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Relaxation oscillations in $\mathbb{R}^3$

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Abstract

The existence of periodic relaxation oscillations in singularly perturbed systems with two slow and one fast variable is analyzed geometrically. It is shown that near a singular periodic orbit a return map can be defined which has a one-dimensional slow manifold with a stable invariant foliation. Under a natural hyperbolicity assumption on the singular periodic orbit this allows to prove the existence of a periodic relaxation orbit for small values of the perturbation parameter. Additionally the existence of an invariant torus is proved for the periodically forced van der Pol oscillator. The analysis is based on methods from geometric singular perturbation theory. The blow-up method is used to analyze the dynamics near the fold-curves.

1. Introduction

Relaxation oscillations (RO), a highly nonlinear type of oscillation, are found in many biological, chemical, physical and neuronal problems. The characteristic feature of RO is a repeated switching between fast and slow motions. In a more narrow sense RO often refers to periodic phenomena of this type. RO occur naturally in singularly perturbed ordinary differential equations, which have dynamics on (at least) two different, e.g. fast and slow, time scales. The prototypical system describing RO in $\mathbb{R}^2$ is the van der Pol oscillator. For more background information and many applications of RO we refer to Grasman [8].

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In this work we study singularly perturbed systems in $\mathbb{R}^3$ of the form

$$\begin{aligned}\dot{x} &= g_1(x,y,z,\varepsilon),\\ \dot{y} &= g_2(x,y,z,\varepsilon),\\ \varepsilon\dot{z} &= f(x,y,z,\varepsilon),\end{aligned} \tag{1}$$

with sufficiently smooth functions g_1, g_2, and f and singular perturbation parameter $\varepsilon \ll 1$. By setting $\varepsilon = 0$ in system (1) we obtain the reduced problem on the critical manifold $S := \{(x,y,z) \in \mathbb{R}^3\colon f(x,y,z,0) = 0\}$. We make the following basic assumption on the geometry of S:

Assumption 1. The critical manifold S is 'S-shaped', i.e.

$$S = S_{\mathrm{a}}^- \cup L^- \cup S_{\mathrm{r}} \cup L^+ \cup S_{\mathrm{a}}^+$$

with attracting upper and lower branches

$$S_{\mathrm{a}}^{\pm}, S_{\mathrm{a}}^+ \cup S_{\mathrm{a}}^- := \{(x,y,z) \in S\colon f_z(x,y,z,0) < 0\},$$

a repelling branch

$$S_{\mathrm{r}} := \{(x,y,z) \in S\colon f_z(x,y,z,0) > 0\}$$

and fold-curves

$$L^{\pm}, L^+ \cup L^- := \{(x,y,z) \in S\colon f_z(x,y,z,0) = 0, f_{zz}(x,y,z,0) \neq 0\}.$$

The limiting problem on the fast time scale $\tau = t/\varepsilon$ is the layer problem

$$\begin{aligned}x' &= 0,\\ y' &= 0,\\ z' &= f(x,y,z,0).\end{aligned} \tag{2}$$

The critical manifold S is a manifold of equilibria for the layer problem. Vertical lines $(x,y) = \mathit{const}$. are called fast fibers of the layer problem. Along these fast fibers a fast transition towards or away from the critical manifold S occurs.

In order to obtain RO in a singularly perturbed system we assume furthermore

Assumption 2. The fold-curves $L^{\pm}$ are given as graphs $(x^{\pm}(y), y, z^{\pm}(y))$, $y \in I^{\pm}$ for certain intervals $I^{\pm}$. The points $p \in L^{\pm}$ of the fold-curves are jump points, i.e.

$$\begin{pmatrix} f_x \\ f_y \end{pmatrix} \cdot \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\Bigg|_{p \in L^{\pm}} \neq 0 \tag{3}$$

and the reduced flow near the fold-curves is directed towards the fold-curves.

Condition (3) is a transversality condition, called normal switching condition in [18], for the reduced flow near the fold-curve $L^{\pm}$ and gives rise to the jumping behavior for solutions reaching the fold-curve $L^{\pm}$.

Let $P(L^{\pm}) \subset S_{\mathrm{a}}^{\mp}$ be the projection along the fast fibers of the fold-curve $L^{\pm}$ on the opposite attracting branch $S_{\mathrm{a}}^{\mp}$.

Assumption 3. The reduced flow is transversal to the curve $P(L^{\pm}|_{I^{\pm}}) \subset S_{\mathrm{a}}^{\mp}$.

The above assumptions describe a natural setting for RO in system (1). In the following we focus on periodic RO. Under Assumptions 1–3 a singular periodic relaxation orbit Γ of system (1) is a piecewise smooth closed curve $\Gamma = \Gamma_{\mathrm{a}}^{-} \cup \Gamma_{f}^{-} \cup \Gamma_{\mathrm{a}}^{+} \cup \Gamma_{f}^{+}$ consisting of solutions $\Gamma_{\mathrm{a}}^{\pm} \subset S_{\mathrm{a}}^{\pm}$ of the reduced system connecting points of the projection-curves $P(L^{\mp}) \subset S_{\mathrm{a}}^{\pm}$ and the fold-curves $L^{\pm}$, where these slow solutions are connected by fast fibers $\Gamma_{f}^{\pm}$ from $L^{\pm}$ to $P(L^{\pm})$.

Assumption 4. There exists a singular periodic orbit Γ for system (1).

The main goal of this work is to show that the existence of a hyperbolic singular relaxation orbit Γ implies the existence of a hyperbolic relaxation orbit Γ_{ε} of system (1). As we will see hyperbolicity in the strongly contracting z direction is already built in, while hyperbolicity in the slow direction is an extra assumption on the reduced flow (see Assumption 5, page 22). A typical situation for a system (1) which satisfies Assumptions 1–4 is shown in Fig. 1.

To detect RO in system (1) near a singular orbit Γ it is sufficient to study a local Poincaré map (return map). We introduce a suitable Poincaré section Σ^{-} near the attracting branch S_{a}^{-} of the critical manifold containing Γ. A possible choice for Σ^{-} is obtained by translating the curve $P(L^{+})$ slightly to the right and by extending the new curve in the vertical direction (see Fig. 1).

We show that under Assumptions 1–4 a return map $\Pi : \mathscr{V} \to \Sigma^{-}$ can be defined for a suitable $\mathscr{V} \subset \Sigma^{-}$ and sufficiently small ε. The map Π is essentially[3] the composition of three different types of maps $\Pi_{S_{\mathrm{a}}}$, Π_{L} and Π_{T}. The map $\Pi_{S_{\mathrm{a}}}$ describes the slow flow near the attracting slow manifold away from the fold, Π_{L} describes the dynamics near the fold, and Π_{T} the fast transition to the other attracting slow manifold.

Outside of an arbitrary small neighborhood $U^{\pm}$ of each fold-curve $L^{\pm}$ the manifolds $S_{\mathrm{a}}^{\pm}$ perturb smoothly to locally invariant manifolds $S_{\mathrm{a},\varepsilon}^{\pm}$ for sufficiently small $\varepsilon > 0$, i.e. they are $O(\varepsilon)$ perturbations of the unperturbed manifolds (see e.g. [6,11]). Moreover, there exists smooth invariant foliations of the manifolds $W^{\mathrm{s}}(S_{\mathrm{a},\varepsilon}^{\pm}) \cap V^{\pm}$ in a neighborhood $V^{\pm}$ of the base $S_{\mathrm{a},\varepsilon}^{\pm}$. Based on these results we have good control of the maps $\Pi_{S_{\mathrm{a}}}$ and Π_{T}.

[3] The map $\Pi_{T} \circ \Pi_{L} \circ \Pi_{S_{\mathrm{a}}}$ is one half of the full return map Π.

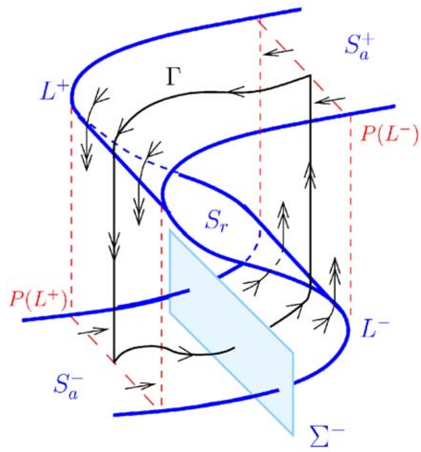

Fig. 1. Singular relaxation orbit Γ and Poincaré section Σ^- near critical manifold S_a^-.

Due to loss of normal hyperbolicity Fenichel theory breaks down near the fold-curves $L^\pm$. In Section 2 we use the recently developed blow-up technique to derive an analytic expression for the map Π_L. Based on this local result we show in Section 3 the existence of a return map $\Pi : \mathscr{V} \to \Sigma^-$ of system (1) under Assumptions 1–4. Not surprisingly the map is a discrete analog of a slow–fast flow with strongly contracting z-direction and slow y-dynamics. These properties allow to prove the existence of an invariant (slow) manifold with an associated invariant strongly contracting foliation (fast fibers) for the return map. These results are fairly standard, however, some care is needed to treat the singular ε-dependence. The proof is given in Appendix A. The dynamics on this slow manifold is described by a 1-d map which is a small perturbation of the unperturbed 'singular' ($\varepsilon = 0$) Poincaré map $G_0 : \mathscr{V} \cap S_\mathrm{a}^- \to \Sigma^- \cap S_\mathrm{a}^-$ induced by the reduced flow on $S_\mathrm{a}^\pm$. Thus hyperbolic fixed points of the 'singular' Poincaré map persist as hyperbolic fixed points of the return map for ε small, resp. as hyperbolic periodic relaxation orbits for system (1).

An interesting type of relaxation oscillations where the return map is defined globally occurs in the forced van der Pol oscillator. In Section 4 we show the existence of an invariant torus for moderate forcing amplitude $A < 1$ and relate the dynamics on the invariant torus to a circle map obtained in the singular limit.

Remark 1. Similar results on the existence of RO can be found in [2,17–19]. In these works the method of matched asymptotic expansions is used. The resulting asymptotic expansions are rather complicated containing fractional powers as well as logarithms of ε. It was shown in [7,13] that at fold points of planar problems the complicated form of the expansions is caused by a resonance phenomenon. As shown in [7] asymptotic expansions can be derived rigorously by means of the blow-up method. In this work we do not try to compute these expansions but focus on the essential geometric features of RO with two slow and one fast variable.

Remark 2. In this work we do not consider phenomena caused by *canard points* at the fold-curves $L^{\pm}$ (Assumption 2) and/or nontransverse reduced flow at the projection-curves $P(L^{\pm})$ of the fold-curves (Assumption 3), although it is possible to obtain very interesting relaxation oscillations of more complicated types in such systems [8–10,14–16]. However, the methods developed in this paper are a step towards a rigorous, geometric analysis of these more complicated phenomena.

2. Blow-up analysis near the fold-curves $L^{\pm}$

We study the transition of the flow of system (1) near the fold-curve L^-. The analysis near the other fold-curve L^+ is completely analogous. We start with a preliminary transformation which brings system (1) in a form suitable for blowing-up near the fold-curve L^-.

Lemma 3. *Under Assumptions* 1 *and* 2 *there exist a smooth change of coordinates which brings system* (1) *locally near the fold-curve* L^- *to*

$$\begin{aligned} x' &= \varepsilon g_1(x,y,z,\varepsilon),\\ y' &= \varepsilon g_2(x,y,z,\varepsilon),\\ z' &= x + z^2 + O(z^3, xyz, x^2z, \varepsilon),\\ \varepsilon' &= 0, \end{aligned} \tag{4}$$

with $g_1(x,y,z,\varepsilon) = 1 + g_{11}(x,y,z,\varepsilon)$ *where* $g_{11}(0,y,0,0) = 0$ *and* $g_2(0,y,0,0) = 0$, $y \in I$. *Here* $'$ *denotes differentiation with respect to the fast time* $\tau = t/\varepsilon$.

Proof. We rectify the fold-curve along the y-axis. Taylor-expansion of the function f with a sequence of linear and near identity transformations gives

$$\begin{aligned} \hat{x}' &= \varepsilon \hat{g}_1(\hat{x},\hat{y},\hat{z},\varepsilon),\\ \hat{y}' &= \varepsilon \hat{g}_2(\hat{x},\hat{y},\hat{z},\varepsilon),\\ \hat{z}' &= \hat{x} + \hat{z}^2 + O(\hat{z}^3, \hat{x}\hat{y}\hat{z}, \hat{x}^2\hat{z}, \varepsilon), \end{aligned}$$

with $\hat{g}_1(0,\hat{y},0,0) > 0$, $\forall \hat{y} \in \hat{I}$. The transversality condition (3) implies $\hat{g}_2(0,\hat{y},0,0) = 0$, $\forall \hat{y} \in \hat{I}$. Let $\hat{g}_1(\hat{x},\hat{y},\hat{z},\varepsilon) = g(\hat{y}) + g_{11}(\hat{x},\hat{y},\hat{z},\varepsilon)$ with $g(\hat{y}) = \hat{g}_1(0,\hat{y},0,0) > 0$ and $g_{11}(0,\hat{y},0,0) = 0$, $\forall \hat{y} \in \hat{I}$. In a last step we stretch the coordinates $\hat{x} = \bar{x}\, g^{\frac{2}{3}}(\bar{y})$ and $\hat{z} = \bar{z} g^{\frac{1}{3}}(\bar{y})$ to obtain

$$\bar{x}' = \varepsilon g^{\frac{1}{3}}(1 + \bar{g}_{11}(\bar{x},\bar{y},\bar{z},\varepsilon)),$$

$$\bar{y}' = \varepsilon g^{\frac{1}{3}} \bar{g}_2(\bar{x}, \bar{y}, \bar{z}, \varepsilon),$$

$$\bar{z}' = g^{\frac{1}{3}}(\bar{x} + \bar{z}^2 + O(\bar{z}^3, \bar{x}\bar{y}\bar{z}, \bar{x}^2\bar{z}, \varepsilon)),$$

where $\bar{y} = \hat{y}$. By the operation of local division we divide out the common factor $g^{\frac{1}{3}}$ (rescaling time) and obtain the assertion by skipping the bars and extending the system by $\varepsilon' = 0$. For details of the proof we refer to [24]. □

System (4) can be viewed as a canonical form for a regular fold. We denote the fold-*curve* of (1) and the fold-*line* of (4) by the same symbol L^-. We distinguish between the two objects by the notion curve and line. Fenichel theory implies the existence of an attracting center-like manifold M_a^- and a repelling center-like manifold M_r of the extended system (4) for sufficiently small $\varepsilon \ll 1$ away from the fold-line L^-. Note, the slow manifold $S_{a,\varepsilon}^-$ resp. $S_{r,\varepsilon}$ is obtained as a section $\varepsilon = const.$ of M_a^- resp. M_r. Near the fold-line L^- Fenichel theory breaks down and we are using the *blow-up* technique to desingularize the flow near the fold-line. For details on this method we refer to [4,5,13] where planar folds are treated. The extension of slow manifolds near regular folds in three-dimensional problems by means of the blow-up method is treated in [24]. The existence of various types of canard solutions at points where the transversality condition (3) is violated is shown in Szmolyan and Wechselberger [21].

We focus our attention on S_a^- and investigate how solutions on $S_{a,\varepsilon}^-$ as well as nearby solutions behave as they pass near the fold-line L^-. We expect that close to the fold-line a transition from slow motion along $S_{a,\varepsilon}^-$ to a fast motion almost parallel to the unstable fibers occurs. In the following analysis we need the notion of exponentially small functions.

Definition 2.1. Let $R(u, \varepsilon) \in \mathbb{R}$ be a function with $u \in \mathbb{R}^k$ and parameter $\varepsilon \ll 1$. We call $R(u, \varepsilon)$ an exponentially small function if $(|R| + ||\nabla R||) \leqslant \exp(-c/\varepsilon)$ for a fixed positive constant c, where ∇R is the gradient of R with respect to $u \in \mathbb{R}^k$.

For small positive $\rho > 0$ and suitable rectangles J_1, $J_2 \in \mathbb{R}^2$ let

$$\Delta^{\text{in}} = \{(-\rho^2, y, z)\colon\ (y, z) \in J_1\}$$

be a section transverse to S_a^- and let

$$\Delta^{\text{out}} = \{(x, y, \rho)\colon\ (x, y) \in J_2\}$$

be a section transverse to the fast fibers (see Fig. 2). Let $\Pi_L : \Delta^{\text{in}} \to \Delta^{\text{out}}$ be the transition map for the flow of (4). The following theorem is the main result for a two-dimensional regular fold.

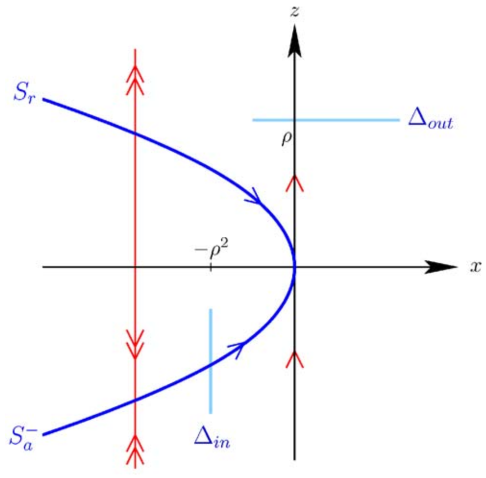

Fig. 2. Projection of critical manifold and sections.

Theorem 1. *For the regular fold written in the canonical form* (4) *there exist* $\rho > 0$ *and* $\varepsilon_0 > 0$ *such that for* $\varepsilon \in (0, \varepsilon_0]$:

(1) *There exists a suitable interval* I^{out} *such that for* $y \in I^{\text{out}}$ *the manifold* $S^-_{\text{a},\varepsilon}$ *intersects* Δ^{out} *in a smooth curve which is a graph, i.e.* $x^{\text{out}} = h^{\text{out}}_{\text{a}}(y^{\text{out}}, \varepsilon)$.

(2) *The section* Δ^{in} *is mapped to an exponentially thin strip around* $S^-_{\text{a},\varepsilon} \cap \Delta^{\text{out}}$, *i.e. its width* R_L *in x-direction is* $O(e^{-c/\varepsilon})$ *where c is a positive constant.*

(3) *The map* $\Pi_L : \Delta^{\text{in}} \to \Delta^{\text{out}}$ *has the form*

$$\Pi_L \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} h^{\text{out}}_{\text{a}}(G_L(y,z,\varepsilon),\varepsilon) + R_L(y,z,\varepsilon) \\ G_L(y,z,\varepsilon) \end{pmatrix}, \tag{5}$$

where $h^{\text{out}}_{\text{a}}(G_L(y,z,\varepsilon),\varepsilon) = O(\varepsilon^{2/3})$, $G_L(y,z,\varepsilon) = G_{L,0}(y) + O(\varepsilon \ln \varepsilon)$, *and the function* $R_L(y,z,\varepsilon)$ *is exponentially small. The function* $G_{L,0}(y) = y + O(\rho^3)$ *is induced by the reduced flow on* S^-_{a} *from* Δ^{in} *to the fold-line* L^-.

The theorem is an extension of results in [24], where the geometric properties 1 and 2 have been proved. Here we obtain additional information on the slow dynamics encoded in the function G_L, which is needed for the analysis of relaxation oscillations.

We define the blow-up transformation $\Phi : B = S^2 \times \mathbb{R}^2 \to \mathbb{R}^4$

$$x = \bar{r}^2 \bar{x}, \quad y = \bar{y}, \quad z = \bar{r}\bar{z}, \quad \varepsilon = \bar{r}^3 \bar{\varepsilon}. \tag{6}$$

This leads to a blow-up manifold $B = S^2 \times \mathbb{R}^2$ with $(\bar{x}, \bar{z}, \bar{\varepsilon}) \in S^2$, i.e. the fold-line is blown-up to a cylinder $S^2 \times I$ with $\bar{y} \in I$.

Remark 4. The rescaling used in [1,17,19] in the analysis of a regular fold point (located at the origin) corresponds to a blow-up

$$x = \bar{r}^2\bar{x}, \quad y = \bar{r}^2\bar{y}, \quad z = \bar{r}\bar{z}, \quad \varepsilon = \bar{r}^3\bar{\varepsilon},$$

which treats the y-variable differently then (6). This inclusion of y in the blow-up means that a single point on the fold-line is blown-up.

For the analysis of the blown-up vector field we need three directional charts, κ_1 for the incoming flow, obtained by $\bar{x} = -1$, κ_2 for the flow on the cylinder, obtained by $\bar{\varepsilon} = 1$, and κ_3 for the outgoing flow, obtained by $\bar{z} = 1$. For the blown-up vector field we obtain special solutions (in the classical chart κ_2) which can be viewed as extensions of the reduced flow on the critical manifold under consideration. The additional charts κ_1 and κ_3 are used to connect the unbounded branches of these special solutions with the reduced problem (backward time) resp. with the fast fibers (forward time). The blow-up is shown in Fig. 3 for fixed $y = const$.

We start our analysis in chart κ_1 where we obtain the extension of the critical manifold M_a^- near the blown-up fold-line L^-.

2.1. Dynamics in chart κ_1

We consider transformation (6) with $\bar{x} = -1$, i.e. we consider a directional blow-up $\Phi_1 : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$\Phi_1(r_1, y_1, z_1, \varepsilon_1) = (-r_1^2, y_1, r_1 z_1, r_1^3\varepsilon_1). \tag{7}$$

After transformation of system (4) and a local division by the factor $r_1\, h_1(r_1, y_1, z_1, \varepsilon_1)$ where $h_1(r_1, y_1, z_1, \varepsilon_1) = 1 + O(r_1)$ we obtain

$$r_1' = -\frac{1}{2} r_1 \varepsilon_1,$$

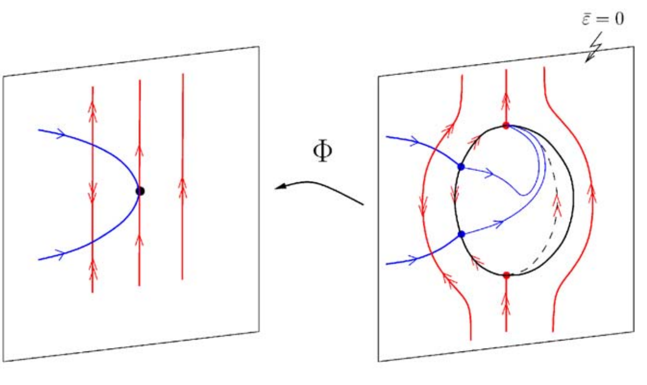

Fig. 3. Blow-up of fold-line for fixed $y = const$.

$$\begin{aligned} y_1' &= O(r_1^3 \varepsilon_1), \\ z_1' &= z_1^2 - 1 + \frac{1}{2} z_1 \varepsilon_1 + O(r_1), \\ \varepsilon_1' &= \frac{3}{2} \varepsilon_1^2. \end{aligned} \tag{8}$$

System (8) has two invariant subspaces, namely the hyperplanes $r_1 = 0$ and $\varepsilon_1 = 0$. In the invariant hyperplane $\varepsilon_1 = 0$ we obtain a normally hyperbolic surface $S_{\mathrm{a},1}$ of equilibria emanating from the line $L_{\mathrm{a},1} = (0, y_1, -1, 0)$ and a normally hyperbolic surface $S_{\mathrm{r},1}$ of equilibria emanating from the line $L_{\mathrm{r},1} = (0, y_1, 1, 0)$. For r_1 small this follows from the implicit function theorem. Actually, $S_{\mathrm{a},1}$ and $S_{\mathrm{r},1}$ are precisely the branches S_{a}^- and S_{r} of the critical manifold S, this also explains the notation. Along the surface $S_{\mathrm{a},1}$ the nonzero eigenvalue is negative and close to -2 for small r_1. Along $S_{\mathrm{r},1}$ the situation is similar, however the nonzero eigenvalue is positive and close to 2 for small r_1. We have gained normal hyperbolicity at the lines $L_{\mathrm{a},1}$ and $L_{\mathrm{r},1}$ due to the blow-up (see Fig. 4).

In the invariant hyperplane $r_1 = 0$ we recover the lines of equilibria $L_{\mathrm{a},1}$ and $L_{\mathrm{r},1}$ and one additional zero eigenvalue due to the third equation. Hence there exist two-dimensional center manifolds $C_{\mathrm{a},1}$, $C_{\mathrm{r},1}$ containing the lines $L_{\mathrm{a},1}$, $L_{\mathrm{r},1}$. Note, ε_1 increases in these manifolds (away from the lines).

In the following we restrict our attention to the attracting objects $S_{\mathrm{a},1}$, $L_{\mathrm{a},1}$ and $C_{\mathrm{a},1}$ and to the set $D_1 = \{(r_1, y_1, z_1, \varepsilon_1)\colon 0 \leqslant r_1 \leqslant \rho,\ 0 \leqslant \varepsilon_1 \leqslant \delta,\ y_1 \in I\}$.

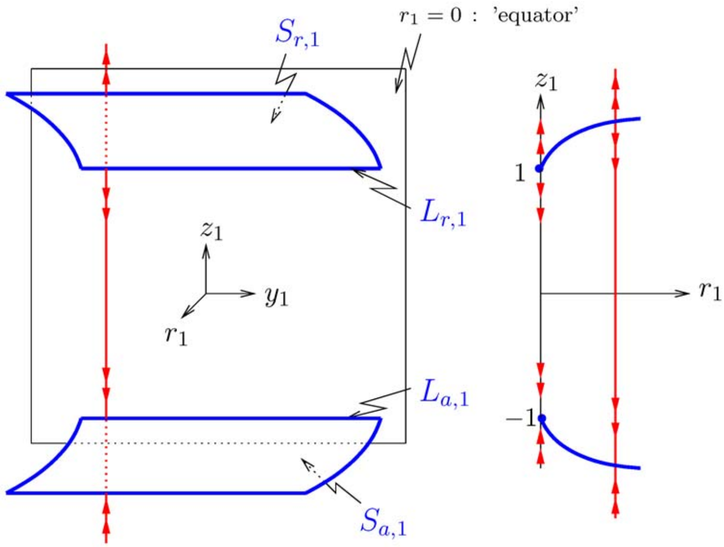

Fig. 4. Invariant manifolds $S_{\mathrm{a},1}$ resp. $S_{\mathrm{r},1}$ in hyperplane $\varepsilon_1 = 0$.

Proposition 2.1. *For ρ and δ sufficiently small the following assertions hold for system* (8):

(1) *There exists an attracting three-dimensional center manifold $M_{a,1}$ of the line of equilibria $L_{a,1} = (0, y_1, -1, 0)$, $y_1 \in I$, containing the surface of equilibria $S_{a,1}$ and the center manifold $C_{a,1}$. In D_1 the manifold $M_{a,1}$ is given as a graph $z_1 = h_{a,1}(r_1, y_1, \varepsilon_1)$. The branch of $C_{a,1}$ in $r_1 = 0$ is unique.*
(2) *There exists a stable invariant foliation F_s with base $M_{a,1}$ and one-dimensional fibers. For any positive $c < 2$ there exists a choice of positive ρ and δ such that the contraction along F_s is stronger than e^{-ct_1}.*

Proof. For system (8) the equilibria of the line $L_{a,1}$ with $y_1 \in I \subset \mathbb{R}$ are nonhyperbolic with triple eigenvalue zero. The nonzero eigenvalue is given by $\lambda = -2$. The assertions follow directly from invariant manifold theory. □

Remark 5. The center directions of the equilibria in the $r_1 = 0$ hyperplane are given by $(y_1, z_1, \varepsilon_1) = (1, 0, 0)$ and $(y_1, z_1, \varepsilon_1) = (0, 1, -4)$.

We now define the following sections

$$\Sigma_1^{\text{in}} = \{(r_1, y_1, z_1, \varepsilon_1) \in D_1\colon\ r_1 = \rho\},$$

$$\Sigma_1^{\text{out}} = \{(r_1, y_1, z_1, \varepsilon_1) \in D_1\colon\ \varepsilon_1 = \delta\},$$

with ρ and δ sufficiently small. Let R_1^{in} be a rectangular box in Σ_1^{in} defined by $|1 + z_1| \leqslant \beta$ for sufficiently small β. The constants ρ, δ and β can be chosen such that $M_{a,1} \cap \Sigma_1^{\text{in}} \subset R_1^{\text{in}}$. Let $\Pi_1 : \Sigma_1^{\text{in}} \to \Sigma_1^{\text{out}}$ be the transition map defined by the flow of (8). The map Π_1 is well defined on R_1^{in} for ρ, δ and β small enough.

We are mostly interested in the evolution of the variable y_1 in $M_{a,1}$. By substituting $z_1 = h_{a,1}(r_1, y_1, \varepsilon_1) = -1 - \varepsilon_1/4 + O(r_1, \varepsilon_1^2)$ into system (8) and rescaling time we obtain the flow on the center manifold $M_{a,1}$ given by

$$\begin{aligned} r_1' &= -\frac{1}{2} r_1, \\ y_1' &= O(r_1^3), \\ \varepsilon_1' &= \frac{3}{2} \varepsilon_1. \end{aligned} \tag{9}$$

The transition time of solutions from Σ_1^{in} to Σ_1^{out} for system (9) (in $M_{a,1}$) is given by

$$T_s = \ln(\delta/\varepsilon_i)^{2/3} \tag{10}$$

with $\varepsilon_i = \varepsilon_1(0)$. We estimate the evolution of y_1 by integrating $y_1' = \rho^3 \exp(-3t/2)O(1)$ and obtain

$$y_1(T_s) = y_i + \rho^3(1 - 2/3(\varepsilon_i/\delta))O(1) = y_i + O(\rho^3) + O(\varepsilon_i) =: G_1(y_i, \rho, \varepsilon_i) \tag{11}$$

with $y_i = y_1(0)$.

Remark 6. The function $G_1(y_i, \rho, 0) = y_i + O(\rho^3)$ describes the flow on $S_{a,1}$ from Σ_1^{in} to the line $L_{a,1}$. In the original problem (4) this corresponds to the reduced flow from Δ^{in} to the fold-line L^-.

Proposition 2.2. *The transition map $\Pi_1 : R_1^{in} \subset \Sigma_1^{in} \to \Sigma_1^{out}$ defined by the flow of system* (8) *has the following properties*: *$\Pi_1(R_1^{in})$ is a three-dimensional wedge-like region in Σ_1^{out}. More precisely, the transition map is given by*

$$\Pi_1 \begin{pmatrix} \rho \\ y_1 \\ z_1 \\ \varepsilon_1 \end{pmatrix} = \begin{pmatrix} \rho(\varepsilon_1/\delta)^{1/3} \\ G_1(y_1, \rho, \varepsilon_1) \\ h_{a,1}^{out}(\rho(\varepsilon_1/\delta)^{1/3}, G_1(y_1, \rho, \varepsilon_1), \delta) + R_1(y_1, z_1, \varepsilon_1) \\ \delta \end{pmatrix}$$

with $h_{a,1}^{out}(\rho(\varepsilon_1/\delta)^{1/3}, G_1(y_1, \rho, \varepsilon_1), \delta) = -1 - \delta/4 + O(\delta^2) + O(\rho(\varepsilon_1/\delta)^{1/3}) = -1 + O(\delta) + O(\varepsilon_1^{1/3})$ *and $R_1(y_1, z_1, \varepsilon_1)$ is exponentially small.*

Proof. Integrating equation $\varepsilon_1' = 3/2\, \varepsilon_1^2$ gives the transition time $T = O(1/\varepsilon_1)$ of a solution of system (8) from $P_1 = (\rho, y_1, z_1, \varepsilon_1) \in R_1^{in}$ to $\Pi_1(P_1) \in \Sigma_1^{out}$. The assertion follows from Proposition 2.1 and from (11). □

The results concerning the dynamics in chart κ_1 are illustrated in Fig. 5. The center manifold $C_{a,1}$ in the hyperplane $r_1 = 0$ can be viewed as the extension of the critical manifold $S_{a,1}$ on the blown-up cylinder which will be further studied in charts κ_2 and κ_3.

2.2. Dynamics in the classical chart κ_2

We consider transformation (6) with $\bar{\varepsilon} = 1$, i.e. we consider the directional blow-up $\Phi_2 : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$\Phi_2(x_2, y_2, z_2, r_2) = (r_2^2 x_2, y_2, r_2 z_2, r_2^3). \tag{12}$$

This transformation is just an ε-dependent rescaling of the variables (x, z) since $r_2 = \sqrt[3]{\varepsilon}$. After transformation and local division by $r_2(1 + O(r_2))$ we obtain the

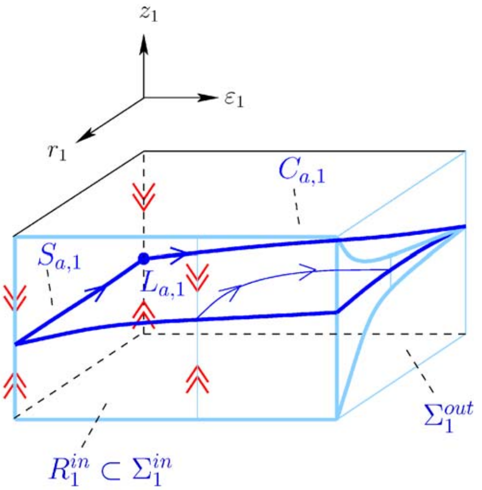

Fig. 5. Dynamics in chart κ_1 projected to $y_1 = const$.

following blown-up vector field:

$$
\begin{aligned}
x_2' &= 1, \\
y_2' &= O(r_2^3), \\
z_2' &= x_2 + z_2^2 + O(r_2), \\
r_2' &= 0.
\end{aligned} \tag{13}
$$

This blown-up system is still a family of vector fields with parameter r_2. Setting $r_2 = 0$ gives the unperturbed problem where the slow variable y_2 is constant and does not affect the dynamics. In chart κ_2 the essential dynamics takes place in the (x_2, z_2) variables. The unperturbed system restricted to the (x_2, z_2) space is a well known Riccati equation, which is of crucial importance for the regular fold (see e.g. [13,17]).

Lemma 7 (Mishchenko and Rozov [17]). *There exists a unique solution γ for the unperturbed system of* (13) *restricted to the* (x_2, z_2) *space with the following asymptotic expansions*

$$
\begin{aligned}
&\tilde{z}_2(x_2) = -(-x_2)^{1/2}\left(1 + \frac{1}{4}(-x_2)^{-3/2} + O(-x_2)^{-3}\right), \quad x_2 \to -\infty, \\
&\tilde{x}_2(z_2) = \Omega - (z_2)^{-1} + O(z_2^{-3}), \quad z_2 \to \infty,
\end{aligned}
$$

i.e. γ is asymptotic for $x_2 \to -\infty$ to the lower branch of the parabola $x_2 + z_2^2 = 0$ and converges to $\Omega > 0$ for $z_2 \to \infty$.

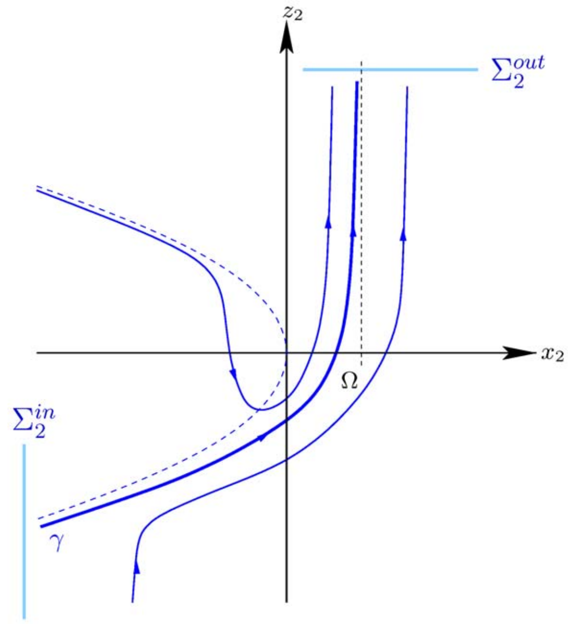

Fig. 6. Solutions of the Riccati equation.

The assertion of Lemma 7 is illustrated in Fig. 6. Notice that γ exists for all y_2 in a suitable neighborhood I of the origin. Thus $\gamma \times I$ is a manifold of solutions for the unperturbed system of (13) with $y_2 \in I$. We connect the flow in the classical chart with the flow in the directional charts κ_1. For this we need to change the coordinates between these charts.

Lemma 8. *The change of coordinates between chart κ_1 and chart κ_2 is given by*

$$\kappa_{12}(x_2, y_2, z_2, r_2) = \left(r_2(-x_2)^{1/2}, y_2, \frac{z_2}{(-x_2)^{1/2}}, \frac{1}{(-x_2)^{3/2}} \right), \quad x_2 < 0, \tag{14}$$

$$\kappa_{21}(r_1, y_1, z_1, \varepsilon_1) = \left(\frac{-1}{\varepsilon_1^{2/3}}, y_1, \frac{z_1}{\varepsilon_1^{1/3}}, r_1 \varepsilon_1^{1/3} \right), \quad \varepsilon_1 > 0. \tag{15}$$

Proposition 2.3. *For $T > 0$ sufficiently large the part of $\kappa_{12}(\gamma \times I)$ corresponding to $x_2 < -T$ is the unique branch of the center manifold $C_{a,1}$.*

Proof. Using the asymptotic parametrization of γ in backward time given in Lemma 7 we show that

$$\left(0, y_2, -\left(1 + \frac{1}{4}(-x_2)^{-3/2} + O(-x_2)^{-3}\right), (-x_2)^{-3/2}\right), x_2 \in (-\infty, -T)$$

is a parametrization of $\kappa_{12}(\gamma \times \{y_2\})$, where $y_2 \in I \subset \mathbb{R}$ is a constant. It is easy to see that this manifold emanates from the line of equilibria $L_{a,1} = (0, y_1, -1, 0)$. Further calculations show that $\kappa_{12}(\gamma \times \{y_2\})$ is tangent to $(0,0,1,-4) \times (0,1,0,0)$ at $L_{a,1}$. These vectors span the tangent-space of the unique center manifold $C_{a,1}$ (see Remark 5). We conclude that $\gamma \times I$ is the unique center manifold $C_{a,1}$ in the hyperplane $r_1 = 0$. □

The importance of these special solutions $\gamma \times I$ is that they lead the attracting slow manifold across the upper half of $S^2 \times I$ to a line from where the take-off in the direction of the fast flow occurs.

We are interested in describing the transition map for system (13) between suitable sections Σ_2^{in} and Σ_2^{out} of $\gamma \times I$ defined for $0 < \delta \ll 1$ by

$$\Sigma_2^{\text{in}} = \{(x_2, y_2, z_2, r_2) \in D_2\colon\ x_2 = -\delta^{-2/3}\},$$

$$\Sigma_2^{\text{out}} = \{(x_2, y_2, z_2, r_2) \in D_2\colon\ z_2 = \delta^{-1/3}\},$$

where D_2 is a bounded domain. Within such a domain we can deduce properties of the flow of (13) from Proposition 7 by using regular perturbation arguments. Note, that under the coordinate transformation κ_{21} the section Σ_1^{out} maps to Σ_2^{in}. Let R_2^{in} be a neighborhood of $(\gamma \times I) \cap \Sigma_2^{\text{in}}$ and $\Pi_2 : R_2^{\text{in}} \subset \Sigma_2^{\text{in}} \to \Sigma_2^{\text{out}}$ be the transition map of the flow (13).

Proposition 2.4. *The transition map Π_2 is a diffeomorphism from R_2^{in} to $\Pi_2(R_2^{\text{in}})$ and has the following properties*:

$$\Pi_2 \begin{pmatrix} -\delta^{-2/3} \\ y_2 \\ z_2 \\ r_2 \end{pmatrix} = \begin{pmatrix} h_{a,2}^{\text{out}}(G_2(y_2, z_2, \delta, r_2), \delta, r_2) + O(z_2 - h_{a,2}^{\text{in}}(y_2, \delta, r_2)) \\ G_2(y_2, z_2, \delta, r_2) \\ \delta^{-1/3} \\ r_2 \end{pmatrix},$$

where

$$G_2(y_2, z_2, \delta, r_2) = y_2 + O(r_2^3),$$

$$h_{a,2}^{\text{in}}(y_2, \delta, r_2) = -\delta^{-2/3}(1 + \delta/4 + O(\delta^2) + O(r_2)),$$

$$h^{\mathrm{out}}_{\mathrm{a},2}(G_2(y_2, z_2, \delta, r_2), \delta, r_2) = \Omega + O(\delta^{1/3}) + O(r_2).$$

Proof. Follows directly from Lemma 7 and regular perturbation theory. $\square$

The dynamics of system (13) for the branch of $\gamma \times I$ along which $z_2 \to \infty$ is studied in chart κ_3.

2.3. Dynamics in chart κ_3

We consider transformation (6) with $\bar{z} = 1$, i.e. we consider a directional blow-up $\Phi_3 : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$\Phi_3(x_3, y_3, r_3, \varepsilon_3) = (r_3^2 x_3, y_3, r_3, r_3^3 \varepsilon_3). \tag{16}$$

After transformation of system (4) and local division by the multiplicative factor $r_3\, h_3(x_3, y_3, r_3, \varepsilon_3)$ with $h_3(x_3, y_3, r_3, \varepsilon_3) = 1 + x_3 + O(r_3)$ we obtain

$$\begin{aligned}
x_3' &= -2x_3 + \varepsilon_3(1 - x_3 + O(x_3^2)) + O(r_3\varepsilon_3),\\
y_3' &= O(r_3^3\varepsilon_3),\\
r_3' &= r_3,\\
\varepsilon_3' &= -3\varepsilon_3.
\end{aligned} \tag{17}$$

System (17) has a line of equilibria $(0, y_3, 0, 0)$, $y_3 \in I \subset \mathbb{R}$, which we denote by L_3.

Lemma 9. *The equilibria of the line $L_3 = (0, y_3, 0, 0)$ with $y_3 \in I \subset \mathbb{R}$ are nonhyperbolic with one zero eigenvalue. The nonzero eigenvalues are given by $\lambda_1 = -2$, $\lambda_2 = 1$ and $\lambda_3 = -3$.*

We connect the flow in the classical chart κ_2 with the flow in the directional chart κ_3.

Lemma 10. *The change of coordinates between chart κ_2 and chart κ_3 is given by*

$$\kappa_{32}(x_2, y_2, z_2, r_2) = \left(\frac{x_2}{z_2^2}, y_2, r_2 z_2, \frac{1}{z_2^3}\right), \quad z_2 > 0, \tag{18}$$

$$\kappa_{23}(x_3, y_3, r_3, \varepsilon_3) = \left(\frac{x_3}{\varepsilon_3^{2/3}}, y_3, \frac{1}{\varepsilon_3^{1/3}}, r_3 \varepsilon_3^{1/3}\right), \quad \varepsilon_3 > 0. \tag{19}$$

Proposition 2.5. *For $T>0$ sufficiently large the part of $\kappa_{32}(\gamma\times I)$ corresponding to $z_2>T$ converges to L_3 as $z_2\to\infty$.*

Proof. Transformation of $\gamma\times\{y_2\}$ by κ_{32} into chart κ_3 gives

$$\left(\frac{\Omega}{z_2^2}+O\left(\frac{1}{z_2^3}\right), y_2, 0, \frac{1}{z_2^3}\right), \quad z_2\in(T,\infty),$$

where we have used the asymptotic parametrization of γ in forward time given in Lemma 7. This shows that $\kappa_{32}(\gamma\times\{y_2\})$ approaches the line of equilibria L_3 tangent to the vector $(1,0,0,0)$. $\square$

We restrict attention to the set

$$D_3=\{(x_3,y_3,r_3,\varepsilon_3)\colon\ 0\leqslant r_3\leqslant\rho,\ \ 0\leqslant\varepsilon_3\leqslant\delta,\ \ y_3\in I\}.$$

For the description of the flow in a neighborhood of L_3 we define sections as follows:

$$\Sigma_3^{\rm in}=\{(x_3,y_3,r_3,\varepsilon_3)\in D_3\colon\ \varepsilon_3=\delta\},$$

$$\Sigma_3^{\rm out}=\{(x_3,y_3,r_3,\varepsilon_3)\in D_3\colon\ r_3=\rho\},$$

where ρ and δ are the same constants as in chart κ_1. Note, that the section $\Sigma_2^{\rm out}$ maps to the section $\Sigma_3^{\rm in}$ under the coordinate transformation κ_{32}. Let Π_3 be the transition map from a suitable neighborhood $R_3^{\rm in}\subset\Sigma_3^{\rm in}$ to $\Sigma_3^{\rm out}$. Our goal is to obtain a formula for the map Π_3. To get a sufficiently precise description of the map Π_3 we have to discuss the structure of system (17) in more detail. Setting $r_3=0$ in (17) we obtain the system

$$x_3'=-2x_3+\varepsilon_3(1-x_3+O(x_3^2)),$$

$$\varepsilon_3'=-3\varepsilon_3,$$

which is decoupled from y_3. For this system the origin is a hyperbolic equilibrium with the eigenvalues -2 and -3. Thus, there exist no resonant terms and we can linearize this system by a suitable transformation $x_3=\tilde\psi(\tilde x_3,\varepsilon_3)-\varepsilon_3$ where $\tilde x_3\mapsto\tilde\psi(\tilde x_3,\varepsilon_3)$ is a C^k near identity transformation (see e.g. [20]). Applying this transformation to system (17) gives

$$\tilde x_3'=-2\tilde x_3+O(r_3\varepsilon_3),$$

$$y_3'=O(r_3^3\varepsilon_3),$$

$$
\begin{aligned}
r_3' &= r_3, \\
\varepsilon_3' &= -3\varepsilon_3.
\end{aligned} \tag{20}
$$

We have the following result:

Proposition 2.6. *The transition map* Π_3 *for system* (17) *has the form*:

$$
\Pi_3 \begin{pmatrix} x_3 \\ y_3 \\ r_3 \\ \delta \end{pmatrix} = \begin{pmatrix} h^{\mathrm{out}}_{\mathrm{a},3}(G_3(x_3, y_3, r_3, \delta), r_3^3, \delta, \rho) + r_3^2\, O(x_3 - h^{\mathrm{in}}_{\mathrm{a},3}(y_3, r_3, \delta)) \\ G_3(x_3, y_3, r_3, \delta) \\ \rho \\ \left(\dfrac{r_3}{\rho}\right)^3 \delta \end{pmatrix},
$$

with

$$
\begin{aligned}
G_3(x_3, y_3, r_3, \delta) &= y_3 + O(r_3^3 \ln r_3) \\
h^{\mathrm{in}}_{\mathrm{a},3}(y_3, r_3, \delta) &= \delta^{2/3}(\Omega - \delta^{1/3} + O(\delta) + O(r_3)) \\
h^{\mathrm{out}}_{\mathrm{a},3}(G_3(x_3, y_3, r_3, \delta), r_3^3, \delta, \rho) &= r_3^2 O(\delta + h^{\mathrm{in}}_{\mathrm{a},3}(y_3, r_3, \delta))
\end{aligned}
$$

Proof. From system (20) we get immediately

$$
\begin{aligned}
r_3(t) &= r_i e^t, \\
\varepsilon_3(t) &= \varepsilon_i e^{-3t},
\end{aligned}
$$

where $r_i = r_3(0)$ and $\varepsilon_i = \varepsilon_3(0)$. The requirement $r_3(T) = r_o$ defines the transition time T as

$$
T = \ln\left(\frac{r_o}{r_i}\right). \tag{21}
$$

Substituting the expressions for r_3 and ε_3 into the equations for $\tilde{x}_3$ and y_3 in system (20) we obtain

$$
\begin{aligned}
\tilde{x}_3' &= -2\tilde{x}_3 + e^{-2t}\, O(r_i), \\
y_3' &= O(r_i^3).
\end{aligned} \tag{22}
$$

We introduce a new variable ξ by setting $\tilde{x}_3 = (\tilde{x}_i + \xi)e^{-2t}$ with $\tilde{x}_i = \tilde{x}_3(0)$ and derive equations for ξ and y_3

$$\begin{aligned} \xi' &= O(r_i), \\ y_3' &= O(r_i^3). \end{aligned} \tag{23}$$

Therefore we obtain the estimates $\xi(T) = O(r_i T)$ and $y_3(T) = y_i + O(r_i^3 T)$ with $y_i = y_3(0)$. Using (21) it follows that

$$\begin{aligned} \tilde{x}_3(T) &= \tilde{x}_i\left(\frac{r_i}{r_o}\right)^2 + O\left(\frac{r_i^3}{r_o^2}\ln\left(\frac{r_o}{r_i}\right)\right), \\ y_3(T) &= y_i + O\left(r_i^3\ln\left(\frac{r_o}{r_i}\right)\right). \end{aligned}$$

Finally we obtain the solution in the original coordinate x_3 by using the inverse transformation of $x_3 = \tilde{\psi}(\tilde{x}_3, \varepsilon_3) - \varepsilon_3$ given by $\tilde{x}_3 = \psi(x_3, \varepsilon_3) + \varepsilon_3$:

$$\begin{aligned} x_3(T) &= (x_i + \varepsilon_i + O((x_i + \varepsilon_i)^2))\left(\frac{r_i}{r_o}\right)^2 + O\left(\frac{r_i^3}{r_o^2}\ln\left(\frac{r_o}{r_i}\right)\right) = r_i^2 O(x_i + \varepsilon_i), \\ y_3(T) &= y_i + O\left(r_i^3\ln\left(\frac{r_o}{r_i}\right)\right). \end{aligned} \tag{24}$$

The result follows immediately by setting $x_i = x_3,\ y_i = y_3,\ r_i = r_3,\ r_o = \rho,\ \varepsilon_i = \delta$ and $\varepsilon_o = (r_3/\rho)^3\delta$. $\square$

The dynamics of system (17) projected to the hyperplane $y_3 = const.$ is shown in Fig. 7.

2.4. Proof of Theorem 1

We define the map $\bar{\Pi}_L : R_1^{\text{in}} \to \Sigma_3^{\text{out}}$ by

$$\bar{\Pi}_L := \Pi_3 \circ \kappa_{32} \circ \Pi_2 \circ \kappa_{21} \circ \Pi_1.$$

The map $\bar{\Pi}_L$ is the transition map from R_1^{in} to Σ_3^{out} for the flow induced by the blown-up vector field. R_1^{in} can be chosen such that the map $\bar{\Pi}_L$ is well defined, i.e. for small enough constants ρ and β and by replacing the interval I by a slightly smaller interval $\tilde{I} \subset I$. Let $P_1 = (\rho, y_1, h_{\text{a},1}(\rho, y_1, \varepsilon_1), \varepsilon_1) \in R_1^{\text{in}}$. Propositions 2.2, 2.4, 2.6 and

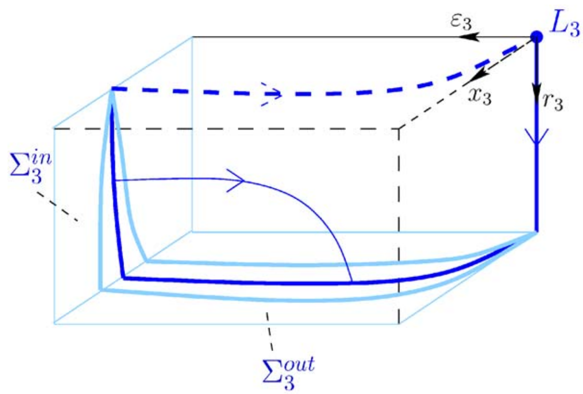

Fig. 7. Dynamics near L_3 in chart κ_3.

Lemmas 8 and 10 imply that

$$\bar{\Pi}_L \begin{pmatrix} \rho \\ y_1 \\ z_1 \\ \varepsilon_1 \end{pmatrix} = \begin{pmatrix} \bar{h}_{\mathrm{a}}^{\mathrm{out}}(\bar{G}_L(y_1, \rho, \varepsilon_1), \varepsilon_1, \rho) + \bar{R}_L(y_1, z_1, \rho, \varepsilon_1) \\ \bar{G}_L(y_1, \rho, \varepsilon_1) \\ \rho \\ \varepsilon_1 \end{pmatrix}$$

with $\bar{G}_L(y_1, \rho, \varepsilon_1) = y_1 + O(\rho^3) + O(\varepsilon_1 \ln \varepsilon_1)$ and $\bar{h}_{\mathrm{a}}^{\mathrm{out}}(\bar{G}_1(y_1, \rho, \varepsilon_1), \varepsilon_1, \rho) = O(\varepsilon_1^{2/3})$. The assertions of Theorem 1 and the formula for the map Π_L (5) follow by applying the appropriate blow-down transformations with $\Delta^{\mathrm{in}} = \Phi_1(R_1^{\mathrm{in}})$. $\square$

3. Reduction to a Poincaré map

We are now ready to describe the Poincaré map Π defined for a suitable section $\mathscr{V} \subset \Sigma^-$ near S_{a}^- in detail. Note that away from the fold-curves $L^\pm$ Fenichel theory applies and the slow manifold $S_{\mathrm{a},\varepsilon}^\pm$ is given as a graph over (x, y). We choose new (local) coordinates in Σ^- such that the slow manifold $S_{\mathrm{a},\varepsilon}^- \cap \Sigma^-$ given as a graph $z = h_{\mathrm{a}}^-(y, \varepsilon)$ corresponds to the y-axis.

Theorem 2. *Let Σ^- be a transverse section near an attracting branch of the critical manifold S of system* (1) *under Assumptions* 1–4. *There exists an open neighborhood $\mathscr{V}$ of the point $\Gamma \cap \Sigma^-$ such that the Poincaré map $\Pi : \mathscr{V} \to \Sigma^-$ induced by the flow of system* (1) *is well defined for small ε. The map is given by*

$$\Pi \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} G(y, z, \varepsilon) \\ R(y, z, \varepsilon) \end{pmatrix}, \tag{25}$$

where $G(y,z,\varepsilon) = G_0(y) + O(\varepsilon \ln \varepsilon)$. The function $R(y,z,\varepsilon)$ is exponentially small and the function $G_0(y)$ describes the return map induced by the reduced flow on $S_{\rm a}^{\pm}$.

Proof. We define the half return map $\Pi_{H^-} := \Pi_T \circ \tilde{\Pi}_L \circ \Pi_{S_{\rm a}} : \mathscr{V} \to \Sigma^+$ where Σ^+ is a section of $S_{\rm a}^+$ transverse to Γ (see Fig. 8).

Remember that we have obtained the local result for the transition map Π_L (5) after preliminary transformations of system (1) to system (4) by a local diffeomorphism. Hence there exists a section $\tilde{\Delta}^{\rm in}$ resp. $\tilde{\Delta}^{\rm out}$ in system (1) near the fold-curve L^- which is the pre-image of the section $\Delta^{\rm in}$ resp. $\Delta^{\rm out}$ in system (4). Note, $\tilde{\Delta}^{\rm in}$ resp. $\tilde{\Delta}^{\rm out}$ is not necessary planar but is a graph over (y,z) resp. over (x,y) (see Fig. 8).

The slow manifold $S_{{\rm a},\varepsilon}^-$ is given as a graph over (x,y). We choose new (local) coordinates in $\tilde{\Delta}^{\rm in}$ such that the slow manifold corresponds to the y-axis. Then Fenichel theory implies the existence of a neighborhood $\mathscr{V} \subset \Sigma^-$ such that the map $\Pi_{S_{\rm a}} : \mathscr{V} \subset \Sigma^- \to \tilde{\Delta}^{\rm in}$ induced by the flow of system (1) is given by

$$\Pi_{S_{\rm a}} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} G_{S_{\rm a}}(y,z,\varepsilon) \\ R_{S_{\rm a}}(y,z,\varepsilon) \end{pmatrix} \tag{26}$$

with $G_{S_{\rm a}}(y,z,\varepsilon) = G_{S_{\rm a},0}(y) + O(\varepsilon)$ where $G_{S_{\rm a},0}(y)$ is induced by the reduced flow on the critical manifold $S_{\rm a}^-$ from Σ^- to $\tilde{\Delta}^{\rm in}$.

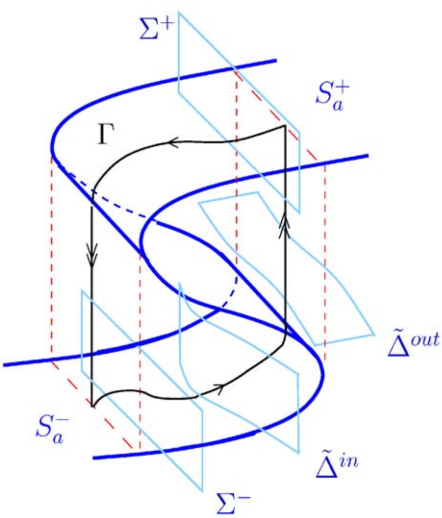

Fig. 8. Sections for analysing the Poincaré map $\Pi : \mathscr{V} \subset \Sigma^- \to \Sigma^-$.

From Theorem 1 follows that the intersection of the slow manifold $S^-_{a,\varepsilon}$ with $\tilde{\Delta}^{out}$ is a graph, i.e. $\tilde{x}^{out} = \tilde{h}^{out}_a(\tilde{y}^{out}, \varepsilon) = O(\varepsilon^{2/3})$. We choose new (local) coordinates in $\tilde{\Delta}^{out}$ such that the slow manifold becomes the y-axis. It follows by (5) that the map $\tilde{\Pi}_L : \tilde{\Delta}^{in} \to \tilde{\Delta}^{out}$ is given by

$$\tilde{\Pi}_L \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \tilde{R}_L(y, z, \varepsilon) \\ \tilde{G}_L(y, z, \varepsilon) \end{pmatrix}, \tag{27}$$

where $\tilde{G}_L(y, z, \varepsilon) = \tilde{G}_{L,0}(y) + O(\varepsilon \ln \varepsilon)$ and $\tilde{G}_{L,0}(y)$ is induced by the reduced flow of system (1) from section $\tilde{\Delta}^{in}$ to the fold-curve L^-.

The map $\Pi_T : \tilde{\Delta}^{out} \to \Sigma^+$ is well defined under Assumptions 3 and 4. We again choose (local) coordinates in Σ^+ such that $S^+_{a,\varepsilon} \cap \Sigma^+$ becomes the y-axis. Fenichel theory implies that the map Π_T induced by the flow of system (1) is given by

$$\Pi_T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} G_T(x, y, \varepsilon) \\ R_T(x, y, \varepsilon) \end{pmatrix} \tag{28}$$

with $G_T(x, y, \varepsilon) = G_{T,0}(x, y) + O(\varepsilon)$ where $G_{T,0}(x, y)$ is induced by the reduced flow on the critical manifold S^+_a between the base point $(x, y) \in P(L^-)$ and Σ^+.

It follows that the half return map $\Pi_{H^-} : \mathcal{V} \to \Sigma^+$ is given by

$$\Pi_{H^-} \begin{pmatrix} y \\ z \end{pmatrix} = \Pi_T \circ \tilde{\Pi}_L \circ \Pi_{S_a} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} G_{H^-}(y, z, \varepsilon) \\ R_{H^-}(y, z, \varepsilon) \end{pmatrix} \tag{29}$$

with $G_{H^-}(y, z, \varepsilon) = G_{T,0} \circ \tilde{G}_{L,0} \circ G_{S_a,0}(y) + O(\varepsilon \ln \varepsilon)$ where $G_{T,0} \circ \tilde{G}_{L,0} \circ G_{S_a,0}(y)$ is induced by the reduced flow from Σ^- to L^- on S^-_a and from the projection-curve $P(L^-)$ on S^+_a to Σ^+.

In a similar way we define a map $\Pi_{H^+} : \mathcal{V}^+ \subset \Sigma^+ \to \Sigma^-$. The analysis of this map is completely analogous to the preceeding analysis and the assertions of the theorem follow for the full return map $\Pi = \Pi_{H^+} \circ \Pi_{H^-}$, i.e.

$$\Pi \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} G(y, z, \varepsilon) \\ R(y, z, \varepsilon) \end{pmatrix}$$

with $G(y, z, \varepsilon) = G_0(y) + O(\varepsilon \ln \varepsilon)$. $\square$

Note that in the limit $\varepsilon \to 0$ the map (25) contracts Σ^- immediately to the invariant manifold $S^-_a \cap \Sigma^-$. Hence we call such a map a singularly perturbed map (SPM). Properties of such a SPM, i.e. existence of an invariant manifold with associated invariant foliation, are shown in Appendix A.

We are now able to state one of our main results.

Theorem 3. *Consider the Poincaré map Π from Theorem 2. There exists a compact neighborhood $\mathcal{K} \subset \mathcal{V}$ of $\Gamma \cap S^-_a$ such that the map $\Pi|_{\mathcal{K}}$ has a one-dimensional* *attracting slow manifold Φ_ε which is C^1 and locally invariant. Furthermore there exists a stable invariant foliation. The slow manifold Φ_ε is given as a graph $z = \varphi(y, \varepsilon), y \in \tilde{I}$. The dynamics of the restriction of Π to Φ_ε is governed by the one-dimensional map*

$$\tilde{I} \to \tilde{I}, \quad y \mapsto G(y, \varphi(y, \varepsilon), \varepsilon) = G_0(y) + O(\varepsilon \ln \varepsilon).$$

Proof. By standard modifications outside of $\mathscr{K}$ the map Π can be extended to a map $\hat{\Pi} : \mathbb{R}^2 \to \mathbb{R}^2$. This map $\hat{\Pi}$ satisfies the assumptions of Theorem A.1 and the existence of the slow manifold Φ_ε and its properties follow. Note that the slow manifold is in general not unique. $\square$

Since the dynamics on the invariant fibers is strongly contracting, the study of the map Π can be reduced to the study of the restriction of Π to Φ_ε which in turn is a $O(\varepsilon \ln \varepsilon)$ perturbation of the map $G_0(y)$. Thus robust dynamical properties, in particular hyperbolic fixed points of G_0 persist under perturbation by ε, i.e. they exist for sufficiently small ε for the 2-d map Π.

Thus, we assume additionally that the singular periodic orbit Γ is hyperbolic for the reduced dynamics, i.e.

Assumption 5. The singular periodic orbit Γ is hyperbolic, i.e. $G_0'(\gamma) \neq 1$, where γ is the y-coordinate of $\Gamma \cap \Sigma^-$.

This assumption can be checked based on the linearization of the reduced flow along Γ. The following result on the existence of periodic relaxation oscillation is an immediate consequence of the above.

Theorem 4. *Assume that system* (1) *satisfies the Assumptions* 1–5. *Then there exists a locally unique, hyperbolic relaxation orbit of system* (1) *close to the singular orbit Γ for sufficiently small values of ε.*

Clearly, the relaxation orbit is asymptotically stable for $|G_0'(\gamma)| < 1$ and of saddle type for $|G_0'(\gamma)| > 1$.

We would like to remark that theorems similar to Theorems 2 and 3 can be easily proved by the same method whenever the singular return map is defined, e.g. $G_0 : \bigcup_{j=1}^n I_j \to \bigcup_{j=1}^n I_j$ with suitable compact disjoint intervals I_j, $j = 1, \ldots, n$. This allows to consider 'k-periodic' relaxation oscillations with $k \in \mathbb{N}$.

4. The forced van der Pol oscillator

Relaxation oscillations were observed the first time by van der Pol [22] who studied properties of a triode circuit. Such a system exhibits self-sustained oscillations with an amplitude independent of the initial conditions. Furthermore, van der Pol and van der Mark [23] investigated that relaxation oscillations get easily entrained by periodic inputs, i.e. the period of the relaxation oscillation is a multiple of the forcing period (subharmonics). They also observed that for certain parameter values two different subharmonics may coexist and there are regions where no subharmonic could be detected (quasiperiodic). This is a transient phenomenon: such quasiperiodic solutions stay close to a chaotic solution before locking in. Such chaotic solutions of the forced van der Pol oscillator have been studied by many people, see e.g. Grasman [8].

A pioneering mathematical investigation of this problem was carried out by Cartwright and Littlewood [3] studying the van der Pol equation

$$\ddot{z} + v(z^2 - 1)\dot{z} + z = vb(v)k\cos kt \tag{30}$$

with period forcing and $v \gg 1$. We analyze this system and show the existence of relaxation oscillations for certain parameter values. Integration of Eq. (30) and the transformations

$$v = 1/\sqrt{\varepsilon}, \quad b = A/\omega, \quad k = \sqrt{\varepsilon}\omega, \quad t = \sqrt{\varepsilon}\tau$$

give the first-order system

$$\begin{aligned} x' &= \varepsilon(-z + A\cos\varphi), \\ \varphi' &= \varepsilon\omega, \\ z' &= x + z - \frac{1}{3}z^3, \end{aligned} \tag{31}$$

which is a singularly perturbed system with perturbation parameter $\varepsilon \ll 1$ on the fast time scale $\tau = t/\sqrt{\varepsilon}$ with $(x, \varphi, z) \in \mathbb{R} \times S^1 \times \mathbb{R}$ and two-dimensional S-shaped critical manifold $x(\varphi, z) = z^3/3 - z$. Thus system (31) fullfills Assumption 1. The reduced flow on this manifold is given by

$$\begin{aligned} \dot{\varphi} &= \omega, \\ (z^2 - 1)\dot{z} &= -z + A\cos\varphi. \end{aligned} \tag{32}$$

At the fold-curves $z = \pm 1$ the reduced flow is singular. The corresponding desingularized system is

$$\begin{aligned} \dot{z} &= -z + A\cos\varphi, \\ \dot{\varphi} &= \omega(z^2 - 1). \end{aligned} \tag{33}$$

The phase portrait of the reduced system is obtained by changing the direction of the flow in the phase portrait of system (33) for $|z| < 1$.

For $A < 1$ system (33) has no equilibrium, just an unstable cycle on the repelling critical manifold. All points on the fold-curves $z = \pm 1$ are jump points (see Fig. 9).

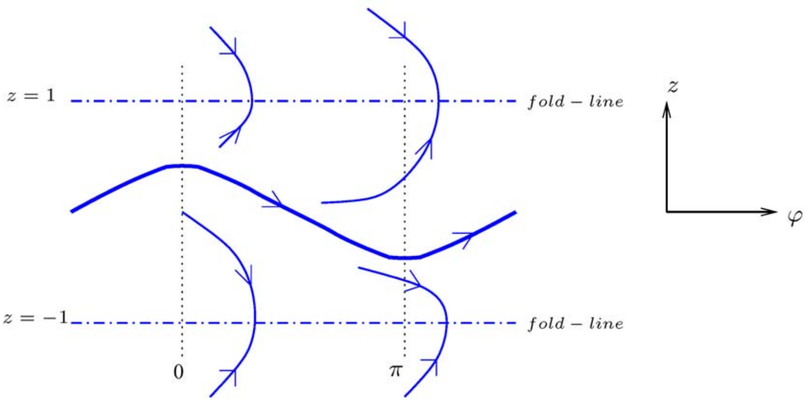

Fig. 9. Reduced flow of (32) for $A < 1$, unstable cycle and jump points.

As the amplitude of the forcing increases to the value $A = 1$ two singular points $(z = 1, \varphi = 0)$ and $(z = -1, \varphi = \pi)$ of system (33) are created which split up in two pairs of singular points at $(z = 1, \varphi = \mp\arccos(1/A))$ and $(z = -1, \varphi = \pi\mp\arccos(1/A))$ for $A > 1$. These singular points are *canard points* for the reduced flow, i.e. solutions pass via these canard points from the attracting to the repelling branch in finite time. Canard points exist for the reduced system for all $A \geqslant 1$. Such points give rise to complicated dynamics nearby the fold-line. For further information on local behavior of solutions nearby canard points we refer to [21].

For $A < 2$ the flow at the projection of a fold-line (at $z = \pm 2$) is directed towards to other fold-line for all $\varphi \in S^1$. For $A = 2$ two points $(z = 2, \varphi = 0)$ and $(z = -2, \varphi = \pi)$ appear where the flow is tangent to the projection fold-line. For $A > 2$ there are two pairs of points $(z = 2, \varphi = \mp\arccos(2/A))$ and $(z = -2, \varphi = \pi\mp\arccos(2/A))$ where the flow is tangent to the fold-line. On the section of the projection of a fold-line between these points the flow is directed away from the other fold-line.

For $A < 1$ the Poincaré map can be defined globally in the variable φ, i.e. for $(\varphi, z) \in \Sigma = S^1 \times [-\delta, \delta]$, and has the properties stated in Theorem 2. Thus Theorem A.1 applies and we conclude the existence of a slow manifold of the return map which implies

Proposition 4.1. *The forced van der Pol oscillator* (31) *possesses for moderate forcing amplitude* $A < 1$ *an attracting invariant torus* T_ε (*see* Fig. 10).

Thus relaxation oscillations in system (31) can be found based on the analysis of the 1-d map $\varphi \mapsto G(\varphi)$ induced by the reduced flow of (32) on the attracting branches $S_a^{\pm}$. This map is a diffeomorphism which maps S^1 onto S^1, i.e. a 1-d circle map, see e.g. [12]. The invariant torus T^ε is destroyed for $A \geqslant 1$ due to the existence of canard

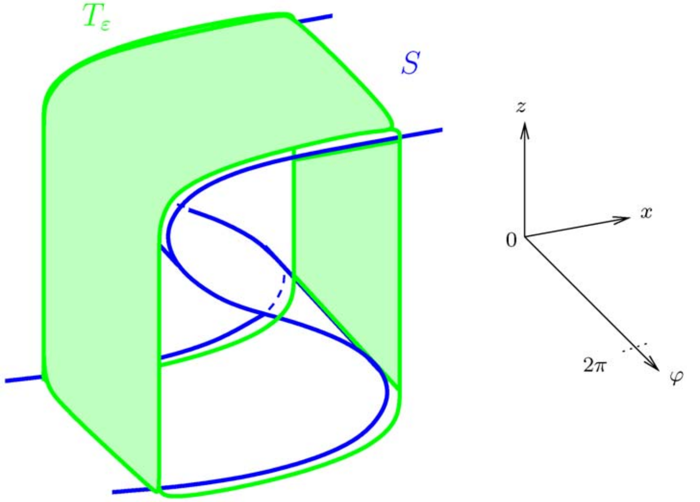

Fig. 10. Invariant torus of forced van der Pol oscillator with moderate forcing amplitude $A < 1$.

solutions. Hence, for $A > 1$ discontinuities develop in the reduced return map and the Poincaré map Π is not globally defined anymore. To our knowledge in this situation rigorous results on the problem of relating the return map to its reduced 'discontinuous' 1-d map do not exist. Nonetheless for $A \geqslant 1$ it is still possible that singular relaxation orbits away from these discontinuities exists. In neighborhoods of these singular orbits the theory developed in this paper is still applicable as outlined at the end of Section 3.

Appendix A. Existence of slow manifolds and invariant foliations of singularly perturbed maps (SPM)

Here we state a theorem about the existence of a slow manifold and a corresponding invariant stable foliation for an ε-dependent family of 2-d maps which is applicable to the return map for relaxation oscillations introduced in Section 3. What follows is the adaptation of standard techniques in invariant manifold theory, e.g. [12] and references therein, to the specific situation, therefore we do not give complete proofs.

Theorem A.1. *We consider a family of diffeomorphisms $\Pi_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$, $r \geqslant 1$ and $\varepsilon \ll 1$ by*

$$\begin{aligned}\Pi_\varepsilon(y,z) &= (G(y,z,\varepsilon), F(y,z,\varepsilon)) \\ &= (G_0(y,\varepsilon) + G_1(y,z,\varepsilon)H_2(\varepsilon), F_1(y,z,\varepsilon)H_1(\varepsilon)),\end{aligned} \tag{A.1}$$

where the functions G, F or equivalently G_0, G_1 and F_1 are C^r functions with respect to (y,z) and the functions $H_i \geqslant 0$, $i = 1,2$ are C^0 with $\lim_{\varepsilon\to 0} H_i(\varepsilon) = 0$. *Assume that the following estimates hold uniformly for sufficiently small values of* ε:

$$\left|\frac{\partial G}{\partial y}\right| \geqslant \mu_1, \quad \left|\frac{\partial G}{\partial y}\right| \leqslant L_1, \quad \left|\frac{\partial G}{\partial z}\right| \leqslant L_2 H_2(\varepsilon), \tag{A.2}$$

$$\left|\frac{\partial F}{\partial y}\right| \leqslant L_3 H_1(\varepsilon), \quad \left|\frac{\partial F}{\partial z}\right| \geqslant \mu_2 H_1(\varepsilon), \quad \left|\frac{\partial F}{\partial z}\right| \leqslant L_4 H_1(\varepsilon), \tag{A.3}$$

where μ_i, $i = 1,2$, L_j, $j = 1,\dots,4$ are positive constants independent of the parameter ε.

Then there exist for all $\varepsilon \in (0,\varepsilon_0)$ with sufficiently small $\varepsilon_0 \ll 1$

(1) *a slow C^r-manifold*

$$\Phi = \{(y, \varphi(y,\varepsilon)) \mid y \in \mathbb{R}\} = graph\, \varphi$$

(2) *and a stable foliation with C^r-fibers,*

$$\Psi_p = \{(\psi_p(z,\varepsilon), z) \mid z \in \mathbb{R}\} = graph\, \psi_p$$

with $\psi_p(\varphi(y,\varepsilon),\varepsilon) = p$ and $\Pi(\Psi_p) \subset \Psi_{\Pi(p)}$.

The assumptions of the theorem imply that $F(y,z,0) = 0$ and $G(y,z,0) = G_0(y,0)$. Hence the singular map is given by $\Pi_0(y,z) = (G_0(y,0),0)$. From now on we skip for convenience the parameter ε in the notation of the map Π. Note, Π is a diffeomorphism, i.e. Π is invertible. Thus

$$|det| := \left|\frac{\partial G}{\partial y}\frac{\partial F}{\partial z} - \frac{\partial G}{\partial z}\frac{\partial F}{\partial y}\right| > 0$$

for $0 < \varepsilon \ll 1$ and we obtain the following upper and lower estimates:

$$|det| \geqslant (\mu_1\mu_2 - L_2 L_3 H_2(\varepsilon)) H_1(\varepsilon) \geqslant L_5 H_1(\varepsilon), \tag{A.4}$$

$$|det| \leqslant (L_1 L_4 + L_2 L_3 H_2(\varepsilon)) H_1(\varepsilon) \leqslant L_6 H_1(\varepsilon). \tag{A.5}$$

Note, that L_5 is bounded away from zero for sufficiently small ε. Since Π is a C^r diffeomorphism it follows that the inverse map Π^{-1} exists and is also C^r for $0 < \varepsilon \ll 1$. Let

$$\Pi^{-1}(y,z) = (g(y,z,\varepsilon), f(y,z,\varepsilon)). \tag{A.6}$$

From the implicit function theorem we obtain the following estimates for the functions f and g:

$$\left|\frac{\partial g}{\partial y}\right| = \left|(det)^{-1}\frac{\partial F}{\partial z}\right| \leqslant L_5^{-1}L_4, \tag{A.7}$$

$$\left|\frac{\partial g}{\partial z}\right| = \left|(det)^{-1}\frac{\partial G}{\partial z}\right| \leqslant L_5^{-1}H_1(\varepsilon)^{-1}L_2H_2(\varepsilon), \tag{A.8}$$

$$\left|\frac{\partial f}{\partial y}\right| = \left|(det)^{-1}\frac{\partial F}{\partial y}\right| \leqslant L_5^{-1}L_3, \tag{A.9}$$

$$\left|\frac{\partial f}{\partial z}\right| = \left|(det)^{-1}\frac{\partial G}{\partial y}\right| \geqslant L_6^{-1}H_1(\varepsilon)^{-1}\mu_1. \tag{A.10}$$

What follows is based on [12]. We start our proof by showing properties of the linear maps $(D\Pi)$ resp. $(D\Pi)^{-1}$.

A.1. Invariant cone-families

Definition A.1. The standard horizontal γ-cone at $p \in \mathbb{R}^2$ is defined by

$$H_p^\gamma = \{(u,v) \in T_p\mathbb{R}^2 \mid |v| \leqslant \gamma |u|\}.$$

The standard vertical γ-cone at $p \in \mathbb{R}^2$ is defined by

$$V_p^\gamma = \{(u,v) \in T_p\mathbb{R}^2 \mid |u| \leqslant \gamma |v|\}.$$

By a *cone-field* we mean a map $K : \mathbb{R}^2 \to T_p\mathbb{R}^2$, $p \mapsto K_p$ which associates to every point $p \in \mathbb{R}^2$ a cone $K_p \in T_p\mathbb{R}^2$ (H_p or V_p in our case). A diffeomorphism $\Pi : \mathbb{R}^2 \to \mathbb{R}^2$ acts naturally on cone fields by

$$(\Pi_* K)_p = (D\Pi)_{\Pi^{-1}(p)}(K_{\Pi^{-1}(p)}).$$

Lemma A.1. *The horizontal and vertical cones H_p^γ and V_p^γ are invariant under the diffeomorphism* (A.1), *i.e.*

$$(D\Pi)_p(H_p^\gamma) \subset Int\, H_{\Pi(p)}^\gamma, \quad (D\Pi)_p^{-1}(V_{\Pi(p)}^\gamma) \subset Int\, V_p^\gamma.$$

(*see* Fig. 11).

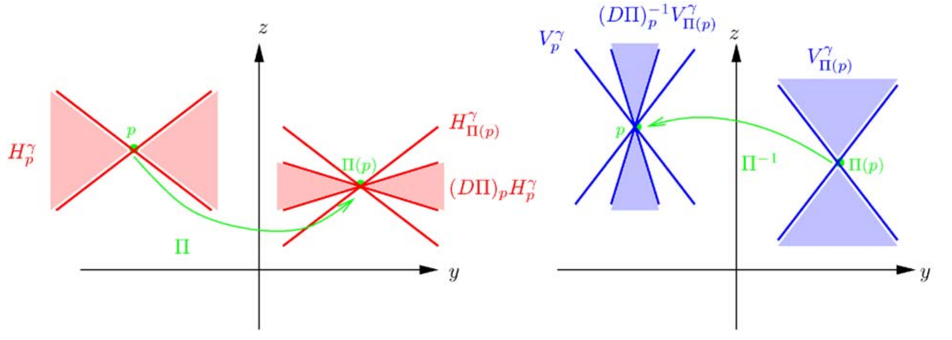

Fig. 11. Invariance of horizontal and vertical cones.

Proof. We denote partial derivatives by subscripts.

(1) $(u, v) \in H^\gamma_p \Rightarrow |v| \leqslant \gamma |u|$, $(u', v') := (D\Pi)_p(u, v)$;

$$|u'| = |(G_y)_p u + (G_z)_p v| \geqslant \mu_1 |u| - L_2 H_2(\varepsilon)|v| \geqslant (\mu_1 - \gamma L_2 H_2(\varepsilon))|u|$$

$$|v'| = |(F_y)_p u + (F_z)_p v| \leqslant H_1(\varepsilon)(L_3 + \gamma L_4)|u|$$

$$\Rightarrow |v'| \leqslant \frac{H_1(\varepsilon)(L_3 + \gamma L_4)}{\mu_1 - \gamma L_2 H_2(\varepsilon)} |u'| =: \gamma'_1(\varepsilon)|u'|.$$

Hence if $\gamma'_1(\varepsilon) < \gamma$ then the horizontal cones H^γ_p are invariant, i.e. $(u', v') \in H^\gamma_{\Pi(p)}$. But this follows immediately for sufficiently small $0 < \varepsilon \ll 1$.

(2) $(u, v) \in V^\gamma_{\Pi(p)} \Rightarrow |u| \leqslant \gamma |v|$, $(u', v') := (D\Pi)^{-1}_p(u, v)$;

$$|u'| = (det)^{-1} |(F_z)_{\Pi(p)} u - (G_z)_{\Pi(p)} v| \leqslant L_5^{-1} H_1(\varepsilon)^{-1}(\gamma L_4 H_1(\varepsilon) + L_2 H_2(\varepsilon))|v|$$

$$|v'| = (det)^{-1} | - (F_y)_{\Pi(p)} u + (G_y)_{\Pi(p)} v| \geqslant L_5^{-1} L_6^{-1} H_1(\varepsilon)^{-1}(L_5 \mu_1 - \gamma L_3 L_6 H_1(\varepsilon))|v|$$

$$\Rightarrow |u'| \leqslant \frac{L_6(L_2 H_2(\varepsilon) + \gamma L_4 H_1(\varepsilon))}{L_5 \mu_1 - \gamma L_3 L_6 H_1(\varepsilon)} |v'| =: \gamma'_2(\varepsilon)|v'|.$$

Hence if $\gamma'_2(\varepsilon) < \gamma$ then the vertical cones $V^\gamma_{\Pi(p)}$ are invariant, i.e. $(u', v') \in V^\gamma_p$. But this follows immediately for sufficiently small $0 < \varepsilon \ll 1$. $\square$

Next we show that vectors in horizontal cones expand and those in vertical cones contract.

Lemma A.2.

$$||(D\Pi)(u,v)||_1 \geqslant \frac{\mu_1 - \gamma L_2 H_2(\varepsilon)}{1+\gamma} ||(u,v)||_1 =: \mu' ||(u,v)||_1 \quad \textit{for } (u,v) \in H^\gamma$$

$$||(D\Pi)(u,v)||_1 \leqslant H_1(\varepsilon) \frac{1+\gamma}{L_6^{-1}\mu_1 - \gamma L_5^{-1} L_3 H_1(\varepsilon)} ||(u,v)||_1 =: \lambda' H_1(\varepsilon) ||(u,v)||_1$$

for $(u,v) \in V^\gamma$

Proof. $||(u',v')||_1 \geq |u'| \geqslant (\mu_1 - \gamma L_2 H_2(\varepsilon))|u| \geqslant (\mu_1 - \gamma L_2 H_2(\varepsilon))(1+\gamma)^{-1} ||(u,v)||_1$ for $(u,v) \in H^\gamma$.

$||(u',v')||_1 \geqslant |v'| \geqslant (L_6^{-1} H_1(\varepsilon)^{-1} \mu_1 - \gamma L_5^{-1} L_3)|v| \geqslant (1+\gamma)^{-1} (L_6^{-1} H_1(\varepsilon)^{-1} \mu_1 - \gamma L_5^{-1} L_3) ||(u,v)||_1$ for $(u,v) \in V^\gamma$. $\square$

These contraction and expansion rates shows that the linear map $(D\Pi)$ admits a $(\lambda' H_1(\varepsilon), \mu')$ splitting which is an exponential splitting with a 'big' gap caused by $\lambda' H_1(\varepsilon) \ll 1$ for sufficiently small $0 < \varepsilon \ll 1$. With that setting we obtain the existence of invariant subspaces within the invariant cones:

Lemma A.3. *There exist subspaces*

$$E_p^+ = \bigcap_{i=0}^{\infty} ((D\Pi)^i H^\gamma)_p$$

$$E_p^- = \bigcap_{i=0}^{\infty} ([(D\Pi)^{-1}]^i V^\gamma)_p$$

inside the cones H^γ *and* V^γ. *These subspaces* E_p^+ *and* E_p^- *are invariant, i.e.* $(D\Pi)_p(E_p^\pm) = E_{\Pi(p)}^\pm$.

Proof. The cones are invariant by Lemma A.1. We have to show the invariance of the subspaces. We define $S_j := (D\Pi)^j(\mathbb{R}^k \times \{0\})$ and $S = \lim_{j\to\infty} S_j$. Since $(u,0) \subset H^\gamma, \forall \gamma$ we have $S \subset E^+$. We need to show $S = E^+$. Let $(u,v) \in E^+$. We can split $(u,v) = (u,v') + (0,v'')$ with $(u,v') \in S$. Furthermore let

$$(u_j, v_j) = ((D\Pi)^{-1})^j (u,v),$$

$$(u_j', v_j') = ((D\Pi)^{-1})^j (u,v'),$$

$$(u_j'', v_j'') = ((D\Pi)^{-1})^j (0,v'').$$

Because of $(u_j, v_j), (u_j', v_j') \in H^\gamma$ it follows by Lemma A.2 that $||(u_j,v_j)||_1 \leqslant (\mu')^{-j} ||(u,v)||_1$ and $||(u_j',v_j')||_1 \leqslant (\mu')^{-j} ||(u,v')||_1$. Since $(u_j'', v_j'') \in V^\gamma$ we have by Lemma A.2 that

$$|v''| \leqslant (\lambda' H_1(\varepsilon))^j ||(u_j'', v_j'')||_1 \leqslant (\lambda' H_1(\varepsilon))^j (||(u_j, v_j)||_1 + ||(u_j', v_j')||_1)$$
$$\leqslant \left(\frac{\lambda' H_1(\varepsilon)}{\mu'}\right)^j (||(u, v)||_1 + ||(u, v')||_1).$$

The limit $j \to \infty$ gives $|v''| \to 0$ which implies $(u, v) \in S = E^+$. The argument for E^- is completely similar using the linear map $(D\Pi)$ instead of $(D\Pi)^{-1}$. $\square$

Remark A.4. The sets $\{E_p^+\}$ and $\{E_p^-\}$ depend continuously on p which follows from the continuity of the map $(D\Pi)$.

A.2. Invariant manifold

On the nonlinear level we want to show the existence of an invariant graph under the action of Π. Let C_γ be a set of functions $\varphi : \mathbb{R} \to \mathbb{R}$ that are Lipschitz continuous with Lipschitz constant γ. The function graph φ is defined as $(y, \varphi(y))$ with $y \in \mathbb{R}$. We show that the map Π acts on the space C_γ.

Lemma A.5. *$\Pi(\text{graph}\,\varphi) = \text{graph}\,\varphi'$ for some $\varphi' \in C_\gamma(\mathbb{R})$.*

Proof. Let $\Pi(\text{graph}\,\varphi) = (G(y, \varphi(y), \varepsilon), F(y, \varphi(y), \varepsilon)) = (y', z')$. Suppose that the image of graph φ is not a graph, i.e. there exist $y_1 \neq y_2$ such that $G(y_1, \varphi(y_1), \varepsilon) = G(y_2, \varphi(y_2), \varepsilon)$. But

$$\begin{aligned}|G(y_1, \varphi(y_1), \varepsilon) - G(y_2, \varphi(y_2), \varepsilon)| &\geqslant |G(y_1, \varphi(y_1), \varepsilon) - G(y_2, \varphi(y_1), \varepsilon)| \\ &\quad - |G(y_2, \varphi(y_2), \varepsilon) - G(y_2, \varphi(y_1), \varepsilon)| \\ &\geqslant \mu_1 |y_1 - y_2| - L_2 H_2(\varepsilon) |\varphi(y_1) - \varphi(y_2)| \\ &\geqslant (\mu_1 - \gamma L_2 H_2(\varepsilon)) |y_1 - y_2| > 0\end{aligned}$$

which contradicts the assumption. Hence $\Pi(\text{graph}\,\varphi) = \text{graph}\,\varphi'$.

Next we show that $|\varphi'(y_1') - \varphi'(y_2')| \leqslant \gamma |y_1' - y_2'|$. From the previous calculation

$$|y_1' - y_2'| \geqslant (\mu_1 - \gamma L_2 H_2(\varepsilon)) |y_1 - y_2|.$$

follows immediately.

Similarly we obtain

$$|\varphi'(y_1') - \varphi'(y_2')| \leqslant |F(y_1, \varphi(y_1), \varepsilon) - F(y_2, \varphi(y_1), \varepsilon)| + |F(y_2, \varphi(y_1), \varepsilon) - F(y_2, \varphi(y_2), \varepsilon)|$$

$$\leqslant H_1(\varepsilon)(L_3 + \gamma L_4)|y_1 - y_2|$$

$$\leqslant \gamma_1'(\varepsilon)|y_1' - y_2'|$$

which proves the assertion. □

This shows the 'invariance' of Π acting on C_γ and can be viewed as the nonlinear counterpart to Lemma A.1. Note that $\gamma_1'(\varepsilon)$ is an improved Lipschitz constant under the action of Π.

We denote by $G_\varphi : \mathbb{R} \to \mathbb{R}$ the y-coordinate of Π acting on graph φ, i.e.

$$\Pi(graph\ \varphi) = \Pi(y, \varphi(y)) =: (G_\varphi(y), \Pi_* \varphi(G_\varphi(y))). \tag{A.11}$$

We introduce a metric

$$d(\varphi_1, \varphi_2) := \max_{y \in \mathbb{R}} |\varphi_1(y) - \varphi_2(y)| \tag{A.12}$$

for $\varphi_1, \varphi_2 \in C_\gamma$. Because of ψ_1, ψ_2 are Lipschtitz continuous this is a well defined metric and $(C_\gamma, d(\cdot\,,\,\cdot))$ is a complete metric space.

Proposition A.1. *The action of Π on C_γ given by*

$$\Pi(graph\ \varphi) = graph(\Pi_* \varphi)$$

is a uniform contraction map with respect to the metric (A.12).

Proof. Let $\varphi_i' = \Pi_* \varphi_i$ for $i = 1, 2$. Definition (A.12) gives

$$d(\varphi_1', \varphi_2') = \max_{G_{\varphi_1}(y) \in \mathbb{R}} |\Pi_* \varphi_1(G_{\varphi_1}(y)) - \Pi_* \varphi_2(G_{\varphi_1}(y))|. \tag{A.13}$$

$$|\varphi_1'(G_{\varphi_1}(y)) - \varphi_2'(G_{\varphi_1}(y))| \leqslant |\varphi_1'(G_{\varphi_1}(y)) - \varphi_2'(G_{\varphi_2}(y))| + |\varphi_2'(G_{\varphi_2}(y)) - \varphi_2'(G_{\varphi_1}(y))|$$

$$\leqslant |F(y, \varphi_1(y), \varepsilon) - F(y, \varphi_2(y), \varepsilon)|$$

$$+\ \gamma_1'(\varepsilon)\,|G(y, \varphi_1(y), \varepsilon) - G(y, \varphi_2(y), \varepsilon)|$$

$$\leqslant (L_4 H_1(\varepsilon) + \gamma_1'(\varepsilon) L_2 H_2(\varepsilon))|\varphi_1(y) - \varphi_2(y)|$$

Hence

$$\Rightarrow d(\varphi_1', \varphi_2') = d(\Pi_* \varphi_1, \Pi_* \varphi_2) \leqslant (L_4 H_1(\varepsilon) + \gamma_1'(\varepsilon) L_2 H_2(\varepsilon))\, d(\varphi_1, \varphi_2) =: \vartheta_1(\varepsilon)\, d(\varphi_1, \varphi_2)$$

To obtain a contraction we need $\vartheta_1(\varepsilon) < 1$. It follows by $\lim_{\varepsilon \to 0} H_i = 0$ that there exists a sufficiently small $\varepsilon_0 \ll 1$ such that $\vartheta_1(\varepsilon) < \hat{\vartheta}_1 < 1$ for all $\varepsilon \in (0, \varepsilon_0)$. Hence we have a uniform contraction for $\varepsilon \in (0, \varepsilon_0)$. The assertion follows. □

The action of Π on the space C_γ given by $\Pi(graph\, \varphi) = graph\, (\Pi_* \varphi)$ is called the *graph transform* which is a uniform contraction with respect to the metric (A.12). The contraction mapping principle yields a unique fixpoint for this action of Π, hence an invariant (horizontal) *graph* Φ.

Remark A.6. The continuous invariant plane fields within the invariant cones H^γ shown in Lemma A.3 are the tangent sets of the invariant graph Φ. Thus, Φ is a C^1 function. Higher degrees of smoothness can be proved in the usual iterative way [12]. We have shown the first part of Theorem A.1.

We straighten the invariant manifold graph Φ to the y-axis by the coordinate transformation $z = \bar{z} + \Phi(y, \varepsilon)$. We omit a new notation for the map Π_ε (A.1) and for the Lipschitz constants. Just keep in mind that $F(y, 0, \varepsilon) = 0$.

A.3. Invariant foliation

Next we prove the existence of an (vertical) invariant foliation for the (horizontal) invariant manifold. Let graph ψ be defined as $(\psi(z), z)$ with $z \in \mathbb{R}$. First we show that the map Π^{-1} acts on the space C_γ.

Lemma A.7. *$\Pi^{-1}(graph\, \psi) = graph\, \psi'$ for some $\psi' \in C_\gamma$.*

Proof. Let $\Pi^{-1}(graph\, \psi) = (g(\psi(z), z, \varepsilon), f(\psi(z), z, \varepsilon)) = (y', z')$. Suppose that the image of graph ψ is not a graph, i.e. there exist $z_1 \neq z_2$ such that $f(\psi(z_1), z_1, \varepsilon) = f(\psi(z_2), z_2, \varepsilon)$. But

$$|f(\psi(z_1), z_1, \varepsilon) - f(\psi(z_2), z_2, \varepsilon)| \geqslant L_5^{-1} L_6^{-1} H_1(\varepsilon)^{-1} (L_5 \mu_1 - \gamma L_3 L_6 H_1(\varepsilon)) |z_1 - z_2| > 0$$

which contradicts the assumption. Hence $\Pi^{-1}(graph\, \psi) = graph\, \psi'$.

Next we show that $|\psi'(z_1') - \psi'(z_2')| \leqslant \gamma |z_1' - z_2'|$. From the previous calculation follows immediately

$$|z_1' - z_2'| \geqslant L_5^{-1} L_6^{-1} H_1(\varepsilon)^{-1} (L_5 \mu_1 - \gamma L_3 L_6 H_1(\varepsilon)) |z_1 - z_2|.$$

Similarly we obtain

$$\begin{aligned}|\psi'(z_1') - \psi'(z_2')| &\leqslant L_5^{-1} H_1(\varepsilon)^{-1}(\gamma L_4 H_1(\varepsilon) + L_2 H_2(\varepsilon))|z_1 - z_2| \\ &\leqslant \gamma_2'(\varepsilon)|z_1' - z_2'|\end{aligned}$$

which proves the assertion. Note that $\gamma_2'(\varepsilon)$ is an improved Lipschitz constant under the action of Π^{-1}. $\square$

We denote by $f_\psi : \mathbb{R} \to \mathbb{R}$ the z-coordinate of Π^{-1} acting on graph ψ, i.e.

$$\Pi^{-1}(graph\,\psi) = \Pi^{-1}(\psi(z), z) =: (\Pi_*^{-1}\psi(f_\psi(z)), f_\psi(z)). \tag{A.14}$$

We cannot expect that this graph transform with respect to the metric (A.12) is a contraction, which would imply the existence of an invariant vertical manifold. But what we expect is an invariant foliation. We introduce $C_{\gamma,p} \subset C_\gamma$ such that

$$C_{\gamma,p} := \{\psi \in C_\gamma \mid \psi(0) = p\}.$$

Note that the functions $\psi \in C_{\gamma,p}$ are within the vertical cone V_p^γ with base point $(p, 0) \in \Phi$ and that $C_\gamma = \bigcup_{p \in \mathbb{R}} C_{\gamma,p}$. We introduce a metric

$$d(\psi_1, \psi_2)_p := \sup_{z \in \mathbb{R}/\{0\}} \frac{|\psi_1(z) - \psi_2(z)|}{|z|} \tag{A.15}$$

for $\psi_1, \psi_2 \in C_{\gamma,p}$. Because of ψ_1, ψ_2 are Lipschtitz continuous this is a well defined metric and $(C_{\gamma,p}, d(\cdot\,,\,\cdot)_p)$ is a complete metric space. This metric takes into account the Lipschitz constant at zero. From Lemma A.7 follows that Π^{-1} maps $\psi \in C_{\gamma,p}$ to $\psi' \in C_{\gamma,p'}$ with $p' = g(p, 0, \varepsilon)$. We introduce a map $\hat{\Pi}^{-1}$ acting on $C_{\gamma,p}$ by

$$\hat{\Pi}^{-1}(graph\,\psi) := \Pi^{-1}(graph\,\psi) - (p' - p, 0).$$

Proposition A.2. *The action of* $\hat{\Pi}^{-1}$ *on* $C_{\gamma,p}$ *given by*

$$\hat{\Pi}^{-1}(graph\,\psi) = graph\,(\hat{\Pi}_*^{-1}\psi) = graph\,(\Pi_*^{-1}\psi) - (p' - p, 0)$$

is a uniform contraction map with respect to the metric (A.15) (*see* Fig. 12).

Proof. Let $\psi_i' = \hat{\Pi}_*^{-1}\psi_i$ for $i = 1, 2$. Definition (A.15) gives

$$d(\psi_1', \psi_2')_p = \sup_{f_{\psi_1}(z) \in \mathbb{R}/\{0\}} \frac{|(\Pi_*^{-1}\psi_1)(f_{\psi_1}(z)) - (\Pi_*^{-1}\psi_2)(f_{\psi_1}(y))|}{|f_{\psi_1}(z)|}. \tag{A.16}$$

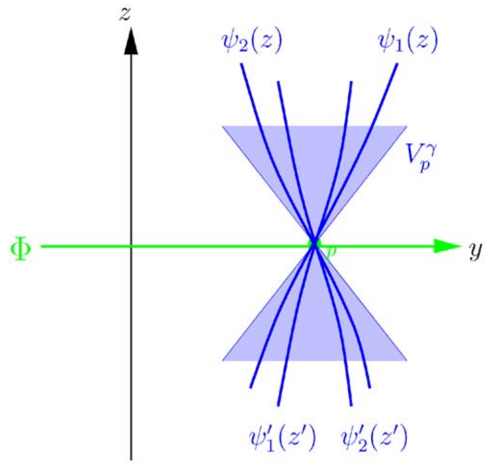

Fig. 12. Contraction of vertical graphs $\psi_i \in C_{\gamma,p}$.

$$
\begin{aligned}
|\psi_1'(f_{\psi_1}(z)) - \psi_2'(f_{\psi_1}(z))| &\leqslant |\psi_1'(f_{\psi_1}(z)) - \psi_2'(f_{\psi_2}(z))| + |\psi_2'(f_{\psi_2}(y)) - \psi_2'(f_{\psi_1}(z))| \\
&\leqslant |g(\psi_1(z), z, \varepsilon) - g(\psi_2(z), z, \varepsilon)| \\
&\quad + \gamma_2'(\varepsilon)|f(\psi_1(z), z, \varepsilon) - f(\psi_2(z), z, \varepsilon)| \\
&\leqslant L_5^{-1}(L_4 + \gamma_2'(\varepsilon)L_3)\,|\psi_1(z) - \psi_2(z)| =: \vartheta_2(\varepsilon)|\psi_1(z) - \psi_2(z)|.
\end{aligned}
$$

Note that $f(y, 0, \varepsilon) = 0$ because of the invariance of graph Φ. Thus

$$
\begin{aligned}
|f_{\psi_1}(z)| &= |f(\psi_1(z), z, \varepsilon) - f(\psi_1(0), 0, \varepsilon)| \\
&\geqslant |f(\psi_1(z), z, \varepsilon) - f(\psi_1(z), 0, \varepsilon)| - |f(\psi_1(z), 0, \varepsilon) - f(\psi_1(0), 0, \varepsilon)| \\
&\geqslant H_1^{-1}(\varepsilon)(L_1^{-1}L_4^{-1}\mu_1 - H_1(\varepsilon)L_5^{-1}L_3\gamma)\,|z| =: H_1^{-1}(\varepsilon)\vartheta_3(\varepsilon)\,|z|.
\end{aligned}
$$

Hence

$$
\Rightarrow d(\psi_1', \psi_2')_p = d(\hat{\Pi}_*^{-1}\psi_1, \hat{\Pi}_*^{-1}\psi_2)_p \leqslant H_1(\varepsilon)\frac{\vartheta_2(\varepsilon)}{\vartheta_3(\varepsilon)} d(\psi_1, \psi_2)_p =: \vartheta_4(\varepsilon)\, d(\psi_1, \psi_2)_p.
$$

It follows that there exists a sufficiently small $\varepsilon_0 \ll 1$ such that $\vartheta_4(\varepsilon) < \hat{\vartheta}_4 < 1$ for all $\varepsilon \in (0, \varepsilon_0)$. Hence the graph transform is a uniform contraction for $\varepsilon \in (0, \varepsilon_0)$. $\square$

We have obtained a unique invariant graph $\Psi \in C_{\gamma,p}$ for each base point $p \in I$. These invariant manifolds depend continuously on the base point p. Thus we have obtained a invariant foliation for the invariant manifold Φ.

Remark A.8. The continuous invariant plane fields within the invariant cones V^{γ} shown in Lemma A.3 are the tangent sets of the invariant graphs Ψ_p. Thus, the family Ψ_p are C^1 functions. Higher degrees of smoothness and smooth dependence of the fibers on their base points can be proved in the usual iterative way [12].

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