Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 260 (2011) 832–851
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Some results on the controllability of forward stochastic heat equations with control on the drift ☆

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Received 2 June 2010; accepted 24 October 2010

Available online 4 November 2010

Communicated by J. Coron

Abstract

In this paper, we establish the null/approximate controllability for forward stochastic heat equations with control on the drift. The null controllability is obtained by a time iteration method and an observability estimate on partial sums of eigenfunctions for elliptic operators. As a consequence of the null controllability, we obtain the observability estimate for backward stochastic heat equations, which leads to a unique continuation property for backward stochastic heat equations, and hence the desired approximate controllability for forward stochastic heat equations. It deserves to point out that one needs to introduce a little stronger assumption on the controller for the approximate controllability of forward stochastic heat equations than that for the null controllability. This is a new phenomenon arising in the study of the controllability problem for stochastic heat equations.

Keywords: Stochastic heat equation; Null controllability; Approximate controllability

☆ This work is partially supported by National Basic Research Program of China (973 Program) under grant 2011CB808000 and the NSF of China under grants 10831007 and 60974035. This paper is an improved version of one chapter of the author's PhD thesis [10] accomplished at Sichuan University under the guidance of Professor Xu Zhang. The author would like to take this opportunity to thank him deeply for his help.

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0022-1236/$ – see front matter
doi:10.1016/j.jfa.2010.10.018

1. Introduction

Let $T > 0$, $G \subset \mathbb{R}^n$ be a given bounded domain with a C^2 boundary ∂G, and G_0 a given nonempty open subset of G. Denote by χ_{G_0} the characteristic function of G_0. Put $Q \triangleq (0,T) \times G$ and $\Sigma \triangleq (0,T) \times \partial G$.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geqslant 0}, P)$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $\{w(t)\}_{t \geqslant 0}$ is defined so that $\{\mathcal{F}_t\}_{t \geqslant 0}$ is its natural filtration augmented by all the P-null sets. Let H be a Banach space. We denote by $L^2_{\mathcal{F}}(0,T;H)$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t \geqslant 0}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0,T;H)}) < \infty$, with the canonical norm; by $L^r_{\mathcal{F}}(0,T;L^2(\Omega;H))$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t \geqslant 0}$-adapted processes X such that $|\mathbb{E}|X|^2_H|_{L^r(0,T)} < \infty$ $(1 \leqslant r \leqslant \infty)$, with the canonical norm. Put $L^\infty_{\mathcal{F}}(0,T;H)$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t \geqslant 0}$-adapted bounded processes. Denote by $L^2_{\mathcal{F}}(\Omega;C([0,T];H))$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t \geqslant 0}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{C(0,T;H)}) < \infty$, with the canonical norm.

Let $a^{ij} \in C^1(\overline{G})$ $(i,j = 1,2,\dots,n)$ satisfy $a^{ij} = a^{ji}$ and for some constant $\mu > 0$,

$$\sum_{i,j=1}^n a^{ij}\xi_i\xi_j \geqslant \mu|\xi|^2, \quad \forall (x,\xi) \in G \times \mathbb{R}^n.$$

This paper is devoted to a study of the null/approximate controllability for the following stochastic heat equation

$$\begin{cases} dy - \displaystyle\sum_{i,j=1}^n \big(a^{ij} y_{x_i}\big)_{x_j}\, dt = a(t) y\, dw + \chi_E \chi_{G_0} f\, dt & \text{in } Q, \\ \tilde{l} \displaystyle\sum_{i,j=1}^n a^{ij} y_{x_i} \nu^j + l y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } G, \end{cases} \tag{1.1}$$

where $a(t) \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R})$, E is a measurable subset in $(0,T)$ with a positive Lebesgue measure (i.e., $m(E) > 0$), χ_E is the characteristic function of E, $\nu = (\nu^1, \nu^2, \dots, \nu^n) = \nu(x)$ is the unit outward normal vector of G at $x \in \partial G$, both l and $\tilde{l}$ belong to $L^\infty(\partial G)$ and satisfy either $\tilde{l} = 1$ and $l \geqslant 0$ or $\tilde{l} = 0$ and $l > 0$, $y_0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(G))$, the control f belongs to $L^\infty_{\mathcal{F}}(0,T;L^2(\Omega;L^2(G)))$. We refer to [2, Chapter 6] for the well-posedness of system (1.1) in the class $y \in L^2_{\mathcal{F}}(\Omega;C([0,T];L^2(G))) \cap L^2(0,T;D(A^{\frac{1}{2}}))$.

The equation we study here is a particular case of stochastic heat equations since $a(\cdot)$ is independent of x. The way to manage the general potential that depends both on t and x is unknown.

Put $\tau = |a|^2_{L^\infty_{\mathcal{F}}(0,T;\mathbb{R})}$. Throughout this paper, we will use C to denote a generic positive constant depending only on G, G_0, T, $(a^{ij})_{n \times n}$, l, $\tilde{l}$ and τ, which may change from one place to another.

In this paper, we will prove the following theorem on the null controllability of system (1.1).

Theorem 1.1. *System* (1.1) *is null controllable at time T, i.e., for each initial datum $y_0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(G))$, there is a control $f \in L^\infty_{\mathcal{F}}(0, T; L^2(\Omega; L^2(G)))$ such that the solution y of system* (1.1) *satisfies $y(T) = 0$ in G, P-a.s. Moreover, the control f satisfies the following estimate*:

$$|f|^2_{L^\infty_{\mathcal{F}}(0,T;L^2(\Omega;L^2(G)))} \leqslant C\mathbb{E}|y_0|^2_{L^2(G)}. \tag{1.2}$$

System (1.1) is said to be *approximately controllable at time T* if for any initial datum $y_0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(G))$, any final state $y_1 \in L^2(\Omega, \mathcal{F}_T, P; L^2(G))$ and any $\varepsilon > 0$, there exists a control $f \in L^2_{\mathcal{F}}(0, T; L^2(G))$ such that the solution of system (1.1) with initial datum y_0 and control f satisfies $|y(T) - y_1|_{L^2(\Omega,\mathcal{F}_T,P;L^2(G))} \leqslant \varepsilon$. Starting from Theorem 1.1, we will show the following approximate controllability result for system (1.1) under a little stronger assumption on the controller than that for the null controllability:

Theorem 1.2. *System* (1.1) *is approximately controllable at time T if and only if $m((s, T) \cap E) > 0$ for any $s \in [0, T)$.*

Remark 1.1. It seems that Theorem 1.2 is unreasonable at the first glance. If $a \equiv 0$, then system (1.1) is like a deterministic heat equation with a random parameter. The readers may guess that one can obtain the approximate controllability by only assuming $m((0, T) \cap E) > 0$. However, this is untrue. The reason for this comes from our definition of the approximate controllability for system (1.1). We want any element belongs to $L^2(\Omega, \mathcal{F}_T, P; L^2(G))$ other than $L^2(\Omega, \mathcal{F}_s, P; L^2(G))$ ($s < T$) can be gained on as close as one wants. Hence we need to put control act until the time T. The more details can be found in the proof of Theorem 1.2.

There are many studies on the controllability of deterministic parabolic equations (e.g. [4,5,7,16,17]). However, very little is known for the stochastic counterpart. To the best of our knowledge, one can find only a very few papers concerned with the controllability problems for stochastic parabolic equations. In [3], the authors announced an approximate controllability result for linear forward stochastic parabolic equations with time-invariant coefficients (it seems that the detailed proof of the result in [3] has never been published). In [1] and [14], the null controllability of both linear forward and backward stochastic parabolic equations was studied. Note however that, in [1], only a reachable set was presented for some linear forward stochastic parabolic equations; while in [14], the authors needed to introduce two controls (one is put on the drift term and the other on the diffusion term) to establish the null controllability result for general linear forward stochastic parabolic equations. In [13], the authors proved that the null controllability of a general class of stochastic parabolic equations can be reduced to suitable deterministic partial differential equations by simple computations on the related Riccati equations. Based on their result, the null controllability of system (1.1) can be established when a is deterministic.

Note that the dual system of system (1.1) is a backward stochastic heat equation. As remarked in [1], it is very hard to establish the observability estimate for this system with only one observer. In [14], the authors introduced two observers to overcome this difficulty, and therefore, they needed to use two controls to achieve the desired null controllability result. The system considered in this paper is simpler than that in [14], but the advantage is that we need to introduce only one control into the system. Moreover, our control is chosen to belong to a small space, i.e., $L^\infty_{\mathcal{F}}(0, T; L^2(\Omega; L^2(G)))$, and also we only put control in the measurable subset E. To do this, we need to borrow some ideas developed in [8] and [16]. As far as we know, Theorem 1.1 is the first null controllability result for forward stochastic parabolic equations with only one control. It would be quite interesting to derive a similar controllability result for linear forward stochastic parabolic equations with general (both time- and space-variant) coefficients or for some nonlinear stochastic parabolic equations but these seem to be challenging open problems.

In some sense, it is surprising that we need a little more assumption in Theorem 1.2 for the approximate controllability of system (1.1) than that in Theorem 1.1 for the null controllability. Indeed, it is well known that in the deterministic setting, the null controllability is usually stronger than the approximate controllability. But this does not remain to be true in the stochastic case. Indeed, from Theorem 1.2, we see that the additional condition (compared to the null controllability) that $m((s,T)\cap E)>0$ for any $s\in[0,T)$ is not only sufficient but also necessary for the approximate controllability of system (1.1). Therefore, for stochastic heat equations, the null controllability does NOT imply the approximate controllability. This indicates that there exists some essential difference between the controllability theory between deterministic heat equations and stochastic heat equations.

The rest of the paper is organized as follows. In Section 2, we show some preliminary results. In Section 3, we will prove Theorem 1.1. In Section 4, we will prove Theorem 1.2.

2. Preliminaries

In this section, we collect some preliminary results that will be used subsequently.

Firstly, we recall the following known and useful property about Lebesgue measurable sets.

Lemma 2.1. *(See [9, pp. 256–257].) For almost all $\tilde{t}\in E$, there exists a sequence of numbers $\{t_i\}_{i=1}^{\infty}\subset(0,T)$ such that*

$$t_1<t_2<\cdots<t_i<t_{i+1}<\cdots<\tilde{t},\quad t_i\to\tilde{t}\text{ as }i\to\infty, \tag{2.1}$$

$$m\big(E\cap[t_i,t_{i+1}]\big)\geqslant\rho(t_{i+1}-t_i),\quad i=1,2,\ldots, \tag{2.2}$$

$$\frac{t_{i+1}-t_i}{t_{i+2}-t_{i+1}}\leqslant C_0,\quad i=1,2,\ldots, \tag{2.3}$$

where ρ and C_0 are two positive constants which are independent of i.

Nextly, let A be an unbounded operator on $L^2(G)$ as follows

$$\begin{cases} D(A)=\left\{u\in H^2(G)\;\middle|\;\tilde{l}\displaystyle\sum_{i,j=1}^{n}a^{ij}u_{x_i}\nu^j+lu=0\text{ on }\partial G\right\},\\ Au=-\displaystyle\sum_{i,j=1}^{n}\big(a^{ij}u_{x_i}\big)_{x_j},\quad\forall u\in D(A).\end{cases} \tag{2.4}$$

Let $\{\lambda_i\}_{i=1}^{\infty}$ be the eigenvalues of A, and $\{e_i\}_{i=1}^{\infty}$ be the corresponding eigenfunctions satisfying $|e_i|_{L^2(G)}=1$, $i=1,2,3,\ldots$. We recall the following explicit observability estimate (for partial sums of the eigenfunctions of A), established in [10] (we refer to [7,8] for a special case of this result).

Lemma 2.2. *There exist two positive constants C_1 and C_2 such that*

$$\sum_{\lambda_i \leqslant r} |a_i|^2 \leqslant C_1 e^{C_2\sqrt{r}} \int_{G_0} \Bigl| \sum_{\lambda_i \leqslant r} a_i e_i(x) \Bigr|^2 dx \tag{2.5}$$

for every finite $r > 0$ and every choice of the coefficients $\{a_i\}_{\lambda_i \leqslant r}$ with $a_i \in \mathbb{C}$.

Further, we need to introduce the following backward stochastic heat equation:

$$\begin{cases} dz + \displaystyle\sum_{i,j=1}^{n} \bigl(a^{ij} z_{x_i}\bigr)_{x_j} dt = -a(t) Z\, dt + Z\, dw & \text{in } Q, \\ \tilde{l} \displaystyle\sum_{i,j=1}^{n} a^{ij} z_{x_i} \nu^j + lz = 0 & \text{on } \Sigma, \\ z(T) = \eta & \text{in } G. \end{cases} \tag{2.6}$$

For any terminal datum $\eta \in L^2(\Omega, \mathcal{F}_T, P; L^2(G))$, according to the well-posedness result for backward stochastic parabolic equations (e.g., [6,12,15]), Eq. (2.6) admits one and only one solution $(z, Z) \in (L^2_{\mathcal{F}}(\Omega; C([0,T]; L^2(G))) \cap L^2_{\mathcal{F}}(0,T; H^1_0(G))) \times L^2_{\mathcal{F}}(0,T; L^2(\Omega))$.

For each $r > 0$, we set $X_r = \operatorname{span}\{e_i(x)\}_{\lambda_i \leqslant r}$ and denote by P_r the orthogonal projection from $L^2(G)$ to X_r. We need to derive some observation results for system (2.6) with the final state belonging to X_r. The desired observation results, with an explicit estimate on the cost of the observation, can be stated as follows.

Proposition 2.1. *For each $r \geqslant \lambda_1$, the solution of Eq.* (2.6) *with $\eta \in L^2(\Omega, \mathcal{F}_T, P; X_r)$ satisfies*:

i) *If $2\lambda_1 > \tau$, then*

$$\mathbb{E}\bigl|z(0)\bigr|^2_{L^2(G)} \leqslant \frac{C_1 e^{C_2\sqrt{r}}}{(m(E))^2} \bigl|\chi_E(t)\chi_{G_0} z\bigr|^2_{L^1_{\mathcal{F}}(0,T; L^2(\Omega; L^2(G)))}; \tag{2.7}$$

ii) *For the general case, it holds that*

$$\mathbb{E}\bigl|z(0)\bigr|^2_{L^2(G)} \leqslant \frac{C_1 e^{C_2\sqrt{r}+\tau T}}{(m(E))^2} \bigl|\chi_E(t)\chi_{G_0} z\bigr|^2_{L^1_{\mathcal{F}}(0,T; L^2(\Omega; L^2(G)))}. \tag{2.8}$$

Proof. Each element η in $L^2(\Omega, \mathcal{F}_T, P; X_r)$ can be written as $\eta = \sum_{\lambda_i \leqslant r} \eta_i e_i(x)$ for a sequence of $\mathcal{F}_T$-measurable random variable $\{\eta_i\}_{\lambda_i \leqslant r}$. In this case, system (2.6) can be reduced to a backward stochastic ordinary differential systems. Indeed, the solution (z, Z) of Eq. (2.6) can be expressed as

$$z = \sum_{\lambda_i \leqslant r} z_i(t) e_i, \qquad Z = \sum_{\lambda_i \leqslant r} Z_i(t) e_i,$$

where $z_i(\cdot) \in L^2_{\mathcal{F}}(\Omega; C[0,T])$ and $Z_i(\cdot) \in L^2_{\mathcal{F}}(0,T)$ ($\lambda_i \leqslant r$), and satisfies the following equation

$$\begin{cases} dz_i - \lambda_i z_i\,dt = -a(t)Z_i\,dt + Z_i\,dw & \text{in } [0,T], \\ z_i(T) = \eta_i. \end{cases}$$

By Lemma 2.2, we have

$$\begin{aligned} \mathbb{E}\sum_{\lambda_i \leqslant r} \big|z_i(t)\big|^2 &\leqslant C_1 e^{C_2\sqrt{r}}\mathbb{E}\int_{G_0} \bigg|\sum_{\lambda_i \leqslant r} z_i(t)e_i\bigg|^2\,dx \\ &= C_1 e^{C_2\sqrt{r}}\mathbb{E}\int_{G_0} |z|^2\,dx, \quad \forall t \in [0,T]. \end{aligned} \tag{2.9}$$

Let us prove first conclusion i). By Itô's formula, we see that $d|z|^2 = 2z\,dz + (dz)^2$. Hence we obtain that

$$\begin{aligned} &\mathbb{E}\int_G \big|z(t)\big|^2\,dx - \mathbb{E}\int_G \big|z(0)\big|^2\,dx \\ &= 2\mathbb{E}\int_0^t \sum_{\lambda_i \leqslant r} \lambda_i \big|z_i(t)\big|^2\,dt + \mathbb{E}\int_0^t\int_G \big(-2a(t)zZ + Z^2\big)\,dx\,dt \\ &\geqslant 2\mathbb{E}\int_0^t \sum_{\lambda_i \leqslant r} \lambda_i \big|z_i(t)\big|^2\,dt - \mathbb{E}\int_0^t\int_G \big|a(t)z\big|^2\,dx\,dt \\ &\geqslant \mathbb{E}\int_0^t \sum_{\lambda_i \leqslant r} (2\lambda_i - \tau)\big|z_i(t)\big|^2\,dt \geqslant 0. \end{aligned} \tag{2.10}$$

From (2.9) and (2.10), we obtain that

$$\mathbb{E}\int_G z^2(x,0)\,dx \leqslant C_1 e^{C_2\sqrt{r}}\mathbb{E}\int_{G_0} \big|z(x,t)\big|^2\,dx, \quad \forall t \in [0,T].$$

Therefore,

$$\int_E \bigg[\mathbb{E}\int_G z^2(x,0)\,dx\bigg]^{\frac{1}{2}}\,dt \leqslant \big(C_1 e^{C_2\sqrt{r}}\big)^{\frac{1}{2}} \int_E \bigg[\mathbb{E}\int_{G_0} \big|z(x,t)\big|^2\,dx\bigg]^{\frac{1}{2}}\,dt.$$

Hence we obtained that for each $\eta \in L^2(G, \mathcal{F}_T, P; X_r)$,

$$\mathbb{E}\int_G z^2(x,0)\,dx \leqslant \frac{C_1 e^{C_2\sqrt{r}}}{(m(E))^2}\left\{\int_0^T \left[\mathbb{E}\int_G \left|\chi_E(t)\chi_{G_0}(x)z(x,t)\right|^2 dx\right]^{\frac{1}{2}} dt\right\}^2$$
$$= \frac{C_1 e^{C_2\sqrt{r}}}{(m(E))^2}|\chi_E\chi_{G_0}z|^2_{L^1_{\mathcal{F}}(0,T;L^2(\Omega;L^2(G)))}. \tag{2.11}$$

This gives the desired estimate (2.7) in conclusion i).

Next, let us prove conclusion ii). By Itô's formula, we find $d(e^{\tau t}|z|^2) = 2e^{\tau t}z\,dz + e^{\tau t}(dz)^2 + \tau e^{\tau t}|z|^2$. Hence we see that

$$\mathbb{E}e^{\tau t}\int_G |z(t)|^2 dx - \mathbb{E}\int_G |z(0)|^2 dx$$
$$= \mathbb{E}\int_0^t \sum_{\lambda_i\leqslant r} 2e^{\tau s}\lambda_i|z_i(s)|^2 ds + \mathbb{E}\int_0^t\int_G e^{\tau s}\left(-2a(s)zZ + Z^2\right)dx\,ds$$
$$+ \mathbb{E}\int_0^t\int_G \tau e^{\tau s}|z(s)|^2 dx\,ds$$
$$\geqslant \mathbb{E}\int_0^t \sum_{\lambda_i\leqslant r} 2e^{\tau s}\lambda_i|z_i(s)|^2 ds \geqslant 0. \tag{2.12}$$

From (2.9) and (2.12), we obtain that

$$\mathbb{E}\int_G z^2(x,0)\,dx \leqslant C_1 e^{C_2\sqrt{r}+\tau T}\mathbb{E}\int_{G_0}|z(x,t)|^2 dx, \quad \forall t\in[0,T].$$

Now, proceeding exactly as in the case considered above, we end up with the desired estimate (2.8). This completes the proof. □

By means of the usual duality argument (e.g., [14,16]), Proposition 2.1 yields the following partial controllability results for system (1.1), with explicit estimates on the control cost.

Proposition 2.2. *For each $r \geqslant \lambda_1$, there exists a control $f_r \in L^\infty_{\mathcal{F}}(0,T;L^2(\Omega;X_r))$ such that the solution y of system* (1.1) *with $f = f_r$ satisfies $P_r(y(\cdot,T)) = 0$ in G, P-a.s. Moreover, f_r verifies*:

i) *If $2\lambda_1 > \tau$, then*

$$|f_r|^2_{L^\infty_{\mathcal{F}}(0,T;L^2(\Omega;X_r))} \leqslant \frac{C_1 e^{C_2\sqrt{r}}}{(m(E))^2}\mathbb{E}|y_0|^2_{L^2(G)}; \tag{2.13}$$

ii) *For the general case, it holds that*

$$|f_r|^2_{L^\infty_{\mathcal{F}}(0,T;L^2(\Omega;X_r))} \leqslant \frac{C_1 e^{C_2\sqrt{r}+\tau T}}{(m(E))^2}\mathbb{E}|y_0|^2_{L^2(G)}. \tag{2.14}$$

Proof. We consider only case i) (case ii) can be analyzed similarly). Let us introduce a linear subspace of $L^1_{\mathcal{F}}(0,T;L^2(\Omega;X_r))$ as follows:

$$\Lambda \equiv \left\{\chi_E\chi_{G_0}z \;\middle|\; z \text{ solves Eq. (2.6) with some } \eta \in L^2(\Omega,\mathcal{F}_T,P;X_r)\right\},$$

and define a linear functional L on Λ as follows:

$$L(\chi_E\chi_{G_0}z) = -\mathbb{E}\int_G y_0 z(0)\,dx,$$

where y_0 is the initial datum of system (1.1). By Proposition 2.1, we see that L is a bounded linear functional (on Λ) whose norm is not larger than $(\frac{C_1e^{C_2\sqrt{r}}}{(m(E))^2}\mathbb{E}|y_0|^2_{L^2(G)})^{\frac{1}{2}}$. By Hahn–Banach Theorem, L can be extended to a bounded linear functional on $L^1_{\mathcal{F}}(0,T;L^2(\Omega;X_r))$ whose norm is not larger than $(\frac{C_1e^{C_2\sqrt{r}}}{(m(E))^2}\mathbb{E}|y_0|^2_{L^2(G)})^{\frac{1}{2}}$. For simplicity, we use the same notation for the extension. Now, by means of a Riesz-type Representation Theorem for general stochastic processes [11], we conclude that there is an $f_r \in L^\infty_{\mathcal{F}}(0,T;L^2(\Omega;X_r))$ such that

$$\mathbb{E}\int_Q \chi_E\chi_{G_0}f_r z\,dx\,dt = -\mathbb{E}\int_G y_0 z(0)\,dx, \tag{2.15}$$

and

$$|f_r|^2_{L^\infty_{\mathcal{F}}(0,T;L^2(\Omega;X_r))} \leqslant \frac{C_1 e^{C_2\sqrt{r}}}{(m(E))^2}\mathbb{E}|y_0|^2_{L^2(G)}.$$

We claim that f_r is the desired control. In fact, a direct computation shows that

$$\begin{aligned}
\mathbb{E}\int_G y(T)\eta\,dx - \mathbb{E}\int_G y_0 z(0)\,dx &= \mathbb{E}\int_Q d(yz)\,dx\\
&= \mathbb{E}\int_Q (z\,dy + y\,dz + dy\,dz)\,dx\\
&= \mathbb{E}\int_Q \left[-\sum_{i,j=1}^n a^{ij}y_{x_i}z_{x_j} + \sum_{i,j=1}^n a^{ij}z_{x_i}y_{x_j} + \chi_E\chi_{G_0}fz\right]dx\,dt\\
&= \mathbb{E}\int_Q \chi_E\chi_{G_0}fz\,dx\,dt.
\end{aligned} \tag{2.16}$$

From (2.15) and (2.16), we know that

$$\mathbb{E}\int_G y(T)\eta\,dx = 0. \tag{2.17}$$

Since η is an arbitrary element in $L^2(\Omega, \mathcal{F}_T, P; X_r)$, equality (2.17) allows us to conclude that $P_r(y(T)) = 0$, P-a.s. □

Finally, we show a decay result for system (1.1) without control.

Proposition 2.3. *If $f \equiv 0$ in system* (1.1)*, then for any $y_0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(G))$ with $P_{\lambda_{k-1}}(y_0) = 0$ for some $k = 2, 3, \ldots,$ the corresponding solution y of system* (1.1) *satisfies*

$$\mathbb{E}\big|y(t)\big|^2_{L^2(G)} \leqslant e^{-(2\lambda_k-\tau)t}\mathbb{E}|y_0|^2_{L^2(G)}. \tag{2.18}$$

Proof. By $y_0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(G))$ satisfying $P_{\lambda_{k-1}}(y_0) = 0$, we see that $y_0 = \sum_{i=k}^{\infty} y_0^i e_i$ for suitable $y_0^i \in L^2(\Omega, \mathcal{F}_0, P)$. Clearly, the solution y of system (1.1) can be expressed as

$$y = \sum_{i=k}^{\infty} y^i(t)e_i,$$

where $y^i(\cdot) \in L^2_{\mathcal{F}}(\Omega; C[0,T])$ solves the following equation

$$\begin{cases} dy^i + \lambda_i y^i\,dt = a(t)y^i\,dw & \text{in } [0,T], \\ y^i(0) = y_0^i. \end{cases}$$

By Itô's formula, we have that

$$d\big(e^{(2\lambda_k-\tau)t}|y|^2\big) = e^{(2\lambda_k-\tau)t}2y\,dy + e^{(2\lambda_k-\tau)t}(dy)^2 + (2\lambda_k-\tau)e^{(2\lambda_k-\tau)t}|y|^2.$$

Hence we know

$$\begin{aligned}
&\mathbb{E}\int_G e^{(2\lambda_k-\tau)t}\big|y(t)\big|^2\,dx - \mathbb{E}\int_G \big|y(0)\big|^2\,dx \\
&\quad = \mathbb{E}\int_0^T e^{(2\lambda_k-\tau)s}\sum_{i=k}^{\infty}(-2\lambda_i)\big|y^i\big|^2\,ds + \mathbb{E}\int_0^T\int_G e^{(2\lambda_k-\tau)s}a^2(s)|y|^2\,dx\,ds \\
&\qquad + (2\lambda_k-\tau)\mathbb{E}\int_0^T e^{(2\lambda_k-\tau)s}|y|^2\,dx\,ds \\
&\quad \leqslant 0,
\end{aligned}$$

which gives the desired estimate (2.18) immediately. □

3. Proof of Theorem 1.1

This section is devoted to giving a proof of Theorem 1.1.

Firstly, we explain the main ideas of our proof, some of which are borrowed from [8,16]. We distinguish two cases. The first case is that $2\lambda_1 > \tau$. In this case, by means of Proposition 2.2, one can show that the projection of solutions of system (1.1) over X_r can be controlled to zero and the control cost is $C_1 e^{C_2\sqrt{r}}/(m(E))^2$. On the other hand, by Proposition 2.3, solutions of system (1.1) without control ($f \equiv 0$) but with a vanishing projection of the initial data over X_r, decay in $L^2(\Omega, \mathcal{F}_t, P; L^2(G))$ at a rate of the order of $\exp(-(2r-\tau)t)$. Therefore, if we divide the set E into two parts $E_1 = (0, T_1) \cap E$ and $E_2 = (T_1, T) \cap E$ where T_1 is a chosen positive number such that $m(E_1) > 0$, we control the projection of the solution over X_r to zero in the first subset and then allow the equation to evolve without control in (T_1, T). It follows that, at time $t = T$, the projection of the solution y over X_r vanishes and the norm of the high frequencies does not exceed the norm of the initial datum y_0. This argument allows us to control to zero the projection of the solutions of (1.1) over X_r for any $r > 0$ but not the whole solution. For the later an iterative argument is needed in which the set E is decomposed into a suitable chosen sequence of subsets $[t_i, t_{i+1}] \cap E$ given by Lemma 2.1 and the argument above is applied in each subset to control an increasing range of frequencies with $\lambda_j \leqslant r_i$ and r_i going to infinity at suitable rate. The difficulty here is reduced to estimate the cost of the control and prove that it is finite. The latter is guaranteed by the energy decay of system (1.1). This is a key point in the proof of Theorem 1.1 in the first case. The second case is that $2\lambda_1 \leqslant \tau$. In this case, noting that $\lambda_i \to \infty$ as $i \to \infty$, we see that there exists a $k \in \mathbb{N}$ such that $2\lambda_k > \tau$. Therefore, by choosing first a control f_0 to make $P_r(y(T_1)) = 0$ (this follows from Proposition 2.2), the problem can be reduced to the first case considered before.

We now prove Theorem 1.1. Without loss of generality, in what follows we assume that $2\lambda_1 > \tau$ and $C_1 \geqslant 1$.

By Lemma 2.1, we can take a number $\tilde{t} \in E$ with $\tilde{t} < T$ and a sequence $\{t_N\}_{N=1}^{\infty}$ in the open interval $(0, T)$ such that (2.1)–(2.3) hold for some positive numbers ρ and C_0, and

$$\tilde{t} - t_1 \leqslant \min\{\lambda_1, 1\}.$$

Let us consider the following equation

$$\begin{cases} d\tilde{y} - \sum_{i,j=1}^{n} \left(a^{ij}\tilde{y}_{x_i}\right)_{x_j} dt = a(t)\tilde{y}\,dw + \chi_E \chi_{G_0} \tilde{f}\,dt & \text{in } G \times (t_1, \tilde{t}), \\ \tilde{l} \sum_{i,j=1}^{n} a^{ij}\tilde{y}_{x_i}\nu^j + l\tilde{y} = 0 & \text{on } \partial G \times (t_1, \tilde{t}), \\ \tilde{y}(t_1) = \tilde{y}_0 & \text{in } G. \end{cases} \tag{3.1}$$

We will show that for any given initial datum $\tilde{y}_0 \in L^2(\Omega, \mathcal{F}_{t_1}, P; L^2(G))$, there exists a control function $\tilde{f} \in L^\infty_{\mathcal{F}}(t_1, \tilde{t}; L^2(\Omega; L^2(G)))$ satisfying $|\tilde{f}|_{L^\infty_{\mathcal{F}}(t_1,\tilde{t};L^2(\Omega;L^2(G)))} \leqslant C\mathbb{E}|\tilde{y}_0|_{L^2(\Omega)}$, such that the solution $\tilde{y}$ of system (3.1) vanishes at time $\tilde{t}$, i.e. $y(\tilde{t}) = 0$ in G, P-a.s.

Set $I_N = [t_{2N-1}, t_{2N}]$, $J_N = [t_{2N}, t_{2N+1}]$ for $N = 1, 2, \ldots$. Then

$$[t_1, \tilde{t}) = \bigcup_{N=1}^{\infty} (I_N \cup J_N).$$

Notice that for each $N \geqslant 1$, it holds that $m(E \cap I_N) > 0$ and $m(E \cap J_N) > 0$. We will put control on I_N and allow the equation to evolve freely on J_N. Also, we fix a strictly monotone increasing sequence $(\lambda_1 \leqslant) r_1 < r_2 < \cdots < r_m \to \infty$ as $m \to \infty$.

Firstly, let us consider the following controlled equation on the interval $I_1 = [t_1, t_2]$,

$$\begin{cases} dy_1 - \sum\limits_{i,j=1}^{n} \big(a^{ij}(y_1)_{x_i}\big)_{x_j}\, dt = a(t) y_1\, dw + \chi_E \chi_{G_0} f_1\, dt & \text{in } G \times (t_1, t_2), \\ \tilde{l} \sum\limits_{i,j=1}^{n} a^{ij}(y_1)_{x_i} \nu^j + l y_1 = 0 & \text{on } \partial G \times (t_1, t_2), \\ y_1(t_1) = \tilde{y}_0 & \text{in } G. \end{cases} \tag{3.2}$$

By Proposition 2.2, there exists a control $f_1 \in L^\infty_{\mathcal{F}}(t_1, t_2; L^2(\Omega; L^2(G)))$ with the estimate:

$$|f_1|^2_{L^\infty_{\mathcal{F}}(t_1,t_2;L^2(\Omega;L^2(G)))} \leqslant \frac{C_1 e^{C_2\sqrt{r_1}}}{(m(E \cap [t_1, t_2]))^2} \mathbb{E}|\tilde{y}_0|^2_{L^2(G)},$$

such that $P_{r_1}(y(\cdot, t_2)) = 0$ in G, P-a.s. Then, by (2.2) and (2.3), we see that

$$|f_1|^2_{L^\infty_{\mathcal{F}}(t_1,t_2;L^2(\Omega;L^2(G)))} \leqslant \frac{C_1 e^{C_2\sqrt{r_1}}}{\rho^2(t_2 - t_1)^2} \mathbb{E}|\tilde{y}_0|^2_{L^2(G)}.$$

Moreover, using Itô's formula, we obtain that

$$\begin{aligned}
\mathbb{E}\big|y_1(\cdot, t_2)\big|^2_{L^2(G)} &\leqslant \mathbb{E}\big|y_1(\cdot, t_1)\big|^2_{L^2(G)} + 2\mathbb{E}\int_{t_1}^{t_2} \Big\langle y_1 \sum_{i,j=1}^{n} \big(a^{ij}(y_1)_{x_i}\big)_{x_j} \Big\rangle_{D(A^{\frac{1}{2}}), D(A^{\frac{1}{2}})^*} ds \\
&\quad + \mathbb{E}\int_{t_1}^{t_2}\int_G a^2(s) y_1^2\, dx\, ds + 2\mathbb{E}\int_{t_1}^{t_2}\int_G f_1 y_1\, dx\, ds \\
&\leqslant \mathbb{E}\big|y_1(\cdot, t_1)\big|^2_{L^2(G)} - 2\lambda_1 \mathbb{E}\int_{t_1}^{t_2}\int_G |y|^2\, dx\, ds + \tau\mathbb{E}\int_{t_1}^{t_2}\int_G |y|^2\, dx\, ds \\
&\quad + \frac{1}{\lambda_1}\mathbb{E}\int_{t_1}^{t_2}\int_G |f_1|^2\, dx\, ds + \lambda_1\mathbb{E}\int_{t_1}^{t_2}\int_G |y|^2\, dx\, ds \\
&\leqslant \mathbb{E}|\tilde{y}_0|^2_{L^2(\Omega)} + \frac{t_2 - t_1}{\lambda_1} |f_1|^2_{L^\infty_{\mathcal{F}}(t_1,t_2;L^2(\Omega;L^2(G)))}.
\end{aligned}$$

Hence

$$\mathbb{E}\big|y_1(\cdot,t_2)\big|^2_{L^2(G)} \leqslant 2\frac{C_1 e^{C_2\sqrt{r_1}}}{\rho^2(t_2-t_1)^2}\mathbb{E}|\tilde{y}_0|^2_{L^2(G)}.$$

Here we have used the facts that $(t_2-t_1)\leqslant \min(\lambda_1,1)$, $\rho\leqslant 1$ and $C_1>1$.

On the interval $J_1\equiv[t_2,t_3]$, we consider the following equation without control:

$$\begin{cases} dz_1-\displaystyle\sum_{i,j=1}^n \big(a^{ij}(z_1)_{x_i}\big)_{x_j}\,dt = a(t)z_1\,dw & \text{in } G\times(t_2,t_3),\\ \tilde{l}\displaystyle\sum_{i,j=1}^n a^{ij}(z_1)_{x_i}\nu^j + lz_1 = 0 & \text{on } \partial G\times(t_2,t_3),\\ z_1(t_2)=y_1(t_2) & \text{in } G. \end{cases}$$

Since $P_r(y_1(\cdot,t_2))=0$ in G, P-a.s., we have

$$\begin{aligned}\mathbb{E}\big|z_1(\cdot,t_3)\big|^2_{L^2(G)} &\leqslant \exp\big((-2r_1+\tau)(t_3-t_2)\big)\mathbb{E}\big|y_1(\cdot,t_2)\big|^2_{L^2(G)}\\ &\leqslant 2\frac{C_1e^{C_2\sqrt{r_1}}}{\rho^2(t_2-t_1)^2}\exp\big((-2r_1+\tau)(t_3-t_2)\big)\mathbb{E}|\tilde{y}_0|^2_{L^2(G)}. \end{aligned} \tag{3.3}$$

Next, we consider the following equation

$$\begin{cases} dy_2-\displaystyle\sum_{i,j=1}^n \big(a^{ij}(y_2)_{x_i}\big)_{x_j}\,dt = a(t)y_2\,dw+\chi_E\chi_{G_0}f_2\,dt & \text{in } G\times(t_3,t_4),\\ \tilde{l}\displaystyle\sum_{i,j=1}^n a^{ij}(y_2)_{x_i}\nu^j + ly_2 = 0 & \text{on } \partial G\times(t_3,t_4),\\ y_2(t_3)=z_1(t_3) & \text{in } G. \end{cases}$$

With a similar argument to system (3.2), one can show that for any $r_2>r_1>0$, there exists a control $f_2\in L^\infty_{\mathcal{F}}(t_3,t_4;L^2(\Omega;L^2(G)))$ satisfying

$$\begin{aligned}|f_2|^2_{L^\infty(t_3,t_4;L^2_{\mathcal{F}}(\Omega;L^2(G)))} &\leqslant \frac{C_1e^{C_2\sqrt{r_2}}}{(m(E\cap[t_3,t_4]))^2}\mathbb{E}\big|z_1(t_3)\big|^2_{L^2(G)}\\ &\leqslant \frac{C_1e^{C_2\sqrt{r_2}}}{\rho^2(t_4-t_3)^2}\mathbb{E}\big|z_1(t_3)\big|^2_{L^2(G)} \end{aligned} \tag{3.4}$$

such that $P_{r_2}(y(\cdot,t_4))=0$ in G, P-a.s.

From (2.3), (3.3) and (3.4), we can get

$$\begin{aligned}&|f_2|^2_{L^\infty_{\mathcal{F}}(t_3,t_4;L^2(\Omega;L^2(G)))}\\ &\quad\leqslant \frac{C_1}{\rho^2(t_2-t_1)^2}C_0^4e^{C_2\sqrt{r_2}}\mathbb{E}\big|z_1(t_3)\big|^2_{L^2(G)}\end{aligned}$$

$$\leqslant 2\left(\frac{C_1}{\rho^2(t_2-t_1)^2}\right)^2 C_0^4 e^{C_2\sqrt{r_1}} e^{C_2\sqrt{r_2}} \exp\bigl((-2r_1+\tau)(t_3-t_2)\bigr)\mathbb{E}|\tilde{y}_0|^2_{L^2(G)}. \tag{3.5}$$

On the interval I_N, we consider the controlled equation:

$$\begin{cases} dy_N - \displaystyle\sum_{i,j=1}^n \bigl(a^{ij}(y_N)_{x_i}\bigr)_{x_j}\,dt = a(t)y_N\,dw + \chi_E\chi_{G_0} f_N\,dt & \text{in } G\times(t_{2N-1},t_{2N}),\\ \tilde{l}\displaystyle\sum_{i,j=1}^n a^{ij}(y_N)_{x_i}\nu^j + l y_N = 0 & \text{on } \partial G\times(t_{2N-1},t_{2N}),\\ y_N(t_{2N-1}) = z_{N-1}(t_{2N-1}) & \text{in } G. \end{cases}$$

On the interval J_N, we consider the following equation without control:

$$\begin{cases} dz_N - \displaystyle\sum_{i,j=1}^n \bigl(a^{ij}(z_N)_{x_i}\bigr)_{x_j}\,dt = a(t)z_N\,dw & \text{in } G\times(t_{2N},t_{2N+1}),\\ \tilde{l}\displaystyle\sum_{i,j=1}^n a^{ij}(z_N)_{x_i}\nu^j + l z_N = 0 & \text{on } \partial G\times(t_{2N},t_{2N+1}),\\ z_N(t_{2N}) = y_N(t_{2N}) & \text{in } G. \end{cases}$$

By induction, utilizing (2.2) and (2.3), we can conclude that, for any given $r_N>0$, there exists a control function $f_N\in L^\infty_{\mathcal{F}}(t_{2N-1},t_{2N};L^2(\Omega;L^2(G)))$ satisfying:

$$\begin{aligned} &|f_N|^2_{L^\infty_{\mathcal{F}}(t_{2N-1},t_{2N};L^2(\Omega;L^2(G)))}\\ &\quad\leqslant 2^{N-1}\left(\frac{C_1}{\rho^2(t_2-t_1)^2}\right)^N C_0^4 C_0^{4\times 2}\cdots C_0^{4(N-1)}\alpha_1\alpha_2\cdots\alpha_N\mathbb{E}|\tilde{y}_0|^2_{L^2(G)}, \end{aligned}$$

where

$$\alpha_N = \begin{cases} \exp(C_2\sqrt{r_1}), & N=1,\\ \exp(C_2\sqrt{r_N})\exp((-2r_{N-1}+\tau)(t_3-t_2)C_0^{-2(N-2)}), & N\geqslant 2, \end{cases} \tag{3.6}$$

and C_0 is defined in (2.3) such that $P_{r_N}(y_N(\cdot,t_{2N}))=0$ in G, P-a.s.

Let

$$\widetilde{C} = \frac{2C_1}{\rho^2(t_2-t_1)^2}C_0^2, \tag{3.7}$$

then we have

$$|f_N|^2_{L^\infty_{\mathcal{F}}(t_{2N-1},t_{2N};L^2(\Omega;L^2(G)))} \leqslant \widetilde{C}^{N(N-1)}\alpha_1\alpha_2\cdots\alpha_N\mathbb{E}|y_0|^2_{L^2(G)}, \tag{3.8}$$

where $N>1$.

Now we choose r_N as:

$$r_N = \left[\overline{C}\widetilde{C}^{N-1}\right]^4 + \tau^4, \quad N \geqslant 1, \tag{3.9}$$

where

$$\overline{C} = \frac{2}{(t_3 - t_2)}.$$

Since $\widetilde{C} > C_0^2 > 1$ and $t_3 - t_2 < 1$, it follows

$$2^4 < r_1 < r_2 < \cdots < r_N < r_{N+1} < \cdots, \quad \text{and} \quad r_N \to \infty \text{ as } N \to \infty.$$

Moreover, we have

$$(r_{N-1})^{\frac{1}{4}}(t_3 - t_2)C_0^{-2(N-2)} - \tau(t_3 - t_2)C_0^{-2(N-2)} \geqslant 2, \quad \forall N \geqslant 2.$$

Therefore

$$\exp\left\{-(2r_{N-1} - \tau)(t_3 - t_2)C_0^{-2(N-2)}\right\} \leqslant \exp\left(-4r_{N-1}^{\frac{3}{4}}\right), \quad \forall N \geqslant 2. \tag{3.10}$$

Note that

$$\begin{aligned}
\widetilde{C}^{N(N+1)} \exp\left(-r_{N-1}^{\frac{3}{4}}\right) = \frac{\widetilde{C}^{N(N+1)}}{(\exp(r_{N-1}^{\frac{1}{4}}))^{r_{N-1}^{\frac{1}{2}}}} &\leqslant \frac{\widetilde{C}^{N(N+1)}}{(\exp(2\widetilde{C}^{N-1}))^{r_{N-1}^{\frac{1}{2}}}} \\
&\leqslant \frac{\widetilde{C}^{N(N-1)}}{\widetilde{C}^{(N-1)\cdot 2r_{N-1}^{\frac{1}{2}}}}
\end{aligned}$$

for each $N \geqslant 2$, we derive from (3.9) that there exists a natural number N_1 with $N_1 \geqslant 2$ such that for each $N \geqslant N_1$, it holds $r_{N-1}^{\frac{1}{2}} \geqslant (\widetilde{C}^{N-1})^2 > N$. Hence we have that for any $N > N_1$, it holds

$$\widetilde{C}^{N(N-1)} \exp\left(-r_{N-1}^{\frac{3}{4}}\right) \leqslant 1. \tag{3.11}$$

By using (3.9) again, we obtain that for each $N \geqslant 2$,

$$\begin{aligned}
\exp(C_2\sqrt{r_N}) \exp\left(-r_{N-1}^{\frac{3}{4}}\right) &= \exp\left(C_2\left(\overline{C}^4\widetilde{C}^{4(N-1)} + \tau^4\right)^{\frac{1}{2}}\right) \exp\left(-\left(\overline{C}^4\widetilde{C}^{4(N-2)} + \tau^4\right)^{\frac{3}{4}}\right) \\
&\geqslant \exp\left(C_2\overline{C}^2\widetilde{C}^{2(N-1)}\right) \exp\left(-\overline{C}^3\widetilde{C}^{3(N-2)}\right) \\
&= \exp\left(C_2\overline{C}^2\widetilde{C}^{2(N-1)} - \overline{C}^3\widetilde{C}^{3(N-2)}\right).
\end{aligned} \tag{3.12}$$

Thus, there exists a natural number $N_2 \geqslant 2$ such that for each $N \geqslant N_2$,

$$\exp(C_2\sqrt{r_N}) \exp\left(-r_{N-1}^{\frac{3}{4}}\right) \leqslant 1. \tag{3.13}$$

Now, put

$$N_0 = \max\{N_1, N_2\}. \tag{3.14}$$

Combining (3.10), (3.11) and (3.13), we see that for all $N \geqslant N_0$,

$$\begin{aligned}\widetilde{C}^{N(N-1)}\alpha_N &= \widetilde{C}^{N(N-1)} \exp(C_2\sqrt{r_N}) \exp\bigl(-(2r_{N-1} - \tau)(t_3 - t_2)C_0^{-2(N-2)}\bigr) \\ &\leqslant \widetilde{C}^{N(N-1)} \exp(C_2\sqrt{r_N}) \exp\bigl(-4r_{N-1}^{\frac{3}{4}}\bigr) \\ &\leqslant \exp\bigl(-2r_{N-1}^{\frac{3}{4}}\bigr).\end{aligned} \tag{3.15}$$

Moreover, it is obviously that

$$\alpha_N \leqslant 1, \quad \forall N \geqslant N_0. \tag{3.16}$$

We set

$$C = \max\bigl\{(\widetilde{C})^{N(N-1)}\alpha_1\alpha_2\cdots\alpha_N,\ 1 \leqslant N \leqslant N_0\bigr\} < \infty. \tag{3.17}$$

It follows from (3.8), (3.15), (3.16), (3.17) that for all $N \geqslant 1$,

$$|f_N|^2_{L^\infty_{\mathcal{F}}(t_{2N-1}, t_{2N}; L^2(\Omega; L^2(G)))} \leqslant C\mathbb{E}|y_0|^2_{L^2(G)}. \tag{3.18}$$

We now construct a control $\tilde{f}$ by setting

$$\tilde{f}(x,t) = \begin{cases} f_N(x,t), & x \in G,\ t \in I_N,\ N \geqslant 1, \\ 0, & x \in G,\ t \in J_N,\ N \geqslant 1, \end{cases} \tag{3.19}$$

from which and by (3.18), we see that the control $\tilde{f} \in L^\infty_{\mathcal{F}}(t_1, \tilde{t}; L^2(\Omega; L^2(G)))$ and satisfies the estimate

$$|\tilde{f}|^2_{L^\infty_{\mathcal{F}}(t_1, \tilde{t}; L^2(\Omega; L^2(G)))} \leqslant C\mathbb{E}|\tilde{y}_0|^2_{L^2(G)}.$$

Let $\tilde{y}$ be the solution of system (3.1) corresponding to the control constructed in (3.19). Then on the interval I_N, $\tilde{y}(\cdot,t) = y_N(\cdot,t)$. Since $P_{r_N}(y_N(\cdot, t_{2N})) = 0$ for all $N \geqslant 1$, we see that

$$P_{r_N}\bigl(y_N(\cdot, t_{2M})\bigr) = 0 \quad \text{for all } M \geqslant N,\ P\text{-a.s.} \tag{3.20}$$

On the other hand, since $t_{2M} \to \tilde{t}$ as $M \to \infty$, we obtain that

$$\tilde{y}(\cdot, t_{2M}) \to \tilde{y}(\cdot, \tilde{t}) \quad \text{strongly in } L^2(G),\ \text{as } M \to \infty,\ P\text{-a.s.},$$

which, combining with (3.20), imply that $P_{r_N}(\tilde{y}(\cdot,\tilde{t})) = 0$ for all $N \geqslant 1$, P-a.s. Since $r_N \to \infty$ as $N \to \infty$, it holds that $\tilde{y}(\cdot,\tilde{t}) = 0$, P-a.s. Thus, we have proved that for each $\tilde{y}_0 \in L^2(\Omega, \mathcal{F}_{t_1}, P; L^2(G))$, there exists a control $f \in L^\infty_{\mathcal{F}}(t_1, \tilde{t}; L^2(\Omega; L^2(G)))$ with the estimate

$|f|^2_{L^\infty_{\mathcal{F}}(t_1,\tilde{t};L^2(\Omega;L^2(G)))} \leqslant C\mathbb{E}|\tilde{y}_0|^2_{L^2(\Omega)}$, where the constant C is given by (3.17), such that the solution $\tilde{y}$ to system (3.1) vanishes at time $\tilde{t}$, namely, $\tilde{y}(\tilde{t}) = 0$ in Ω, P-a.s.

Next, we take $\tilde{y}_0$ to be $\psi(x, t_1)$, where $\psi(x, t)$ is the solution to the following equation

$$\begin{cases} d\psi - \displaystyle\sum_{i,j=1}^{n} \left(a^{ij}\psi_{x_i}\right)_{x_j} dt = a(t)\psi\, dw & \text{in } G \times (0, t_1), \\ \tilde{l} \displaystyle\sum_{i,j=1}^{n} a^{ij}\psi_{x_i}\nu^j + l\psi = 0 & \text{on } \partial G \times (0, t_1), \\ \psi(0) = y_0 & \text{in } G, \end{cases}$$

and construct a control f by setting

$$f(x,t) = \begin{cases} 0 & \text{in } G \times (0, t_1), \\ \tilde{f}(x,t) & \text{in } G \times (t_1, \tilde{t}), \\ 0 & \text{in } G \times (\tilde{t}, T). \end{cases} \tag{3.21}$$

It is clear that the control f belongs to $L^\infty_{\mathcal{F}}(0, T; L^2(\Omega; L^2(G)))$ and that the corresponding solution y of system (1.1) verifies $y(T) = 0$ in Ω, P-a.s. Moreover, the control f constructed in (3.21) satisfies the following estimate:

$$|f|^2_{L^\infty_{\mathcal{F}}(0,T;L^2(\Omega;L^2(G)))} \leqslant C\mathbb{E}|y_0|^2_{L^2(\Omega)},$$

where C is given by (3.17). This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

In this section, we shall give a proof of Theorem 1.2. In the sequel, C is a generic positive constant depending also on $s \in [0, T)$ (as before, it may change from line to line).

As a preliminary, we first show the two following propositions which have their independent interests in the theory of stochastic partial differential equations.

Proposition 4.1. *If $m((s, T) \cap E) > 0$ for any $s \in [0, T)$, then for arbitrary given $\eta \in L^2(\Omega, \mathcal{F}_T, P; L^2(G))$, the corresponding solution of Eq.* (2.6) *satisfies*

$$\left|z(s)\right|^2_{L^2(\Omega,\mathcal{F}_s,P;L^2(G))} \leqslant C\mathbb{E} \int\limits_{(s,T)\cap E} \int\limits_{G_0} \left|z(t)\right|^2 dx\, dt. \tag{4.1}$$

Remark 4.1. Proposition 4.1 is an observability inequality for Eq. (2.6) with only one observer. It seems that it is very difficult (if is not impossible) to establish it directly (as remarked at pp. 99 and 108–110 in [1]). Here we use a duality argument to derive this inequality from the null controllability result.

Proof of Proposition 4.1. Consider the following controlled system

$$\begin{cases} dy - \displaystyle\sum_{i,j=1}^{n} \left(a^{ij} y_{x_i}\right)_{x_j} dt = a(t) y\, dw + \chi_{(s,T)\cap E}\chi_{G_0} f\, dt & \text{in } (s,T)\times G, \\ \tilde{l} \displaystyle\sum_{i,j=1}^{n} a^{ij} y_{x_i} \nu^j + l(x) y = 0 & \text{on } (s,T)\times \partial G, \\ y(s) = y_s & \text{in } G, \end{cases} \tag{4.2}$$

where the state variable $y_s \in L^2(\Omega, \mathcal{F}_s, P; L^2(G))$ and the control variable $f \in L^2_{\mathcal{F}}(s,T; L^2(G))$. By Theorem 1.1, system (4.2) is null controllable, i.e., for any $y_s \in L^2(\Omega, \mathcal{F}_s, P; L^2(G))$, we can find a control $f \in L^2_{\mathcal{F}}(s,T; L^2(G))$ such that $y(T) = 0$ in G, P-a.s. Moreover, by (1.2), it holds

$$|f|^2_{L^2_{\mathcal{F}}(s,T;L^2(G))} \leqslant C|f|^2_{L^\infty_{\mathcal{F}}(s,T;L^2(\Omega;L^2(G)))} \leqslant C|y_s|^2_{L^2(\Omega,\mathcal{F}_s,P;L^2(G))}. \tag{4.3}$$

Applying Itô's formula to $d(yz)$, where y and z solve respectively systems (4.2) and (2.6), we end up with

$$\mathbb{E}\int_G y(T)z(T)\,dx - \mathbb{E}\int_G y_s z(s)\,dx = \mathbb{E}\int_{(s,T)\cap E}\int_{G_0} fz\,dx\,dt.$$

Since $y(T) = 0$ in G, P-a.s., we arrive at

$$-\mathbb{E}\int_G y_s z(s)\,dx = \mathbb{E}\int_{(s,T)\cap E}\int_{G_0} fz\,dx\,dt.$$

Choosing $y_s = -z(s)$, it follows that

$$\begin{aligned} \mathbb{E}\int_G \left|z(s)\right|^2 dx &= \mathbb{E}\int_{(s,T)\cap E}\int_{G_0} fz\,dx\,dt \\ &\leqslant C\left(\mathbb{E}\int_{(s,T)\cap E}\int_{G_0} |f|^2\,dx\,dt\right)^{\frac{1}{2}}\left(\mathbb{E}\int_{(s,T)\cap E}\int_{G_0} |z|^2\,dx\,dt\right)^{\frac{1}{2}} \\ &\leqslant C\left(\mathbb{E}\int_G \left|z(s)\right|^2 dx\right)^{\frac{1}{2}}\left(\mathbb{E}\int_{(s,T)\cap E}\int_{G_0} |z|^2\,dx\,dt\right)^{\frac{1}{2}}, \end{aligned}$$

which gives immediately the desired estimate (4.1). □

As an easy corollary of Proposition 4.1, we have the following unique continuation property of solutions to Eq. (2.6).

Proposition 4.2. *If $m((s,T)\cap E)>0$ for any $s\in[0,T)$, then any solution (z,Z) of Eq.* (2.6) *vanishes identically in G provided that $z=0$ in $G_0\times E$, P-a.s.*

Remark 4.2. If the condition $m((s,T)\cap E)>0$ for any $s\in[0,T)$ is not assumed, Proposition 4.2 may fail to be true. This can be shown by the following counterexample. Let E satisfy that $m(E)>0$ and $m((s_0,T)\cap E)=0$ for some $s_0\in[0,T)$. Let $(z_1,Z_1)=0$ in $G\times(0,s_0)$, P-a.s. and let ξ_2 be a nonzero process belonging to $L^2_{\mathcal{F}}(s_0,T)$ (then $Z_2\equiv\xi_2 e_1$ is a nonzero process in $L^2_{\mathcal{F}}(s_0,T;L^2(G))$). Solving the following forward stochastic differential equation

$$\begin{cases} d\zeta_1-\lambda_1\zeta_1\,dt=-a(t)\xi_2\,dt+\xi_2\,dw & \text{in } [s_0,T],\\ \zeta_1(s_0)=0, \end{cases}$$

we find a nonzero $\zeta_1\in L^2_{\mathcal{F}}(\Omega;C[s_0,T])$. In this way, we find a nonzero solution $(z_2,Z_2)\equiv(\zeta_1e_1,\xi_2e_1)\in L^2_{\mathcal{F}}(\Omega;C([s_0,T];L^2(G)))\times L^2_{\mathcal{F}}(s_0,T;L^2(G))$ to the following forward stochastic partial differential equation

$$\begin{cases} dz_2+\displaystyle\sum_{i,j=1}^n\big(a^{ij}(z_2)_{x_i}\big)_{x_j}\,dt=-a(t)Z_2\,dt+Z_2\,dw & \text{in } (s_0,T)\times G,\\ \tilde{l}\displaystyle\sum_{i,j=1}^n a^{ij}(z_2)_{x_i}\nu^j+l(x)z_2=0 & \text{on } (s_0,T)\times\partial G,\\ z_2(s_0)=0 & \text{in } G. \end{cases}\tag{4.4}$$

(Note however that one cannot solve system (4.4) directly because this system is non-wellposed.) Put

$$(z,Z)=\begin{cases}(z_1,Z_1) & \text{in } G\times(0,s_0),\\ (z_2,Z_2) & \text{in } G\times(s_0,T).\end{cases}$$

Then, (z,Z) is a nonzero solution of system, for which $z=0$ in $G_0\times E$, P-a.s. Note also that, the nonzero solution constructed for system (4.4) indicates that forward uniqueness does NOT hold for backward stochastic differential equations.

Proof of Proposition 4.2. Since $z=0$ in $G_0\times E$, P-a.s., we have $\mathbb{E}\int_{(s,T)\cap E}\int_{G_0}|z|^2\,dx\,dt=0$ for any $s\in[0,T)$. By Proposition 4.1, we know that for any $s\in[0,T)$, it holds $z(s)=0$ in G, P-a.s. Therefore, $Z=0$ in G, P-a.s. and for a.e. $t\in[0,T]$. □

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. The "if" part. Since system (1.1) is linear, it suffices to show that its attainable set A_T at time T with initial datum $y(0)=0$ is dense in $L^2(\Omega,\mathcal{F}_T,P;L^2(G))$. Let us prove this by the contradiction argument. Assume that there exists an $\eta\in L^2(\Omega,\mathcal{F}_T,P;L^2(G))$ such that $\eta\neq0$ and $\mathbb{E}\int_G y(T)\eta\,dx=0$ for any $y(T)\in A_T$. Using $d(yz)=y\,dz+z\,dy+dy\,dz$

again (where y solves system (1.1) with $y_0 = 0$ and arbitrarily given $f \in L^2_{\mathcal{F}}(0, T; L^2(G))$; while z solves Eq. (2.6) with the above given final datum η), we obtain

$$\mathbb{E}\int_G y(T)\eta\,dx = \mathbb{E}\int_E\int_{G_0} fz\,dx\,dt. \tag{4.5}$$

Hence $\mathbb{E}\int_E\int_{G_0} fz\,dx\,dt = 0$ for any $f \in L^2_{\mathcal{F}}(0, T; L^2(G))$. Therefore we get $z = 0$ in $G_0 \times E$, P-a.s. By Proposition 4.2, we arrive at $\eta = 0$, a contradiction.

The "only if" part. We use the contradiction argument again. Assume that $m((s_0, T) \cap E) = 0$ for some $s_0 \in [0, T)$ and system (1.1) is approximately controllable at time T. If $z = 0$ in $G_0 \times E$, P-a.s., from (4.5) (since (4.5) is obtained by integration by parts, it holds for any E), we know that $\mathbb{E}\int_G y(T)\eta\,dx = 0$ for any $y(T) \in A_T$. Since A_T is dense in $L^2(\Omega, \mathcal{F}_T, P; L^2(G))$, for any $\varepsilon > 0$, we can find a $y_T^\varepsilon \in A_T$ such that $|\eta - y_T^\varepsilon|_{L^2(\Omega,\mathcal{F}_T,P;L^2(G))} < \varepsilon$. Therefore we have

$$0 = \mathbb{E}\int_G y_T^\varepsilon\eta\,dx = \mathbb{E}\int_G \eta^2\,dx - \mathbb{E}\int_G \big(\eta - y_T^\varepsilon\big)\eta\,dx.$$

Hence it holds that $\mathbb{E}\int_G \eta^2\,dx \leqslant \varepsilon(\mathbb{E}\int_G \eta^2\,dx)^{\frac{1}{2}}$, which implies that $(\mathbb{E}\int_G \eta^2\,dx)^{\frac{1}{2}} \leqslant \varepsilon$. Since ε is an arbitrarily positive number, we have $\mathbb{E}\int_G \eta^2\,dx = 0$, which, in turn, contradicts the counterexample in Remark 4.2. This completes the proof of Theorem 1.2. □

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