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Lawson topology of the space of formal balls and the hyperbolic topology

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ABSTRACT

Let (X, d) be a metric space and $\mathbf{B}X = X \times \mathbb{R}$ denote the partially ordered set of (generalized) formal balls in X. We investigate the topological structures of $\mathbf{B}X$, in particular the relations between the Lawson topology and the product topology. We show that the Lawson topology coincides with the product topology if (X, d) is a totally bounded metric space, and show examples of spaces for which the two topologies do not coincide in the spaces of their formal balls. Then, we introduce a hyperbolic topology, which is a topology defined on a metric space other than the metric topology. We show that the hyperbolic topology and the metric topology coincide on X if and only if the Lawson topology and the product topology coincide on $\mathbf{B}X$.

1. Introduction

Let $\mathbb{R}_+$ denote the set of non-negative real numbers. From a metric space (X, d), we can construct a partially ordered set $(\mathbf{B}^+X, \sqsubseteq)$ where $\mathbf{B}^+X = X \times \mathbb{R}_+$ and $(x, r) \sqsubseteq (y, s)$ if $d(x, y) \leq r - s$. An element of $\mathbf{B}^+X$ is called a *formal ball* in (X, d). Formal balls were first introduced by Weihrauch and Schreiber to represent a metric space in a domain [15], and the poset of formal balls has been studied and used as an approximating structure of a metric space [1,2,7,10]. In this paper, we also consider formal balls with negative radii and study the partially ordered set $\mathbf{B}X = X \times \mathbb{R}$ of such generalized formal balls with the same order relation.

The sets $\mathbf{B}X$ and $\mathbf{B}^+X$ have the Lawson topology, which is a Hausdorff topology defined on a partially ordered set. Edalat and Heckmann [1] investigated further properties of $\mathbf{B}^+X$ as a computational model for (X, d) and showed that the set of maximal elements of $\mathbf{B}^+X$ with the relative Lawson topology is homeomorphic to X with the metric topology. Moreover, as we will show in Proposition 5, the relative Lawson topology on every slice $X \times \{t\} \subset \mathbf{B}X$ ($t \in \mathbb{R}$) is homeomorphic to the metric topology of X. Therefore, it is natural to ask whether the Lawson topology and the product topology coincide on $\mathbf{B}^+X$, and on $\mathbf{B}X$.

In the first half of this paper (Section 3), we give sufficient conditions for the two topologies to coincide and give examples of spaces for which the two topologies do not coincide both for the cases of $\mathbf{B}X$ and $\mathbf{B}^+X$.

In the second half (Section 4), we relate this problem on $\mathbf{B}X$ with a topological problem on X. We introduce the hyperbolic topology of a metric space (X, d), which is generated by those sets $\{y : d(a, y) - d(b, y) < s\}$ for $a, b \in X$ and $-d(a, b) < s$ and is in general different from the metric topology. We show that the hyperbolic topology and the metric topology coincide on X if and only if the Lawson topology and the product topology coincide on $\mathbf{B}X$.

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2. Preliminaries and notation

For each point x of a metric space (X, d) and each $r \in \mathbb{R}_+$ we denote the *r-open ball* of x by $S_r(x) = \{y \in X : d(x, y) < r\}$ and the *r-closed ball* in X by $B_r(x) = \{y \in X : d(x, y) \leq r\}$.

A non-empty subset D of a partially ordered set (abbrev. poset) $(L, \leq)$ is called *directed* if every finite subset of D has an upper bound. A poset L is called a *directed complete poset* (abbrev. *dcpo*) if every directed subset of L has a least upper bound.

Let $(L, \leq)$ be a poset and $x, y \in L$. We say that x is *way below* y and write $x \ll y$ if for every directed subset D of L for which $\sup D$ exists and $y \leq \sup D$, there exists $d \in D$ such that $x \leq d$. For a poset $(L, \leq)$, $x \in L$ and $A \subset L$ we use the following notation:

$\Uparrow x = \{y \in L : x \ll y\}$,
$\Downarrow x = \{y \in L : x \gg y\}$,
$\Uparrow A = \{y \in L : x \ll y$ for some $x \in A\}$, and
$\Downarrow A = \{y \in L : x \gg y$ for some $x \in A\}$.

Similarly, we write

$\uparrow x = \{y \in L : x \leq y\}$,
$\downarrow x = \{y \in L : x \geq y\}$,
$\uparrow A = \{y \in L : x \leq y$ for some $x \in A\}$, and
$\downarrow A = \{y \in L : x \geq y$ for some $x \in A\}$.

A poset L is said to be *continuous*, if for each $y \in L$ $\Downarrow y$ is directed and $y = \sup \Downarrow y$.

Let L be a poset and U a subset of L. Then U is said to be *Scott open* if $U = \uparrow U$ and, for every directed set D of L with $\sup D \in U$, there is $d \in D$ such that $d \in U$. The family $\sigma(L)$ of all Scott open sets of L is a topology of L and we say it is the *Scott topology*. It is well known that $\Uparrow x$ is a Scott open set for every $x \in L$ and $\{\Uparrow x : x \in L\}$ forms an open base for the Scott topology if L is continuous (cf. [6,11]).

We call the topology of a poset L generated by $\{L - \uparrow x : x \in L\}$ the *lower topology* and we denote it by $\omega(L)$. The join $\sigma(L) \vee \omega(L)$ of the Scott topology $\sigma(L)$ and the lower topology $\omega(L)$ is called the *Lawson topology*. The Lawson topology of L is denoted by $\lambda(L)$. If L is continuous, then the Lawson topology of L is generated by the sets $\Uparrow x$ and $L - \uparrow x$ for $x \in L$.

In this paper, we also deal with dual notions. The dual Scott topology $\sigma^{\mathrm{op}}(L)$ and the upper topology $\nu(L)$ are the Scott topology and the lower topology of the opposite order relation L^{op}, respectively. The dual Scott topology has the base $\{\Downarrow x : x \in L\}$ when L^{op} is continuous, and the upper topology is generated by $\{L - \downarrow x : x \in L\}$. The upper topology is weaker than the Scott topology, and the lower topology is weaker than the dual Scott topology.

We describe some auxiliary results about the poset $(\mathbf{B}^+X, \sqsubseteq)$ which are due to Edalat and Heckmann [1].

Lemma 1 ([1]). *Let (X, d) be a metric space and $(x, r), (y, s) \in \mathbf{B}^+X$. Then $(x, r) \ll (y, s)$ if and only if $d(x, y) < r - s$.*

Theorem 2 ([1]). *For a metric space (X, d) $\mathbf{B}^+X$ is a continuous poset. Furthermore, $\mathbf{B}^+X$ is a dcpo if and only if (X, d) is complete.*

Theorem 3 ([1]). *Let (X, d) be a metric space. Then $\mathbf{B}^+X$ is a computational model for (X, d), i.e., the relative Scott and the relative Lawson topologies on the set* Max $(\mathbf{B}^+X)$ *of all maximal elements of $\mathbf{B}^+X$ coincide and X is homeomorphic to* Max $(\mathbf{B}^+X)$.

Further, we refer the reader to [6] for domain theory and [4] for topology.

3. Lawson and product topologies in the space of formal balls

3.1. Lawson and product topologies on BX

For a metric space (X, d) we consider the poset $\mathbf{B}^+(X, d) = X \times \mathbb{R}_+$ of formal balls in X with the order relation

$$(x, r) \sqsubseteq (y, s) \quad \text{if } d(x, y) \leq r - s. \tag{1}$$

In this paper, we also deal with formal balls with negative radii, and therefore define the poset $\mathbf{B}(X, d) = X \times \mathbb{R}$ with the same order relation as (1). When the metric function is clear from the context, we simply denote $\mathbf{B}^+(X, d)$ and $\mathbf{B}(X, d)$ by $\mathbf{B}^+X$ and $\mathbf{B}X$, respectively.

Remark 4. Properties corresponding to Lemma 1 and the first part of Theorem 2 hold on $\mathbf{B}X$. However, the second part of Theorem 2 does not hold for $\mathbf{B}X$. Further, there are no maximal elements in $\mathbf{B}X$ and we cannot generalize Theorem 3 to $\mathbf{B}X$.

Instead, we have the following property.

Proposition 5. *Let (X, d) be a metric space. For each $t \in \mathbb{R}$, the relative Scott and the relative Lawson topologies on $X \times \{t\} \subset \mathbf{B}X$ coincide and X is homeomorphic to $X \times \{t\}$.*

Proof. From the definition, $(x, r) \sqsubseteq (y, s)$ if and only if $(x, r+t) \sqsubseteq (y, s+t)$. This means that the mapping $(x, r) \mapsto (x, r+t)$ from $\mathbf{B}X$ to $\mathbf{B}X$ is an order isomorphism and therefore $X \times \{t\}$ $(t \in \mathbb{R})$ are all homeomorphic with respect to both the relative Scott and the relative Lawson topologies. On the other hand, $\mathbf{B}^+X$ is a subspace of $\mathbf{B}X$ with respect to both topologies, and therefore, $X \times \{0\}$ is homeomorphic to the set of maximal elements of $\mathbf{B}^+X$, which is homeomorphic to X with respect to both topologies by Theorem 3. □

The sets $\mathbf{B}^+X$ and $\mathbf{B}X$ naturally have the product topology of the metric topology of X and the Euclidean topology of $\mathbb{R}_+$ and $\mathbb{R}$, respectively. On the other hand, they also have topologies defined through the order relation $\sqsubseteq$, that is, the Scott topology, the lower topology, and the Lawson topology. Among them, the Scott topology and the lower topology satisfy only the T_0-separation axiom. On the other hand, the Lawson topology of a continuous poset is always a Hausdorff topology. Considering Proposition 5, we have natural questions as to whether the Lawson topologies of $\mathbf{B}^+X$ and $\mathbf{B}X$ coincide with the product topologies of $X \times \mathbb{R}_+$ and $X \times \mathbb{R}$, respectively.

We first consider the case of $\mathbf{B}X$.

When $\Uparrow(y, s) \ni (x, r)$, we have $\Uparrow(y, s) \supset \Uparrow(x, s - d(x, y)) \ni (x, r)$. Therefore, for a point $(x, r) \in \mathbf{B}X$, sets of the form $\Uparrow(x, r + \varepsilon)$ for $\varepsilon > 0$ form a σ-neighbourhood base of (x, r), and sets of the form $\Uparrow(x, r + \varepsilon) - \cup_{j=1}^{m} \uparrow(y_j, s_j)$ where $\varepsilon > 0$ and $d(y_j, x) + r > s_j$ for $j \le m$ form a λ-neighbourhood base of (x, r).

We begin with a simple observation.

Lemma 6. *For a metric space (X, d), the product topology π coincides with the join $\sigma \vee \sigma^{\mathrm{op}}$ of the Scott topology σ and the dual Scott topology σ^{op} on $\mathbf{B}X$.*

Proof. First, we show that σ is weaker than π. It suffices to show that $\Uparrow(x, r) \in \pi$ for each $(x, r) \in \mathbf{B}X$. Let $(x, r) \in \mathbf{B}X$ and $(y, s) \in \Uparrow(x, r)$. Then it follows from Lemma 1 and Remark 4 that $d(x, y) < r - s$. Put $\delta = \frac{r-s-d(x,y)}{2} > 0$. Then $U = S_\delta(y) \times (s - \delta, s + \delta)$ is a π-neighbourhood of (y, s) such that $U \subset \Uparrow(x, r)$. Hence $\Uparrow(x, r) \in \pi$.

Next, we show that σ^{op} is weaker than π. The map f from $\mathbf{B}X$ to $\mathbf{B}X$ defined as $f(x, r) = (x, -r)$ is an order isomorphism from $(\mathbf{B}X, \sqsubseteq)$ to $(\mathbf{B}X, \sqsubseteq^{\mathrm{op}})$. It is also a homeomorphism from the product topology $(\mathbf{B}X, \pi)$ to itself. Therefore, when U is a σ^{op}-open set, $f^{-1}(U)$ is a σ-open set and therefore is a π-open set, and thus U is a π-open set.

To show that $\sigma \vee \sigma^{\mathrm{op}}$ is stronger than π, let $U = S_\varepsilon(x) \times (r - \varepsilon, r + \varepsilon)$ be a π-neighbourhood of (x, r). Then, for $V = \Uparrow(x, r + \varepsilon) \cap \Downarrow(x, r - \varepsilon)$, we have $(x, r) \in V \subset U$. □

Corollary 7. *For a metric space (X, d), the Lawson topology λ is weaker than the product topology π on $\mathbf{B}X$.*

Proof. Since the lower topology ω is weaker than σ^{op}, $\lambda = \sigma \vee \omega$ is weaker than $\pi = \sigma \vee \sigma^{\mathrm{op}}$. □

For the converse of Corollary 7, we show the following.

Proposition 8. *Let (X, d) be a metric space. If (X, d) is totally bounded, then*
(1) the upper topology ν and the Scott topology σ coincide on $\mathbf{B}X$, and
(2) the lower topology ω and the dual Scott topology σ^{op} coincide on $\mathbf{B}X$.

Proof. (1) Let $(x, r) \in \mathbf{B}X$ and $U = \Uparrow(x, r + \varepsilon)$ be its σ-neighbourhood. Since d is totally bounded, there are finitely many points $x_1, x_2, \dots, x_n$ of X such that $\cup_{i=1}^{n} S_{\varepsilon/4}(x_i) = X$. We put $V = \mathbf{B}X - \cup_{i=1}^{n} \downarrow(x_i, r - d(x, x_i) + \varepsilon/2)$. Then, since $d(x, x_i) > r - (r - d(x, x_i) + \varepsilon/2)$, we have $(x, r) \notin \downarrow(x_i, r - d(x, x_i) + \varepsilon/2)$ for every $i \le n$. Therefore, $(x, r) \in V$ and V is a ν-neighbourhood of (x, r).

Suppose that $(y, s) \in V$. There is x_i such that $d(x_i, y) < \varepsilon/4$. Since $(y, s) \notin \downarrow(x_i, r - d(x, x_i) + \varepsilon/2)$, $d(y, x_i) > s - (r - d(x, x_i) + \varepsilon/2)$. Therefore, $\varepsilon/4 > s - (r - d(x, x_i) + \varepsilon/2)$ and thus $r - s + 3\varepsilon/4 > d(x, x_i)$. Therefore, $r - s + \varepsilon > d(x, x_i) + \varepsilon/4 > d(x, x_i) + d(x_i, y) \ge d(x, y)$. This means $(y, s) \in \Uparrow(x, r + \varepsilon)$.

Thus, we proved $(x, r) \in V \subset U$ and therefore ν is stronger than σ. Since ν is weaker than σ in general, they coincide.

By definition, (2) means that the upper topology and the Scott topology coincide on $(\mathbf{B}X, \sqsubseteq^{\mathrm{op}})$. Since $f(x, r) = (x, -r)$ is an order isomorphism from $(\mathbf{B}X, \sqsubseteq)$ to $(\mathbf{B}X, \sqsubseteq^{\mathrm{op}})$, it is equivalent to (1). □

As a corollary, we have a sufficient condition for the Lawson topology and the product topology to coincide on $\mathbf{B}X$.

Corollary 9. *If (X, d) is a totally bounded metric space (in particular, (X, d) is a compact metric space), then the Lawson topology λ and the product topology π coincide on $\mathbf{B}X$.*

Proof. Since $\lambda = \sigma \vee \omega$ and $\pi = \sigma \vee \sigma^{\mathrm{op}}$, $\omega = \sigma^{\mathrm{op}}$ implies $\lambda = \pi$. □

We have thus proved that (X is totally bounded) $\Rightarrow$ ($\sigma = \nu$ and $\sigma^{\mathrm{op}} = \omega$ on $\mathbf{B}X$) $\Rightarrow$ ($\pi = \lambda$ on $\mathbf{B}X$). In the following two examples, we show that both of the implications are strict and the opposite implications do not hold.

Example 10. We give an example of a metric space X which is not totally bounded and the upper topology and the Scott topology coincide on $\mathbf{B}X$. Let $N_0 = \{a_0, a_1, a_2, \dots, b_0, b_1, b_2, \dots\}$ and d_0 be the following metric function on N_0.

$$d_0(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 2, & \text{if } (x, y) \text{ is } (a_n, b_n) \text{ or } (b_n, a_n), \\ 1, & \text{otherwise.} \end{cases}$$

Obviously, the metric space (N_0, d_0) is not totally bounded. For a Scott neighbourhood base $U = \Uparrow(a_n, r)$ of $(a_n, r - \varepsilon)$ where $\varepsilon > 0$, consider the set $V = \mathbf{B}X - \downarrow(b_n, r - 2)$. We have $(a_n, r - \varepsilon) \in V$. On the other hand, when $(y, s) \in V$, it is easy to show that $(y, s) \in U$. Therefore, $V \subset U$.

Example 11. For many spaces of formal balls, $\lambda = \pi$ holds but $\omega = \sigma^{\mathrm{op}}$ does not hold. For example, from Corollary 19 below, $\lambda = \pi$ and $\omega \ne \sigma^{\mathrm{op}}$ hold in $\mathbf{B}\mathbb{R}$.

We give an example of a metric space for which the Lawson and the product topologies do not coincide on the space of its formal balls.

Example 12. Let X_0 be an infinite set and fix a point $x_0 \in X_0$. Consider the following metric function d_s on X_0.

$$d_s(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x_0 \in \{x,y\} \text{ and } x \neq y, \\ 2, & \text{otherwise.} \end{cases}$$

In the poset $\mathbf{B}(X_0, d_s)$, if $\Uparrow(x, r)$ contains $(x_0, 1)$, then we have $1 < r - 1$ when $x \neq x_0$, and $0 < r - 1$ when $x = x_0$. Therefore, in both cases, $\Uparrow(x, r)$ also contains $(y, 0)$ for all $y \in X_0$. On the other hand, if $X - \uparrow(x, r)$ contains $(x_0, 1)$, it also contains $(y, 0)$ for all $y \neq x$. Therefore, with the Lawson topology, every open set U which contains $(x_0, 1)$ also contains $(y, 0)$ for infinitely many $y \in X_0$. Since the metric topology of (X_0, d_s) is the discrete topology, the Lawson topology is different from the product topology on $\mathbf{B}(X_0, d_s)$, and also on $\mathbf{B}^+(X_0, d_s)$.

If we use the following metric function d_f on X_0,

$$d_f(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } \varepsilon \neq y, \end{cases}$$

then the metric topology of (X_0, d_f) is the discrete topology and the Lawson topology of $\mathbf{B}(X_0, d_f)$ is the product topology. Therefore, we have the following.

Proposition 13. *The Lawson topology of $\mathbf{B}(X, d)$ is not determined by the metric topology of (X, d), and depends on the metric function d. This is also the case for $\mathbf{B}^+(X, d)$.*

3.2. Lawson and product topologies on $\mathbf{B}^+X$

When the Lawson topology and the product topology coincide on $\mathbf{B}X$, they also coincide on $\mathbf{B}^+X$ because $\mathbf{B}^+X$ is a subspace of $\mathbf{B}X$. However, the converse is not true, in general. We will first show that, when we consider $\mathbf{B}^+X$ instead of $\mathbf{B}X$, the two topologies coincide for a wider class of metric spaces.

Theorem 14. *Let (X, d) be a metric space. If for each bounded subset A of X the restriction of d on A is totally bounded, then the Lawson topology λ and the product topology π coincide on $\mathbf{B}^+X$.*

Proof. Let $(x, r) \in \mathbf{B}^+X$ and $\varepsilon > 0$.
Case 1. Let $r = 0$ and $U = S_\varepsilon(x) \times [0, \varepsilon)$ a π-neighbourhood of $(x, 0)$. One can easily show that $(x, 0) \in \Uparrow(x, \varepsilon) \subset U$.
Case 2. Let $r > 0$. Without loss of generality, we may assume that $\varepsilon < r$. Let $U = S_\varepsilon(x) \times (r - \varepsilon, r + \varepsilon)$ be a π-neighbourhood of (x, r). Since d is totally bounded on $B_{r+\varepsilon}(x)$, there are finitely many points $x_1, x_2, \ldots, x_n \in B_{r+\varepsilon}(x)$ such that $\cup_{i=1}^n S_{\varepsilon/4}(x_i) \supset B_{r+\varepsilon}(x)$. We put $V = \Uparrow(x, r + \varepsilon/4) - \cup_{i=1}^n \uparrow(x_i, r - \varepsilon/4)$. Then V is a λ-neighbourhood of (x, r). Let $(y, s) \in V$. Since $(y, s) \in \Uparrow(x, r + \varepsilon/4)$, we have

$$0 \le d(x, y) < r + \varepsilon/4 - s \ \ (< r + \varepsilon). \tag{2}$$

Thus $y \in B_{r+\varepsilon}(x)$ and hence there is x_i such that $d(x_i, y) < \varepsilon/4$. Since $(y, s) \notin \uparrow(x_i, r - \varepsilon/4)$, we have $\varepsilon/4 > d(x_i, y) > r - \varepsilon/4 - s$ and hence $s > r - \varepsilon/2$. On the other hand, by (2), it follows that $s < r + \varepsilon/4$. Thus, we have $|r - s| < \varepsilon/2 < \varepsilon$. It also follows from (2) that $d(x, y) < r + \varepsilon/4 - s < r + \varepsilon/4 - (r - \varepsilon/2) < \varepsilon$. Hence $(y, s) \in U$ and hence $V \subset U$. □

The metric space (X_0, d_s) in Example 12 does not satisfy the condition in Theorem 14 and the two topologies do not coincide on $\mathbf{B}^+(X_0, d_s)$. We will show, in the next example, a metric space $(\mathbb{N}, d_p)$ which satisfies the condition of Theorem 14 and thus the two topologies coincide on $\mathbf{B}^+(\mathbb{N}, d_p)$, but do not coincide on $\mathbf{B}(\mathbb{N}, d_p)$.

Example 15. Consider the following metric function d_p on the set of non-negative integers $\mathbb{N}$.

$$d_p(x,y) = \begin{cases} 0, & \text{if } x = y, \\ x + y, & \text{if } x \neq y. \end{cases}$$

Since $(\mathbb{N}, d_p)$ satisfies the condition of Theorem 14, the two topologies coincide on $\mathbf{B}^+(\mathbb{N}, d_p)$. In the poset $\mathbf{B}(\mathbb{N}, d_p)$, consider a λ-neighbourhood U of $(0, 0)$, of the form $\Uparrow(0, \varepsilon) - \cup_{j=1}^m \uparrow(y_j, s_j)$ for $\varepsilon > 0$ and $y_j > s_j$ $(j \le m)$. $\Uparrow(0, \varepsilon)$ contains $(x, -x)$ for all $x \in \mathbb{N}$. On the other hand, $\uparrow(y_j, s_j)$ does not contain $(x, -x)$ for $x \neq y_j$, because $d_p(x, y_j) = x + y_j > s_j + x$. Thus, U contains $(x, -x)$ for $x \notin \{y_1, \ldots y_m\}$. Since the metric topology of $(\mathbb{N}, d_p)$ is the discrete topology, the Lawson topology is different from the product topology on $\mathbf{B}(\mathbb{N}, d_p)$.

3.3. The space of formal balls of normed linear spaces

As we have noted, even when the two topologies coincide on $\mathbf{B}^+X$, they do not coincide on $\mathbf{B}X$, in general. Here, we show that they coincide when X is a normed linear space.

Proposition 16. *Let $(X, \|\cdot\|)$ be a normed linear space and d the metric induced by the norm $\|\cdot\|$. If the Lawson topology and the product topology coincide on $\mathbf{B}^+X$, the two topologies coincide also on $\mathbf{B}X$.*

Proof. Consider the point $(\mathbf{0}, 1) \in \mathbf{B}X$, where $\mathbf{0} \in X$ is the origin of $(X, \|\cdot\|)$. Let $U = S_\varepsilon(\mathbf{0}) \times (1-\varepsilon, 1+\varepsilon)$ be a π-neighbourhood of $(\mathbf{0}, 1)$, where $0 < \varepsilon < 1$. Since the Lawson topology and the product topology coincide on $\mathbf{B}^+X$, there is a λ-neighbourhood V of $\mathbf{B}X$ such that $V \cap \mathbf{B}^+X \subset U$. We can assume $V = \Uparrow(\mathbf{0}, 1+\delta) - \cup_{i=1}^n \uparrow(y_i, s_i)$ for $1 > \delta > 0$ and $(y_i, s_i) \in \mathbf{B}X$.

Suppose that $V \not\subset U$. There is a point $(x, r) \in \mathbf{B}X$ such that $(x, r) \in V$ and $(x, r) \notin U$. Since $V \cap \mathbf{B}^+X \subset U$, we have $r < 0$. Since $(x, r) \in V$, we have $d(x, \mathbf{0}) = \|x\| < 1 + \delta - r$ and $d(x, y_i) > s_i - r$. Since $(\mathbf{0}, 0) \in \Uparrow(\mathbf{0}, 1+\delta)$ and $(\mathbf{0}, 0) \notin V$, $(\mathbf{0}, 0) \in \uparrow(y_i, s_i)$ for some i and therefore $(\mathbf{0}, r) \in \uparrow(y_i, s_i)$ because $r < 0$. Thus, $(\mathbf{0}, r) \notin V$ and we have $x \neq \mathbf{0}$. Let $z = x + \frac{r}{\|x\|}x$ and consider the point $(z, 0) \in \mathbf{B}^+X$. We have $d(z, \mathbf{0}) = \|x\| + r < 1 + \delta$ and $d(z, y_i) \geq d(x, y_i) - d(z, x) > s_i - r - (-r) = s_i$. Therefore, $(z, 0) \in V$. Since $(z, 0) \notin U$, it contradicts with $(V \cap \mathbf{B}^+X) \subset U$.

Thus, we have proved that, in $\mathbf{B}X$, any π-neighbourhood of $(\mathbf{0}, 1)$ contains a λ-neighbourhood of $(\mathbf{0}, 1)$. Since the mapping $(x, r) \mapsto (x + z, r + t)$ for $z \in X$ and $t \in \mathbb{R}$ is an order isomorphism from $\mathbf{B}X$ to $\mathbf{B}X$, this mapping is a λ-homeomorphism on $\mathbf{B}X$, and at the same time, it is a π-homeomorphism on $\mathbf{B}X$. Therefore, for any point $q \in \mathbf{B}X$, any π-neighbourhood of q contains a λ-neighbourhood of q. □

Therefore, for normed linear spaces, the two topologies coincide for $\mathbf{B}X$ under the condition of Theorem 14. On the other hand, when $(X, \|\cdot\|)$ is a normed linear space, the condition of Theorem 14 is equivalent to X being finite-dimensional.

Lemma 17. *Let $(X, \|\cdot\|)$ be a normed linear space and d the metric induced by the norm $\|\cdot\|$. The restriction of d on A is totally bounded for each bounded subset A of X if and only if $(X, \|\cdot\|)$ is finite-dimensional.*

Proof. First, the restriction of d on A is totally bounded for each bounded subset A of X if and only if the unit ball $B = \{x : \|x\| \leq 1\}$ is totally bounded. It is known that if a normed linear space X is finite-dimensional, then B is compact and thus totally bounded. On the other hand, it is also known that if a normed linear space X is infinite-dimensional, then there is a linearly independent sequence (x_n) in B such that $\|x_n - x_m\| \geq 1$ when $n \neq m$. This means that X is not totally bounded. See, for example, [3,12] for these fundamental properties of normed linear spaces. □

Theorem 18. *If $(X, \|\cdot\|)$ is a finite-dimensional normed linear space, then the Lawson topology and the product topology coincide on $\mathbf{B}X$. This is also the case on $\mathbf{B}^+X$.*

Corollary 19. *Let $(\mathbb{R}^n, d)$ be the n-dimensional Euclidean space. Then the Lawson topology coincides with the Euclidean topology on $\mathbf{B}\mathbb{R}^n = \mathbb{R}^{n+1}$. This is also the case on $\mathbf{B}^+\mathbb{R}^n$.*

4. Hyperbolic topology of a metric space

4.1. Boundaries of subbasic open sets in BX

For each $(a, u) \in \mathbf{B}X$, we define $\mathrm{Bd}(a, u) = \{(y, s) \in \mathbf{B}X : d(a, y) = u - s\}$. For $A \subset \mathbf{B}X$, we denote by $\mathrm{Bd}_\lambda A$ and $\mathrm{Cl}_\lambda A$ the boundary and the closure of A with respect to the Lawson topology, respectively.

Proposition 20. *The sets $\Uparrow(a, u)$ and $\mathbf{B}X - \uparrow(a, u)$ are regular open in $(\mathbf{B}X, \lambda)$, and $\mathrm{Bd}(a, u)$ is the topological boundary between them.*

Proof. First, note that $\Uparrow(a, u)$, $\mathrm{Bd}(a, u)$, and $\mathbf{B}X - \uparrow(a, u)$ are pairwise disjoint and $\Uparrow(a, u) \cup \mathrm{Bd}(a, u) \cup (\mathbf{B}X - \uparrow(a, u)) = \mathbf{B}X$. Since $\Uparrow(a, u)$ and $\mathbf{B}X - \uparrow(a, u)$ are λ-open sets, we have $\mathrm{Bd}_\lambda \Uparrow(a, u) \subset \mathrm{Bd}(a, u)$ and $\mathrm{Bd}_\lambda(\mathbf{B}X - \uparrow(a, u)) \subset \mathrm{Bd}(a, u)$.

Let $(x, r) \in \mathrm{Bd}(a, u)$ and $U = \Uparrow(x, r+\varepsilon) - \cup_{j=1}^m \uparrow(y_j, s_j)$ be a λ-open neighbourhood of (x, r), where $\varepsilon > 0$ and $d(y_j, x) > s_j - r$ for $j \leq m$. To show $(x, r) \in \mathrm{Bd}_\lambda(\mathbf{B}X - \uparrow(a, u))$, let $t \in \mathbb{R}$ such that $r < t < r + \varepsilon$. Then $d(a, x) = u - r > u - t$. Hence $(x, t) \notin \uparrow(a, u)$. On the other hand, for each $j \leq m$ we have $d(x, y_j) > s_j - r > s_j - t$. Thus, $(x, t) \notin \uparrow(y_j, s_j)$. It is clear that $(x, t) \in \Uparrow(x, r+\varepsilon)$. Hence $(x, t) \in U \cap (\mathbf{B}X - \uparrow(a, u))$. This implies that $(x, r) \in \mathrm{Cl}_\lambda(\mathbf{B}X - \uparrow(a, u))$, and hence $(x, r) \in \mathrm{Bd}_\lambda(\mathbf{B}X - \uparrow(a, u))$.

To prove $(x, r) \in \mathrm{Bd}_\lambda \Uparrow(a, u)$, let $\delta = \min\{r + d(y_j, x) - s_j : j = 1, 2, \ldots, m\} > 0$. We take $t \in \mathbb{R}$ with $r - \delta < t < r$. Then for each $j \leq m$, we have $t > r - \delta \geq r - (r + d(y_j, x) - s_j) = s_j - d(y_j, x)$. Hence $d(y_j, x) > s_j - t$ and hence $(x, t) \notin \uparrow(y_j, s_j)$. On the other hand, since $d(x, x) = 0 < r + \varepsilon - t$, we have $(x, t) \in \Uparrow(x, r+\varepsilon)$. Hence $(x, t) \in U$. Since $d(a, x) = u - r < u - t$, we have $(x, t) \in \Uparrow(a, u)$. Thus $(x, t) \in U \cap \Uparrow(a, u)$. This implies that $(x, r) \in \mathrm{Cl}_\lambda \Uparrow(a, u)$, and hence $(x, r) \in \mathrm{Bd}_\lambda \Uparrow(a, u)$. □

Remark 21. The counterpart of Proposition 20 for $\mathbf{B}^+X$ does not hold in general. Let $\mathrm{Bd}^+(a, u) = \{(x, r) \in \mathbf{B}^+X : d(a, x) = u - r\} \subset \mathbf{B}^+X$. Then $\mathrm{Bd}^+_\lambda(\mathbf{B}^+X - \uparrow(a, u)) = \mathrm{Bd}^+(a, u)$ holds but $\mathrm{Bd}^+_\lambda \Uparrow(a, u) = \mathrm{Bd}^+(a, u)$ does not hold in general. Here, $\mathrm{Bd}^+_\lambda A$ is the boundary of A in $\mathbf{B}^+X$ with respect to the Lawson topology.

In fact, let (X, d) be a discrete metric space which has at least two points such that

$$d(x, y) = \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Fix a point $x_0 \in X$. Then $\mathrm{Bd}^+_\lambda \Uparrow(x_0, 1) = \{(x_0, 1)\}$. On the other hand, $\mathrm{Bd}^+(x_0, 1) = ((X - \{x_0\}) \times \{0\}) \cup \{(x_0, 1)\}$.

4.2. The hyperbolic topology of a metric space

For a metric space (X, d), we call the topology θ generated by those sets $\theta(a, b, s) = \{y : d(a, y) - d(b, y) < s\}$ for $a, b \in X$ and $-d(a, b) < s$ the *hyperbolic topology* of (X, d). Note that $\{y : d(x, y) - d(a, y) = s\}$ is a hyperbolic curve when (X, d) is the Euclidean plane $\mathbb{R}^2$ and $-d(a, b) < s < d(a, b)$. Therefore, $\theta(a, b, s)$ is one side of this "generalized hyperbolic curve" in a metric space X. Note also that the other side of this generalized hyperbolic curve is $\theta(b, a, -s)$.

Proposition 22. *θ is a Hausdorff topology.*

Proof. This is immediate because for every $a, b \in X$ with $a \neq b$, $\theta(a, b, 0) \ni a$, $\theta(b, a, 0) \ni b$, and $\theta(a, b, 0) \cap \theta(b, a, 0) = \emptyset$. □

A natural question is whether the hyperbolic topology θ and the metric topology μ coincide on X. First, since $\theta(a, b, s)$ for $a, b \in X$ and $-d(a, b) < s$ are open sets of the metric topology, we have the following.

Proposition 23. *μ is stronger than θ.*

We study more about the hyperbolic topology on X through its relation with the Lawson topology on $\mathbf{B}X$. First, note that the projection π_1 from $\mathbf{B}X$ onto X causes a bijection from $\mathrm{Bd}(a, u)$ onto X with the converse defined as $y \mapsto (y, u - d(a, y))$. Therefore, for each $a \in X$ and $u \in \mathbb{R}$, the relative Lawson topology on $\mathrm{Bd}(a, u)$ induces a topology on X. This topology does not depend on the choice of u, because, for $t \in \mathbb{R}$, the mapping $(x, r) \mapsto (x, r + t)$ from $\mathbf{B}X$ onto $\mathbf{B}X$ is an order isomorphism and $\mathrm{Bd}(a, u)$ is homeomorphically mapped onto $\mathrm{Bd}(a, u + t)$ by this mapping with respect to their relative Lawson topologies. We denote this topology on X by θ_a.

For a basic open set $\Uparrow(x, r) \cap \mathrm{Bd}(a, u)$ of $\mathrm{Bd}(a, u)$ and $(y, s) \in \mathrm{Bd}(a, u)$, $(y, s) \in \Uparrow(x, r)$ if and only if $d(y, x) < r - s$ if and only if $d(x, y) - d(a, y) < r - u$. In the same way, for $(y, s) \in \mathrm{Bd}(a, u)$, $(y, s) \in \mathbf{B}X - \uparrow(x, r)$ if and only if $d(x, y) - d(a, y) > r - u$. Therefore, by putting $t = r - u$, the topology θ_a is generated by the sets $\theta_{a,+}(b, t) = \{y : d(b, y) - d(a, y) < t\}$ for $b \in X$ and $-d(a, b) < t$ and $\theta_{a,-}(b, t) = \{y : d(b, y) - d(a, y) > t\}$ for $b \in X$ and $t < d(a, b)$. Therefore, we call θ_a the *hyperbolic topology with the pole a*. From this explanation, it is obvious that the hyperbolic topology θ is the join of θ_a for $a \in X$. We show that θ_a $(a \in X)$ are all identical and thus θ is equal to θ_a for every $a \in X$.

Lemma 24. *Let $a, b \in X$. The topologies θ_a and θ_b have the same neighbourhood system at b.*

Proof. We fix a real number $u \in \mathbb{R}$ and consider the relative Lawson topology on $\mathrm{Bd}(a, u)$. Let $v = u - d(a, b)$ and consider the point (b, v) on $\mathrm{Bd}(a, u)$, which corresponds to $b \in X$ through the bijection between X and $\mathrm{Bd}(a, u)$.

Since open sets of the form $U = \Uparrow(b, v + \varepsilon) - \cup_{j=1}^{m} \uparrow(y_j, s_j)$ with $\varepsilon > 0$ and $d(b, y_j) > s_j - v$ for $j \leq m$ constitute a neighbourhood base of (b, v), we consider a θ_a-neighbourhood $B = \pi_1(U \cap \mathrm{Bd}(a, u))$ of $b \in X$. We define $V = \mathbf{B}X - \cup_{j=1}^{m} \uparrow(y_j, s_j) - \uparrow(a, u - \varepsilon)$. V is a λ-neighbourhood of (b, v), and therefore $C = \pi_1(V \cap \mathrm{Bd}(b, v))$ is a θ_b-neighbourhood of b. We will show that $C \subset B$.

Suppose that $x \in C$, that is, $(x, r) \in V \cap \mathrm{Bd}(b, v)$ for $r = v - d(x, b)$. Then, $r = v - d(x, b) = u - d(a, b) - d(x, b) \leq u - d(a, x)$. Let $t = u - d(a, x)$. We have $r \leq t$ and $(x, t) \in \mathrm{Bd}(a, u)$. Since $(x, r) \notin \uparrow(y_j, s_j)$ and $r \leq t$, $(x, t) \notin \uparrow(y_j, s_j)$ holds. On the other hand, since $(x, r) \notin \uparrow(a, u - \varepsilon)$, we have $u - \varepsilon - d(a, x) < r$. Therefore, $t = u - d(a, x) < r + \varepsilon = v + \varepsilon - d(x, b)$. This implies that $(x, t) \in \Uparrow(b, v + \varepsilon)$. In this way, we have $(x, t) \in U$, and thus $x \in B$.

Next, suppose that a θ_b-neighbourhood C of $b \in X$ is given. That is, for a real number v, consider a point $(b, v) \in \mathbf{B}X$ and a λ-neighbourhood $V = \mathbf{B}X - \cup_{j=1}^{m} \uparrow(y_j, s_j)$ of (b, v) such that $\pi_1(V \cap \mathrm{Bd}(b, v)) = C$. Note that when $(x, r) \in \mathbf{B}X$ satisfies $\Uparrow(x, r) \ni (b, v)$, $\Uparrow(x, r) \supset \mathrm{Bd}(b, v)$ holds and therefore we only need to consider a λ-neighbourhood V of this form. We put $\delta = \min\{v + d(y_j, b) - s_j : j \leq m\} > 0$. Consider $\mathrm{Bd}(a, u)$ for $u = v - \delta/2 + d(a, b)$. Let $U = \Uparrow(b, v) - \cup_{j=1}^{m} \uparrow(y_j, s_j)$. Then U is a λ-neighbourhood of $(b, v - \delta/2)$, and we have $(b, v - \delta/2) \in \mathrm{Bd}(a, u)$. Therefore, $B = \pi_1(U \cap \mathrm{Bd}(a, u))$ is a θ_a-neighbourhood of $b \in X$. Then we have

$$
\begin{aligned}
B &= \pi_1(U \cap \mathrm{Bd}(a, u)) \\
&\subset \pi_1(U) \\
&= \pi_1(\Uparrow(b, v) - \cup_{j=1}^{m} \uparrow(y_j, s_j)) \\
&\subset \pi_1(\mathrm{Bd}(b, v) - \cup_{j=1}^{m} \uparrow(y_j, s_j)) \\
&= \pi_1(V \cap \mathrm{Bd}(b, v)) \\
&= C.
\end{aligned}
$$

This completes the proof. □

Theorem 25. (1) *All of the topologies θ_a $(a \in X)$ are identical.*
(2) *For each $a \in X$, θ_a is equal to θ.*

Proof. (1) Let $a, b \in X$. For each $x \in X$, if B is a θ_a-neighbourhood of x, it is also a θ_x-neighbourhood of x by Lemma 24, and therefore, it is a θ_b-neighbourhood of x again by Lemma 24.
(2) Obvious because θ is the join of θ_a for $a \in X$. □

Corollary 26. *In the hyperbolic topology θ, the sets $\theta_{a,-}(b,s)$ for $b \in X$ and $s < d(a,b)$ form a neighbourhood subbase of $a \in X$.*

This theorem can be stated on $(\mathbf{B}X, \lambda)$ as follows.

Corollary 27. *The topological boundaries* Bd q *($q \in \mathbf{B}X$) with the relative Lawson topology are all homeomorphic.*

4.3. The relation between the Lawson topology on **B***X and the hyperbolic topology on X*

We will denote by ϵ the Euclidean topology on $\mathbb{R}$. We are interested in the relation between the two topologies λ and $\theta \times \epsilon$ on $\mathbf{B}X$, where λ is the Lawson topology on $\mathbf{B}X$ and θ is the hyperbolic topology on X. They do not coincide when θ and μ are different, because the restriction of λ to $X \times \{t\}$ is the metric topology by Proposition 5. However, we have the following.

Theorem 28. *Let (X,d) be a metric space. For each point $(a,u) \in \mathbf{B}X$, the map f from $X \times \mathbb{R}$ $(= \mathbf{B}X)$ to $\mathbf{B}X$ defined as $f(x,r) = (x, r+u-d(x,a))$ is a homeomorphism from $(X \times \mathbb{R}, \theta \times \epsilon)$ to $(\mathbf{B}X, \lambda)$.*

Proof. The map f is obviously a bijection from $X \times \mathbb{R}$ onto $\mathbf{B}X$. We first prove that f is an open map. For a subbasic open set $U_1 = X \times \{s : s < r\}$ of $(X \times \mathbb{R}, \theta \times \epsilon)$, $f(U_1) = \{(x,t) : x \in X, t < u + r - d(x,a)\} = \Uparrow(a, u+r)$, which is an open set. In the same way, for $U_2 = X \times \{s : s > r\}$, $f(U_2) = \mathbf{B}X - \uparrow(a, u+r)$ is an open set. Therefore, we only need to consider the image of $B \times \mathbb{R}$ for a subbasic open set B of (X, θ). As a subbase of (X,θ), we consider the subbase of θ_a, that is, $\theta_{a,+}(b,s) = \{y : d(b,y) - d(a,y) < s\}$ for $b \in X$ and $-d(a,b) < s$ and $\theta_{a,-}(b,s) = \{y : d(b,y) - d(a,y) > s\}$ for $b \in X$ and $s < d(a,b)$. We have $f(\theta_{a,-}(b,s) \times \mathbb{R}) = \theta_{a,-}(b,s) \times \mathbb{R}$. We show that it is λ-open. Let $(y,t) \in \theta_{a,-}(b,s) \times \mathbb{R}$ and $U = \Uparrow(a, (d(b,y)+d(a,y)-s)/2+t) - \uparrow(b, (d(b,y)+d(a,y)+s)/2+t)$. Since $y \in \theta_{a,-}(b,s)$, we have $d(b,y)-d(a,y) > s$. Hence, $d(a,y) + t < (d(b,y)+d(a,y)-s)/2 + t$. This means $(y,t) \in \Uparrow(a, (d(b,y)+d(a,y)-s)/2+t)$. In the same way, we have $d(b,y) + t > (d(b,y)+d(a,y)+s)/2 + t$, which means $(y,t) \notin \uparrow(b, (d(b,y)+d(a,y)+s)/2+t)$. Therefore, we have $(y,t) \in U$. On the other hand, when $(x,r) \in U$, we have $d(a,x) + r < (d(b,x)+d(a,x)-s)/2 + r$ and $d(b,x) + r > (d(b,x)+d(a,x)+s)/2 + r$, and therefore, $d(b,x) - d(a,x) > s$. Hence, we have $(x,r) \in \theta_{a,-}(b,s) \times \mathbb{R}$, and we have $U \subset \theta_{a,-}(b,s) \times \mathbb{R}$. Thus, f is an open map.

Next, we show that f is continuous. Since f is a bijection, we only need to show that for any point (x,r) of $\mathbf{B}X$ and for any subbasic open neighbourhood V of (x,r), there exists a $\theta \times \epsilon$-open set U such that $(x,r) \in f(U) \subset V$.
Case 1. Consider the case $V = \Uparrow(x, r+\delta)$ for $\delta > 0$.
Case 1.1. If $x = a$, as we have seen, for a $\theta \times \epsilon$-open set $U' = X \times \{s : s < r + \delta - u\}$, we have $f(U') = V$.
Case 1.2. If $x \neq a$, we only need to consider the case $0 < \delta < d(a,x)$, because $\Uparrow(x, r+\delta) \supset \Uparrow(x, r+\delta')$ when $\delta > \delta'$. Let $B = \theta(x, a, -d(a,x) + \delta/2)$ and $U = B \times (r + d(a,x) - u - \delta/2, r + d(a,x) - u + \delta/2)$. Then $f(U) = \{(z,t) : z \in B$ and $r + d(a,x) - d(z,a) - \delta/2 < t < r + d(a,x) - d(z,a) + \delta/2\}$. We have $(x,r) \in f(U)$. On the other hand, since $\theta(x, a, -d(a,x) + \delta/2) = \{z : d(x,z) - d(a,z) < -d(a,x) + \delta/2\}$, we have $t < r + \delta - d(x,z)$ whenever $(z,t) \in f(U)$. Therefore, $f(U) \subset \Uparrow(x, r+\delta) = V$.
Case 2. Consider the case $V = \mathbf{B}X - \uparrow(y,s)$. Let $\delta = d(x,y) + r - s > 0$.
Case 2.1. If $d(x,a) = d(x,y) + d(y,a)$, consider the set $V' = \mathbf{B}X - \uparrow(a, s + d(a,y))$. As we have seen, $V' = f(X \times \{t : t > s + d(a,y) - u\})$. If $(z,t) \in V'$, we have $t + d(z,a) > s + d(a,y)$ and thus $t > s + d(a,y) - d(z,a) \geq s - d(y,z)$, which means $(z,t) \in V$. Therefore, $V' \subset V$. On the other hand, since $d(x,y) + r - s > 0$ and $d(x,a) = d(x,y) + d(y,a)$, we have $d(x,a) + r > s + d(y,a)$, which means $(x,r) \in V'$. Therefore, we have $(x,r) \in f(X \times \{t : t > s + d(a,y) - u\}) \subset V$.
Case 2.2. If $d(x,a) < d(x,y) + d(y,a)$, we only need to consider the case $d(x,y) + d(y,a) - d(x,a) > \delta > 0$. In fact, since $s = d(x,y) + r - \delta$, when δ decreases, s increases and therefore the set $V = \mathbf{B}X - \uparrow(y,s)$ decreases, and V includes the point (x,r) only when $\delta > 0$. Define $B = \theta(a, y, -d(x,y) + d(x,a) + \delta/2)$, and $U = B \times (r + d(a,x) - u - \delta/2, r + d(a,x) - u + \delta/2)$. Then $f(U) = \{(z,t) : z \in B$ and $r + d(a,x) - d(z,a) - \delta/2 < t < r + d(a,x) - d(z,a) + \delta/2\}$. Suppose that $(z,t) \in f(U)$. Since $z \in B$, we have $d(z,a) - d(z,y) < -d(x,y) + d(x,a) + \delta/2$. Therefore, $t > r + d(a,x) - d(z,a) - \delta/2 > r - \delta + d(x,y) - d(z,y) = r - (d(x,y) + r - s) + d(x,y) - d(z,y) = s - d(z,y)$, which means $(z,t) \in \mathbf{B}X - \uparrow(y,s) = V$. Therefore, $f(U) \subset V$. $\square$

Theorem 29. *For a metric space (X,d), the following are equivalent:*
(1) *The Lawson topology λ and the product topology π coincide on* **B***X.*
(2) *The hyperbolic topology θ and the metric topology μ coincide on X.*

Proof. It is immediate to prove (1) implies (2), because, for each $p \in \mathbf{B}X$, the restriction of π to Bd p induces the metric topology on X through the first projection.

For the converse, suppose that $(a,u) \in \mathbf{B}X$. First, note that the map $g : X \times \mathbb{R} \to X \times \mathbb{R}$ defined as $g(x,r) = (x, r - u + d(x,a))$ is a homeomorphism from $(X \times \mathbb{R}, \mu \times \epsilon) = (\mathbf{B}X, \pi)$ to itself. On the other hand, since θ and μ coincide on X, the map f in Theorem 28 is a homeomorphism from $(\mathbf{B}X, \pi)$ to $(\mathbf{B}X, \lambda)$. Therefore, $f \circ g$, which is the identity map on $\mathbf{B}X$, is a homeomorphism from $(\mathbf{B}X, \pi)$ to $(\mathbf{B}X, \lambda)$. $\square$

Example 30. Consider the metric space (X_0, d_s) in Example 12. The hyperbolic topology of (X_0, d_s) is generated by those sets $\{x\}$ $(x \in X_0 - \{x_0\})$ and $X_0 - A$, where A is a finite subset of X_0 which does not contain x_0. We observe that the hyperbolic topology of (X_0, d) is non-metrizable if X_0 is uncountable.

5. Concluding remarks

In this paper, we gave an example of a metric space for which the Lawson topology and the product topology do not coincide on the space of its formal balls. However, it is rather an artificial example and a natural question is whether they coincide when restricted to a more natural class of spaces, like normed linear spaces. As we have shown in Theorem 18, they coincide for finite-dimensional normed linear spaces. This problem for the infinite-dimensional case will be discussed in a forthcoming paper with examples of spaces for which the two topologies do or do not coincide.

As we have shown, both the metric topology (X, μ) and the hyperbolic topology (X, θ) are homeomorphic to certain subspaces of $(\mathbf{B}X, \lambda)$. This means that, within the order relation on $\mathbf{B}X$, not only the information of the metric topology of X, but also the information of the hyperbolic topology of X is encoded. In addition, the subset $\mathrm{Bd}\, p$ $(p \in \mathbf{B}X)$ is itself defined only from the order structure of $\mathbf{B}X$, and therefore we can obtain a homeomorphic copy of (X, θ) only from the order structure of $\mathbf{B}X$.

Since the Lawson topology $\lambda(\mathbf{B}X)$ is the join of the Scott topology $\sigma(\mathbf{B}X)$ and the lower topology $\omega(\mathbf{B}X)$, the restriction of $\lambda(\mathbf{B}X)$ to $\mathrm{Bd}(a, u)$ is also the join of these two topologies. We denote by $\theta_{a,+}$ and $\theta_{a,-}$ the topologies on X defined by the relative Scott topology and by the relative lower topology on $\mathrm{Bd}(a, u)$, respectively, through the bijection between X and $\mathrm{Bd}(a, u)$. The topology $\theta_{a,+}$ is generated by the sets $\theta_{a,+}(x, s)$ for $x \in X$ and $-d(a, x) < s$, and $\theta_{a,-}$ is generated by the sets $\theta_{a,-}(x, s)$ for $x \in X$ and $s < d(a, x)$. Therefore, for every $a \in X$ the hyperbolic topology θ is the join of the bitopological space [9] $(X, \theta_{a,+}, \theta_{a,-})$. This bitopological space coincides with the one induced from the bottomed partial metric of a based metric space (i.e., metric space with a base point), introduced by Steve Mathhews and Ralph Kopperman (personal communication). The authors think that partial metrics [13] would be an effective tool for investigating the structure of the hyperbolic topology.

As we showed in Proposition 20, for $p \in \mathbf{B}X$, $\Uparrow p$ and $\mathbf{B}X - \uparrow p$ are regular open sets which are exteriors of each other. However, when they are restricted to $\mathrm{Bd}\, q$ $(q \in \mathbf{B}X)$, this does not hold in general. Actually, in (X, θ), $\theta(a, b, s) = \{y : d(a, y) - d(b, y) < s\}$ and $\theta(b, a, -s) = \{y : d(a, y) - d(b, y) > s\}$ may not be exteriors of each other. For example, when X is the normed linear space $\mathbb{R}^2$ with the maximal norm $\|(x, y)\| = \max(|x|, |y|)$, the set $S = \{z : d((-1, 0), z) = d((1, 0), z)\}$ is equal to $\{(x, y) : y \geq \max(x + 1, -x + 1) \text{ or } y \leq \min(x - 1, -x - 1)\} \cup \{0\} \times [-1, 1]$, which is a closed set containing an open set. On the other hand, since $(\mathbb{R}^2, \|\cdot\|)$ is a finite-dimensional normed linear space, its hyperbolic topology θ is equal to the metric topology by Theorems 18 and 29. Therefore, $\mathrm{Bd}_\theta \theta((-1, 0), (1, 0), 0) = \mathrm{Bd}_\mu \theta((-1, 0), (1, 0), 0)$, which is not equal to S. In this way, "generalized hyperbolic curves" in metric spaces are not curves in general, but they may be "regions".

When X is a separable metric space, $\mathbf{B}X$ is an ω-continuous poset and when x ranges over a dense subset of X and r ranges over $\mathbb{Q}$, $\Uparrow(x, r)$ and $\mathbf{B}X - \uparrow(x, r)$ form a countable subbase of $\mathbf{B}X$ [1]. Since $\Uparrow(x, r)$ and $\mathbf{B}X - \uparrow(x, r)$ are regular open sets which are exteriors of each other, they form a dyadic subbase defined in [14]. However, as we mentioned above, this does not hold for (X, θ).

The current work started with the aim of showing that the space of formal balls is, in some sense, a very simple domain environment, and in order to evaluate the simplicity, we tried to calculate the dimension of the Lawson topology of $\mathbf{B}X$. When the Lawson topology and the product topology coincide on $\mathbf{B}X$ and X is n-dimensional, $\mathbf{B}X$ is $n + 1$-dimensional with the weak inductive dimension and other dimension functions. We refer the reader to [5], [8] for dimension theory. The investigation of the dimension of the Lawson topology of $\mathbf{B}X$ is left as an open problem.

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