JOURNAL OF DIFFERENTIAL EQUATIONS **155**, 153–176 (1999)
Article ID jdeq.1998.3580, available online at http://www.idealibrary.com on IDEAL®

Stable Vortex Solutions to the Ginzburg–Landau Equation with a Variable Coefficient in a Disk

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DEDICATED TO PROFESSOR KYUYA MASUDA ON THE OCCASION OF HIS 60TH BIRTHDAY

This paper deals with stable solutions with a single vortex to the Ginzburg–Landau equation having a variable coefficient subject to the Neumann boundary condition in a planar disk. The equation has a positive parameter, say λ, which will play an important role for the stability of the solution. We consider the equation with a radially symmetric coefficient in the disk and suppose that the coefficient is monotone increasing in a radial direction. Then the equation possesses a pair of solutions with a single vortex for large λ. Although these solutions for the constant coefficient are unstable, they can be stable for a suitable variable coefficient and large λ. The purpose of this article is to give a sufficient condition for the coefficient to allow those solutions being stable for any sufficiently large λ. As an application we show an example of the coefficient enjoying the condition, which has an arbitrarily small total variation.

1. INTRODUCTION

We are concerned with the following Ginzburg–Landau equation with a variable coefficient in a disk of $\mathbb{R}^2$ subject to Neumann boundary condition,

$$\begin{cases} a(x)^{-1}\operatorname{div}(a(x)\nabla\Phi)+\lambda(1-|\Phi|^2)\,\Phi=0, & x\in D:=\{|x|<1\},\\ \dfrac{\partial\Phi}{\partial\nu}=0, & x\in\partial D, \end{cases} \tag{1.1}$$

* This work was supported in part by the Ryukoku Joint Research Center for Science and Technology.

0022-0396/99 $30.00

where $a(x)$ is a positive smooth function, $\partial/\partial \nu$ denotes the outer normal derivative on the boundary $\partial D = \{|x| = 1\}$, and $\Phi(x)$ is a complex valued function, say $\Phi(x) = u(x) + iv(x)$. We always identify $\Phi(x)$ with the two-component real vector function $(u(x), v(x))$. Equation (1.1) is a simplified model to describe a superconducting phenomenon in a thin (superconducting) film with a variable thickness. Indeed $\Phi(x)$ is the order parameter describing a superconducting state and $a(x)$ denotes the variable thickness of the film with the bottom D. For the detail of the physical background of the model, one can refer to the introduction of the previous work [2] (see also [6] for the derivation of $a(x)^{-1} \operatorname{div}(a(x) \nabla \cdot)$ in a thin domain).

Equation (1.1) is the Euler–Lagrange equation for the energy functional

$$E(\Phi) := \int_D \left\{ |\nabla \Phi|^2 + \frac{\lambda}{2} (1 - |\Phi|^2)^2 \right\} a(x)\, dx. \tag{1.2}$$

We say that a solution of (1.1) is stable if it is a local minimizer of (1.2). On the other hand we may regard (1.1) as the stationary equation of the parabolic equation:

$$\begin{cases} \dfrac{\partial \Phi}{\partial t} = \dfrac{1}{a(x)} \operatorname{div}(a(x) \nabla \Phi) + \lambda (1 - |\Phi|^2)\, \Phi, & (x, t) \in D \times (0, \infty), \\ \dfrac{\partial \Phi}{\partial \nu} = 0, & (x, t) \in \partial D \times (0, \infty), \\ \Phi(x, 0) = \Phi_0(x), \end{cases} \tag{1.3}$$

where Φ_0 is chosen in an appropriate function space, for instance, $C^0(\bar{D}; \mathbb{C})$, where $\bar{D}$ denotes the closure of D. Then the solutions generate a smooth semiflow there. Thanks to the result in [15] the Lyapunov's stability for an equilibrium solution of (1.3) coincides with the above stability for the energy functional (1.2). Indeed the nonlinear term of (1.2) is real analytic (for details, see [15]).

In this paper we discuss the existence of a stable solution of (1.1) with a zero, which is called a "vortex"; henceforth we simply call such a solution a "vortex solution." Before stating the result, we observe some features of Eq. (1.1). As a specific aspect for the Neumann condition case, it is clearly verified that all the global minimizers of (1.2) are constant with modulus one, that is, they are of the form $\Phi = e^{ic}$ $(c \in \mathbb{R})$. Moreover the previous result of [8] revealed that any nonconstant solution is unstable when $a(x)$ is constant. Indeed, for the constant $a(x)$, there is no stable nonconstant solution to the Ginzburg–Landau equation in any convex domain with Neumann boundary condition.[1] We remark that nonconstant solution

[1] For the three- or higher-dimensional case, it was found that there are contractible domains allowing stable nonconstant solutions (see [3, 10]).

must have a zero, i.e., a vortex if the domain is simply connected (see [10]). Fortunately, by the previous work [2], there is a stable vortex solution for an appropriate choice of λ and $a(x)$. More precisely for sufficiently large but fixed λ there is a function $a(x)$ admitting a stable vortex solution. Then, corresponding to the size of λ, we have to make up $a(x)$ carefully so that the vortices can be trapped around prescribed points. Indeed the profile of $a(x)$ has a sharp layer around each vortex.

Here we assume that $a(x)$ is radially symmetric and monotone increasing in $|x|$, that is,

$$a=a(r), \qquad r=|x|, \qquad \text{and} \qquad a'(r)\geqslant 0 \quad (0\leqslant r\leqslant 1), \qquad \text{where} \quad '=d/dr.$$

Then (1.1) is written as

$$\frac{1}{ar}(ar\Phi_r)_r+\Phi_{\theta\theta}+\lambda(1-|\Phi|^2)\,\Phi=0 \qquad \text{in } D, \quad \Phi_{r\,|\,r=1}=0, \tag{1.4}$$

where $\Phi_r=\partial\Phi/\partial r$, $\Phi_{\theta\theta}=\partial^2\Phi/\partial\theta^2$. Under this condition, there is a solution in the form $\Phi=f(r)\,e^{i\theta}$ (or $f(r)\,e^{-i\theta}$) satisfying $f(0)=0$ for sufficiently large λ. In fact, putting it into (1.4) yields

$$f''+\frac{(ar)'}{ar}f'-\frac{1}{r^2}f+\lambda(1-f^2)\,f=0, \qquad r\in(0,1), \quad f(0)=0, \quad f'(1)=0. \tag{1.5}$$

It can be proved that a positive solution of (1.5) is uniquely determined and it satisfies $f'(r)>0$ $(0<r<1)$ (see Lemmas 2.1 and 3.3). Hence $\Phi=f(r)\,e^{i\theta}$, $f(r)\,e^{-i\theta}$ are vortex solutions (with vortex $x=(0,0)$).

Our main purpose here is to give a sufficient condition for $a(x)$ to allow that the above vortex solutions are stable for large λ (Theorem 2.2). Moreover, as an application, we show that even though the total variation of $a(r)$ is arbitrarily small, the vortex solutions can be stable for large λ when the variation is sufficiently localized around the vortex (see Corollary 2.3 and Remark 2.4). Note that the total variation of $a(x)$ in this case is just the difference, $a(1)-a(0)$, because of the monotonicity of the function.

Compared with the result in [2], one sees that not only the stable vortex solutions are explicitly given but the strong restriction of $a(x)$ for the previous work is certainly relaxed in this specific case.

We also consider a parametrized family of coefficients whose profiles are step-function-like; more precisely, those coefficients have the same total variation and are constant except for middle intervals, which can be shifted by the parameter. We show the solutions are stable when the interval is very close to the origin by applying the main theorem, while they are unstable when it is far from the origin (see Section 5).

We note our approach to prove the stability of the vortex solutions. We consider the linearized eigenvalue problem around the solutions (or the second variational of the energy functional). Using Fourier expansion, it turns out that the problem can be reduced to an eigenvalue problem of 2-component ordinary differential equations on an interval $0 \leqslant r \leqslant 1$ (see Subsections 3.2 and 3.3). Thereby the least eigenvalue of the problem determines the stability of the solutions. Moreover we obtain a nice formula for the least eigenvalue that is due to the nice work by Mironescu [14].[2] To investigate it, some asymptotic behavior of a pair of eigenfunctions as $\lambda \to \infty$ is required. A natural way to study the eigenvalues is to characterize the limit eigenvalue problem as $\lambda \to \infty$. In what follows, however, it doesn't work well in this case. To be clear, let $a(x) \equiv 1$. By the change of the variable $s = r\sqrt{\lambda}$ and taking $\lambda \to \infty$, the limit problem is converted to the one on the infinite interval $[0, \infty)$. Then $(f'_\infty(r), f_\infty(r)/r)$ formally gives a pair of eigenfunctions corresponding to zero eigenvalue, where f_∞ is the solution of (1.5) as $\lambda \to \infty$. Since $f_\infty(r) \to 1$ as $r \to \infty$, $f_\infty(r)/r$ doesn't belong to $L^2(0, \infty)$. Hence it is impossible to formulate the limit eigenvalue problem in the usual sense. (For a nonconstant $a(x)$ a similar problem would happen to the limit problem.)

Here we take another approach to investigate the stability for large λ. We characterize some qualitative property of the eigenfunctions for large λ rather than the limit as $\lambda \to \infty$ (see Lemma 3.4). To do it, we use an elaborate asymptotic behavior of the solution of (1.5) as $\lambda \to \infty$, which is presented in Lemma 3.3.

In the next section we state the main theorem and a corollary. Section 3 completes the proof except for Lemma 3.3, which will be proved in Section 4. In the final section we consider a parametrized family of the coefficients and discuss a stability change of the vortex solutions as the coefficient varies. It suggests a bifurcation problem that is left to future study. In the last part of the final section we discuss the condition of the monotonicity of $a(r)$ and how it plays a role in the stability of the vortex solution.

2. MAIN THEOREM

Let $a(r)$ be a C^2 function in $r \in [0, 1]$ satisfying

$$\begin{cases} a(r) > 0 & a'(r) \geqslant 0 \quad (0 \leqslant r \leqslant 1) \\ a(1) = 1, & a'(0) = a'(1) = 0. \end{cases} \tag{2.1}$$

[2] He used it to prove the stability of a single vortex solution for the Dirichlet boundary condition $\Phi_{|r=1} = e^{i\theta}$ (with the constant coefficient). See Remark 3.5.

LEMMA 2.1. *Assume the condition* (2.1). *Then there is a* $\lambda_* > 0$ *such that for each* $\lambda > \lambda_*$, *Eq.* (1.5) *has a unique positive solution* $f = f_\lambda(r)$. *Thus Eq.* (1.4) *has a pair of solutions*

$$\Phi = f_\lambda(r)\, e^{i\theta}, \qquad f_\lambda(r)\, e^{-i\theta} \tag{2.2}$$

for $\lambda > \lambda_*$.

Proof. In the case $a(r) \equiv 1$, the unique existence of the positive solution to (1.5) is known (for instance see [1]). Let $\hat{f}_\lambda$ be such a unique positive solution for $a \equiv 1$ and let $g \equiv 1$. We can easily check that g and $\hat{f}_\lambda$ are an upper and lower solutions to (1.5), respectively. Hence it guarantees the existence of a positive solution to (1.5). The uniqueness follows from the same argument as in the proof of Lemma 3.1 in [8]. ∎

To discuss the stability of Φ_λ, we further assume that there exists $0 \leqslant r_* < r_* \leqslant 1$ such that

$$\begin{cases} a'(r) \text{ has at most a finite number of zeros in } I_0 := [r_*, r^*], \\ a'(r) = 0 \qquad \text{in} \quad r \in [0, r_*] \cup [r^*, 1], \\ a''(r) \geqslant 0 \qquad \text{in a neighborhood of } r = r_*. \end{cases} \tag{2.3}$$

The following theorem is the main result of this article.

THEOREM 2.2. *In addition to the conditions* (2.1), (2.3), *if*

$$\int_{I_0} \frac{a'(r)}{r}\, dr > 1, \tag{2.4}$$

then there is a λ_0 $(> \lambda_*)$ *such that for* $\lambda > \lambda_0$ *the solutions* (2.2) *are stable.*

COROLLARY 2.3. *Under the conditions* (2.1), (2.3), *suppose that there is a* $\beta \in (0, 1]$ *such that*

$$\frac{a(\beta) - a(0)}{\beta} > 1. \tag{2.5}$$

Then the same assertion of Theorem 2.2 *is true.*

This corollary immediately follows from Theorem 2.2 and the fact

$$\int_{I_0} \frac{a'(r)}{r}\, dr = \int_0^1 \frac{a'(r)}{r}\, dr \geqslant \frac{1}{\beta} \int_0^\beta a'(r)\, dr.$$

Remark 2.4. The condition (2.5) implies that if the mean value in $[0, \beta]$ is larger than one, any smallness of the total variation of $a(r)$ doesn't matter with the vortex solutions to be stable for large λ. The following $a(r)$ is a simple case to enjoy the conditions (2.3) and (2.5).

$$a'(r) \geqslant 0, \qquad r \in (0, \beta), \qquad a(r) = 1, \qquad r \in [\beta, 1],$$
$$a''(r) \geqslant 0 \qquad \text{in a neighborhood of } r = 0, \tag{2.6}$$

and

$$\frac{1 - a(0)}{\beta} > 1.$$

3. PROOF OF THE MAIN THEOREM

3.1. *Decomposition of the Linearized Eigenvalue Problem*

First we note that Eq. (1.4) is invariant under the transformation

$$\Phi \mapsto \Phi e^{ic}$$

for an arbitrarily given real number c. Hence given a solution $\tilde{\Phi}(x)$, which is not identically zero, the set

$$\{\tilde{\Phi} e^{ic} : c \in \mathbb{R}\} \tag{3.1}$$

is a continuum of the solutions. The tangential direction of this continuum at $c = 0$ is given by $i\tilde{\Phi}$, thus the corresponding tangent space at $\tilde{\Phi}$ is

$$T(\tilde{\Phi}) = \{si\tilde{\Phi} : s \in \mathbb{R}\}.$$

Considering this fact, it suffices for the proof of Theorem 2.2 to show that there is a $\mu > 0$ such that

$$\frac{d^2}{ds^2} E(\Phi_\lambda + s\Psi)_{|\, s=0}$$
$$\geqslant \mu \int_D |\Psi|^2 \, a \, dx \qquad \text{for any} \quad \Psi \in H^1(D; \mathbb{C}), \quad \operatorname{Re} \int_D \Psi (i\Phi_\lambda)^* \, dx = 0, \tag{3.2}$$

where we put $\Phi_\lambda = f_\lambda e^{i\theta}$ or $f_\lambda e^{-i\theta}$ and * denotes the complex conjugate (recall that $\mathbb{C}$ is identified with $\mathbb{R}^2$). Indeed this implies that the second variational is positive for any nonzero Ψ orthogonal to $T(\Phi_\lambda)$ and that Φ_λ is a local minimizer (for instance, see Lemma 2.1 of [10]).

We only consider the case $\Phi_\lambda = f_\lambda e^{i\theta}$ since the other case is also treated literally in the same way. Substituting $\Phi = \Phi_\lambda + \Psi$ and putting $\Psi = \psi e^{i\theta}$ yield

$$F(\psi) := E(\Phi_\lambda + \Psi) - E(\Phi_\lambda)$$
$$= \int_D \left\{ |\nabla\psi|^2 + \frac{i}{r^2}\left(\psi \frac{\partial\psi^*}{\partial\theta} - \psi^* \frac{\partial\psi}{\partial\theta}\right) + \frac{|\psi|^2}{r^2} \right.$$
$$\left. -\lambda(1-f_\lambda^2)\,|\psi|^2 + \frac{\lambda}{2}(|\psi|^2 + 2f_\lambda \operatorname{Re}\psi)^2 \right\} a\,dx. \tag{3.3}$$

Using Fourier expansion

$$\psi = \sum_{n=-\infty}^{+\infty} \psi_n e^{in\theta}$$

we obtain

$$F(\psi) = 2\pi \sum_{n=-\infty}^{+\infty} \tilde{F}_n(\psi_n) + \frac{\lambda}{2}\int_D \{|\psi_n|^2 + 2f_\lambda \operatorname{Re}\psi\}^2\, a\,dx, \tag{3.4}$$

where

$$\tilde{F}_n(\psi_n) := \int_0^1 \left\{ |\psi_n'|^2 + \frac{(n+1)^2}{r^2}\,|\psi_n|^2 - \lambda(1-f_\lambda^2)\,|\psi_n|^2 \right\} ar\,dr. \tag{3.5}$$

Because of

$$2\operatorname{Re}\psi = \sum_{n=-\infty}^{+\infty} (\psi_n e^{in\theta} + \psi_n^* e^{-in\theta})$$

we have

$$\int_0^{2\pi} (2\operatorname{Re}\psi)^2\, d\theta = 2\pi \sum_{n=-\infty}^{+\infty} 2\{\operatorname{Re}(\psi_n\psi_{-n}) + |\psi_n|^2\}.$$

Thus (3.4) can be written as

$$F(\psi) = 2\pi \sum_{n=-\infty}^{+\infty} \left\{ \tilde{F}_n(\psi_n) + \lambda \int_0^1 \{\operatorname{Re}(\psi_n\psi_{-n}) + |\psi_n|^2\}\, f_\lambda^2 ar\,dr \right\}$$
$$+ \frac{\lambda}{2}\int_D (|\psi|^4 + 4f_\lambda \operatorname{Re}\psi\,|\psi|^2)\, a\,dx. \tag{3.6}$$

To verify the inequality (3.2), we can drop the higher order terms than the quadratic ones of (3.6) and reduce the minimizing problem of the infinitely many decoupled energy functionals,

$$
\begin{aligned}
F_0(\psi_0) &:= \tilde{F}_0(\psi_0) + 2\lambda \int_0^1 (\mathrm{Re}\,\psi_0)^2 f_\lambda^2 ar\,dr,\\
&= \int_0^1 \left\{ |\psi_0'|^2 + \frac{1}{r^2} |\psi_0|^2 - \lambda(1 - f_\lambda{}^2)\,|\psi_0|^2 \right.\\
&\qquad \left. + 2\lambda(\mathrm{Re}\,\psi_0)^2 f_\lambda^2 \right\} ar\,dr,
\end{aligned}
$$

$$
\begin{aligned}
F_n(\psi_n, \psi_{-n}) &:= \tilde{F}_n(\psi_n) + \tilde{F}_{-n}(\psi_{-n}) + \lambda \int_0^1 \{(|\psi_n|^2 + |\psi_{-n}|^2)\\
&\qquad + 2\,\mathrm{Re}(\psi_n \psi_{-n})\} f_\lambda^2 ar\,dr,\\
&= \int_0^1 \left\{ |\psi_n'|^2 + |\psi'_{-n}|^2 + \frac{(n+1)^2}{r^2} |\psi_n|^2 + \frac{(n-1)^2}{r^2} |\psi_{-n}|^2 \right.\\
&\qquad - \lambda(1 - 2f_\lambda^2)(|\psi_n|^2 + |\psi_{-n}|^2)\\
&\qquad \left. + 2\,\mathrm{Re}(\psi_n \psi_{-n})\, f_\lambda^2 \right\} ar\,dr, \qquad n = 1, 2, ... \qquad (3.7)
\end{aligned}
$$

(note that $\mathrm{Re}(\psi_0^2) + |\psi_0|^2 = 2(\mathrm{Re}\,\psi_0)^2$). By virtue of the next lemma, however, it turns out that the functional F_1 determines the stability.

LEMMA 3.1. (i) *Given λ, there is a poitive number μ_0 such that*

$$
F_0(\varphi) \geqslant \mu_0 \int_0^1 |\varphi|^2 ar\,dr, \qquad \varphi \in H_r^1(0, 1), \quad \mathrm{Re} \int_0^1 \varphi(-if_\lambda)\, ar\,dr = 0,
$$

where

$$
\begin{aligned}
H_r^1(0, 1) := \Big\{ &\varphi \in L^2((0, 1); \mathbb{C}) : \varphi \text{ is differentiable in the distribution sense}\\
&\text{and} \int_0^1 (|\varphi|^2 + |\varphi'|^2)\, ar\,dr < \infty \Big\}.
\end{aligned}
$$

(ii) *Given $n \geqslant 2$,*

$$
F_n(\varphi, \phi) > F_1(\varphi, \phi) \qquad \text{for} \quad \varphi, \phi \in H_r^1(0, 1), \quad \varphi, \phi \not\equiv 0
$$

holds.

Proof. Since $(n+1)^2 > 4$ and $(n-1)^2 > 0$ for $n \geqslant 2$, the proof of (ii) is clear. We prove (i).

With the real form $\varphi = g + ih$ we can decouple F_0 as

$$F_0(\varphi) = F_{01}(g) + F_{02}(h),$$
$$F_{01}(g) := \int_0^1 \left\{ (g')^2 + \frac{1}{r^2} g^2 - \lambda(1 - 3f_\lambda^2)\, g^2 \right\} ar\, dr,$$
$$F_{02}(h) := \int_0^1 \left\{ (h')^2 + \frac{1}{r^2} h^2 - \lambda(1 - f_\lambda^2)\, h^2 \right\} ar\, dr.$$

From this decomposition it follows that the minimizing problem of F_0 is reduced to the decoupled eigenvalue problems

$$\begin{aligned} g'' + \frac{(ar)'}{ar} g' - \frac{1}{r^2} g + \lambda(1 - 3f_\lambda^2)\, g &= -\mu g, \\ h'' + \frac{(ar)'}{ar} h' - \frac{1}{r^2} h + \lambda(1 - f_\lambda^2)\, h &= -\mu h. \end{aligned} \tag{3.8}$$

Namely the minimum of F_{01} (resp. F_{02}) is the least eigenvalue μ of the first (resp. second) problem and a minimizer is attained by the corresponding eigenfunction.

We easily check that there is a zero eigenvalue and the corresponding eigenfunction is given by $(g, h) = (0, f_\lambda)$ (recall $\psi = i\Phi_\lambda$). Since $f_\lambda > 0$ in $(0, 1]$, the zero is the least eigenvalue of the second problem. Moreover $F_{01}(\varphi) > F_{02}(\varphi)$ for $\varphi \not\equiv 0$. Hence we obtain the assertion of the lemma. ∎

The next corollary immediately follows from the above lemma.

COROLLARY 2.6. *Suppose*

$$\inf \left\{ \frac{F_1(\varphi, \phi)}{|\varphi|_{L_r^2}^2 + |\phi|_{L_r^2}^2};\ (\varphi, \phi) \in (H_r^1((0, 1); \mathbb{R}))^2,\ (\varphi, \phi) \not\equiv (0, 0) \right\} > 0,$$

where

$$|\cdot|_{L_r^2} := \left\{ \int_0^1 |\cdot|^2\, ar\, dr \right\}^{1/2}.$$

Then (3.2) *holds.*

3.2. *Reduction of the Problem and a Key Lemma*

First we list some properties of the positive solution to (1.5), which will be necessary for the later argument.

LEMMA 3.3. *The solution f_λ to* (1.5) *satisfies the following*:

(i) $0 < f_\lambda(r) < 1$ *and* $f'_\lambda(r) > 0$, $r \in (0, 1)$.

(ii) $(f_\lambda(r)/r) > f'_\lambda(r)$, $r \in (0, 1)$.

(iii) *For an arbitrarily given and fixed* $\alpha > 0$ *there are* $\lambda_1 > 0$ *and* $C_1 > 0$ *such that for each* $\lambda > \lambda_1$

$$\|f_\lambda - 1\|_{C^1[\alpha, 1]} \leqslant \frac{C_1}{\lambda} \tag{3.9}$$

holds. Moreover

$$\lim_{\lambda \to \infty} \|f''\|_{C^0[\alpha, 1]} = 0, \tag{3.10}$$

thus

$$\lim_{\lambda \to \infty} \left\| -\frac{1}{r^2} + \lambda(1 - f_\lambda^2) \right\|_{C^0[\alpha, 1]} = 0. \tag{3.11}$$

The proof of Lemma 3.3 is stated in Section 4.

Next consider $F_1(\varphi, \phi)$. We rewrite F_1,

$$F_1(\varphi, \phi) = \int_0^1 \left\{ |\varphi'|^2 + |\phi'|^2 + \frac{4}{r^2} |\varphi|^2 - \lambda(1 - 2f_\lambda^2)(|\varphi|^2 + |\phi|^2) + 2\lambda f_\lambda^2 \operatorname{Re}(\varphi\phi) \right\} ar\, dr.$$

Putting $\varphi = g_1 + ih_1$, $\phi = g_2 + ih_2$, we write

$$F_1(\varphi, \phi) = \mathscr{E}(g_1, -g_2) + \mathscr{E}(h_1, h_2),$$

where

$$\mathscr{E}(v, w) := \int_0^1 \left\{ (v')^2 + (w')^2 + \frac{4}{r^2} v^2 - \lambda(1 - 2f_\lambda^2)(v^2 + w^2) - 2\lambda f_\lambda^2 vw \right\} ar\, dr. \tag{3.12}$$

Thus the problem is reduced to the minimizing problem of $\mathscr{E}(v, w)$. Using the change of variables

$$p = (w - v)/\sqrt{2}, \qquad q = (v + w)/\sqrt{2} \tag{3.13}$$

we obtain

$$\mathscr{E}(v, w) = \mathscr{F}(p, q) := \int_0^1 \Big\{ (p')^2 + (q')^2 + \frac{2}{r^2}(p-q)^2 - \lambda(1-f_\lambda^2)(p^2+q^2) + 2\lambda f_\lambda^2 p^2 \Big\}\, ar\, dr. \tag{3.14}$$

The corresponding eigenvalue problem to the energy functional $\mathscr{F}$ is

$$-\mathscr{L}\begin{pmatrix} p \\ q \end{pmatrix} = \mu \begin{pmatrix} p \\ q \end{pmatrix}$$
$$\mathrm{Dom}(\mathscr{L}) = \{(p, q) \in (H_r^2(0, 1); \mathbb{R})^2 : p'(1) = q'(1) = 0\}, \tag{3.15}$$

where

$$\mathscr{L}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p'' + \dfrac{(ar)'}{ar}\, p' - \dfrac{2}{r^2}(p-q) + \lambda(1-3f_\lambda^2)\, p \\ q'' + \dfrac{(ar)'}{ar}\, q' - \dfrac{2}{r^2}(q-p) + \lambda(1-f_\lambda^2)\, q \end{pmatrix}. \tag{3.16}$$

We use $(f'_\lambda, f_\lambda/r)$ as a test function to investigate the least eigenvalue of $-\mathscr{L}$. Indeed differentiating (1.5) with respect to r, we can check

$$-\mathscr{L}\begin{pmatrix} f'_\lambda \\ \dfrac{f_\lambda}{r} \end{pmatrix} = \begin{pmatrix} \left(\dfrac{a'}{a}\right)' f'_\lambda \\ \left(\dfrac{a'}{a}\right) \dfrac{f_\lambda}{r^2} \end{pmatrix}. \tag{3.17}$$

Multiplying $f'_\lambda ar$ and $(f_\lambda/r)\, ar$ with the first and the second components of (3.15) respectively and integrating from 0 to 1 by parts yield

$$\int_0^1 \left(\frac{a'}{a}\right)' f'_\lambda par\, dr + \int_0^1 \left(\frac{a'}{a}\right) \frac{f_\lambda}{r^2}\, qar\, dr + a(1)\, f''_\lambda(1)\, p(1) + a(1) \left(\frac{f_\lambda}{r}\right)'_{|r=1} q(1) = \mu \left\{ \int_0^1 f'_\lambda par\, dr + \int_0^1 \frac{f_\lambda}{r}\, qar\, dr \right\},$$

where we used (3.17). Hence we obtain

$$\mu = \frac{f''_\lambda(1)\, p(1) - f_\lambda(1)\, q(1) + \langle (a'/a)'\, f'_\lambda, p\rangle + \langle (a'/a)\, f_\lambda/r^2, q\rangle}{\langle f'_\lambda, p\rangle + \langle f_\lambda/r, q\rangle}$$

$$\langle v, w\rangle := \int_0^1 v(r)\, w(r)\, a(r)\, r\, dr. \tag{3.18}$$

The following lemma will play a key role to prove the positivity of μ in the next subsection.

LEMMA 3.4. *Let $(p(r), q(r))$ be an eigenfunction corresponding to the least eigenvalue of $-\mathscr{L}$.*

(i) *The eigenfunction satisfies*

$$q(r) > p(r) > 0 \qquad (\text{or } q(r) < p(r) < 0), \quad r \in (0, 1].$$

(ii) *Let μ be the least eigenvalue of $-\mathscr{L}$ and let μ^* be any number satisfying $\mu^* < 1$. Then arbitrarily given α, $0 < \alpha < 1$, there are positive numbers λ_2 and C such that for each $\lambda > \lambda_2$*

$$\left(\frac{C}{\lambda}\, p(r) + q(r)\right)' < 0, \qquad r \in (\alpha, 1),$$

provided that μ belongs to $(-\infty, \mu^*]$, where λ_2 and C can be chosen independently of μ.

Proof. (i) We prove the positivity of $p(r), q(r)$. Recall the eigenfunctions are C^2 in $[0, 1]$. We see that there is no point $r \in (0, 1]$ at which one of these eigenfunctions and its first derivative vanish simultaneously; indeed it contradicts the uniqueness of the initial value problem of ordinary differential equations. Considering

$$(p - q)^2 = p^2 - 2pq + q^2 \geqslant p^2 + q^2 - 2\, |p|\, |q|, \tag{3.19}$$

and that the eigenfunction $(p(r), q(r))$ attains the minimum of $\mathscr{F}$, we assert

$$\mathscr{F}(p, q) = \mathscr{F}(|p|, |q|).$$

This implies that $(|p|, |q|)$ must be also an eigenfunction corresponding to the least eigenvalue. A contradiction follows from the smoothness of the eigenfunction unless the both p, q retain the same sign in $(0, 1]$. Indeed if p or q changes its sign, the modulus of the function loses C^1 smoothness at the nodal point. By (3.19), we can also exclude the case that p and q have opposite signs each other. Thus the both p and q are chosen to be positive.

Next we show $q > p$. Apply the same argument just mentioned above for the positivity of p, q to that of $\mathscr{E}(v, w)$. Then the both minimizing functions v, w have the same sign in $(0, 1]$. By the definition (3.13) we obtain the desired result.

(ii) Given $0 < \alpha < 1$, consider a family of test functions $\{w(r; \xi)\}_{\alpha \leqslant \xi < 1}$ which are the normalized eigenfunction of the first eigenvalue of

$$L_\xi[w] := -(ar)^{-1}(arw')' + \frac{1}{r^2} w,$$
$$w \in \{w \in H^2((\xi, 1); \mathbb{R}) : w'(\xi) = w'(1) = 0\}. \tag{3.20}$$

Let σ_ξ be the first eigenvalue of L_ξ. Since

$$\int_\xi^1 \left(|w'|^2 + \frac{1}{r^2} w^2\right) ar\, dr \geqslant \int_\xi^1 w^2 ar\, dr$$

and $\lim_{\xi \to 1} \sigma_\xi = 1$ hold, we have

$$\inf_{\xi \in [\alpha, 1)} \sigma_\xi = 1. \tag{3.21}$$

Define

$$\langle u, v \rangle_\xi := \int_\xi^1 u(r)\, v(r)\, a(r)\, r\, dr,$$

and

$$B_\lambda(r) := -\frac{1}{r^2} + \lambda(1 - f_\lambda(r)^2). \tag{3.22}$$

Multiplying the first and second components of (3.16) by $w(\cdot; \xi)$ and integrating by parts yield

$$\xi a(\xi)\, w(\xi; \xi)\, p'(\xi) + \sigma_\xi \langle p, w(\cdot; \xi)\rangle_\xi - \langle B_\lambda p, w(\cdot; \xi)\rangle_\xi - \left\langle \frac{2}{r^2} q, w(\cdot; \xi)\right\rangle_\xi + 2\lambda \langle f_\lambda^2 p, w(\cdot; \xi)\rangle_\xi = \mu \langle p, w(\cdot; \xi)\rangle_\xi \tag{3.23}$$

$$\xi a(\xi)\, w(\xi; \xi)\, q'(\xi) + \sigma_\xi \langle q, w(\cdot; \xi)\rangle_\xi - \langle B_\lambda q, w(\cdot; \xi)\rangle_\xi - \left\langle \frac{2}{r^2} p, w(\cdot; \xi)\right\rangle_\xi = \mu \langle q, w(\cdot; \xi)\rangle_\xi. \tag{3.24}$$

The next inequality immediately follows from multiplying (3.23) by C/λ and adding it to (3.24):

$$\begin{aligned}\xi a(\xi)\, w(\xi;\xi)\{Cp'(\xi)/\lambda+q'(\xi)\} \\ &= -(\sigma_\xi-\mu)\langle Cp/\lambda+q, w(\cdot;\xi)\rangle_\xi+\langle B_\lambda\{Cp/\lambda+q\}, w(\cdot;\xi)\rangle_\xi \\ &\quad+2\langle(1/r^2-Cf_\lambda^2)\, p, w(\cdot;\xi)\rangle_\xi+\frac{2C}{\lambda}\langle(1/r^2)\, q, w(\cdot;\xi)\rangle_\xi. \qquad (3.25)\end{aligned}$$

Take

$$C>\frac{2}{\alpha^2}.$$

By virtue of Lemma 3.3 (iii), we may assume $1/2<f_\lambda$. Thus for each $r\in[\xi, 1]$,

$$\frac{1}{r^2}-Cf_\lambda(r)^2<0.$$

Using this fact, we can evaluate (3.25) as

$$\begin{aligned}\xi a(\xi)\, w(\xi;\xi)\{Cp'(\xi)/\lambda+q'(\xi)\} \\ &\leqslant -\{\sigma_\xi-\mu-\|B_\lambda\|_{C^0[\alpha,1]}\}\langle Cp/\lambda+q, w(\cdot;\xi)\rangle_\xi+\frac{2C}{\alpha^2\lambda}\langle q, w(\cdot;\xi)\rangle_\xi \\ &< -\{\sigma_\xi-\mu-\|B_\lambda\|_{C^0[\alpha,1]}-2C/(\alpha^2\lambda)\}\langle Cp/\lambda+q, w(\cdot,\xi)\rangle_\xi. \qquad (3.26)\end{aligned}$$

On the other hand from the assumption on μ and (3.21),

$$\sigma_\xi-\mu-\|B_\lambda\|_{C^0[\alpha,1]}-\frac{2C}{\alpha^2\lambda}\geqslant 1-\mu^*-\|B_\lambda\|_{C^0[\alpha,1]}-2C/\alpha^2\lambda.$$

By this inequality and Lemma 3.3 (iii), we conclude that the right hand side of (3.26) is negative for large λ. ∎

3.3. *The Proof of the Positivity of* F_1

We prove the positivity of the least eigenvalue μ of $-\mathscr{L}$. We show a contradiction under the assumption that there is a sequence $\{\lambda_j\}$, $\lim_{j\to\infty}\lambda_j=\infty$, such that for each $\lambda=\lambda_j$ the least eigenvalue is nonpositive, that is, $\mu \leqslant 0$. Indeed under this assumption, Lemma 3.4 (ii) tells that for any $\alpha > 0$ there is a j_0 such that for $\lambda = \lambda_j$, $j \geqslant j_0$

$$\left(\frac{C}{\lambda} p(r) + q(r) \right)' < 0, \qquad r \in [\alpha, 1] \tag{3.27}$$

holds. Then we will show the contradiction, $\mu > 0$, for sufficiently large j.

For simplicity of notations we don't specify the sequence $\{\lambda_j\}$ in the argument below, simply write λ, as long as there is no confusion.

Recall the conditions (2.3) and (2.4) again. Let $\{r_k\}_{k=0, \ldots, n} \subset I_0$ be the set of zeros of $a'(r)$ with increasing order, that is,

$$r_* = r_0 < r_1 < r_2 < \cdots < r_n = r^*.$$

Then there is a $\eta_0 > 0$ such that $2\eta_0 < r_{k+1} - r_k (k = 0, \ldots, n)$ and

$$\int_{I_0 \setminus U(\eta)} \frac{a'(r)}{r} \, dr > 1 \tag{3.28}$$

hold, where we put for $\eta \in (0, \eta_0)$

$$U(\eta) := \bigcup_{k=0}^{n} U_k(\eta), \qquad U_k(\eta) := \{ |r - r_k| < \eta \} \cap I_0.$$

Since $a'(r)$ vanishes nowhere in $I \setminus U(\eta)$, there is a $M_0 = M_0(\eta) > 0$ such that

$$\left\{ a''(r) - \frac{(a'(r))^2}{a(r)} \right\} r \geqslant -M_0 a'(r), \qquad r \in I_0 \setminus U(\eta). \tag{3.29}$$

We may also assume

$$a''(r) \geqslant 0, \quad r \in [r_0, r_0 + \eta], \qquad \text{and} \qquad \frac{a'(r)}{a(r_0)} = \frac{a'(r)}{a(0)} \leqslant 1, \quad r \in U(\eta). \tag{3.30}$$

Set

$$J_1 := (1 + C/\lambda) \int_{I_0} \frac{a'}{a} \frac{f}{r^2} qar \, dr, \qquad J_2 := (1 + C/\lambda) \int_{I_0} \left(\frac{a'}{a} \right)' f'par \, dr, \tag{3.31}$$

where we dropped the subscript λ of f_λ for simplicity of notation.

First evaluate J_1 from below. Recalling $q > p > 0$ in Lemma 3.4 (i), we obtain

$$J_1 \geqslant (1+\lambda/C) \int_{I_0 \setminus U(\eta)} a'(r)(f/r)\, q\, dr + (1+C/\lambda) \int_{U(\eta)} a'(r)(f/r)\, q\, dr$$

$$\geqslant \int_{I_0 \setminus U(\eta)} a'(r)(f/r)(Cp/\lambda + q)\, dr + (1+C/\lambda) \int_{U(\eta)} a'(r)(f/r)\, q\, dr. \qquad (3.32)$$

Next we carry it out for J_2. With the aid of (3.29) and (3.30),

$$J_2 = \int_{I_0 \setminus U(\eta)} \left\{ a'' - \frac{(a')^2}{a} \right\} r(1+C/\lambda)\, pf'\, dr$$

$$+ (1+C/\lambda) \int_{U(\eta)} \left\{ a'' - \frac{(a')^2}{a} \right\} rf'p\, dr$$

$$\geqslant -M_0 \int_{I_0 \setminus U(\eta)} (Cp/\lambda + q)\, a'f'\, dr$$

$$-(1+C/\lambda) \left\{ \int_{\bigcup_{k=1}^n U_k(\eta)} |a''|\, f'p\, dr + \int_{U(\eta)} a'f'p\, dr \right\}. \qquad (3.33)$$

Combining (3.32) with (3.33) and using Lemma 3.3 yield

$$J_1 + J_2 \geqslant \int_{I_0 \setminus U(\eta)} (f/r - M_0 f')(Cp/\lambda + q)\, a'\, dr$$

$$+ (1+C/\lambda) \int_{U(\eta)} (fq/r - f'p)\, a'\, dr$$

$$- \int_{\bigcup_{k=1}^n U_k(\eta)} |a''|\, f'(r)(Cp/\lambda + q)\, dr$$

$$\geqslant \int_{I_0 \setminus U(\eta)} (f/r - M_0 C_1/\lambda)(Cp/\lambda + q)\, a'\, dr$$

$$-(\kappa C_1/\lambda) \int_{\bigcup_{k=1}^n U_k(\eta)} (Cp/\lambda + q)\, dr, \qquad (3.34)$$

where C_1 is as in (3.9) and put

$$\kappa := \sup_{r \in I_0} |a''(r)|.$$

Moreover we used $fq/r - f'p > (f/r - f')\, p > 0$ in the above computation.

We evaluate the second term of the last inequality of (3.34). Using (3.27), we have

$$\int_{[r_k-\eta,\, r_k+\eta]\cap I_0} (Cp/\lambda+q)\, dr \leqslant Cp(r_k-\eta)/\lambda + q(r_k-\eta)$$

$$\leqslant \frac{1}{r_k - r_{k-1} - 2\eta} \int_{[r_{k-1}+\eta,\, r_k-\eta]} (Cp/\lambda+q)\, dr$$

for each k, $1 \leqslant k \leqslant n$, thus

$$\int_{\bigcup_{k=1}^n U_k(\eta)} (Cp/\lambda+q)\, dr \leqslant \frac{1}{\min_{1\leqslant k\leqslant n}\{r_k - r_{k-1} - 2\eta\}} \int_{I_0\setminus U(\eta)} (Cp/\lambda+q)\, dr.$$

We see from the above inequalities that

$$J_1 + J_2 \geqslant \int_{I_0\setminus U(\eta)} (f/r - M_1/\lambda)(Cp/\lambda+q)\, a'\, dr$$

holds, where

$$M_1 := M_0 C_1 + \frac{\kappa C_1}{\min_{1\leqslant k\leqslant n}\{r_k - r_{k-1} - 2\eta\}\ \min_{r\in I\setminus U(\eta)} a'(r)}.$$

Since $f/r - M_1/\lambda$ is positive on $[\eta, 1]$ for large λ, we obtain

$$J_1 + J_2 \geqslant \int_{I_0\setminus U(\eta)} \left\{\frac{a'}{r} - \frac{M_2}{\lambda}\right\} dr \left(\frac{C}{\lambda}\, p(1) + q(1)\right) \tag{3.35}$$

for $M_2 = C_1/\eta + M_1$ (recall $f \geqslant 1 - C_1/\lambda$). Here we used (3.27) again.

Finally we show

$$S := f''(1)\, p(1) - f(1)\, q(1) + (J_1+J_2)/(1+C/\lambda) > 0$$

which implies $\mu > 0$. Note that

$$f''(1)\, p(1) - f(1)\, q(1) \geqslant -|f''(1)|\, p(1) - (Cp(1)/\lambda + q(1)).$$

Therefore

$$S \geqslant \left[\left\{\int_{I_0\setminus U(\eta)} \left(\frac{a'}{r} - \frac{M_2}{\lambda}\right) dr\right\} (1+C/\lambda)^{-1} - (1+|f''(1)|)\right] (Cp(1)/\lambda + q(1)).$$

When $\lambda \to \infty$, the coefficient of $(Cp(1)/\lambda + q(1))$ tends to

$$\int_{I_0 \setminus U(\eta)} \frac{a'}{r}\, dr - 1.$$

This is positive by the condition (3.28); hence for sufficiently large λ the sign of S is positive. It, however, contradicts the first assumption $\mu \leqslant 0$ for any $\lambda = \lambda_j$ satisfying $\lim_{j \to \infty} \lambda_j = \infty$. This completes the proof of Theorem 2.2.

Remark 3.5. As seen in the preceding subsections, Fourier expansion is very useful for the studying of the stability. Indeed the underlying symmetry of the domain makes this approach effective. In a radially symmetric domain, such an approach was taken in various papers including [8, 13, 14]. On the other hand the idea to use the test function $(f'_\lambda, f_\lambda/r)$ for deriving the formula (3.2) is due to [14], where the stability of the single vortex solution with Dirichlet boundary data $\Phi_{|r=1} = e^{i\theta}$ for $a \equiv 1$ is discussed. It is a very clever approach to prove it. In the Neumann case with the variable coefficient, however, we needed to develop the study of the eigenvalue problem for the desired result.

4. PROOF OF LEMMA 3.3

In this section we prove Lemma 3.3.

Proof of (i). The former inequality has been already proved in Lemma 2.1. We show the latter positivity of f'_λ by following the argument found in [1], where it is proved for the case $a \equiv 1$. We will drop the subscription λ of f_λ for simplicity of notations throughout the proof of (i) and (ii). Let

$$z_0 := \inf\{r \in (0, 1] : f'(r) = 0\}.$$

Then $f'(z_0) = 0$ and

$$f''(z_0) = \{1/z_0^2 - \lambda(1 - f(z_0)^2)\}\ f(z_0) \leqslant 0.$$

We claim $f''(z_0) < 0$. If not, $f''(z_0) = 0$ implies

$$1/z_0^2 = \lambda(1 - f(z_0)^2).$$

Setting

$$U_\lambda(r) := 1/r^2 - \lambda(1 - f(r)^2), \tag{4.1}$$

we obtain

$$U'_\lambda(z_0) = \{-2/r^3 + 2\lambda f(r)\ f'(r)\}_{|r = z_0} = -2/z_0^3 < 0$$

and $U_\lambda(z_0)=0$. This implies that there is a $\delta>0$ such that

$$(ar)^{-1}(arf')'=U_\lambda(r)\,f(r)>0, \qquad r\in(z_0-\delta, z_0),$$

hence,

$$a(z_0-\delta)\cdot(z_0-\delta)\,f'(z_0-\delta)<a(z_0)\,z_0 f'(z_0)=0,$$

which contradicts the definition of z_0.

Next we show that there is no other zero of f' in $(0, 1]$. Let $z_1>z_0$ be the second zero of f'. Then

$$f''(z_1)=U_\lambda(z_1)\,f(z_1)\geqslant 0.$$

Considering the fact $f'<0$ in (z_0, z_1), we have

$$U_\lambda'(r)=-2/r^3+2\lambda f(r)\,f'(r)<0, \qquad r\in(z_0, z_1].$$

It, however, contradicts the signs of $U_\lambda(r)$ at $r=z_0, z_1$ (recall $U_\lambda(z_0)<0$). Therefore the interval $(0, 1]$ allows the only one zero of f' which must be 1.

Proof of (ii). When $a\equiv 1$, the inequality of (ii) is known (for instance, see [13]). We can also directly apply the same argument in [13] to the case $a'(r)\geqslant 0$. In fact rewrite (1.5) as

$$(rf'-f)'+\frac{1}{r}(rf'-f)=-\lambda(1-f^2)\,rf-\frac{a'}{a}rf'=:V(r).$$

The right hand side $V(r)$ is negative in $(0, 1)$ by the assumption of $a(r)$ and (i) of Lemma 3.3. Since $rf'(r)-f(r)=0$ at $r=0$, we have the expression

$$rf'-f=-\int_0^r (s/r)\,V(s)\,ds.$$

This yields (ii).

Proof of (iii). The estimate (3.11) immediately follows from (3.10). We prove (3.9) and (3.10) under the assumption

$$\|f_\lambda-1\|_{C^0[\alpha, 1]}<\frac{C_1}{\lambda}. \tag{4.2}$$

Then we show (4.2).

Under (4.2), $U_\lambda(r)$ defined in (4.1) is uniformly bounded on $[\alpha, 1]$ with respect to λ. The Schauder estimate tells that there is a $C_a>0$ such that

$$\|f_\lambda\|_{C^2[\alpha+\delta, 1]}<C_a$$

for arbitrarily chosen small $\delta > 0$. Differentiating Eq. (1.5) yields

$$(f'_\lambda)'' - 2\lambda(f'_\lambda) = W_\lambda(r) := -\{(1/r + a'/a)\, f'_\lambda\}' + (f_\lambda/r^2)' - 3\lambda(1 - f_\lambda^2)\, f'_\lambda. \tag{4.3}$$

We can evaluate W_λ as

$$\| W_\lambda \|_{C^0[\alpha+\delta,\, 1]} < C_b$$

for a constant $C_b > C_a > 0$. Let $\chi(\cdot) \in C^\infty(\mathbb{R})$ be a non-negative function satisfying

$$\chi(r) = \begin{cases} 0 & (r \leqslant \alpha) \\ 1 & (r \geqslant \alpha + \delta). \end{cases}$$

Then $g := \chi f'_\lambda$ satisfies

$$g'' - 2\lambda g = \chi W_\lambda + 2\chi' f''_\lambda + \chi'' f'_\lambda, \qquad g(\alpha) = g(1) = 0.$$

Since g attains a maximum at a point, say $r^* \in (\alpha, 1)$, we obtain

$$0 \leqslant \max_{\alpha \leqslant r \leqslant 1} g(r) \leqslant -\frac{\chi W_\lambda + 2\chi' f''_\lambda + \chi'' f'_\lambda}{2\lambda},$$

where we used $g''(r^*) \leqslant 0$. Hence there is a $C_d > 0$ such that

$$\max_{\alpha+\delta \leqslant r \leqslant 1} f'_\lambda(r) \leqslant \frac{C_d}{\lambda}. \tag{4.4}$$

A uniform estimate of $\| f'''_\lambda \|_{C^0[\alpha+\delta,\, 1]}$ in λ follows from (4.3) with the aid of (4.4). Thus the compactness argument tells

$$\| f''_\lambda \|_{C^0[\alpha+\delta,\, 1]} \to 0 \qquad \text{as} \quad \lambda \to \infty.$$

By virtue of the arbitrariness of α and δ, we can replace $\alpha + \delta$ by a new α to obtain the desired result.

Now we prove (4.2). Since the solution $\hat{f}_\lambda$ for $a \equiv 1$ is a lower solution to (1.5) (see the proof of Lemma 2.1), it suffices to prove the estimate (4.2) for $\hat{f}_\lambda$. Let us consider

$$g'' + \frac{1}{r}\, g' - \frac{1}{r^2}\, g + (1 - g^2)\, g = 0, \qquad r \in (0, \ell), \quad g(0) = g(\ell) = 0. \tag{4.5}$$

We denote a unique positive solution to (4.5) by g_ℓ. Note that there is a unique $\ell=\ell(\lambda)$ for which $g'_{\ell(\lambda)}(\sqrt{\lambda})=0$ (see [1]), thus $\hat{f}_\lambda(r)=g_{\ell(\lambda)}(\sqrt{\lambda}\, r)$. It suffices for (4.2) to prove that arbitrarily given η, $0<\eta\leqslant 1$,

$$1-g_{\ell(\lambda)}(\sqrt{\eta\lambda})<\frac{C_1}{\lambda} \tag{4.6}$$

holds for a constant C_1 independent of λ.

First we show (4.6) for $\eta=1$. Let us introduce a positive solution of the eigenvalue problem:

$$-\frac{1}{r}(rw')'+\frac{1}{r^2}w=\sigma w, \qquad r\in(0,\ell), \quad w(0)=w(\ell)=0. \tag{4.7}$$

Using the first order Bessel function $J_1(r)$, the normalized solution w_ℓ and the eigenvalue σ_ℓ are given by

$$w_\ell=\frac{J_1(j_1 r/\ell)}{J_1(k_1)}, \qquad \sigma_\ell=\frac{j_1^2}{\ell^2},$$

where j_1 and k_1 are the first zeros of J_1 and J_1', respectively. Note that w_ℓ attains the maximum 1 at $r=\ell k_1/j_1$. We let

$$\delta_0:=(1-\sigma_\ell)^{1/2}=(1-j_1^2/\ell^2)^{1/2}.$$

Then $\delta_0 w_\ell$ is a lower solution of g_ℓ, which follows from

$$\begin{aligned}\left\{\frac{1}{r}(rw_\ell')'-\frac{1}{r^2}w_\ell+(1-\delta_0^2 w_\ell^2)\,w_\ell\right\}\delta_0 &\\ =\{-\sigma_\ell+(1-\delta_0^2 w_\ell^2)\}\,\delta_0 w_\ell &\\ =(1-\sigma_\ell)(1-w_\ell^2)\,\delta_0 w_\ell>0.&\end{aligned}$$

Therefore

$$\delta_0 w_\ell(\ell k_1/j_1)=\delta_0<g_\ell(\ell k_1/j_1)\leqslant \max_{0\leqslant r\leqslant \ell} g_\ell(r).$$

This yields

$$\sqrt{1-j_1^2/\lambda}\leqslant\sqrt{1-\frac{j_1^2}{\ell(\lambda)^2}}<g_{\ell(\lambda)}(\sqrt{\lambda}).$$

Hence

$$1-g_{\ell(\lambda)}(\sqrt{\lambda})<\frac{C_1}{\lambda}$$

holds for a constant $C_1 > 0$. (The above argument for $\eta = 1$ was borrowed from [12].)

Finally we show (4.6) for any given $\eta \in (0, 1)$. When $\ell = \ell(\eta\lambda)$, the above argument still works for $g_\ell = g_{\ell(\eta\lambda)}$. It thereby turns out that

$$1 - g_{\ell(\eta\lambda)}(\sqrt{\eta\lambda}) < \frac{C_\eta}{\lambda}, \qquad C_\eta := C_1/\eta$$

holds. On the other hand by applying Lemma 4.1 in [1], we obtain

$$g_{\ell(\eta\lambda)}(r) < g_{\ell(\lambda)}(r) \qquad \text{for} \quad r \in (0, \ell(\eta\lambda)).$$

Put $C_1 = C_\eta$. Then combining the above inequalities leads us to (4.6). This completes the proof of (iii).

5. REMARK

Given $\eta, \delta \in (0, 1)$, consider a family of coefficients $\{a_\beta\}_{\eta \leqslant \beta \leqslant 1}$ such that

$$\begin{aligned} &a_\beta(r) = \begin{cases} 1 - \delta, & r \in [0, \beta - \eta] \\ 1, & r \in [\beta, 1] \end{cases} \\ &a'_\beta(r) > 0, \qquad r \in (\beta - \eta, \beta) \\ &a''(r) \geqslant 0 \qquad \text{in a neighborhood of } r = \beta - \eta, \end{aligned} \tag{5.1}$$

where it is assumed that a_β $(\eta \leqslant \beta \leqslant 1)$ have a fixed profile in the interval $[\beta - \eta, \beta]$. It is clear that these coefficients satisfy the assumptions (2.1), (2.3). Therefore, if $\delta/\beta > 1$, then the assertion of Theorem 2.2 holds.

On the other hand when δ is small and β is close to one, the solutions are unstable. This is done by evaluating $\mathscr{F}(p, q)$ of (3.14) with the test function $(p, q) = (f'_\lambda, f_\lambda/r)$ as

$$\begin{aligned} \mathscr{F}(f'_\lambda, f_\lambda/r) &= \int_0^1 [(f''_\lambda)^2 + \{(f_\lambda/r)'\}^2 + (2/r^2)(f'_\lambda - f_\lambda/r)^2 \\ &\quad - \lambda(1 - f_\lambda^2)\{(f'_\lambda)^2 + (f_\lambda/r)^2\} + 2\lambda f_\lambda^2(f'_\lambda)]\, ar\, dr \\ &= f''_\lambda(1)\, f'_\lambda(1) - f_\lambda(1)^2 + \langle (a'/a)'\, f'_\lambda, f'_\lambda \rangle + \langle (a'/a)\, f_\lambda/r^2, f_\lambda/r \rangle \\ &= -f_\lambda(1)^2 + \int_{\beta-\eta}^{\beta} \{a'' - (a')^2/a\}\, f'^2_\lambda r\, dr + \int_{\beta-\eta}^{\beta} (f_\lambda/r)^2\, a'\, dr, \end{aligned} \tag{5.2}$$

where we used the integration by parts and (3.17). Since

$$\sup_{r\in[\beta-\eta,\,\beta]} f_\lambda'(r) = O(1/\lambda), \qquad f_\lambda(1)\to 1 \quad (\lambda\to\infty)$$

and

$$\int_{\beta-\eta}^{\beta} \{a''-(a')^2/a\}\, f_\lambda'^2 r\, dr \leqslant (C_1/\lambda)^2 \int_{\beta-\eta}^{\beta} |a''|\, dr$$

$$\int_{\beta-\eta}^{\beta} (f_\lambda/r)^2\, a'\, dr \leqslant \frac{1}{(\beta-\eta)^2}\int_{\beta-\eta}^{\beta} a'\, dr \leqslant \frac{\delta}{(\beta-\eta)^2},$$

$\mathscr{F}(f_\lambda', f_\lambda/r)$ is negative for sufficiently large λ provided $\delta/(\beta-\eta)^2<1$. Take δ satisfying

$$\delta<(1-\delta/2)^2$$

and $\eta=\delta/2$. Then the solutions (2.2) are stable for $\beta=\delta/2$ while unstable for $\beta=1$. This suggests a bifurcation as β varies in $(\delta/2, 1)$, which will be a future problem.

We finally remark on the condition of the monotonicity of $a(r)$. We can construct the single vortex solution $\Phi=f_\lambda(r)\, e^{i\theta}$ (or $f_\lambda(r)\, e^{-i\theta}$) for sufficiently large $\lambda>0$ even if the monotonicity $a'(r)\geqslant 0$ is absent in Lemma 1.5. Indeed, let g be an eigenfunction with $0<g(r)<1$ $(r\in(0,1])$ to the first eigenvalue of the problem

$$g''+\frac{(ar)'}{ar}\,g'-\frac{1}{r^2}\,g=-\sigma g \qquad (0<r<1), \quad g(0)=g'(1)=0,$$

and, instead of $\hat{f}_\lambda$ in the proof of Lemma 1.5, use

$$f_*=\delta g(r), \qquad \delta:=(1-\sigma/\lambda)^{1/2}$$

as a lower solution for large λ. Then we obtain the solution. As for this vortex solution we may consider the stability or instability of it. Recall that Eq. (1.1) is a 2-dimensional simplified model to describe the superconducting phenomenon in a 3-dimensional thin film and that $a(x)$ represents the variable thickness of the thin film. By [8] every nonconstant solution is unstable for any convex domain. In our situation, $a''(r)\leqslant 0$, $r\in I:=[0,1]$, implies the convexity of the thin film. Since $a'(0)=0$, we have $a'(r)\leqslant 0$ in I under this condition. Observing $\mathscr{F}(f_\lambda', f_\lambda/r)$ as in (5.2), we see that it is negative in this case. This fact is consistent with the instability result by [8]. However, for general $a(r)$, whose second derivative changes the sign, there is no direct way to distinct the stability. Note that $a'(r) \geqslant 0$, $r \in I$ provided that $a''(r) \geqslant 0$ in I. Hence, in our conditions the nonnegativity of $a''(r)$ is relaxed.

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