An upper bound for Cubicity in terms of Boxicity

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Abstract

An axis-parallel $b$-dimensional box is a Cartesian product $R_1 \times R_2 \times \cdots \times R_b$ where each $R_i$ (for $1 \leq i \leq b$) is a closed interval of the form $[a_i, b_i]$ on the real line. The boxicity of any graph $G$, box($G$) is the minimum positive integer $b$ such that $G$ can be represented as the intersection graph of axis-parallel $b$-dimensional boxes. A $b$-dimensional cube is a Cartesian product $R_1 \times R_2 \times \cdots \times R_b$, where each $R_i$ (for $1 \leq i \leq b$) is a closed interval of the form $[a_i, a_i+1]$ on the real line. When the boxes are restricted to be axis-parallel cubes in $b$-dimension, the minimum dimension $b$ required to represent the graph is called the cubicity of the graph (denoted by cub($G$)). In this paper we prove that cub($G$) $\leq \lceil \log_2 n \rceil \cdot \text{box}($G$)$, where $n$ is the number of vertices in the graph. We also show that this upper bound is tight.

Some immediate consequences of the above result are listed below:

1. Planar graphs have cubicity at most $3\lceil \log_2 n \rceil$.
2. Outer planar graphs have cubicity at most $2\lceil \log_2 n \rceil$.
3. Any graph of treewidth $tw$ has cubicity at most $(tw + 2)\lceil \log_2 n \rceil$. Thus, chordal graphs have cubicity at most $(\omega + 1)\lceil \log_2 n \rceil$ and circular arc graphs have cubicity at most $(2\omega + 1)\lceil \log_2 n \rceil$, where $\omega$ is the clique number.

The above upper bounds are tight, but for small constant factors.

1. Introduction

Let $\mathcal{F} = \{S_x \subseteq U : x \in V\}$ be a family of subsets of $U$, where $V$ is an index set. The intersection graph $\mathcal{I}(\mathcal{F})$ of $\mathcal{F}$ has $V$ as vertex set, and two distinct vertices $x$ and $y$ are adjacent if and only if $S_x \cap S_y \neq \emptyset$. Representations of graphs as the intersection graphs of various geometric objects is a well-studied area in graph theory. A prime example of a graph class defined in this way is the class of interval graphs. A graph $I(V, E)$ is an interval graph if and only if there exists a function $\Pi$ which maps each vertex $u \in V$ to a closed interval of the form $[l(u), r(u)]$ on the real line such that $(u, v) \in E$ if and only if $\Pi(u) \cap \Pi(v) \neq \emptyset$. We will call $\Pi$ an interval representation of $I(V, E)$. An indifference graph is an interval graph which has an interval representation in which each of the intervals is of the same length. We will call such an interval representation a unit interval representation of the graph. Indifference graphs are also known as unit interval graphs. See Chapter 8 of [15] for more information on interval graphs and indifference graphs.

Motivated by theoretical as well as practical considerations, graph theorists have tried to generalize the concept of interval graphs in many ways. In many cases, a graphical representation of the intersection graph of a family of geometric objects, which are generalizations of intervals, is sought. Concepts such as boxicity and interval number are examples.

In this paper we only consider simple, finite, undirected graphs. $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of $G$, respectively. For a vertex $v \in V(G)$, $N(v)$ denotes its neighbours, i.e. $N(v) = \{u \in V : (u, v) \in E\}$. For a graph $G$, the boxicity box($G$) is the minimum positive integer $b$ such that $G$ can be represented as the intersection graph of axis-parallel...
Lemma 1 (Roberts [19]). Given a graph $G$, the minimum positive integer $b$ such that there exist interval graphs $G_1, G_2, \ldots, G_b$ with $V(G) = V(G_i)$ for $1 \leq i \leq b$ and satisfying $E(G) = E(G_1) \cap E(G_2) \cap \cdots \cap E(G_b)$ is equal to box$(G)$.

Lemma 2 (Roberts [19]). Given a graph $G$, the minimum positive integer $b$ such that there exist indifference graphs $G_1, G_2, \ldots, G_b$ with $V(G) = V(G_i)$ for $1 \leq i \leq b$ and satisfying $E(G) = E(G_1) \cap E(G_2) \cap \cdots \cap E(G_b)$ is equal to cub$(G)$.

The concepts of cubicity and boxicity were introduced by Roberts [19]. They find applications in niche overlap in ecology and in solving problems of fleet maintenance in operations research. (See [11].) It was shown by Cozzens [10] that computing the boxicity of a graph is a nondeterministic polynomial (NP)-hard problem. Later, this was improved by Yannakakis [23], and finally by Kratochvil [17] who showed that deciding whether the boxicity of a graph is at most 2 itself is an NP-complete problem. The complexity of finding the maximum independent set in bounded boxicity graphs was considered by [16, 14]. Some NP-hard problems are known to be either polynomial time solvable or have much better approximation ratio on low boxicity graphs. For example, the max-clique problem is polynomial time solvable on bounded boxicity graphs and the maximum independent set problem has log $n$ approximation ratio for graphs with boxicity 2 [13].

There have been many attempts to find the cubicity and boxicity of graphs with special structures. In his pioneering work, Roberts [19] proved that the boxicity of a complete $k$-partite graph (where each part has at least 2 vertices) is $k$. He also proved that the cubicity of any graph on $n$ vertices cannot be greater than $2n/3$ and the boxicity cannot be greater than $\lceil n/2 \rceil$. Scheinerman [20] showed that the boxicity of outer planar graphs is at most 2. Thomassen [21] proved that the boxicity of planar graphs is bounded above by 3. The boxicity of split graphs is investigated by Cozzens and Roberts [11]. Chandran and Sivadasan [6] proved that the cubicity of the $d$-dimensional hypercube $H_d$ is $\Theta\left(\frac{d}{\log d}\right)$. They also proved that for any graph $G$, box$(G) \leq tw(G) + 2$ where tw$(G)$ is the treewidth of $G$ [7]. This in turn throws light on the boxicity of various other graph classes. The boxicity of series-parallel graphs was studied in [4]. It was shown in [5] that for any graph on $n$ vertices with maximum degree $\Delta$, the boxicity is $O(\Delta \log n)$.

Researchers have also tried to generalize or extend the concept of boxicity in various ways. The poset boxicity [22], the rectangular number [9], grid dimension [2], circular dimension [13] and the boxicity of digraphs [8] are some examples. Roberts and Cozzens proposed a theory of dimensional properties, attempting to generalize the concepts of cubicity and boxicity [12]. These concepts were further developed by Kratochvil and Tuza [18].

2. Our results

It is easy to see that for any graph $G$, box$(G) \leq$ cub$(G)$. In this paper we prove an upper bound for cubicity in terms of boxicity. We need the following lemmas.

Lemma 3 (Roberts [19]). Let $G$ be a graph and let $G_1, G_2, \ldots, G_j$ be graphs such that (1) $V(G) = V(G_p)$ for $1 \leq p \leq j$ and (2) $E(G) = E(G_1) \cap E(G_2) \cap \cdots \cap E(G_j)$. Then cub$(G) \leq$ cub$(G_1) +$ cub$(G_2) + \cdots +$ cub$(G_j)$.

Lemma 4. Let $r(n)$ denote the largest real number such that there exists a non-complete graph $G$ (i.e. a graph $G$ such that box$(G) > 0$) on $n$ vertices such that cub$(G) = r(n)$box$(G)$. Then, there exists an interval graph $G'$ on $n$ vertices such that cub$(G') = r(n)$.

Proof. Let $G$ be a graph on $n$ vertices such that box$(G) = b$ and cub$(G) = b \cdot r(n)$. Then by Lemma 1, there exists interval graphs $G_1, G_2, \ldots, G_b$ such that $V(G) = V(G_i)$ for $1 \leq i \leq b$ and $E(G) = E(G_1) \cap E(G_2) \cap \cdots \cap E(G_b)$. By Lemma 3, $r(n) \cdot b =$ cub$(G) \leq \sum_{i=1}^{b}$ cub$(G_i)$. It follows that there exists at least one $i$, $(1 \leq i \leq b)$ such that cub$(G_i) \geq r(n)$. Recalling that $G_i$ is a (non-complete) interval graph and thus box$(G_i) = 1$ we have cub$(G_i) \geq r(n) \cdot$ box$(G_i)$. From the definition of $r(n)$, it follows that cub$(G_i) = r(n) \cdot$ box$(G_i) = r(n)$, as required. \qed

Lemma 5. For every interval graph $G$ on $n$ vertices, there exists an ordering $f : V(G) \rightarrow \{0, 1, 2, \ldots, n - 1\}$ of its vertices such that if $u, v, w \in V(G)$ satisfy $f(u) < f(w) < f(v)$ and $(u, v) \in E(G)$ then $(u, w) \in E(G)$, also.

Proof. Consider an interval representation of $G$ and order the vertices in the non-decreasing order of the left end-points of the intervals. It is easy to verify that this order satisfies the required property. \qed

Theorem 1. For a graph $G$ on $n$ vertices, cub$(G) \leq \lceil \log_2 n \rceil$box$(G)$. Moreover, this upper bound is tight.
By Lemma 4, it is enough to show that for any interval graph $G$ on $n$ vertices, $\text{cub}(G) \leq \lceil \log_2 n \rceil$. Let $k = \lceil \log_2 n \rceil$. Then by Lemma 2, we only have to show that there exists $k$ indifference graphs $I_1, I_2, \ldots, I_k$ such that $V(I_i) = V(G)$ for $1 \leq i \leq k$ and $E(G) \subseteq \bigcap_{i=1}^k E(I_i)$. Let $f$ be an ordering of $V$ as described in Lemma 5. For any vertex $u$, let $f_i(u)$ denote the $i$th least significant bit in the binary representation of $f(u)$ using $k$ bits (note that only $k = \lceil \log_2 n \rceil$ bits will be required for this binary representation as $u, f(u) < n$). Thus, $f(u) = \sum_{i=1}^k f_i(u)2^{i-1}$.

For constructing each indifference graph $I_i$, we define a partition of $V$ into two sets $A_i$ and $B_i$ as follows.

$$A_i = \{u \in V : f_i(u) = 0\} \quad \text{and} \quad B_i = \{u \in V : f_i(u) = 1\}.$$ Clearly $(A_i, B_i)$ is a partition of $V$. Now we define the indifference graph $I_i$ by defining its unit interval representation $I_i$ as follows:

- For $v \in B_i$, $I_i(v) = [n + f(v) + 1, 2n + f(v) + 1]$.
- For $v \in A_i$, if $N(v) \cap B_i = \emptyset$, $I_i(v) = [0, n]$.
- For $v \in A_i$, if $N(v) \cap B_i \neq \emptyset$: (Let $t = \max_{x \in (N(v) \cap B_i)} f(x)$. $I_i(v) = [t + 1, n + t + 1]$.)

**Claim 1.** $E(I_i) \supseteq E(G)$ for $1 \leq i \leq k$.

Let $(u, v) \in E(G)$. We only have to consider the following three cases.

**Case 1:** $u \in A_i$ and $v \in A_i$. Then $I_i(u) \cap I_i(v) = \emptyset$ since the $n \in I_i(u) \cap I_i(v)$.

**Case 2:** $u \in B_i$ and $v \in B_i$. Here also $I_i(u) \cap I_i(v) = \emptyset$ since $2n \in I_i(u) \cap I_i(v)$.

**Case 3:** $u \in A_i$ and $v \in B_i$. In this case, let $z = \max\{f(x) : x \in N(u) \cap B_i\}$. Now, $f(v) \leq z$, since $v \in N(u) \cap B_i$. Now recall that $I_i(v) = [n + f(v) + 1, 2n + f(v) + 1]$ and $I_i(u) = [z + 1, n + z + 1]$. Clearly, $n + z + 1 \in I_i(u) \cap I_i(v)$, and thus $I_i(u) \cap I_i(v) = \emptyset$. □

**Claim 2.** If $(u, v) \notin E(G)$ then there exists an $i, 1 \leq i \leq k$ such that $(u, v) \notin E(I_i)$.

Let us assume without loss of generality that $f_i(u) < f_i(v)$. Define $t = \max_{x \in \{2, 3, \ldots, k\}} f_i(x)$. $t$ therefore is the most significant bit position at which $f_i(u)$ and $f_i(v)$ differ. Note that since $f_i(u) \neq f_i(v)$ there is at least one bit position at which $f_i(u)$ and $f_i(v)$ differ. Also note that since $f_i(u) < f_i(v)$, $f_i(u) = 0$ and $f_i(v) = 1$.

Now, we will show that $(u, v) \notin E(I_i)$. It follows from our earlier observation that $u \in A_i$ and $v \in B_i$. If $N(u) \cap B_i = \emptyset$, clearly $(u, v) \notin E(I_i)$, since in that case $I_i(u) = [0, n]$ and $I_i(v) = [n + f(v) + 1, 2n + f(v) + 1]$ and these two intervals are disjointed. So, we can assume that $N(u) \cap B_i \neq \emptyset$. Now, let $w \in B_i$ be such that $f(w) = \max\{f(x) : x \in N(u) \cap B_i\}$. We claim that $f(w) < f(v)$. Suppose not. Then clearly $f_i(u) < f_i(v)$. $f_i(w)$. Now by Lemma 5, $(u, v) \in E(G)$, since $(u, w) \in E(G)$, contradicting the assumption that $(u, v) \notin E(G)$. Now, recall that $I_i(u) = [f(w) + 1, n + f(w) + 1]$, and $I_i(v) = [n + f(v) + 1, 2n + f(v) + 1]$. Since $f(w) < f(v)$ we have $I_i(u) \cap I_i(v) = \emptyset$ and thus $(u, v) \notin E(I_i)$. □

From Claims 1 and 2 we have, $E(G) = E(I_1) \cap E(I_2) \cap \cdots \cap E(I_k)$ as required. So by Lemma 2, $\text{cub}(G) \leq k = \lceil \log_2 n \rceil$.

Finally, the tightness of our result can be verified by considering the star graph on $n$ vertices, $S(n)$. (Note: The star graph $S(n)$ is the complete bipartite graph $K_{1,n-1}$, with a single vertex on one side and the remaining $n-1$ vertices on the other side.) Its boxicity equals 1, since it is an interval graph. It is also known that [19] $\text{cub}(S(n)) = \lceil \log_2 (n-1) \rceil$. Note that when $n \neq 2^k + 1$, we have $\lceil \log_2 (n-1) \rceil = \lceil \log_2 n \rceil$ and thus our upper bound is tight. □

### 2.1. Consequences of our result

The upper bound that we developed should be useful in many cases where a bound for one of the two quantities (boxicity and cubicity) is already known. Combining our theorem with previously known upper bounds for boxicity, we get various upper bounds for cubicity, which we list in the following table. Here $n$ denotes the number of vertices in the graph, $tw = \text{treewidth}(G)$ is the treewidth of $G$, $\Delta = \Delta(G)$ is the maximum degree and $\omega = \omega(G)$ is the clique number, i.e. the number of vertices in the biggest clique in $G$. Each of the references given corresponds to the paper in which the corresponding upper bound for boxicity was proved.

<table>
<thead>
<tr>
<th>Graph class</th>
<th>Upper bound for box($G$)</th>
<th>Upper bound for cub($G$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chordal graphs [7]</td>
<td>$\omega + 1$</td>
<td>$\lceil \omega + 1 \rceil \lceil \log_2 n \rceil$</td>
</tr>
<tr>
<td>Circular arc graphs [7]</td>
<td>$2\omega + 1$</td>
<td>$2 (\omega + 1) \lceil \log_2 n \rceil$</td>
</tr>
<tr>
<td>Planar graphs [17]</td>
<td>3</td>
<td>$3 \lceil \log_2 n \rceil$</td>
</tr>
<tr>
<td>Outer planar graphs [20]</td>
<td>2</td>
<td>$2 \lceil \log_2 n \rceil$</td>
</tr>
<tr>
<td>Any graph [7]</td>
<td>$tw + 2$</td>
<td>$(tw + 2) \lceil \log_2 n \rceil$</td>
</tr>
</tbody>
</table>

**Remark 1.** It may be noted that all the upper bounds given in the above table are tight, but for a small constant factor. The tight example is provided by the star graph $S(n)$.

**Remark 2.** Since $\omega(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree, we also get that $\text{cub}(G) \leq (\Delta + 2) \lceil \log_2 n \rceil$ for chordal graphs, and $\text{cub}(G) \leq (2\Delta + 3) \lceil \log_2 n \rceil$ for circular arc graphs.
2.1.1. Algorithmic consequences

Our proof provides an $O(n^2 \log n)$ algorithm to represent any interval graph $G$ (on $n$ vertices) in a $\log_2 n$-dimensional space as the intersection graph of $n$ axis-parallel $\log_2 n$-dimensional cubes, when the interval representation of $G$ is given. Also following from this, a polynomial time algorithm to translate any given box representation of a graph in a $b$-dimensional space to a cube representation in $b \log_2 n$-dimensional space.

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References


