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Gaston Darboux and the History of Complex Dynamics

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As documented in a recent paper by Hélène Gispert (*Archive for History of Exact Sciences* **28** (1983), 37–106), Gaston Darboux's calls in the mid-1870s for a more rigorous approach to analysis were largely ignored by the French mathematical community for quite some time. Gabriel Koenigs's 1884–1885 papers on the iteration of complex functions mark perhaps the earliest instance where Darboux directly influenced a French mathematician to develop a rigorous approach to his subject. © 1995 Academic Press, Inc.

Comme l'a fait remarquer Hélène Gispert dans un de ses articles (*Archive for History of Exact Sciences* **28** (1983), 37–106), Gaston Darboux propose une approche plus rigoureuse de l'analyze que la plupart des mathématiciens français ont négligé jusqu'aux années 1880. Les articles de Gabriel Koenigs (1884–1885) sur l'itération des fonctions analytiques, constituent peut-être la toute première occasion où Darboux a directement influencé un mathématicien français pour développer une approche rigoureuse à ce sujet. © 1995 Academic Press, Inc.

Wie neulich in einer Arbeit von Hélène Gispert (*Archive for History of Exact Sciences* **28** (1983), 37–106) dokumentiert wurde, hat die französische Mathematiker-Community die Aufforderungen Gaston Darbouxs Mitte der 1870er Jahren bezüglich strengeren Arbeitsmethoden in der Analysis meistens ignoriert. Die Arbeiten Gabriel Koenigs 1884–1885 über iterierte komplexe Funktionen stellen möglicherweise den ersten Fall dar, in dem ein französischer Mathematiker den Aufforderungen Darbouxs nachkam. © 1995 Academic Press, Inc.

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In her paper [10] Hélène Gispert describes how Gaston Darboux's mid-1870s pleas for a more rigorous approach to analysis found virtually no acceptance among French mathematicians until the latter half of the 1880s. Without disputing the substance of Gispert's claims, the purpose of this note is to document the direct influence Darboux had on a young French mathematician's attempts to develop a rigorous approach to his subject in 1884. The mathematician in question is Gabriel Koenigs, and his field was the iteration of complex functions, a subject which has come to be known as complex dynamics.

Although the bulk of Darboux's output concerned differential geometry, he wrote a series of three papers, [5; 6; 7] which are informed by the conviction that French analysis was not practiced with sufficient rigor. In these three papers Darboux artfully blends insightful counterexamples with carefully argued proofs to ground rigorously several fundamental concepts pertaining to the theory of real functions, including the term-by-term integration of series. A key element of Darboux's approach to the integration of series is the concept of uniform convergence, and in

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his paper [6] he proves several theorems regarding the uniform convergence of a series of single variable real-valued functions. Uniform convergence also animates many of the counterexamples Darboux provides. For example, one of the most important counterexamples in [6] is a continuous real function g which is nowhere differentiable. Darboux proves the continuity of g by noting that g is the sum of a uniformly convergent series of continuous functions.

Relying on a vast number of previously unpublished letters exchanged by Darboux and Jacques Hoüel between 1872 and 1883, Gispert describes how the chilly reception afforded Darboux's three papers by the Paris mathematical community influenced Darboux's decision to publish no further articles on foundational matters. It is evident from the correspondence Gispert includes that Hoüel, who was one of Darboux's principal collaborators in the editorship of the *Bulletin des sciences mathématiques et astronomiques*, thought Darboux's approach overly restrictive because many of the counterexamples Darboux offered, such as a nowhere differentiable continuous function, were in Hoüel's opinion the result of bizarre or absurd circumstance [10, 58]. In one particular exchange from early 1875 Darboux discussed the "abyss" between uniform convergence and regular convergence [10, 54] while Hoüel countered that he was blind to the existence of this abyss [10, 56]. Gispert suggests that Hoüel's attitude toward foundational issues typified that of his contemporaries in France [10, 58].

According to Gispert one must "wait until 1885 to find in France an echo of his [Darboux's] work regarding the foundations of analysis" [10, 63]. Although Hoüel's attitude may have been typical of French mathematicians, there were some echoes of Darboux's views before 1885. In 1883 Camille Jordan in his *Cours d'analyse* presented theorems on the term-by-term integration and differentiation of uniformly convergent series of real functions [12, 92–93] which are identical to theorems stated and proved by Darboux [6, 82–83]. Although it is not certain that Darboux influenced Jordan's approach, there was a second echo of Darboux's work before 1885 in which his influence is indisputable. In 1884, young Gabriel Koenigs (1858–1931) used results from Darboux's paper [6] to lay the foundations for his work on the iteration of complex functions. My intent is to document Darboux's influence on Koenigs and to indicate the historical significance this had for the development of complex dynamics, as the study of the iteration of complex functions is often called.

From 1879 to 1882 Koenigs studied at the École normale supérieure, where Darboux taught until 1881, and he received his doctorate from the University of Paris in 1882. Koenigs is often linked mathematically with Darboux and wrote his doctoral thesis, entitled "Sur les propriétés infinitésimales de éspace réglé," under Darboux's direction. Rene Taton, in fact, refers to Koenigs as a "disciple" of Darboux in his entry on Koenigs in the *Dictionary of Scientific Biography* [20, 446]. The relationship between Koenigs and Darboux evolved into a more collaborative one as time went on, as indicated by the fact that Darboux appended a lengthy note by Koenigs, entitled "Sur les géodésiques à intégrales quadratiques," to the fourth volume of his *Leçons sur la théorie générale des surfaces*.

After receiving his doctorate, Koenigs enjoyed a distinguished career as a mathematician. He lectured on mechanics at the University of Besançon from 1883 to 1885 and then served as professor of mathematical analysis at the University of Toulouse for a year, before returning to Paris in 1886 as an assistant professor at the École normale. In 1896 he became professor of mechanics at the Sorbonne. Koenigs was awarded several prizes by the French Academy of Sciences and was honored in 1918 with membership in the mechanics section of the Academy.

Koenigs's papers [14; 15] together compose the first rigorous study of the iteration of complex analytic functions. The 19th-century study of complex iteration concentrated on the examination of the point sets $\{f^n(z): n \in \mathbf{Z}, n > 0\}$ where z is near a fixed point x of f, that is, a point satisfying $f(x) = x$. This study is based in large measure on the fact that when $|f'(x)| < 1$ there exists a disc D surrounding the fixed point x such that for all z in D, $f^n(z)$ converges to x as n approaches infinity.

Koenigs's study grew out of a body of work written between 1870 and 1884 concerning Newton's method and its generalizations, on the one hand (Schröder [18; 19] and Cayley [4]), and the solution of certain functional equations, on the other (Abel [1], Korkine [16], and Farkas [8]). The commonality between these two strains resides in the fact that certain functional equations are very useful in the study of iteration. Schröder contended in [19] that iteration of complex functions would be best studied through the solution of functional equations. These functional equations are typified by the equation, commonly known as the Schröder functional equation,

$$B \circ f = mB, \tag{1}$$

where f is a given complex analytic function and m a given complex constant. If there exists an invertible function B satisfying (1) then, as Schröder pointed out, iteration of f reduces to repeated multiplication by m since (1) implies that

$$B \circ f^n = m^n B.$$

At the time of Koenigs's first paper on iteration in 1883 [13], the body of work just described did not present a rigorous, general approach to the study of complex iteration. Schröder in fact doubted that one could find a general method to solve functional equations of the sort he considered. Korkine attempted a general solution of the Abel equation $b \circ f = 1 + b$, also considered by Schröder, but his reasoning lacked rigor. Farkas's contributions, on the other hand, were fairly rigorous but not general. In his paper [14] he proved the existence of a convergent power series solution to an important special case of (1),

$$B \circ f = f'(x)B, \tag{2}$$

where x is a fixed point of f. However, Farkas's result applies only to functions f which satisfy a peculiar set of conditions dictated by his proof. Farkas's solution can be generalized, but he gave no indication that he was aware of this fact. Despite the flaws in Farkas's approach, it seems that he was wise to focus his attention on (2) since, beginning with the work of Koenigs, this particular version of the Schröder equation would play an important role in the study of the iteration of complex functions.

In contrast to the work of Farkas and Korkine, Koenigs's approach to complex iteration in his papers [14; 15] is both rigorous and general. At the center of Koenigs's study of iteration is his solution of (2). To solve it he required only that f be analytic in a neighborhood of a fixed point x with $|f'(x)| \neq 0,1$, which reduces the case where $|f'(x)| > 1$ to that of $0 < |f'(x)| < 1$. For such a function f Koenigs defined the function B as:

$$B(z) = \lim_{n\to\infty} \frac{f^n(z) - x}{(f'(x))^n}. \tag{3}$$

That B formally satisfies (2) is seen as follows:

$$\begin{aligned} B(f(z)) &= \lim_{n\to\infty} \frac{f^n(f(z)) - x}{(f'(x))^n} \\ &= \lim_{n\to\infty} \frac{f^{n+1}(z) - x}{(f'(x))^{n+1}} f'(x) \\ &= f'(x)B(z). \end{aligned}$$

Koenigs presented two proofs that the limit (3) converges on a disc D surrounding the fixed point x. The initial one, presented in his paper [13], which was published in Darboux's *Bulletin* in 1883, is characterized by a certain vagueness and lacks the rigor and clarity of the second proof, given one year later in his paper [14].

In both demonstrations the convergence of the limit (3) on D is reduced to the convergence of a series of analytic functions $\sum u_i$ to its limit function u. The series used in both demonstrations are virtually identical, except for a reindexing of the series in the latter proof which reflects an increased attention to rigor on the part of Koenigs. The convergence of $\sum u_i$ to u, however, is treated much more rigorously in the second proof than it is in the first, and therein lies the influence of Darboux, as will be shown below.

In the 1883 proof from [13] Koenigs heuristically showed that $\sum u_i$ converges absolutely to u on D, but he seemed to be unconcerned with whether B was analytic or not. Such flaws are entirely absent in the 1884 proof presented in [14]. In this latter proof he modified his previous argument in the light of Darboux's treatment of uniform convergence, showing that B is analytic on D via a rigorous argument based on the uniform, rather than absolute, convergence of $\sum u_i$ to u on D.

That Darboux directly influenced the change in Koenigs's approach to the convergence of the limit (3) is made clear by the following quotation from the introduction to Koenigs's 1884 paper [14]:

> The nature of the topic demands the use of the most general theorems from the theory of functions. I was principally inspired by the excellent memoir *Sur les fonctions discontinues* which Darboux published in Volume IV of the second series of *Annales de l'École normale.*
>
> An easy extension of the results from this memoir to complex quantities ... yields the following theorems, which serve as the base of my work [14, 4 (my translation)].

The two theorems which Koenigs then listed can be summarized as follows: Let the u_i be analytic in a region D. Then, if the infinite series $\sum u_i$ is uniformly convergent in D, its limit function u is continuous on D. If, in addition, $\sum u_i'$ converges uniformly in D then it converges to u', and u is therefore analytic in D. Although Koenigs did not seem to realize it, the application of the Cauchy integral formula leads to the stronger result that if a series of analytic functions $\sum u_i$ converges uniformly on D to u, then u is analytic on D and $\sum u_i'$ also converges uniformly to u' on D.[1] As Koenigs himself indicated, his theorems are routine extensions of theorems Darboux had proved for one-variable real functions in [6].

Koenigs's use of Darboux's theory of uniform convergence is of historical significance not only as evidence that at least one French mathematician heeded Darboux's calls for increased rigor in the first half of the 1880s, but also because Koenigs was the dominant figure in the study of complex iteration in the late 19th century. With help from Darboux's theory of uniform convergence, Koenigs put the study of complex iteration on a rigorous footing. He also guided the development of the study throughout the remainder of the 19th century. In the 1890s he helped direct the doctoral dissertations of two mathematicians, Auguste Grévy and Leopold Leau, each of whom made significant contributions to complex iteration with their respective examinations of two special cases Koenigs had not treated.[2] Koenigs and his successors established a fairly complete local theory of iteration which in turn laid the groundwork for the global studies on the iteration of rational complex functions by Pierre Fatou and Gaston Julia towards the end of World War I.[3]

Koenigs's work also surfaced in Paul Appell's paper [3], in which Appell used Koenigs's function B to find a complex analytic solution for a special case of the Hill differential equation $d^2u/dz^2 - uf = 0$. After invoking uniform convergence to demonstrate first the continuity and then the analyticity of a series of complex analytic functions, Appell noted that this use of uniform convergence was "analogous to that of Koenigs" [3, 286].

Koenigs's adaptation of Darboux's theory of uniform convergence to complex functions in 1884 is important, in light of Gispert's paper, for several reasons. It documents perhaps the first direct influence on a French mathematician of Darboux's ideas regarding the foundations of analysis. Moreover, Koenigs's use of uniform convergence provided an opportunity for Darboux's ideas to percolate

[1] Weierstrass first published a proof of the stronger result in 1880 in his paper [23]. Weierstrass also proved the result in a manuscript from 1841 which went unpublished until the 1890s, see [22]. That neither Koenigs nor Paul Appell, whose work is discussed below, realized that Koenigs's two theorems could be condensed into one suggests that even in the 1880s complex function theory had not yet become an integral part of analysis in France.

[2] The first special case is the one where $f'(x) = 0$, which Grévy studied in his thesis [11]; the second is the case $|f'(x)| = 1$ which Leau investigated in his thesis [17].

[3] Fatou's note [9] from 1906 was actually the first global study of iteration, but the only functions he explicitly treated were of the form $f(z) = z^n/(z^n + 2)$, where $n \geq 2$. Fatou, like Julia, needed the theory of normal families (articulated by Paul Montel in three papers between 1912 and 1917) to develop a general global theory of iteration. A detailed investigation of the historical background to the works of Julia and Fatou is given in my book [2].

down to other mathematicians, such as Appell. Finally, Koenigs's sensitivity toward the issues that Darboux raised in his three papers from the 1870s foreshadowed the generational shift in the French mathematical community toward the sort of views espoused by Darboux, a shift which gained momentum in the 1890s. Indeed, it was the successors of Koenigs in the theory of iteration, in particular Julia and Fatou, who would show that one of the objects of Darboux's investigation which received the most criticism, continuous functions without derivatives, occur naturally in the study of iteration.

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