T^* = S^* - iI$, and assertion (ii) of the Lemma implies that $\mathfrak{R}(T_0^*) = \mathfrak{H}$; then $T_0^* \subset T^*$ gives $\mathfrak{R}(T_0^*) = \mathfrak{R}(T^*) = \mathfrak{H}$, thus verifying condition (c) of Theorem 1.

Applying Theorem 1 to T_0 and T, we find that (iv) of Theorem 1 gives $T^* = T_0^* \dotplus (T^*)_\infty$. But $(T^*)_\infty = (S^* - iI)_\infty = (S^*)_\infty$ from (3.4), and thus

$$(S^* - iI) = (S_0^* - iI) \dotplus (S^*)_\infty .$$

It is easy to see that this implies $S^* = S_0^* \dotplus (S^*)_\infty$, proving (ii).

The last conclusion of Theorem 2 is a consequence of (iii) of Theorem 1. We have

$$\nu(T^*) = \nu(S^* - iI) = \mathfrak{E}(+i) = \mathfrak{D}(M^+),$$

$$\nu(T_0^*) = \nu(S_0^* - iI) = \mathfrak{E}_0(+i) = \mathfrak{D}((M_0)^+),$$

$$\dim M^+ = \dim \mathfrak{D}(M^+), \qquad \dim(M_0)^+ = \dim \mathfrak{D}((M_0)^+),$$

and (iii) of Theorem 1 then yields $\dim M^+ = \dim(M_0)^+ + \dim \mathfrak{H}_0$. Applying Theorem 1 to $T_0 = S_0 - iI$, $T = S - iI$, we obtain $\dim M^- = \dim(M_0)^- + \dim \mathfrak{H}_0$, completing the proof of Theorem 2.

COROLLARY. *The symmetric operator S has a self-adjoint subspace extension in $\mathfrak{H}^2$ if and only if*

$$\dim(M_0)^+ = \dim(M_0)^-,$$

that is, if and only if S_0 has a self-adjoint extension in $\mathfrak{H}^2$.

Proof. This is a direct consequence of Theorem B and (iii) of Theorem 2.

4. SELF-ADJOINT EXTENSIONS OF A NONDENSELY DEFINED SYMMETRIC OPERATOR

In this section we assume that S_0 is a closed densely defined symmetric operator in $\mathfrak{H}$, S is defined by (3.3), and the following are satisfied:

$$\dim \mathfrak{H}_0 = p < \infty, \qquad \dim(M_0)^+ = \dim(M_0)^- = \omega < \infty. \tag{4.1}$$

Then $\dim M^+ = \dim M^- = p + \omega = q$, say, and from the Corollary to Theorem 2 we know that S has self-adjoint subspace extensions in $\mathfrak{H}$. Our aim is describe each such H, with given $\dim H(0)$, in terms of "generalized boundary conditions". Since any self-adjoint extension H. of S in $\mathfrak{H}^2$ satisfies $S \subset H \subset S^*$, we have $H(0) \subset S^*(0)$, and so $\dim H(0) \leqslant \dim \mathfrak{H}_0 = p$.

Our final result will be obtained via several mutations of Theorem A, denoted by Theorems A_1 through A_4, with the final version being Theorem 3.

THEOREM A_1. *Let H be a self-adjoint subspace extension of S in $\mathfrak{H}^2$. Then there exist q elements $\{\alpha_j, \beta_j\}$, $j = 1,..., q$, in S^* such that*

(1a) *the $\{\alpha_j, \beta_j\}$ are linearly independent* mod S,

(1b) $(\beta_k, \alpha_j) - (\alpha_k, \beta_j) = 0, j, k = 1,..., q,$

(1i) $H = \{\{h, k\} \in S^* \mid (k, \alpha_j) - (h, \beta_j) = 0, j = 1,..., q\},$

(1ii) $H = S \dotplus N_1, N_1 = \text{span}\{\{\alpha_j, \beta_j\}\}.$

Conversely, if $\{\alpha_j, \beta_j\}$, $j = 1,..., q$, are q elements in S^* satisfying (1a), (1b), *then H defined by* (1i) *is a self-adjoint extension of S in $\mathfrak{H}^2$ and* (ii) *is valid.*

Proof. The first half of the result follows from Theorem A applied to the operator S. If $H = S \oplus M_1 = S^* \ominus JM_1$, and $\{\alpha_j, \beta_j\}$, $j = 1,\ldots, q$, is a basis for M_1, then (1a) is valid since

$$\{\alpha, \beta\} = \sum_{j=1}^{q} c_j\{\alpha_j, \beta_j\} \in S, \qquad c_j \in \mathbf{C},$$

implies $\{\alpha, \beta\} \in S \cap M_1 = \{\{0, 0\}\}$, and hence all $c_j = 0$. The assertion (1i) is equivalent to $H = S^* \ominus JM_1$, and (1ii) is true with $N_1 = M_1$. Item (1b) follows from the fact that $M_1 \subset H$.

For the proof of the converse, let

$$\{\alpha_j, \beta_j\} = \{f_j, g_j\} + \{\alpha_j', \beta_j'\}, \qquad j = 1,\ldots, q,$$

be the unique decomposition such that $\{f_j, g_j\} \in S$, $\{\alpha_j', \beta_j'\} \in M$, and define M_1 to be the span of the $\{\alpha_j', \beta_j'\}$. We claim that $JM_1 = M \ominus M_1$ and $H = S \oplus M_1$. From Theorem A it then follows that H is self-adjoint.

If $\{h, k\} \in S^*$, then the symmetry of S implies

$$(k, \alpha_j) - (h, \beta_j) = (k, \alpha_j') - (h, \beta_j'), \tag{4.2}$$

and applying this twice to the relation (1b) we obtain

$$(\beta_k', \alpha_j') - (\alpha_k', \beta_j') = 0, \qquad j, k = 1,\ldots, q. \tag{4.3}$$

Now (1i) and (4.2) show that

$$H = \{\{h, k\} \in S^* \mid (k, \alpha_j') - (h, \beta_j') = 0, j = 1,\ldots, q\} = S^* \ominus JM_1. \tag{4.4}$$

The $\{\alpha_j', \beta_j'\}$ are linearly independent for

$$0 = \sum c_j\{\alpha_j', \beta_j'\} = \sum c_j\{\alpha_j, \beta_j\} - \sum c_j\{f_j, g_j\}$$

implies that

$$\sum c_j\{\alpha_j, \beta_j\} \in S,$$

and (1a) then implies all $c_j = 0$. Clearly $M_1 \subset M$, and from (4.3) we have $JM_1 \subset M \ominus M_1$. Now $\dim M_1 = q = (\dim M)/2$ and hence $\dim(M \ominus M_1) = q$. Since J is unitary, $\dim JM_1 = \dim M_1 = \dim(M \ominus M_1)$, and thus $JM_1 = M \ominus M_1$. Therefore from (4.4) we have $H = S^* \ominus JM_1 = S^* \ominus (M \ominus M_1) = S^* \oplus M_1$, and H is self-adjoint. Clearly, $H = S \dotplus M_1 = S \dotplus N_1$, and so H satisfies (1ii).

THEOREM A_2. *Let H be a self-adjoint extension of S in $\mathfrak{H}^2$, with* $\dim H(0) = s$ *and $\varphi_1, \ldots, \varphi_s$ a basis for $H(0)$. Then there exist $\{\alpha_k, \beta_k\}$, $k = s+1, \ldots, q$, in S^* such that*

(2a) *the $\{\alpha_k, \beta_k\}$ are linearly independent* $\operatorname{mod}(S \dotplus (S^*)_\infty)$,

(2b) $(\alpha_k, \varphi_j) = 0, j = 1, \ldots, s, k = s+1, \ldots, q,$

$(\beta_k, \alpha_j) - (\alpha_k, \beta_j) = 0, j, k = s+1, \ldots, q,$

and

(2i) *H is the set of all $\{h, k\} \in S^*$ such that*

$$(h, \varphi_j) = 0, \qquad j = 1, \ldots, s,$$
$$(k, \alpha_j) - (h, \beta_j) = 0, \qquad j = s+1, \ldots, q,$$

(2ii) $H = S \dotplus N_2, N_2 = \operatorname{span}\{\{0, \varphi_j\}, \{\alpha_k, \beta_k\}\}$.

Conversely, if $\varphi_1, \ldots, \varphi_s$ are linearly independent elements of $S^*(0) = \mathfrak{H}_0$, and $\{\alpha_k, \beta_k\}$, $k = s+1, \ldots, q$, are in S^* satisfying (2a), (2b), *then H defined by* (2i) *is a self-adjoint extension of S such that* $H(0) = \operatorname{span}\{\varphi_1, \ldots, \varphi_s\}$, *and* (2ii) *is valid.*

Proof. Suppose H is a self-adjoint extension of S given by $H = S \oplus M_1$, as in Theorem A. Consider $P_M\{0, \varphi_j\} = \{\alpha_j, \beta_j\}$, $j = 1, \ldots, s$, where P_M is the orthogonal projection of $\mathfrak{H}^2$ onto M. Since $\{0, \varphi_j\} \in H$, $P_M\{0, \varphi_j\} \in M_1$, and these are linearly independent. Indeed, if

$$0 = \sum c_j P_M\{0, \varphi_j\} = P_M\left\{0, \sum c_j \varphi_j\right\},$$

then if $\varphi = \sum c_j \varphi_j \in H(0)$, we have $\{0, \varphi\} \in S$. But since S is an operator, $\varphi = 0$ and all $c_j = 0$. We can now add to these elements $q - s$ other elements $\{\alpha_k, \beta_k\}$, $k = s+1, \ldots, q$, to obtain a basis for M_1. Clearly, if N_2 is defined as in (2ii), then $H = S \oplus M_1 = S \dotplus N_2$. Moreover, as in the proof of the first part of Theorem A_1, H is the set of all $\{h, k\} \in S^*$ satisfying

$$(k, \alpha_j) - (h, \beta_j) = 0, \qquad j = 1, \ldots, q.$$

But for $j = 1, \ldots, s$,

$$(k, 0) - (h, \varphi_j) = (k, \alpha_j) - (h, \beta_j) = 0$$

(see (4.2)), which gives (2i). Since $\{h, k\} = \{\alpha_k, \beta_k\} \in M_1 \subset H$, it satisfies (2i), and this is just (2b). It remains to check (2a). This can be done by first verifying that the elements $\{0, \varphi_1\},\ldots, \{0, \varphi_s\}, \{\alpha_{s+1}, \beta_{s+1}\},\ldots, \{\alpha_q, \beta_q\}$ are linearly independent mod S and then noticing that this is equivalent to $\{\alpha_{s+1}, \beta_{s+1}\},\ldots, \{\alpha_q, \beta_q\}$ being linearly independent $\mathrm{mod}(S \dotplus (S^*)_\infty)$.

Turning to the proof of the converse, we have just noted that (2a) implies condition (1a) of Theorem A_1 for the q elements $\{0, \varphi_1\},\ldots, \{0, \varphi_s\}, \{\alpha_{s+1}, \beta_{s+1}\},\ldots, \{\alpha_q, \beta_q\}$ in S^*. The hypothesis (2b) amounts to (1b) of Theorem A_1. Thus the latter theorem implies that H as given in (2i) (which is (1i) for this case) is a self-adjoint extension of S, and (2ii) is valid. In order to complete the proof, we must show that $H(0) = \mathrm{span}\{\varphi_1,\ldots, \varphi_s\}$. Since $\{0, \varphi_j\} \in H$, $j = 1,\ldots, s$, and these elements are linearly independent, we have $\mathrm{span}\{\varphi_1,\ldots, \varphi_s\} \subset H(0)$ and $\dim H(0) \geqslant s$. Suppose $\{0, \varphi\} \in H = S \dotplus N_2$ and for some $b_j, c_j \in \mathbf{C}$ we have

$$\{0, \varphi\} = \{f, g\} + \sum_{j=1}^{s} b_j\{0, \varphi_j\} + \sum_{j=s+1}^{q} c_j\{\alpha_j, \beta_j\}.$$

Then

$$\sum_{j=s+1}^{q} c_j\{\alpha_j, \beta_j\} \in S \dotplus (S^*)_\infty,$$

and (2a) implies that all $c_j = 0$. This means that $f = 0$ and then $g = 0$, since S is an operator, and thus

$$\varphi = \sum_{j=1}^{s} b_j \varphi_j,$$

yielding $H(0) = \mathrm{span}\{\varphi_1,\ldots, \varphi_s\}$.

We now exploit the precise nature of S^* as given in Theorem 2, namely, $S^* = S_0^* \dotplus (S^*)_\infty$, where $(S^*)_\infty = \{0\} \oplus \mathfrak{H}_0$. Thus $\{h, k\} \in S^*$ if and only if $h \in \mathfrak{D}(S_0^*)$ and $k = S_0^*h + \varphi$ for some $\varphi \in \mathfrak{H}_0$. For the $\{\alpha_k, \beta_k\} \in S^*$ of Theorem A_2, we put

$$\{\alpha_k, \beta_k\} = \{\alpha_k, S_0^*\alpha_k + \varphi_k'\}, \qquad k = s+1,\ldots, q, \varphi_k' \in \mathfrak{H}_0. \tag{4.5}$$

In these terms it is easy to see that the $\{\alpha_k, \beta_k\}$ are linearly independent $\mathrm{mod}(S \dotplus (S^*)_\infty)$ if and only if $\alpha_{s+1},\ldots, \alpha_q$ are linearly independent mod $\mathfrak{D}(S)$.

For any $h, h' \in \mathfrak{D}(S_0^*)$ we let

$$\langle hh' \rangle = (S_0^*h, h') - (h, S_0^*h').$$

This is a semi-bilinear skew-hermitian form on $\mathfrak{D}(S_0^*) \times \mathfrak{D}(S_0^*)$, which can be considered also as a form on $[\mathfrak{D}(S_0^*)/\mathfrak{D}(S_0)] \times [\mathfrak{D}(S_0^*)/\mathfrak{D}(S_0)]$, that is,

$$\langle (h+f)(h'+f') \rangle = \langle hh' \rangle = -\overline{\langle h'h \rangle}$$

for all $f, f' \in \mathfrak{D}(S_0)$. For $\{h, k\} = \{h, S_0^*h + \varphi\} \in S^*$ and $\{\alpha_k, \beta_k\}$ given by (4.5), we then have

$$(k, \alpha_j) - (h, \beta_j) = \langle h\alpha_j \rangle - (h, \varphi_j') + (\varphi, \alpha_j)$$

and

$$(\beta_k, \alpha_j) - (\alpha_k, \beta_j) = \langle \alpha_k \alpha_j \rangle - (\alpha_k, \varphi_j') + (\varphi_k', \alpha_j).$$

Using these notations, Theorem A_2 becomes the following result.

THEOREM A_3. *Let H be a self-adjoint extension of S in $\mathfrak{H}^2$, with* $\dim H(0) = s$ *and $\varphi_1, \ldots, \varphi_s$ a basis for $H(0)$. Then there exist $\alpha_k \in \mathfrak{D}(S_0^*)$, $\varphi_k' \in \mathfrak{H}_0$, $k = s+1, \ldots, q$, such that*

(3a) $\alpha_{s+1}, \ldots, \alpha_q$ *are linearly independent* mod $\mathfrak{D}(S)$,

(3b) $(\alpha_k, \varphi_j) = 0, j = 1, \ldots, s, k = s+1, \ldots, q,$
$\langle \alpha_k \alpha_j \rangle - (\alpha_k, \varphi_j') + (\varphi_k', \alpha_j) = 0, j, k = s+1, \ldots, q,$

and

(3i) *H is the set of all $\{h, S_0^*h + \varphi\} \in S^*$ such that*

$$(h, \varphi_j) = 0, \qquad j = 1, \ldots, s,$$

$$\langle h\alpha_j \rangle - (h, \varphi_j') + (\varphi, \alpha_j) = 0, \qquad j = s+1, \ldots, q,$$

(3ii) $H = S \dotplus N_3$, $N_3 = \operatorname{span}\{\{0, \varphi_j\}, \{\alpha_k, S_0^*\alpha_k + \varphi_k'\}\}$.

Conversely, if $\varphi_1, \ldots, \varphi_s$ are linearly independent elements of $\mathfrak{H}_0$, and $\alpha_k \in \mathfrak{D}(S_0^*)$, $\varphi_k' \in \mathfrak{H}_0$, $k = s+1, \ldots, q$, satisfy (3a), (3b), *then H defined by* (3i) *is a self-adjoint extension of S such that $H(0) =$* $\operatorname{span}\{\varphi_1, \ldots, \varphi_s\}$, *and* (3ii) *is valid.*

Now, given an H as described by (3i), (3ii) above, we are going to introduce a new basis for N_3. In terms of this new basis, the specification of $h \in \mathfrak{D}(H)$ can be separated from the specification of the values $S_0^*h + \varphi$ for $\{h, S_0^*h + \varphi\} \in H$. We start by looking at the second equality in (3i). If $\varphi_1,..., \varphi_s$ is a basis for $H(0)$, and $\varphi_1,..., \varphi_s, \varphi_{s+1},..., \varphi_p$ is a basis for $\mathfrak{H}_0$, any $\varphi \in \mathfrak{H}_0$ can be written as

$$\varphi = \sum_{k=1}^{p} c_k \varphi_k, \qquad c_k \in \mathbf{C},$$

and thus (3i) gives, for $j = s + 1,..., q$,

$$(\varphi, \alpha_j) = \sum_{k=1}^{p} c_k(\varphi_k, \alpha_j) = \sum_{k=s+1}^{p} c_k(\varphi_k, \alpha_j) = (h, \varphi_j') - \langle h\alpha_j \rangle,$$

using (3b). This is a set of $q - s$ equations for the $p - s$ constants $c_{s+1},..., c_p$. The $c_1,..., c_s$ are arbitrary. We analyze the coefficients (φ_k, α_j). Let

$$\{\alpha_j, S_0^*\alpha_j + \varphi_j'\} = \{0, \varphi_j\}, \qquad j = 1,..., s, \tag{4.5}$$

and

$$\{\alpha_j, S_0^*\alpha_j + \varphi_j'\} = \{f_j, Sf_j\} + \{\alpha_j', S_0^*\alpha_j' + \varphi_j''\}, \qquad j = 1,..., q,$$

where

$$\{f_j, Sf_j\} \in S, \qquad \{\alpha_j', S_0^*\alpha_j' + \varphi_j''\} \in M.$$

Then $H = S \oplus M_1$, where $M_1 = P_M N_3 = \text{span}\{\{\alpha_j', S_0^*\alpha_j' + \varphi_j''\}\}$ and

$$(\varphi_k, \alpha_j) = (\varphi_k, f_j) + (\varphi_k, \alpha_j') = (\varphi_k, \alpha_j'), \qquad j = 1,..., q, \quad k = 1,..., p,$$

since $f_j \in \mathfrak{D}(S) \subset \mathfrak{H}_0^{\perp}$. Let $C = (C_{jk})$ be the $q \times p$ matrix defined by

$$C_{jk} = (\varphi_k, \alpha_j), \qquad j = 1,..., q, \quad k = 1,..., p.$$

Then $C_{jk} = (\varphi_k, \alpha_j')$, where $\alpha_1',..., \alpha_k'$ is a basis for $\mathfrak{D}(M_1)$. The null space of C, $\nu(C)$, is the set of all $\{c_1,..., c_p\}$ such that

$$0 = \sum_{k=1}^{p} C_{jk} c_k = \sum_{k=1}^{p} c_k(\varphi_k, \alpha_j) = \left(\sum_{k=1}^{p} c_k \varphi_k, \alpha_j'\right), \qquad j = 1,..., q.$$

Thus

$$\dim \nu(C) = \dim((\mathfrak{D}(M_1))^{\perp} \cap \mathfrak{H}_0) = \dim H(0) = s,$$

and rank $C = p - s$. Here we have used the fact that $H(0) = (\mathfrak{D}(M_1))^{\perp} \cap \mathfrak{H}_0)$; see Theorem 8′ of [3]. From (4.5) $\alpha_j = 0$, $j = 1,..., s$, and from (3a) of Theorem A_3 we have $(\varphi_k, \alpha_j) = 0$, for $j = s + 1,..., q$, $k = 1,..., s$. Thus $C_{jk} = 0$ if either j or k is between 1 and s. If

$$C^0 = (C^0_{jk}), \qquad C^0_{jk} = (\varphi_k, \alpha_j), \qquad j = s + 1,..., q, \quad k = s + 1,..., p,$$

then rank C = rank $C^0 = p - s$, and C^0 has maximum rank. By relabeling the α_j's, we can assume, and do, that the upper left corner of C^0, namely,

$$C^1 = (C^1_{jk}), \qquad C^1_{jk} = (\varphi_k, \alpha_j), \qquad j, k = s + 1,..., p,$$

is nonsingular.

The conditions (3i) of Theorem A_3 now become the following for $c_{s+1},..., c_p$:

$$\sum_{k=s+1}^{p} c_k(\varphi_k, \alpha_j) = (h, \varphi_j') - \langle h\alpha_j \rangle, \qquad j = s + 1,..., p, \tag{4.6}$$

$$\sum_{k=s+1}^{p} c_k(\varphi_k, \alpha_j) = (h, \varphi_j') - \langle h\alpha_j \rangle, \qquad j = p + 1,..., q. \tag{4.7}$$

The constants $c_{s+1},..., c_p$ are uniquely determined by (4.6). Let

$$C^2 = (C^2_{jk}), \qquad C^2_{jk} = (\varphi_k, \alpha_j), \qquad j = p + 1,..., q, \quad k = s + 1,..., p,$$

and

$$c = \begin{bmatrix} c_{s+1} \\ \vdots \\ c_p \end{bmatrix}, \qquad \varphi^1 = \begin{bmatrix} \varphi'_{s+1} \\ \vdots \\ \varphi_p' \end{bmatrix}, \qquad \alpha^1 = \begin{bmatrix} \alpha_{s+1} \\ \vdots \\ \alpha_p \end{bmatrix},$$

$$\varphi^2 = \begin{bmatrix} \varphi'_{p+1} \\ \vdots \\ \varphi_q' \end{bmatrix}, \qquad \alpha^2 = \begin{bmatrix} \alpha_{p+1} \\ \vdots \\ \alpha_q \end{bmatrix}.$$

Then (4.6), (4.7) may be written in vector form as

$$\begin{aligned} C^1 c &= (h, \varphi^1) - \langle h\alpha^1 \rangle, \\ C^2 c &= (h, \varphi^2) - \langle h\alpha^2 \rangle, \end{aligned} \tag{4.9}$$

and thus

$$c = (h, \psi) - \langle h\gamma \rangle, \tag{4.10}$$

where

$$\gamma = (\bar{C}^1)^{-1}\alpha^1, \qquad \psi = (\bar{C}^1)^{-1}\varphi^1. \tag{4.11}$$

The condition (4.9) now becomes

$$\langle h\delta \rangle - (h, \zeta) = 0, \tag{4.12}$$

where

$$\delta = \alpha^2 - \bar{C}^2(\bar{C}^1)^{-1}\alpha^1, \qquad \zeta = \varphi^2 - \bar{C}^2(\bar{C}^1)^{-1}\varphi^1. \tag{4.13}$$

Using an obvious notation we have

$$\begin{bmatrix}\{\gamma, S_0{}^*\gamma + \psi\}\\ \{\delta, S_0{}^*\delta + \zeta\}\end{bmatrix} = \mathscr{C}\begin{bmatrix}\{\alpha^1, S_0{}^*\alpha^1 + \varphi^1\}\\ \{\alpha^2, S_0{}^*\alpha^2 + \varphi^2\}\end{bmatrix}, \tag{4.14}$$

where $\mathscr{C}$ is an invertible matrix:

$$\mathscr{C} = \begin{bmatrix}(\bar{C}^1)^{-1} & 0\\ -\bar{C}^2(\bar{C}^1)^{-1} & I\end{bmatrix}, \qquad \mathscr{C}^{-1} = \begin{bmatrix}\bar{C}^1 & 0\\ \bar{C}^2 & I\end{bmatrix}.$$

Thus

$$\begin{aligned} N_3 &= \operatorname{span}\{\{0, \varphi_j\}, \{\alpha_k , S_0{}^*\alpha_k + \varphi_k'\}\}, \qquad j = 1,..., s, \quad k = s + 1,..., q,\\ &= \operatorname{span}\{\{0, \varphi_j\}, \{\gamma_k , S_0{}^*\gamma_k + \psi_k\}, \{\delta_l , S_0{}^*\delta_l + \zeta_l\}\},\\ &\qquad\qquad j = 1,..., s, \quad k = s + 1,..., p, \quad l = p + 1,..., q,\end{aligned}$$

and (3i) becomes (using (4.10) and (4.12)):

$$(h, \varphi_j) = 0, \qquad j = 1,..., s,$$

$$\varphi = c_1\varphi_1 + \cdots + c_s\varphi_s + \sum_{k=s+1}^{p} [(h, \psi_k) - \langle h\gamma_k\rangle]\, \varphi_k , \tag{4.15}$$

$$\langle h\delta_l\rangle - (h, \zeta_l) = 0, \qquad l = p + 1,..., q,$$

where $c_1 ,..., c_s$ are arbitrary complex constants.

We now interpret the conditions (3a), (3b) of Theorem A_3 in terms of the $\gamma_k , \psi_k , \delta_l , \zeta_l$. From (3a) and (4.14) follows that $\gamma_{s+1} ,..., \gamma_p$, $\delta_{p+1} ,..., \delta_q$ are linearly independent mod $\mathfrak{D}(S)$. Using the semi-bilinearity of the relations (3b), and (4.14), we find that (3b) becomes:

$$(\gamma_k, \varphi_j) = 0, \quad j = 1,..., s, \quad k = s + 1,..., p, \tag{4.16}$$

$$(\delta_l, \varphi_j) = 0, \quad j = 1,..., s, \quad l = p + 1,..., q, \tag{4.17}$$

$$\langle \gamma_k \gamma_j \rangle - (\gamma_k, \psi_j) + (\psi_k, \gamma_j) = 0, \quad j, k = s + 1,..., p, \tag{4.18}$$

$$\langle \delta_l \gamma_j \rangle - (\delta_l, \psi_j) + (\zeta_l, \gamma_j) = 0, \quad j = s + 1,..., p, \; l = p + 1,..., q, \tag{4.19}$$

$$\langle \delta_l \delta_j \rangle - (\delta_l, \zeta_j) + (\zeta_l, \delta_j) = 0, \quad j, l = p + 1,..., q. \tag{4.20}$$

These can be simplified once we note that

$$(\gamma_k, \varphi_r) = \delta_{kr}, \qquad k, r = s + 1,..., p, \tag{4.21}$$

$$(\delta_l, \varphi_m) = 0, \qquad l = p + 1,..., q, \qquad m = s + 1,..., p; \tag{4.22}$$

they are a direct consequence of the definitions of γ_k, δ_l given in (4.11), (4.13) and the definitions of the matrices C^1, C^2 (δ_{kr} is the Kronecker symbol). Now (4.17) and (4.22) imply that

$$(\delta_l, \varphi_j) = 0, \qquad j = 1,..., p,$$

that is, $\delta_l \in \mathfrak{H}_0^{\perp}$, and hence $(\delta_l, \psi_j) = 0$ in (4.19) and $(\zeta_l, \delta_j) = 0 = (\delta_l, \zeta_j)$ in (4.20).

One further simplication can be introduced into (4.16)–(4.20). We know that $\{\gamma_k, S_0{}^*\gamma_k + \psi_k\} \in H$ and (4.15) implies $\psi_k = \psi_{k0} + \psi_k'$, where $\psi_{k0} \in H(0)$ and

$$\psi_k' = \sum_{r=s+1}^{p} [(\gamma_k, \psi_r) - \langle \gamma_k \gamma_r \rangle] \varphi_r = \sum_{r=s+1}^{p} (\psi_k, \gamma_r) \varphi_r,$$

where we have used (4.18). Now we observe that we can replace ψ_k everywhere by ψ_k'. This can be done in the description of N_3. Also in (4.15), $(h, \psi_k) = (h, \psi_k')$ since $(h, \psi_{k0}) = 0$, due to the first relation in (4.15). Similarly, in (4.18), $(\gamma_k, \psi_j) = (\gamma_k, \psi_j')$ for $\gamma_k \in \mathfrak{D}(H)$, $\psi_{j0} \in H(0) = (\mathfrak{D}(H))^{\perp}$. So we can assume $\psi_k = \psi_k'$. Then

$$\psi_k = \sum_{r=s+1}^{p} D_{kr}\varphi_r, \qquad D_{kr} = (\psi_k, \gamma_r),$$

where by (4.18)

$$D_{kr} = (\gamma_k, \psi_r) - \langle \gamma_k \gamma_r \rangle = \bar{D}_{rk} - \langle \gamma_k \gamma_r \rangle.$$

Hence

$$D_{kr} = E_{kr} - (1/2)\langle\gamma_k\gamma_r\rangle,$$

where $E = (E_{kr})$ is a $(p - s) \times (p - s)$ hermitian matrix of constants, and

$$\psi_k = \sum_{r=s+1}^{p} [E_{kr} - (1/2)\langle\gamma_k\gamma_r\rangle]\,\varphi_r\,, \qquad E = E^*. \tag{4.23}$$

Conversely, for any $E = E^*$, if (4.21) is valid, then (4.18) is true for the ψ_k defined by (4.23).

In an entirely similar fashion we can replace (4.19), which is

$$\langle\delta_l\gamma_j\rangle + (\zeta_l\,, \gamma_j) = 0,$$

by

$$\zeta_k = -\sum_{r=s+1}^{p} \langle\delta_k\gamma_r\rangle\,\varphi_r\,,$$

provided we retain (4.21) and (4.22). Theorem A_3 now becomes the following result in terms of the γ_k, ψ_k, δ_l, ζ_l.

THEOREM A_4. *Let H be a self-adjoint extension of S in $\mathfrak{H}^2$, with* $\dim H(0) = s$. *Let $\varphi_1, ..., \varphi_s$ be a basis for $H(0)$ and $\varphi_1, ..., \varphi_s, \varphi_{s+1}, ..., \varphi_p$ a basis for $\mathfrak{H}_0$. Then there exist $\gamma_{s+1}, ..., \gamma_p, \delta_{p+1}, ..., \delta_q$ in $\mathfrak{D}(S_0^*)$ and $E_{kr} \in \mathbf{C}$ such that*

(4a) $\gamma_{s+1}, ..., \gamma_p, \delta_{p+1}, ..., \delta_q$ *are linearly independent* mod $\mathfrak{D}(S)$,

(4b) $\begin{cases} (\gamma_k\,, \varphi_j) = 0, & j = 1,..., s \\ (\gamma_k\,, \varphi_j) = \delta_{kj}\,, & j = s+1,..., p \end{cases} \quad k = s+1,..., p,$

(4c) $\begin{cases} (\delta_l\,, \varphi_j) = 0, & l = p+1,..., q, \quad j = 1,..., p, \\ \langle\delta_l\delta_j\rangle = 0, & j, l = p+1,..., q, \end{cases}$

and if

(4d) $$\begin{cases} \psi_k = \sum_{r+s+1}^{p} [E_{kr} - (1/2)\langle\gamma_k\gamma_r\rangle]\,\varphi_r\,, & k = s+1,..., p, \\ E_{kr} \in \mathbf{C}, \quad E = (E_{kr}) = E^*, & \\ \zeta_k = -\sum_{r=s+1}^{p} \langle\delta_k\gamma_r\rangle\,\varphi_r\,, & k = p+1,..., q, \end{cases}$$

then

(4i) *H is the set of all* $\{h, S_0{}^*h + \varphi\}$, $h \in \mathfrak{D}(S_0{}^*)$, $\varphi \in \mathfrak{H}_0$, *such that*

$$(h, \varphi_j) = 0, \qquad j = 1,\ldots, s,$$

$$\langle h\delta_l\rangle - (h, \zeta_l) = 0, \qquad l = p + 1,\ldots, q,$$

$$\varphi = c_1\varphi_1 + \cdots + c_s\varphi_s + \sum_{k=s+1}^{p} [(h, \psi_k) - \langle h\gamma_k\rangle]\, \varphi_k\,, \qquad c_j \in \mathbf{C},$$

(4ii) $H = S \dotplus N_4$,

$$N_4 = \operatorname{span}\{\{0, \varphi_j\}, \{\gamma_k\,, S_0{}^*\gamma_k + \psi_k\}, \{\delta_l\,, S_0{}^*\delta_l + \zeta_l\}\}.$$

Conversely, if $\varphi_1,\ldots, \varphi_s, \varphi_{s+1},\ldots, \varphi_p$ *is a basis for* $\mathfrak{H}_0$, *and* γ_k, $\delta_l \in \mathfrak{D}(S_0{}^*)$ *exist satisfying* (4a)–(4c), *and* ψ_k, ζ_k *are defined by* (4d), *then H defined by* (4i) *is a self-adjoint extension of S such that* $H(0) = \operatorname{span}\{\varphi_1,\ldots, \varphi_s\}$, *and* (4ii) *is valid.*

Now we shall show that it is possible to choose the γ_k quite arbitrarily in $\mathfrak{D}(S_0{}^*)$ and to assume that the $\delta_l \in \mathfrak{D}(S_0{}^*)$ are linearly independent mod $\mathfrak{D}(S_0)$ (instead of mod $\mathfrak{D}(S)$). The only sacrifice we make in this process is the description of H as in (4ii). The final result is Theorem 3 below.

Recall that S_0 is a closed densely defined symmetric operator in the Hilbert space $\mathfrak{H}$, and S is the symmetric operator in $\mathfrak{H}$ defined by

$$\mathfrak{D}(S) = \mathfrak{D}(S_0) \cap \mathfrak{H}_0{}^\perp, \qquad S \subset S_0\,,$$

where we assume

$$\dim \mathfrak{H}_0 = p < \infty, \qquad \dim(M_0)^+ = \dim(M_0)^- = \omega < \infty,$$

$$(M_0)^\pm = \{\{h, S_0{}^*h\} \in S_0{}^* \mid S_0{}^*h = \pm ih\},$$

$$q = p + \omega.$$

THEOREM 3. *Let H be a self-adjoint extension of S in* $\mathfrak{H}^2$, *with* $\dim H(0) = s$. *Let* $\varphi_1,\ldots, \varphi_s$ *be an orthonormal basis for* $H(0)$ *and* $\varphi_1,\ldots, \varphi_s$, $\varphi_{s+1},\ldots, \varphi_p$ *an orthonormal basis for* $\mathfrak{H}_0$. *Then there exist* $\gamma_{s+1},\ldots, \gamma_p$, $\delta_{p+1},\ldots, \delta_q$ *in* $\mathfrak{D}(S_0{}^*)$ *and* $E_{kr} \in \mathbf{C}$ *such that*

(a) $\delta_{p+1},\ldots, \delta_q$ *are linearly independent* mod $\mathfrak{D}(S_0)$,

(b) $\langle\delta_l\delta_j\rangle = 0$, $j, l = p + 1,\ldots, q$, *and if*

$$\text{(c)}\quad\begin{cases}\psi_k = \sum_{r=s+1}^{p} [E_{kr} - (1/2)\langle\gamma_k\gamma_r\rangle]\,\varphi_r\,, & k = s+1,\dots,p,\\ E_{kr}\in\mathbf{C}, \quad E = (E_{kr}) = E^*, & \\ \zeta_k = -\sum_{r=s+1}^{p}\langle\delta_k\gamma_r\rangle\,\varphi_r\,, & k = p+1,\dots,q,\end{cases}$$

then

(i) *H is the set of all $\{h,\ S_0{}^*h + \varphi\}$, $h \in \mathfrak{D}(S_0{}^*)$, $\varphi \in \mathfrak{H}_0$, such that*

$$(h, \varphi_j) = 0, \qquad j = 1,\dots, s,$$
$$\langle h\delta_l\rangle - (h, \zeta_l) = 0, \qquad l = p+1,\dots, q,$$

$$\varphi = c_1\varphi_1 + \cdots + c_s\varphi_s + \sum_{k=s+1}^{p} [(h, \psi_k) - \langle h\gamma_k\rangle]\,\varphi_k\,, \qquad c_j\in\mathbf{C},$$

(ii) $H_s h = Q_0 S_0{}^*h + \sum_{k=s+1}^{p} [(h, \psi_k) - \langle h\gamma_k\rangle]\varphi_k$, *where Q_0 is the orthogonal projection of $\mathfrak{H}$ onto $(H(0))^{\perp}$.*

Conversely, if $\varphi_1,\dots,\varphi_s, \varphi_{s+1},\dots,\varphi_p$ is an orthonormal basis for $\mathfrak{H}_0$, and $\gamma_k, \delta_l \in \mathfrak{D}(S_0{}^*)$ exist satisfying (a), (b), *and ψ_k, ζ_k are defined by* (c), *then H defined by* (i) *is a self-adjoint extension of S such that $H(0) =$* span$\{\varphi_1,\dots,\varphi_s\}$, *and H_s is given by* (ii).

Proof of Theorem 3. If H is a self-adjoint extension of S, then Theorem A_4 guarantees the existence of γ_k, δ_l satisfying (b), (c), and (i) is valid. Let us check (a). If for some $a_l \in \mathbf{C}$,

$$\delta = \sum_{l=p+1}^{q} a_l\delta_l \in \mathfrak{D}(S_0),$$

we would have, from (4c) of Theorem A_4, that $\delta \in \mathfrak{D}(S_0) \cap \mathfrak{H}_0{}^{\perp} = \mathfrak{D}(S)$, and then (4a) implies that all the $a_l = 0$. This proves (a).

The formula for H_s given in (ii) is a direct consequence of (i) and the fact that $\{h, H_s h\}$ is orthogonal to $\{0, \varphi_j\}$, $j = 1,\dots, s$. It is here we are using the orthonormal character of the φ_j.

As to the converse, we shall show that for the given γ_k, δ_l, we can find $\gamma_k{}'$, $\delta_l{}' \in \mathfrak{D}(S_0{}^*)$ satisfying (4a)–(4c) of Theorem A_4. We seek such elements of the form

$$\gamma_k{}' = \gamma_k + \gamma_k^{(0)}, \qquad \delta_l{}' = \delta_l + \delta_l^{(0)}, \qquad \gamma_k^{(0)}, \delta_l^{(0)} \in \mathfrak{D}(S_0),$$

which will satisfy

$$\begin{aligned} &(\gamma_k', \varphi_j) = 0, \qquad j = 1,\ldots, s, \qquad k = s + 1,\ldots, p, \\ &(\gamma_k', \varphi_j) = \delta_{kj}, \qquad j = s + 1,\ldots, p \\ &(\delta_l', \varphi_j) = 0, \qquad l = p + 1\ldots, q, \qquad j = 1,\ldots, p. \end{aligned} \tag{4.24}$$

We note that $\det((\varphi_k, \varphi_j)) = 1$, and since $\mathfrak{D}(S_0)$ is dense in $\mathfrak{H}$, there exist $\varphi_1^{(0)},\ldots, \varphi_p^{(0)} \in \mathfrak{D}(S_0)$ such that

$$\det((\varphi_r^{(0)}, \varphi_j)) \neq 0, \qquad j, r = 1,\ldots, p. \tag{4.25}$$

Let

$$\gamma_k^{(0)} = \sum_{r=1}^{p} a_{kr}\varphi_r^{(0)} \in \mathfrak{D}(S_0), \qquad k = s + 1,\ldots, p,$$

$$\delta_l^{(0)} = \sum_{r=1}^{p} b_{lr}\varphi_r^{(0)} \in \mathfrak{D}(S_0), \qquad l = p + 1,\ldots, q,$$

be chosen so that the equations in (4.24) are valid. Because of (4.25), these equations are uniquely solvable for the a_{kr} and b_{kr}. Then the γ_k', δ_l' will satisfy (4b) and the first relation in (4c) of Theorem A_4. The second relation in (4c) is also valid since $\langle\delta_l'\delta_j'\rangle = \langle\delta_l\delta_j\rangle = 0$ by (b). As to (4a), suppose

$$\eta = \sum_{k=s+1}^{p} a_k\gamma_k' + \sum_{l=p+1}^{q} b_l\delta_l' \in \mathfrak{D}(S) \subset \mathfrak{H}_0^{\perp}.$$

Then $(\eta, \varphi_j) = 0$, $j = 1,\ldots, p$, and from (4.24) we obtain $a_k = 0$ for $k = s + 1,\ldots, p$. Thus $\eta \in \mathfrak{D}(S) \subset \mathfrak{D}(S_0)$ and

$$\eta = \sum_{l=p+1}^{q} b_l\delta_l' = \sum_{l=p+1}^{q} b_l\delta_l + \sum_{l=p+1}^{q} b_l\delta_l^{(0)},$$

or

$$\sum_{l=p+1}^{q} b_l\delta_l \in \mathfrak{D}(S_0).$$

Using (a), we find all $b_l = 0$, completing the proof of (4a) for the γ_k', δ_l'.

From the converse of Theorem A_4, we now have H is described as in (4i), with γ_k', δ_l' replacing γ_k, δ_l in (4i) and (4d). But $\langle hh^{(0)}\rangle = 0$

for all $h \in \mathfrak{D}(S_0^*)$, $h^{(0)} \in \mathfrak{D}(S_0)$. Thus in (4d) we have $\langle \gamma_k' \gamma_r' \rangle = \langle \gamma_k \gamma_r \rangle$, $\langle \delta_k' \gamma_r' \rangle = \langle \delta_k \gamma_r \rangle$, and in (4i) we have $\langle h\delta_l' \rangle = \langle h\delta_l \rangle$, $\langle h\gamma_k' \rangle = \langle h\gamma_k \rangle$. Hence, if ψ_k, ζ_k are defined by (c), then H as given by (i) is a self-adjoint extension of S. This completes the proof of Theorem 3.

5. Special Cases of Theorem 3

The statement of Theorem 3 needs interpreting in certain special cases. If $s = 0$, then $H(0) = \{0\}$ and $H = H_s$ is an *operator* extension of S. Then in (c), the sums run from $r = 1$ to $r = p$, and from (i), (ii) we have H is the set of all $\{h, S_0^*h + \varphi\} = \{h, Hh\}$, $h \in \mathfrak{D}(S_0^*)$, such that

$$\begin{gathered} \langle h\delta_l \rangle - (h, \zeta_l) = 0, \qquad l = p+1, \dots, q = p + \omega, \\ Hh = S_0^*h + \sum_{k=1}^{p} [(h, \psi_k) - \langle h\gamma_k \rangle] \varphi_k , \end{gathered} \tag{5.1}$$

where ψ_k, ζ_k are defined by (c). If all $\gamma_k = 0$ and all $E_{kr} = 0$, then from (c) we obtain $\psi_k = 0$, $\zeta_k = 0$, and hence the conditions (5.1) reduce to

$$\begin{gathered} \langle h\delta_l \rangle = 0, \qquad l = p+1, \dots, q = p + \omega, \\ Hh = S_0^*h, \end{gathered} \tag{5.2}$$

where the $\delta_l \in \mathfrak{D}(S_0^*)$ are linearly independent mod $\mathfrak{D}(S_0)$ and satisfy

$$\langle \delta_l \delta_j \rangle = 0, \qquad j, l = p+1, \dots, q = p + \omega. \tag{5.3}$$

Such operator extensions satisfy $S_0 \subset H \subset S_0^*$, and it is known (see, for example [2, Theorem 3] that (5.2), (5.3) characterize such extensions. The latter fact can also be deduced from Theorem 3 itself.

Theorem 3 implies the *existence* of self-adjoint extensions H of S such that $\dim H(0) = s$ for any given integer s satisfying $0 \leqslant s \leqslant p$. Indeed, let H_1 be any self-adjoint extension of S_0 described as above via $\delta_{p+1}, \dots, \delta_q \in \mathfrak{D}(S_0^*)$ which are linearly independent mod $\mathfrak{D}(S_0)$ and such that $\langle \delta_l \delta_j \rangle = 0$, $(j, l = p+1, \dots, q)$. Choose $\gamma_k = 0$, $E_{kr} = 0$, and hence $\psi_k = \zeta_k = 0$ in (c). Then Theorem 3 asserts that the set of all $\{h, S_0^*h + \varphi\}$, where $h \in \mathfrak{D}(S_0^*)$, $\varphi \in \mathfrak{H}_0$, and

$$\begin{gathered} (h, \varphi_j) = 0, \qquad j = 1, \dots, s, \\ \langle h\delta_l \rangle = 0, \qquad l = p+1, \dots, q = p + \omega, \\ \varphi \in \operatorname{span}\{\varphi_1, \dots, \varphi_s\}, \end{gathered}$$

is a self-adjoint extension of S such that $H(0) = \text{span}\{\varphi_1, ..., \varphi_s\}$, and $H_s h = Q_0 S_0{}^* h = Q_0 H_1 h$, where Q_0 is the orthogonal projection of $\mathfrak{H}$ onto $(H(0))^\perp$. Thus if $\Phi_s = \text{span}\{\varphi_1, ..., \varphi_s\}$,

$$H = \{\{h, H_1 h + \varphi\} \mid h \in \mathfrak{D}(H_1) \cap \Phi_s{}^\perp, \varphi \in \Phi_s\}.$$

If $s = p$, we have $H(0) = \mathfrak{H}_0$, there are no γ_k involved in Theorem 3, and the sums from $s + 1$ to p are vacuous. Thus there are no ψ_k, ζ_k, and $\mathfrak{D}(H)$ is described by

$$\langle h\delta_l \rangle = 0, \qquad l = p + 1, ..., q = p + \omega, \quad h \in \mathfrak{D}(S_0{}^*) \cap \mathfrak{H}_0{}^\perp,$$

where $\delta_{p+1}, ..., \delta_q$ are linearly independent elements of $\mathfrak{D}(S_0{}^*)$ satisfying (5.3). In this case, $H_s h = Q_0 S_0{}^* h$, where Q_0 is the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{H}_0{}^\perp$.

If $\omega = 0$, then $S_0 = S_0{}^*$ is a self-adjoint operator. Consequently, $p = q$, and there are no δ_l or ζ_l involved in the statement of Theorem 3. If $\omega = 0$ *and* $s = p$, then S_0 is self-adjoint, $H(0) = \mathfrak{H}_0$, and $H_s h = Q_0 S_0 h$, where Q_0 is the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{H}_0{}^\perp$. Hence H is the set of all $\{h, Q_0 S_0 h + \varphi\}$ such that $h \in \mathfrak{D}(S_0) \cap \mathfrak{H}_0{}^\perp$, $\varphi \in \mathfrak{H}_0$. Thus, given any self-adjoint operator S_0 in $\mathfrak{H}$, with $\mathfrak{D}(S_0)$ dense in $\mathfrak{H}$, and subspace $\mathfrak{H}_0 \subset \mathfrak{H}$, $\dim \mathfrak{H}_0 < \infty$, the operator H_s on $\mathfrak{H}_0{}^\perp$ defined by $H_s h = Q_0 S_0 h$ is a densely defined self-adjoint operator. This is a result due to W. Stenger ([7, Lemma 1]).

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References

1. R. Arens, Operational calculus of linear relations, *Pacific J. Math.* **11** (1961), 9–23.
2. E. A. Coddington, The spectral representation of ordinary self-adjoint differential operators, *Ann. of Math.* **60** (1954), 192–211.
3. E. A. Coddington, Extension theory of formally normal and symmetric subspaces, *Mem. Amer. Math. Soc.* **134** (1973).
4. E. A. Coddington, Self-adjoint subspace extensions of non-densely defined symmetric operators, *Bull. Amer. Math. Soc.* **79** (1973), 712–715.
5. E. A. Coddington, Eigenfunction expansions for non-densely defined operators generated by symmetric ordinary differential expressions, *Bull. Amer. Math. Soc.* **79** (1973), 964–968.

6. I. C. Gohberg and M. G. Krein, The basic propositions on defect numbers, root numbers and indices of linear operators, *Uspehi Mat. Nauk* (N.S.) **12** (1957), No. 2 (74), 43–118; English version in *Amer. Math. Soc. Transl.* (Series 2) **13** (1960), 185–264.
7. W. Stenger, On the projection of a self-adjoint operator, *Bull. Amer. Math. Soc.* **74** (1968), 369–372.