Elastoplastic analysis of soil-structures strengthened by two-directional reinforcements using a multiphase approach

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Article history:
Received 5 June 2009
Received in revised form 15 October 2009
Available online 4 January 2010

Keywords:
Multiphase model
Membrane-reinforced structures
Elastoplasticity
Finite elements
Yield design

1. Introduction

One of the reinforcement techniques used to improve soil-structures stiffness and strength consists in incorporating into the soil mass a distribution of permeable membranes displaying better mechanical characteristics than those of the native soil. These reinforcements are frequently made of a high strength polymer such as geosynthetics or geogrids which display in general an anisotropic behavior along with a dissymmetric behavior in tension–compression since the compression strength of such a polymer materials is generally equal to zero.

Looking forward to designing an improved soil-structure by membranes such as steepened slopes, retaining walls or embankments over soft soils (for example, Michalowski, 1997; Skinner and Rowe, 2005), the direct simulation by means of a classical finite element or finite difference numerical tool appears to be non-feasible due to the negligible thickness of the reinforcing membranes compared to the distance separating two neighboring inclusions requiring a very fine meshing of the reinforced structure and leading to oversized numerical problems. This phenomenon is accentuated by the considerable difference between the stiffness and strength properties of the materials involved in such a composite.

As an alternative approach to direct numerical simulations, the classical periodic homogenization technique (see Suquet, 1985, for example) appears to be a good way to overcome such difficulties since the heterogeneous medium could be replaced by an equivalent anisotropic homogeneous one. As already pointed by de Buhan and Hassen (2008) for soil reinforced by linear inclusions and more recently by Ben Hassine et al. (2009) for soil-structures reinforced by membranes, the periodic homogenization fails to capture the interaction between the soil and the reinforcing inclusions whereas the multiphase model initially developed for unidirectional reinforcements (de Buhan and Sudret, 2000; Hassen and de Buhan, 2005; de Buhan and Hassen, 2008) and extended for two-dimensional inclusions (Ben Hassine et al., 2008) appears as a generalized homogenization procedure allowing to take into account a possible soil-membrane interface law.

The present paper is devoted to the extension of the multiphase model for soil-structures reinforced by flexible membranes to elastoplastic behavior of the different constituents, the equilibrium equations are presented in Section 2 and the elastoplastic behavior of the different constituents is introduced with the appropriate plastic flow-rule for each constituent and for the interaction prevailing between them. The corresponding stiffness and strength parameters of the model are then identified and the elastoplastic boundary value problem is described in Section 3. Section 4 concerns the development of a numerical procedure based on the classical return mapping algorithm for the elastoplastic analysis of two-phase systems. Application of the presented model and related numerical tool is then performed on an illustrative example of a membrane reinforced-earth curved retaining wall in Section 5, and the so-obtained results are favorably compared to yield design results (Ben Hassine et al., 2009). This study could be considered as a continuation of the works of the latter, which contributions is devoted to the elastic analysis (Ben Hassine et al., 2008) and more recently to failure analysis (Ben Hassine et al., 2009), using a yield design approach, of such a reinforced structures. The elastoplastic

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extension of the model and the developed numerical tool, validated by comparison to analytical solutions, is not limited to elasticity or failure of cylindrical-shaped structures since it could evaluate the macroscopic behavior of any two-dimensional reinforced earth-structure at any loading level.

2. Problem statement and principle of the multiphase approach

The problem under consideration is the one of a soil-structure of height \( H \), made of a purely cohesive soil (dry sand), reinforced by a regularly spaced horizontal membranes as shown in Fig. 1. The multiphase modeling of such a soil-structure consists in replacing the reinforced media by the superposition of two interacting phases, named the matrix and the reinforcement phases, respectively. The kinematics of the two-phase system is described by two different displacement fields \( z^m \) and \( z^r \) associated to the matrix and the reinforcement phases, respectively. Starting from this kinematical description, the constitutive equations of the model could be derived by means of the virtual work method, as already developed in Buhan and Sudret (2000) for soil-structures reinforced by one-dimensional inclusions, the equilibrium equations and constitutive behavior are given for each phase separately.

2.1. Statics of the two-phase system (Ben Hassine et al., 2008)

As a result of the model construction, the internal efforts are represented by the Cauchy stress tensor as regards the matrix phase and the density of in-plane stresses for the reinforcement phase (Ben Hassine et al., 2008):

\[
\sigma^m = \sigma^r_{\alpha\beta} \varepsilon_{\alpha\beta}, \quad \alpha, \beta = 1, 2
\]

where \( \varepsilon_{\alpha\beta} \) and \( \varepsilon_{\alpha\beta} \) denote the unit vectors of the reinforcement plane.

The equilibrium equation of the matrix phase writes:

\[
div \sigma^m + \rho^m \dot{E}^m = 0
\]

where \( \sigma^m \) denotes the classical Cauchy stress tensor defined in every point of the matrix phase, \( \rho^m \dot{E}^m \) is the volume density of the external body forces applied to the matrix, while \( I \) is the volume density of interaction forces exerted by the reinforcement phase on the matrix phase.

As regards the reinforcement phase, the equilibrium equation is the following:

\[
div \sigma^r + \rho^r \dot{E}^r = 0
\]

where \( \sigma^r \) and \( \rho^r \dot{E}^r \) represent the external body force volume density applied to the reinforcement phase and \( (-I) \) denotes the volume density of the interaction forces exerted by the matrix phase onto the reinforcement phase.

The components \( n_{\alpha\beta}^{r\cdot} \) of the in-plane stress tensor density could be interpreted as their analogous stress component \( N_{\alpha\beta}^{r\cdot} \) observed in the reinforcing membrane, divided by the spacing \( h \) between two neighboring inclusions, or by an equivalent manner, as the product of the corresponding Cauchy-stress component \( \sigma_{\alpha\beta}^{r\cdot} \) by the volume fraction of reinforcement \( \eta \) defined as (see Ben Hassine et al., 2008, for more details):

\[
n_{\alpha\beta}^{r\cdot} = \frac{N_{\alpha\beta}^{r\cdot}}{h} = \eta \sigma_{\alpha\beta}^{r\cdot}, \quad \text{with} \quad \eta = \frac{d}{h}
\]

where \( d \) denotes the thickness of one reinforcing membrane.

It is worth mentioning here that, since \( \eta \) is a plane stress field, it comes from the equilibrium of the reinforcement phase (3) that all body forces \( \rho^r \dot{E}^r - I \) must be in the membrane plane which is equivalent to:

\[
\rho^r \dot{E}^r - I = 0
\]

As it will be shown later on, the external body force volume density \( \rho^r \dot{E}^r \) could be neglected since the volume fraction of reinforcement is very small. The last relationship thus becomes:

\[
I = 0
\]

The first and second component \((- I_1 \) and \(- I_2 \) of the volume density of the interaction forces \((- I) \) could be identified at the microscopic scale to the density of shear stresses \( \sigma_{12} \) and \( \sigma_{23} \) exerted by the soil onto the reinforcing membrane per unit length transverse to the reinforcement plane (Fig. 2):

\[
-I_1 = \frac{\sigma_{13} - \sigma_{13}}{h} \quad \text{and} \quad -I_2 = \frac{\sigma_{23} - \sigma_{23}}{h}
\]

where \( \sigma_{13} \) (resp., \( -\sigma_{23} \)) is the ith component of the stress vector exerted on the upper (resp., lower) surface of the membrane of unit normal \( \varepsilon_i \) (resp., \( -\varepsilon_i \)).

2.2. Elastoplastic constitutive behavior

In the context of small perturbations and under the assumption of elastic perfectly plastic behavior of the different materials, the constitutive behavior equations are defined as follows.

For the matrix phase, strains are classically described by the linearised strain tensor defined as:

\[
\varepsilon^m = \frac{1}{2} \left( \text{grad}^m + (\text{grad}^m)^T \right)
\]

and the elastoplastic constitutive law is expressed as:

\[
\sigma^m = C^m : (\varepsilon^m - \varepsilon_p^m)
\]

where \( C^m \) denotes the fourth order elastic-moduli tensor, whereas \( \varepsilon_p^m \) is the plastic strain tensor. The evolution of \( \varepsilon_p^m \) is governed by the following flow rule:
\[ \dot{\varepsilon}_m^c = \dot{\lambda} \frac{\partial g_m^c}{\partial \varepsilon_m^c} \quad \text{with} \quad \dot{\lambda} = \begin{cases} 0 & \text{if } f_m^c(\varepsilon_m^c) = f_m^c(\varepsilon_m^c) = 0 \\ 1 & \text{otherwise} \end{cases} \]  

(10)

where \( \dot{\lambda} \) is the plastic multiplier, \( f_m^c \) and \( g_m^c \) are the yield function and the plastic potential of the matrix phase, respectively.

As regards the reinforcement phase, strains are described by the symmetric part of the gradient of the reinforcement phase displacement field:

\[ \varepsilon_r^p = \frac{1}{2} \left( \text{grad} \varepsilon_r^p + \text{grad} \varepsilon_r^p \right) \]  

(11)

the constitutive equation is the following:

\[ \varepsilon_r^p = C_r^{ijkl} \varepsilon_j^r \otimes \varepsilon_k^r \quad \alpha, \beta = 1, 2 \]  

(12)

and \( C_r^{ijkl} \) is the ‘plane stress’ stiffness tensor which could be expressed as:

\[ C_r^{ijkl} = C_r^{ijkl} + \varepsilon_r^p \otimes \varepsilon_r^p \quad \text{with} \quad \alpha, \beta, \gamma, \delta = 1, 2 \]  

(13)

and the evolution of the reinforcement phase plastic strains obey a plastic flow-rule given by:

\[ \dot{\varepsilon}_r^p = \dot{\lambda} \frac{\partial f_r^p}{\partial \varepsilon_r^p} \quad \text{with} \quad \dot{\lambda} = \begin{cases} 0 & \text{if } f_r^p(\varepsilon_r^p) = f_r^p(\varepsilon_r^p) = 0 \\ 1 & \text{else} \end{cases} \]  

(15)

where \( \dot{\lambda} \) is the plastic multiplier, \( f_r^p \) and \( g_r^p \) are the yield function and the plastic potential of the reinforcement phase, respectively.

It comes out from the construction of the model, that the strain variable in duality with the interaction efforts density is the difference of displacement between the two phases:

\[ \Delta = \varepsilon_m^c - \varepsilon_r^p \]  

(16)

and the corresponding constitutive law is expressed as:

\[ l = C_m^{ijkl} \left( \Delta - \Delta^p \right) \]  

(17)

In the last equation, \( C_r^{ijkl} \) corresponds to the matrix-reinforcement interaction stiffness tensor and \( \Delta^p \) denotes the plastic interaction strain variable verifying:

\[ \Delta^p = \dot{\lambda} \frac{\partial f_r^p}{\partial \varepsilon_r^p} \quad \text{with} \quad \dot{\lambda} = \begin{cases} 0 & \text{if } f_r^p(l) = f_r^p(l) = 0 \\ 1 & \text{else} \end{cases} \]  

(18)

2.3. Identification of the constitutive parameters

The constitutive parameters introduced here-above are determined from the mechanical and geometrical characteristics of the reinforced structure.

- **Matrix phase:** due to the fact that the volume fraction of reinforcement defined by (4) is very small, and thus the soil volume fraction is close to unity, the mechanical characteristics of the matrix phase are identified to those of the soil. It comes that, for purely frictional native material, isotropic and elastic perfectly plastic with frictional angle \( \phi \), dilatancy angle \( \psi \) and Lamé coefficients \( (\lambda, \mu) \), the constitutive behavior law could be expressed as:

\[ \sigma_m^c = \lambda \text{tr} \left( \varepsilon_m^c - \varepsilon_m^p \right) + 2\mu \left( \varepsilon_m^c - \varepsilon_m^p \right) \]  

(19)

whereas the yield function and plastic potential are

\[ \begin{cases} f_m^c(\sigma_m^c) = \sigma_m^c(1 + \sin \phi) - \sigma_m^c(1 - \sin \phi) < 0 \\ g_m^c(\sigma_m^c) = \sigma_m^c(1 + \sin \psi) - \sigma_m^c(1 - \sin \psi) = C \end{cases} \]  

(20)

where \( \sigma_m^*, \sigma_m^{**} \) denotes the maximal (resp., minimal) principal stress and \( C \) is an arbitrary constant.

- **Reinforcement phase:** the constitutive parameters of this phase are identified to those of one reinforcing membrane divided by the distance \( h \) separating two neighboring inclusions. Under the assumption of isotropy, the reinforcement phase constitutive law writes:

\[ \varepsilon_r^p = \beta^p \text{tr} \left( \varepsilon_r^p - \varepsilon_r^p \right) + \beta^p \left( \varepsilon_r^p - \varepsilon_r^p \right) \]  

(21)

where the rigidities \( \beta^p \) and \( \beta^p \) are the following functions of the Young’s moduli \( E \) and Poisson coefficient \( ν \) of the membrane constitutive material [Ben Hassine et al., 2008]:

\[ \beta^p = \nu \frac{\nu - 1}{1 - 2\nu} \eta E \]  

and \( \beta^p = \frac{1}{1 - \nu} \eta E \)  

(22)

Assuming that the membranes constitutive material obey a Tresca yield criterion with a corresponding traction resistance \( \sigma_0 \) and a compression resistance equal to zero, one can prove that the yield condition involving the in-plane membrane internal efforts is given by [Ben Hassine et al., 2009]:

\[ 0 \leq n_r^c, \ n_r^c, \ n_r^c = \eta \sigma_0 \]  

(23)

where \( n_r^c, n_r^c, n_r^c \) denote the principal values of the reinforcement stress tensor \( \sigma_r^c \).

The plastic flow-rule expressed in the general case by (15), becomes with respect to (23) (Fig. 3):

\[ \dot{\varepsilon}_r^p = \dot{\lambda} \frac{\partial f_r^p}{\partial \varepsilon_r^p} \quad \text{with} \quad \dot{\lambda} = \begin{cases} 0 & \text{if } n_r^c = n_r^c \text{ and } n_r^c = n_r^c = 0 \\ 1 & \text{else} \end{cases} \]  

(24)

The Fig. 3 displays the yield surface and the plastic flow-rule. The particular cases 1–4 correspond to a ‘face regime’ flow for which the direction of plastic strains is completely defined by the normality condition, while the cases 5–9 are relative to the singularities of the yield surface.
3. Elastoplastic boundary value problem

Let $\Omega$ denote the geometrical domain occupied by an elastoplastic two-phase system, the kinematics of which is completely described by two displacement vectors verifying the geometrical compatibility condition (25), so that:

$$\varepsilon^m = \varepsilon^m_1 + \varepsilon^m_2 + \varepsilon^m_3$$

and

$$\varepsilon^n = \varepsilon^n_1 + \varepsilon^n_2 + \varepsilon^n_3$$

Such a couple of displacement fields are kinematically admissible (K.A.) if they are piecewise continuously differentiable and verifying the following boundary conditions:

$$\varepsilon^n_1 = \varepsilon^{n(d)}_1$$

on $\partial\Omega^{m}_{r}$ and

$$\varepsilon^n_2 = \varepsilon^{n(d)}_2$$

on $\partial\Omega^{m}_{r}$

where $\partial\Omega_{m}$ (resp., $\partial\Omega_{r}$) denotes the part of the boundary of $\Omega$ where the component $i$ of the displacement vector attached to the matrix (resp., reinforcement) phase is prescribed.

On the other hand, a generalized stress field $(\sigma^m, n^r)$ is said to be statically admissible if the matrix and the reinforcement stress fields are piecewise continuously differentiable verifying the equilibrium equations and the boundary conditions. The equilibrium equations are expressed as:

$$\text{div}\sigma^m + \rho^m \dot{E}^m + f = 0$$

and

$$\text{div}n^r + \rho^r \dot{E}^r - I = 0$$

where $(\sigma^m)$ (resp., $(n^r)$) denotes the discontinuity of $\sigma^m$ (resp. $n^r$) across any discontinuity surface $\Sigma_m$ (resp., $\Sigma_r$), of normal unit vector $\hat{n}$.

Denoting by $\partial\Omega_{m}$ the part of $\partial\Omega$ where a force density $T_{i}^{n(d)}$ is prescribed on the constituent $\alpha$, the boundary conditions for the two-phase system could be written as:

$$T_{i}^{n} = T_{i}^{n(d)}$$

on $\partial\Omega^{m}_{r}$ and

$$T_{i}^{r} = T_{i}^{n(d)}$$

on $\partial\Omega^{r}_{r}$

with the following complementarity conditions:

$$\partial\Omega^{m}_{r} \cap \partial\Omega^{r}_{r} = 0$$

and

$$\partial\Omega^{m}_{r} \cup \partial\Omega^{r}_{r} = \partial\Omega$$

for $\alpha = m, r$.

The generalized stress field $(\sigma^m, n^r)$ is said to be plastically admissible (P.A.), if the matrix, reinforcement and interaction strength conditions are satisfied:

$$f^m(\sigma^m) = \sigma^m(1 + \sin \varphi) - \sigma^{m(d)}(1 - \sin \varphi) \leq 0$$

for $0 \leq n_0 \leq n_0'$

and

$$f^r(\sigma^r) = |\sigma^r| - \sigma^{r(d)} \leq 0$$

Solving a two-phase elastoplastic boundary problem consists in exhibiting at any time $t$ a statically and plastically admissible generalized stress field $(\sigma^m, n^r)$ along with a kinematically admissible displacement fields $\varepsilon^m$ and $\varepsilon^n$, verifying the elastoplastic constitutive behavior of the matrix phase, the reinforcement phase and the interaction.

4. Numerical implementation of the model

The problem under consideration is the one of a two-phase system which loading depends on several parameters denoted $\{Q\}$. In order to assess the elastoplastic evolution of the structure, the
loading is classically applied as a succession of a sufficiently small increments such as \( \delta \{ Q \} = \{ Q(t + \delta t) \} - \{ Q(t) \} \). Assuming known the elastoplastic solution up to the loading \( \{ Q(t) \} \), in terms of displacement fields \( \{ u^m, e^m \} (t) \), generalized stress field \( \{ \sigma^m + \delta \sigma^m \} (t) \) and plastic strain fields \( \{ \varepsilon^p, \varepsilon^p \} (t) \), the problem is to update the latter solution at time \( t + \delta t \) corresponding to the application of the load increment \( \delta \{ Q \} \) such as:

\[
\begin{align*}
\delta u^m (t + \delta t) &= \delta \sigma^m (t) + \delta \sigma^m (t) \\
\delta e^m (t + \delta t) &= \delta \sigma^m (t) + \delta \sigma^m (t) \\
\delta f (t + \delta t) &= \delta \sigma^m (t) + \delta \sigma^m (t)
\end{align*}
\]

(36)

for stresses,

\[
\begin{align*}
\delta u^m (t + \delta t) &= \delta \sigma^m (t) + \delta \sigma^m (t) \\
\delta e^m (t + \delta t) &= \delta \sigma^m (t) + \delta \sigma^m (t)
\end{align*}
\]

(37)

as concerns the displacement fields, and:

\[
\begin{align*}
\delta u^m (t + \delta t) &= \delta \sigma^m (t) + \delta \sigma^m (t) \\
\delta e^m (t + \delta t) &= \delta \sigma^m (t) + \delta \sigma^m (t) \\
\delta f (t + \delta t) &= \delta \sigma^m (t) + \delta \sigma^m (t)
\end{align*}
\]

(38)

for the three plastic strain fields of the two-phase system.

The increments of stresses and displacements introduced here before being the solution of an elastic problem relative to the application of the increment of loading \( \delta \{ Q \} \) with the prescribed non-elastic (plastic) strains \( \{ \delta \sigma^m, \delta e^m, \delta f \} (t) \):

\[
\begin{align*}
\{ \delta \sigma^m, \delta e^m, \delta f \} (t) &= \text{ELAS} \left\{ \{ Q \}; \{ \delta \sigma^m, \delta e^m, \delta f \} (t) \right\}
\end{align*}
\]

(39)

The prescribed plastic strains have to satisfy the plastic flow rules expressed in the incremental form:

\[
\delta e^m = \delta \frac{f}{\sigma^m} (\sigma^m_m + \delta \sigma^m) \quad \text{with} \quad \delta i = \begin{cases} 0 & \text{if} \ f_m (\sigma^m_m) = f_m (\sigma^m_m) = 0 \\ i & \text{else} \end{cases}
\]

(40)

\[
\delta e^m = \delta \frac{f}{\sigma^m} (\sigma^m_m + \delta \sigma^m) \quad \text{with} \quad \delta i = \begin{cases} 0 & \text{if} \ f_m (\sigma^m_m) = f_m (\sigma^m_m) = 0 \\ i & \text{else} \end{cases}
\]

(41)

\[
\delta e^m = \delta \frac{f}{\sigma^m} (\sigma^m_m + \delta \sigma^m) \quad \text{with} \quad \delta i = \begin{cases} 0 & \text{if} \ f_m (\sigma^m_m) = f_m (\sigma^m_m) = 0 \\ i & \text{else} \end{cases}
\]

(42)

Combining the elastoplastic constitutive law and the plastic flow rule for each phase and for the interaction, it comes:

\[
\begin{align*}
\sigma^m + \delta \sigma^m &= \text{proj} \left\{ \sigma^m + C_m : \delta \sigma^m \right\} \\
\gamma^p + \delta \gamma^p &= \text{proj} \left\{ \gamma^p + C_p : \delta \gamma^p \right\} \\
I + \delta I &= \text{proj} \left\{ I + C_I : \delta I \right\}
\end{align*}
\]

(43)

\[
\begin{align*}
\sigma^m + \delta \sigma^m &= \text{proj} \left\{ \sigma^m + C_m : \delta \sigma^m \right\} \\
\gamma^p + \delta \gamma^p &= \text{proj} \left\{ \gamma^p + C_p : \delta \gamma^p \right\} \\
I + \delta I &= \text{proj} \left\{ I + C_I : \delta I \right\}
\end{align*}
\]

(44)

\[
\begin{align*}
\sigma^m + \delta \sigma^m &= \text{proj} \left\{ \sigma^m + C_m : \delta \sigma^m \right\} \\
\gamma^p + \delta \gamma^p &= \text{proj} \left\{ \gamma^p + C_p : \delta \gamma^p \right\} \\
I + \delta I &= \text{proj} \left\{ I + C_I : \delta I \right\}
\end{align*}
\]

(45)

where \( C_m, C_p, C_I \) denote the convexes of elasticity of the matrix phase, the reinforcement phase and the interaction respectively. Those domains are defined as follows:

\[
\begin{align*}
\sigma^m \in C_m &\iff f_m (\sigma^m) \leq 0 \\
\sigma^m \in C_m &\iff f_m (\sigma^m) \leq 0 \\
\sigma^m \in C_m &\iff f_m (\sigma^m) \leq 0
\end{align*}
\]

(46)

\[
\begin{align*}
\sigma^m \in C_m &\iff f_m (\sigma^m) \leq 0 \\
\sigma^m \in C_m &\iff f_m (\sigma^m) \leq 0 \\
\sigma^m \in C_m &\iff f_m (\sigma^m) \leq 0
\end{align*}
\]

proj. denotes the operator of projection onto the convex of elasticity \( C^p (z = m, r, l) \), defined with respect to the elastic energy scalar product defined as:

\[
\langle \sigma, \varepsilon \rangle = \frac{1}{2} \varepsilon : \left( \sigma \right){\quad}^{-1} : \varepsilon
\]

(47)

for the matrix phase.
algorithm is calculated by means of the finite element method which is based on the following minimum principle of the potential energy:

\[
(W - \Phi) \{ \delta_u^m, \delta_u^r \} = \text{Min} \{ (W - \Phi) \{ \delta_u^m, \delta_u^r \} \}
\]  

(60)

where \( \{ \delta_u^m, \delta_u^r \} \) denotes the solution of the problem of elasticity (50), whereas \( \{ \delta_u^m, \delta_u^r \} \) denotes any kinematically admissible matrix and reinforcement displacement fields. The introduced functional \( (W - \Phi) \) is the potential energy of the two-phase system, defined as the difference between the elastic potential \( W \) and the work of prescribed external efforts \( \Phi \). The elastic potential is defined as the sum of three terms corresponding to the contributions of the different constituents of the two-phase system, namely the matrix phase, the reinforcement phase and the interaction:

\[
W[\{ \delta_u^m, \delta_u^r \}] = \int_\Omega \left( \frac{1}{2} \left( \delta_u^m : \mathbf{C}^m : \delta_u^m \right) - \delta_u^m : \mathbf{M} : \delta_u^m \right) \, d\Omega \\
+ \int_\Omega \left( \frac{1}{2} \left( \delta_u^r : \mathbf{C}^r : \delta_u^r \right) - \delta_u^r : \mathbf{M} : \delta_u^r \right) \, d\Omega \\
+ \int_\Omega \frac{1}{2} \left( \delta_u^m : \mathbf{C}^m : \delta_u^r \right) \, d\Omega
\]

(61)

The potential \( \Phi \) corresponds to the work of body forces, which reduces to \( \rho^E \mathbf{F}^m \) since \( \rho^E \) is neglected, along with the work of prescribed external effort densities:

\[
\Phi[\{ \delta_u^m, \delta_u^r \}] = \int_\Omega \left( \rho^E \mathbf{F}^m \right) : \delta_u^m \, d\Omega + \int_{\partial \Omega} \mathbf{T}^{\mathbf{m}d} : \delta_u^m \, dS \\
+ \int_{\partial \Omega} \mathbf{T}^{\mathbf{r}d} : \delta_u^r \, dS
\]

(62)

The finite element formulation of the minimum principle (60) leads classically to the following minimization problem expressed in a matrix form:

\[
\text{Min} \{ \delta \mathbf{U} \} [K] \{ \delta \mathbf{U} \} - \{ \delta \mathbf{F}_p \} \{ \delta \mathbf{U} \} \\
= \{ \delta \mathbf{F} \}
\]

(63)

where \( [K] \) is the stiffness matrix and corresponds to the quadratic part of the elastic potential \( W \), \( \{ \delta \mathbf{U} \} \) is the unknown nodal displacement vector, whereas \( \{ \delta \mathbf{F} \} \) and \( \{ \delta \mathbf{F}_p \} \) are the nodal forces vectors associated to the work of prescribed external efforts \( \Phi \) and to the linear part of \( W \) associated to plastic strains, respectively.

5. Illustrative example: stability analysis of a membrane-reinforced earth structure

As a first application of the multiphase model and the correlatively developed numerical code, the stability analysis of a cylindrical membrane-reinforced retaining wall, already considered by Ben Hassine et al. (2009) within the context of the yield design theory, is analyzed under the assumption of elastic perfectly plastic behavior with an associated flow rule for the different constituents.

The cylindrical retaining wall, of height \( H \) and radius \( R \) (Fig. 5), composed of a purely frictional soil, is reinforced by a distribution of ring-shaped membranes of internal and external radii \( R-L \) and \( R \), respectively, displayed regularly into the soil (Fig. 5).

The following geometrical characteristics have been selected for the numerical analysis:

\[
R = 10 \, \text{m}, \quad L = 5 \, \text{m}, \quad \eta = \frac{d}{h} = 1\% \quad (64)
\]

where \( d \) denotes the thickness of one reinforcing membrane and \( h \) the distance between two neighboring inclusions (Fig. 5).

Assuming that the contact between the reinforced structure and the substratum is perfectly adherent and that the lateral \( (z = R) \) and higher \( (z = H) \) surfaces are stress free, the sole load parameter of the structure is the soil specific weight.

The mechanical characteristics of the different constituents are the followings:

\[
E^m = 10 \, \text{MPa}, \quad \nu^m = 0.4, \quad \phi^m = 35^\circ, \quad \gamma^m = 20 \, \text{KN/m}^3 \quad (65)
\]

\[
E^r = 1000 \, \text{MPa}, \quad \nu^r = 0.4, \quad \sigma_0 = 100 \, \text{MPa} \quad (66)
\]

5.1. Identification of the elastoplastic parameters of the two-phase model

In order to simulate the elastoplastic behavior of the considered structure, the reinforced zone comprised between the cylindrical surfaces of equations \( r = R - L \) and \( r = R \) is replaced by a two-phase system (Fig. 5). The input data to be used for calculations is determined as follows.

Indeed, the volume fraction of soil is close to unity \( (1 - \eta = 99\%) \), the elastoplastic characteristics of the soil (65) could be assigned to those of the matrix phase:

\[
E^m = 1 \, \text{MPa}, \quad \nu^m = 0.4, \quad \phi^m = 35^\circ, \quad \gamma^m = 20 \, \text{KN/m}^3 \quad (67)
\]

As regards the reinforcement phase, the rigidities and resistance parameters \( \alpha', \beta' \) and \( n'_0 \) are given by (22) and (23):

\[
\alpha' = \frac{\nu^r}{1 - \nu^r} \eta E^r = 11.9 \, \text{MPa}, \quad \beta' = \frac{1}{1 + \nu^r} \eta E^r = 0.71 \, \text{MPa}, \quad n'_0 = \eta \sigma_0 = 10 \, \text{MPa}, \quad \gamma' = 0
\]

(68)

It has been shown in Section 2 that the stiffness tensor and the yield function related to the volume density of interaction are given by (27) and (28), \( c_i' \) and \( l_0' \) could be evaluated by means of numerical simulation performed on a representative elementary volume and they depend on the non-dimensional parameter \( \varepsilon \) defined as (see Ben Hassine et al., 2008, for more details):

\[
\varepsilon = \frac{h}{H}
\]

(69)

For the next the value of \( \varepsilon \) will not be fixed, and a parametric study as a function of this non-dimensional factor will be performed.

6. Confrontation of elastoplastic and yield design results

The elastoplastic analysis of the structure stability is performed using a f.e.m numerical tool based on the iterative procedure de-
tailed in Section 4, taking into account of the axisymmetric geometry and loading, the problem is treated here under axisymmetric conditions and the structure has been discretized into 424 6-noded triangular elements (907 nodes) as sketched in Fig. 6. The calculations are performed in the three following situations:

- The matrix and reinforcement phases are perfectly bonded which corresponds to:
  \[ c' \to 0 \quad \text{and} \quad \chi = \frac{l_0 l}{2N_0} \to 0 \quad (71) \]

- The third considered situation is the one of an intermediate case with the following numerical values:
  \[ c' = 106 \text{ MPa/m}^2, \quad \text{and} \quad \chi = 0.5 \quad (72) \]

The value of the interaction stiffness \( c' \) corresponds to the following parameters proposed by Ben Hassine et al. (2008):

\[ c_0 = 4.24 \text{ MPa/m}^2, \quad \varepsilon = 0.2 \quad (73) \]

The results of these three calculations are represented in Fig. 7 which displays the evolution of the non-dimensional parameter \( K = \frac{c_m H}{\gamma} \) as a function of the displacement \( \delta(A) \) of one representative point (point A in Fig. 6). These curves show that the stability of the soil-reinforced structure is a function of the interaction prevailing between the different constituents. The ultimate stability factor \( K \) being increased from 1.9 for a smooth interface (\( \chi \to 0 \)) to 6.4 for perfectly bonded phases (\( \chi \to \infty \)). The intermediate case (\( \chi = 0.5 \)) corresponds to an intermediate stability factor of 4.6. These results could be compared to those obtained by Ben Hassine et al. (2009) by means of the kinematic approach of yield design leading to an upper bound for the stability factor: \( K^{\text{UB}} \). Those authors developed an axisymmetric failure mechanism with one conical discontinuity surface in the matrix phase, along with a failure zone for the interaction (failure by ‘slippage’) and a discontinuity surface in the reinforcement phase which corresponds to a failure zone by ‘breakage’. The optimization of such a failure mechanism leads to the following upper bound values of the stability factor:

\[
\begin{align*}
K^{\text{UB}}(\chi = 100) &= 6.95 \\
K^{\text{UB}}(\chi = 0.5) &= 4.8 \\
K^{\text{UB}}(\chi = 10^{-4}) &= 2
\end{align*}
\quad (74)
\]

These results are in quite good agreement with those obtained by means the numerical simulations in plasticity. The failure mechanism obtained for the intermediate case (\( \chi = 0.5 \)) is represented in Fig. 8, which displays the shadings of plastic strain rates at failure relative to the matrix phase (a), the reinforcement phase (b) and the interaction (c), showing clearly the presence of a ‘breakage’ (b) failure zone and a ‘slippage’ failure zone (c) as predicted by the kinematic approach of yield design. Besides, it seems that the elastoplastic failure mechanism (a) involves a rotational mechanism.
instead of the translation one adopted by Ben Hassine et al. (2009). It appears that the upper bounds (74) could be further improved by considering such a generalized rotational failure mechanism (Fig. 9) (see Thai Son et al., 2009, for more details).

7. Conclusion

The modified multiphase model aimed at assessing the macroscopic behavior of soil-structures reinforced by membranes, developed by Ben Hassine et al. (2008) and Ben Hassine et al. (2009) in the context of elasticity and yield design is extended here to elastoplastic behavior of the different constituents with the appropriate flow rule. A finite element numerical tool based on an adapted return mapping algorithm for the multiphase model with a local projection on the different yield surfaces has been developed, providing an alternative approach to the direct simulation of the ‘real’ reinforced soil-structure which appears to be very complex and leading to time consuming numerical models.

The results of the finite element simulations presented in this paper are in very good agreement with the results of the yield design study performed by Ben Hassine et al. (2009), using a mechanism with a linear failure surface, which corresponds to the particular case of a rotational failure mechanism with an infinite radius, could be improved by considering a rotational failure mechanism with the appropriate logspiral failure surface.

Acknowledgement

The author thanks Professor Patrick de Buhan for discussions, interest and his different comments.

References


