Annals of Mathematical Logic 16 (1979) 101–143.

FILTER SPACES AND CONTINUOUS FUNCTIONALS

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Received 16 March 1977

0. Introduction and preliminaries

The aim of this paper is to provide a mathematically civilized introduction to the continuous functionals. The continuous functionals are a finite type structure over the natural members and were first discussed in the papers of Kreisel [17] and of Kleene [15] (where they were called countable functionals). These original treatments were very concrete and well-suited to immediate applications (see Kreisel [17, 18]). But the far more abstract approach of the present paper is not simply of interest in its own right. It admits of useful applications in topos theory (for which see Hyland [12]) and provides the conceptual background for much recent work in recursion theory (Norman [22], Wainer [29]).

The first five sections provide the basic theory of the continuous functionals. This covers properties of the bases (finite bits of information) and concludes with a discussion of closure properties of the continuous functionals.

The remainder of the paper proper deals with two other approaches to the continuous functionals. The first is the original one of Kleene's via associates. This concrete equivalent of our abstract approach is useful because associates at type 2 are very easy to visualize. We use them to provide information about the induced topology introduced in Section 2. The second approach we consider is via sequence convergence. It first appeared as a model for Bar Recursion in Scarpellini [24]. We make use of sequence convergence to provide information about topologies on the spaces of continuous functionals. In particular it emerges that in order to reflect the continuity of maps successfully, one must resort to topologies which are not 1st countable. This shows the need for something like our approach via filter spaces in order to express the constructive kind of recursion theory we discuss in Appendix A.

Fundamental results for the study of recursion theory on the continuous functionals are organized in three appendices. Appendix A considers recursion on filter spaces from a general point of view and gives a formulation of (what I believe is) the fundamental notion of "recursive in" for coded filter spaces. Appendix B shows what simpler formulations of the recursion theory can be made for the special case of the continuous functionals. (A degree structure on the continuous functionals arises out of our definition of "recursive in". We do not discuss its properties here, but note that it differs from the degree structure arising from the notion of computable by S1–S9 (for which see Gandy, Hyland [10]).) Appendix C proves the recursive density theorem; this involves an effectivization of material from Section 4.

Two approaches are not covered by this paper: Kreisel's original formulation and its mathematically civilized version using equivalence classes in lattices. These have been studied in some detail by Ershov (see in particular Ershov [5]). The equivalence of the lattice theoretic approach to that of the present paper will be discussed in full generality in a further paper which I am preparing. The main abstract result is quoted in Hyland [12].

The main non-logical prerequisite for an understanding of this paper is an appreciation of basic category theory. The reader should know what adjoint functors are and what a cartesian closed category (often hereafter, c.c.c.) is. A good reference is MacLane [21]. In addition the following concept plays a fundamental role in many definitions. Assume that in a c.c.c., we have a definite choice of terminal objects, products and function spaces (that is, the category comes equipped with the appropriate adjoint functors). A *sub-c.c.c.* of a given c.c.c. is then just a full subcategory containing the terminal object and closed under the taking of products and function spaces. A sub-c.c.c. of a c.c.c. is automatically cartesian closed itself. Of particular importance is the *sub-c.c.c. generated by an object* of a c.c.c.; namely the least sub-c.c.c. of the c.c.c. containing the object. This has a description of a kind familiar to logicians. We define *type symbols* as follows:

(i) 0 is a type symbol;

(ii) if σ and τ are type symbols, so are $(\sigma\times\tau)$ and $(\sigma\to\tau)$.

Let A be an object in a c.c.c.; we define objects A_σ for each type symbol σ inductively:

(i) A_0 is A;

(ii) $A_{\sigma\times\tau}$ is $A_\sigma\times A_\tau$, the product of A_σ and A_τ;

(iii) $A_{\sigma\to\tau}$ is $[A_\sigma, A_\tau]$, the function space from A_σ to A_τ.

Then the sub-c.c.c. generated by A is the full sub-category with objects the A_σ's together with the terminal object.

The structure of a c.c.c. induces various isomorphisms between the objects of the sub-c.c.c. generated by A. In particular it is easy to see that for any σ there is a τ such that A_σ is isomorphic to A_τ and such that τ is the product of type symbols of the form $(\rho\to 0)$ or 0. But we only need to consider such special types at one point in this paper (the proof of Theorem 3 of Appendix B).

The significance of the notion of a c.c.c. is that it is the categorical formulation of "closure under explicit definition and λ-abstraction". Generally one value of a categorical formulation is that it makes explicit the importance of closure conditions on the maps one is interested in. Another value it has in this paper, is that it makes sense of the choice of definitions needed in proofs, in particular in Section

4; the strategy of Section 4 involves exploiting a pair of adjoint functors and this is what motivates the definitions.

This paper concentrates on the continuous functionals and the corresponding c.c.c. FIL of filter spaces. However other finite type structures (most notably the effective operations) may be obtained by considering other c.c.c.'s involving filter spaces (but restricting the maps). The form of these generalizations is sketched in the discussion at the end of Section 1; the material of the paper (apart from that on sequence convergence) applies equally well to the generalizations.

Some of the material in this paper appeared in my thesis (Hyland [11]), but its formulation is radically different.

1. Filter spaces

A *filter* Φ on a set X is a non-empty collection of non-empty subsets of X, satisfying,

(i) if $X \supseteq A \supseteq B$ and $B \in \Phi$, then $A \in \Phi$, and

(ii) if $A \in \Phi$ and $B \in \Phi$, then $A \cap B \in \Phi$.

A *filter base* Φ on a set X is a non-empty collection of non-empty subsets of X, such that if $A \in \Phi$ and $B \in \Phi$, then for some $C \subseteq A \cap B$, $C \in \Phi$. In particular if a non-empty collection of non-empty subsets of X is closed under finite intersection, it is a filter base. The filter bases with which we deal will usually be of this simple sort.

A filter base Φ on X, *generates* a unique filter, which we write $[\Phi]$, on X, defined by

$$[\Phi] = \{A \subseteq X \mid \text{for some } B \in \Phi, B \subseteq A\}.$$

A *filter space* (X, F) is a set X together with *filter structure*, which is an operation F which associates to each point $x \in X$, a collection $F(x)$ of filters on X, such that,

(i) if $\Phi \supseteq \Psi$ and $\Psi \in F(X)$, then $\Phi \in F(x)$, and

(ii) the principal ultrafilter $[\{x\}]$ at x is in $F(x)$.

The idea behind the definition of a filter space is that the Φ in $F(x)$ converge to x; that is, they are ways of approaching, or approximating to x. In terms of this idea, conditions (i) and (ii) have very natural interpretations. It has been customary to put additional conditions on a notion of filter convergence: in particular those for a limit space (Binz and Keller [1]) and those for a convergence space (Choquet [2]). But we will not need to consider these conditions, though the spaces which we consider will all satisfy them.

For a given filter space (X, F) we write "$\Phi \downarrow x$" (read "Φ converges to x") for "$[\Phi] \in F(x)$", where Φ is a filter base on X. Much of the theory of filter convergence can be written most elegantly in terms of filter bases. But it is more usual not to give the definition in these terms, and we have adhered to this practice to avoid confusion with the "canonical filter bases" which we shall introduce later.

A map $f:(X,F)\to(Y,G)$ from one filter space to another (i.e. a map between the underlying sets) is *continuous* iff whenever $\Phi\downarrow x$ in X, then $f(\Phi)\downarrow f(x)$ in Y. ($f(\Phi)$ naturally consists of those subsets of Y which are images under f of elements of Φ). This definition is equivalent to the more usual one that whenever $\Phi\in F(x)$, $[f(\Phi)]\in G(f(x))$. The collection of filter spaces with the continuous maps as morphisms, forms the category FIL of filter spaces.

There is a natural injection of the category TOP of topological spaces into FIL. To each point of a topological space we can associate the collection of all filters which include the neighbourhood filter at the point: this gives rise to a corresponding filter space (in fact, a convergence space). A map between topological spaces is continuous iff the map between the corresponding filter spaces is continuous (in the sense defined above). Thus the image of TOP under the injection (i.e. faithful functor, which is (1-1) on objects) is a full subcategory of FIL. We consider the left adjoint to this injection in Section 2.

Given filter spaces (X,F) and (Y,G), there is a natural filter structure $(F\times G)$ on $(X\times Y)$; we define it by stipulating that $\Theta\downarrow(x,y)$ in $X\times Y$ iff $p(\Theta)\downarrow x$ in X and $q(\Theta)\downarrow y$ in Y, where p and q are the projections from $(X\times Y)$ to X and Y respectively; this notion of filter base convergence determines the operation $(F\times G)$. Clearly $(F\times G)$ is a filter structure and is the coarsest such that the projections are continuous. Thus it gives rise to a product (the *canonical product*), in the category FIL.

We let $[X,Y]$ denote the set of continuous maps from (X,F) to (Y,G). (Note that the dependence on F and G is overlooked by our notation; this should cause no confusion.) We define a filter structure $[F,G]$ on $[X,Y]$ by stipulating that $\Theta\downarrow f$ in $[X,Y]$ iff whenever $\Phi\downarrow x$ in X, $\Theta(\Phi)\downarrow f(x)$ in Y. (Here $\Theta(\Phi)$ consists of all $W(U)$ with $W\in\Theta$ and $U\in\Phi$; $W(U)$ is the union of all images of U under elements of W.) Again the notion of filter base convergence determines the operation $[F,G]$, which is a filter structure. We call $([X,Y],[F,G])$ the *canonical function space* (from (X,F) to (Y,G)) in the category FIL. This terminology is justified by the following proposition.

Proposition 1.1. *FIL is cartesian closed; and the right adjoint to the product is provided by the canonical function space.*

Proof. The proof is trivial, and is in effect in Binz and Keller [1] (for the case of limit spaces). So we restrict ourselves to observing that there is just one place in the proof where we use a condition on the filter structure. We use condition (i) in the definition of filter space, to show that the evaluation map (the co-unit of the adjunction between product and function space) is continuous.

Remark. (1) The categories of limit spaces and of convergence spaces are full sub-c.c.c.'s of FIL.

(2) If we drop condition (ii) in the definition of filter space, we get an even larger cartesian closed category. But we have no use for it here; we need the fact that constant maps (i.e. maps whose range is a singleton) are continuous for which we need (ii).

(3) Of course FIL is much more than cartesian closed. It is a closed span category (cf. Day [4]) and even a quasi-topos (cf. Wyler [30]). But this further structure is not much use to us.

We recall from Section 0, the notion of the sub-c.c.c. generated by an object of a cartesian closed category. *The category of the continuous functionals* $\mathscr{C}$, is the sub-c.c.c. of the c.c.c. FIL generated by the space of natural numbers (with the filter structure corresponding to the usual discrete topology). For each type symbol σ, we have an object C_σ of $\mathscr{C}$, where C_0 is the natural numbers as above, $C_{\sigma\times\tau}$ is the product $C_\sigma \times C_\tau$, and $C_{\sigma\to\tau}$ is the function space $[C_\sigma, C_\tau]$. C_σ is the *space of continuous functionals* of type σ, and the elements of the C_σ's are the *continuous functionals*. (We abuse notation by letting C_σ denote both the object of $\mathscr{C}$ and the corresponding underlying set.) The objects of $\mathscr{C}$ consist of the C_σ's together with the terminal object of FIL (the one point space); the presence of the latter is of no great significance.

Remark. Since $\mathscr{C}$ is a c.c.c., there are many isomorphisms among the C_σ's; and since C_0 is isomorphic with $C_{0\times 0}$, there are some additional isomorphisms. As far as I know, no complete characterization of the isomorphisms amongst the C_σ's is known (see Hyland [13] and Norman [22]).

Discussion of other type structures

From any c.c.c. with natural number object, we can obtain a type structure (collection of spaces of finite type) over the natural numbers. The category of all sets and mappings gives rise to the maximal type structure. The continuous functionals can be obtained from a variety of cartesian closed coreflective subcategories of TOP (see Sections 8 and 9).

Further examples can be obtained as follows. Take as objects the coded filter spaces but as maps only those recursive in some restricted collection of functions (see Appendix A for definitions). We always obtain a c.c.c., and thence a type structure over the natural numbers. In the case that we only consider recursive maps, we obtain the effective operations (see the discussion in Gandy, Hyland [10].) Apart from the material on sequence convergence, the results of this paper go through readily for this family of generalizations of the continuous functionals: we refer to them as *the Kreisel generalizations of the continuous functionals.* Further important generalizations along these lines are involved in Wainer's work on the 1-sections of non-normal type 2 objects (Wainer [29]), but we do not try to describe them here.

It is worth remarking that though the above generalizations were first considered in Kreisel [17], what he in fact considers is too general. One can't allow the complexity considered to vary between the levels and still have a c.c.c. (closure under explicit definition and λ-abstraction). After all, suitable type-changing maps are elementary.

There are of course many other important cartesian closed type structures over the natural numbers (Gödel's primitive recursive functionals, Kreisel's intensional continuous functionals), but no attempt is made here to give a general theory for them.

2. The induced topology

Let (X, F) be a filter space. A subset O of X is *open* (with respect to F) iff whenever $x \in O$ and $\Phi \in F(x)$, then $O \in \Phi$. It is easy to see that the collection of open sets is a topology on X, the *topology induced by* F. When there is no chance of confusion, we simply say "the *induced topology*", and "O is *open in* (X, F)".

Taking the induced topology gives rise to an obvious functor $T:\mathrm{FIL} \to \mathrm{TOP}$. Let $F:\mathrm{TOP} \to \mathrm{FIL}$ denote the injection described in Section 1. Then the following proposition belongs to folklore.

Proposition 2.1. *T is left adjoint to F.*

Proof. The proof is trivial.

Remark. Since F is a full injection, we may identify TOP with its image in FIL. Then Proposition 2.1 says that TOP is a reflective subcategory of FIL. We can express this most suggestively by saying that a map from a filter space to a topological space is continuous iff it is continuous with respect to the induced topology. We use this simple idea to considerable effect in Section 4.

The identity map is continuous from a filter space to its induced topological space. So there are more filters and hence less structure in the induced topology. Thus knowing the induced topology on (X, F) and (Y, G) tells us rather little about the induced topology on the product $(X \times Y, F \times G)$ or function space $([X, Y], [F, G])$. But what little can be said is sufficient for our purposes.

Proposition 2.2. (a) *If O and O' are open in (X, F) and (Y, G) respectively, then $O \times O'$ is open in $(X \times Y, F \times G)$.*

(b) *If O is open in $(X \times Y, F \times G)$ and $x \in X$, then $O_x = \{y \mid (x, y) \in O\}$ is open in (Y, G).*

Corollary 2.3. *Given x a point of (X, F) and O open in (Y, G), $\{f \mid f \in [X, Y]$ and $f(x) \in O\}$ is open in $[X, Y]$.*

Proof. The proof of the proposition is trivial. For the corollary we argue as follows. The evaluation map, $\mathrm{ev}:[X, Y]\times X\to Y$ is continuous (by Proposition 1.1). Since constant maps are continuous (cf. Remark (2) following Proposition 1.1), evaluation at x, $\mathrm{ev}(-,x):[X, Y]\to Y; f\to f(x)$, is continuous with respect to the filter structures, and hence with respect to the induced topologies. $\{f \mid f\in[X, Y] \text{ and } f(x)\in O\}$ is the inverse image of an open set under a continuous map and is therefore itself open.

Proposition 2.2 and Corollary 2.3 give us some information about separation conditions on the induced topology.

Proposition 2.4. *If (X, F) and (Y, G) are filter spaces and the induced topology on (Y, G) is Hausdorff, then so is the induced topology on $([X, Y], [F, G])$.*

Proof. Take f, g distinct elements of $[X, Y]$. There is $x\in X$ such that $f(x)$ and $g(x)$ are distinct elements of Y. Let O_1 and O_2 be open sets in Y separating $f(x)$ and $g(x)$. Then $\{h \mid h\in[X, Y] \text{ and } h(x)\in O_i\}$ separate f and g, for $i=1,2$ and are open by Corollary 2.3.

Corollary 2.5. *The category of filter spaces with Hausdorff induced topology is a full sub-c.c.c. of FIL.*

Proof. Closure under product by Proposition 2.2(a), and under function space by Proposition 2.4.

In particular, the spaces of continuous functionals defined in Section 1, all have Hausdorff induced topologies.

The rest of this section is designed to give the reader a more complete picture of the induced topology on function spaces. We do not use the results in the rest of the paper (though we do use the notation introduced). However the results are interesting in themselves, and are connected with important questions about the intrinsic recursion theory on the continuous functionals (see Appendix B).

We first consider the definition of compactness for filter spaces. Let $\mathscr{X}$ be a set of subsets of X, and Γ a collection of filters on X: we say that $\mathscr{X}$ covers Γ iff for every $\Phi\in\Gamma$ there is $A\in\mathscr{X}$ with $A\in\Phi$. (Of course, we also have the usual notion of a collection of sets covering another.)

Proposition 2.6. *The following conditions on a filter space (X, F) are equivalent:*

(a) *Every ultrafilter on X converges in (X, F).*

(b) *There is a collection of filters on X, $\Gamma\subseteq\bigcup\{F(x)\mid x\in X\}$, such that if $\mathscr{X}$ covers Γ then some finite $\mathscr{Y}\subseteq\mathscr{X}$ covers X.*

Proof. Suppose (a), and let Γ be the collection of ultrafilters. If $\mathscr{X}$ covers Γ, consider the finite intersections of the complements of elements of $\mathscr{X}$. The collection of these cannot form a filter base, as if it did, it would be included in some ultrafilter Φ; since $\mathscr{X}$ covers Γ, there would be $A\in\mathscr{X}$ with $A\in\Phi$; but by stipulation $(X-A)\in\Phi$, a contradiction. Hence there is some finite intersection of the complements of elements of $\mathscr{X}$ which is empty; that is some finite $\mathscr{Y}\subseteq\mathscr{X}$ with $\mathscr{Y}$ covering X. Thus the collection Γ of all ultrafilters satisfies (b). Conversely, suppose (b) holds, suppose that Φ is an ultrafilter which fails to converge. No element of Γ can be included in Φ, so that $\{X-U \mid U\in\Phi\}$ covers Γ. But by (b), this means that some finite intersection of elements of Φ is empty, a contradiction. Thus every ultrafilter must converge, and (a) holds.

We say that a filter space which satisfies the equivalent conditions of Proposition 2.6 is *compact.*

Remarks. (1) The definition coincides for topological spaces with the usual one.

(2) The conditions of Proposition 2.6 are also equivalent to a strengthened form of (b) where one demands that the finite $\mathscr{Y}$ covers Γ as well as X.

(3) A compact convergence space whose induced topology is Hausdorff is necessarily topological. (A convergence space is a filter space where $\Phi\downarrow x$ iff for all ultrafilters Ψ with $\Psi\supseteq\Phi$, $\Psi\downarrow x$.) This remark has relevance for the countable functionals (cf. Remark (1) following Proposition 1.1); all the compact subspaces (in the sense to be defined) of the spaces of countable functionals, are topological. (In fact, they look like the closed subspaces of Cantor space.)

If A is a subset of a filter space (X,F), there is a natural filter structure F_A on A. For $x\in A$, $\Phi\in F_A(x)$ iff for some $\Psi\in F(x)$, $\Phi=\{U\cap A \mid U\in\Psi\}$ (i.e. Φ is the *trace* of Ψ on A). (Note that one pays no attention to those $\Psi\in F(x)$ whose trace on A is not a filter.) (A,F_A) is the subspace of (X,F) determined by A.

Suppose A is a subset of a filter space (X,F). A is *compact* (in (X,F)) iff the subspace (A,F_A) is a compact filter space. We now show that "compact-open" sets are always open in the induced topology.

First we introduce a notation for subsets of function spaces. Let (X,F) and (Y,G) be filter spaces, and let $U\subseteq X$ and $V\subseteq Y$. We define $[U,V]$ by,

$$[U,V]=\{f \mid f\in[X,Y] \quad\text{and}\quad f(U)\subseteq V\}.$$

This notation will recur throughout this paper.

Proposition 2.7. *Let A be compact in (X,F) and O open in (Y,G). Then $[A,O]$ is open in $([X,Y],[F,G])$.*

Proof. By the definition of compactness, we may take a collection of filters $\Gamma\subseteq\bigcup\{F(x)\mid x\in A\}$, such that if $\mathscr{X}$ covers Γ then some finite subset $\mathscr{Y}$ of $\mathscr{X}$ covers A. Now let $f\in[A,O]$ and let Θ be a filter converging to f. For each Φ_i in Γ (i from some suitable index set I), there is $x_i\in A$ with $\Phi_i\downarrow x_i$. Then $\Theta(\Phi_i)\downarrow f(x_i)$ and $f(x_i)$ is in O; so there exist $W_i\in\Theta$ and $U_i\in\Phi_i$ such that $W_i(U_i)\subseteq O$. $\{U_i \mid i\in I\}$ covers Γ, so for some finite $J\subseteq I$, $\{U_i\mid i\in J\}\supseteq A$. If $W=\bigcap\{W_i\mid i\in J\}$, then $W\in\Theta$ and $W(A)\subseteq O$. Hence $[A,O]$ is in Θ. Since this occurs for an arbitrary Θ converging to an arbitrary f in $[A,O]$, $[A,O]$ is open in the induced topology.

The induced topology on a pair of spaces bears no simple relation to that on their product or function space. (It does not even determine them.) The induced topology on a product may contain more open sets than the product of the induced topologies, and similarly, the induced topology on a function space may properly include the compact-open topology. This is only to be expected as TOP is not cartesian closed: explicit examples are given in Section 7. The situation is not improved by considering coreflective subcategories of TOP (e.g. making use of compactly generated topologies) but a full discussion is beyond the scope of this paper.

3. Bases for filter spaces

The idea of a basis for a filter space is that it should be a collection of subsets of the space, in terms of which we can completely determine the filter structure.

Let (X, F) be a filter space and $\mathcal{U}$ a collection of subsets of X. $\mathcal{U}$ is a *basis* for (X, F) iff whenever $\Phi \in F(x)$, then $\mathcal{U} \cap \Phi$ is a filter base and $\mathcal{U} \cap \Phi \downarrow x$. (The bases with which we shall deal, will be closed under finite intersection; so $\mathcal{U} \cap \Phi$ will automatically be a filter base.) In the context of a given basis $\mathcal{U}$ for (X, F), we will write Φ' for $\mathcal{U} \cap \Phi$ (where $\Phi \in F(x)$, some $x \in X$). Such a Φ' is a *canonical filter base* introduced by $\mathcal{U}$. The canonical filter bases determine the filter structure in the following sense. A filter Φ is in $F(x)$ iff it includes some canonical filter base converging to x; that is to say, the canonical filter bases generate the filter structure F on X.

Remark. The notion of a basis used here, is more general, for topological filter spaces, than the usual notion of a basis for a topological space. For example the power set of a space is always a basis.

It is important for what we do later that we can readily describe how to construct bases for the products and function spaces of filter spaces for which we already have bases. The case of products is easy.

Proposition 3.1. *Let $\mathcal{U}$ be a basis for (X, F) and $\mathcal{V}$ a basis for (Y, G); then $\{U \times V \mid U \in \mathcal{U}$ and $V \in \mathcal{V}\}$ is a basis for $(X \times Y, F \times G)$.*

Proof. The proof is straightforward.

The case of function spaces is more interesting. Let $\mathcal{U}$ be a basis for (X, F) and $\mathcal{V}$ a basis for (Y, G). Let $\mathcal{W}$ consist of all finite intersections of sets of the form $[U, V]$ where $U \in \mathcal{U}$ and $V \in \mathcal{V}$. (The notation $[U, V]$ was introduced in Section 2.)

Proposition 3.2. *In the above situation, $\mathcal{W}$ is a basis for $([X, Y], [F, G])$.*

Proof. Let $f \in [X, Y]$ and $\Theta \in [F, G](f)$. We wish to show that $\Theta' = \mathscr{W} \cap \Theta$ converges to f. For this it is sufficient to show for any canonical filter base Φ' with $\Phi' \downarrow x$ in X, that $\Theta'(\Phi) \downarrow f(x)$. But $\Theta(\Phi') \downarrow f(x)$ so $[\Theta(\Phi')]' = \mathscr{V} \cap [\Theta(\Phi')] \downarrow f(x)$; if $V \in [\Theta(\Phi')]'$, there is $W \in \Theta$ and $U \in \Phi'$ such that $W(U) \subseteq V$, i.e. there is $U \in \Phi'$ with $[U, V] \in \Theta$: if $[U, V] \in \Theta$, $[U, V] \in \Theta'$ and so if $V \in [\Theta(\Phi')]'$, then $V \in [\Theta'(\Phi')]'$. This shows that $\Theta'(\Phi') \downarrow f(x)$, and completes the proof.

The bases of Propositions 3.1 and 3.2 are the *canonical* bases for a product and a function space, respectively.

We next consider a natural topological condition on the basis $\mathscr{U}$ for a filter space (X, F). This condition is needed in many proofs in Section 4, and also has a computational significance (for which see Appendix B).

Proposition 3.3. *Let $\mathscr{U}$ be a basis for (X, F); the following conditions are equivalent.*

(a) *If a canonical Φ' (introduced by $\mathscr{U}$) tends to x, then every member of Φ' contains x.*

(b) *Every member of $\mathscr{U}$ is closed in the induced topology.*

Proof. That (b) implies (a) is obvious. Suppose on the other hand that $U \in \mathscr{U}$ and U is not closed. Then there is $x \in U$ with $\Phi \in F(x)$ such that $(X \setminus U) \notin \Phi$. Thus if $V \in \Phi$, $U \cap V \neq \emptyset$, so we can consider the filter Ψ generated by $\{V \cap U \mid V \in \Phi\}$. $\Psi \supseteq \Phi$ so $\Psi \in F(x)$, and $x \notin U$ while $U \in \Psi'$. This contradicts (a).

A basis satisfying the equivalent conditions of Proposition 3.3 will be called a *regular basis*. This is a sensible terminology as the limit space analogue of regularity (as considered by Cook and Fisher [3]) is equivalent to admitting a basis of closed sets. Of course most bases in the original topological sense are not regular bases in our sense; and many spaces do not admit regular bases at all.

The constructions we have given for bases for products and function spaces, preserve the notion of a regular basis.

Proposition 3.4. *Let the bases $\mathscr{U}$ for (X, F) and $\mathscr{V}$ for (Y, G) be regular. Then the bases of Proposition 3.1 for $(X \times Y, F \times G)$ and Proposition 3.2 for $([X, Y], [F, G])$ are regular.*

Proof. For the product, the proposition follows from Proposition 2.2 (a) and the fact that the product of closed sets is closed in the product topology. For the function space note that $[X, Y] \setminus [U, V] = \bigcup\{[\{x\}, Y \setminus V] \mid x \in U\}$ which by Corollary 2.3 is a union of open sets and so is open.

Corollary 3.5. *The spaces of continuous functionals all admit regular bases.*

Proof. C_0 the space of natural numbers clearly has a regular basis.

We discuss the natural bases for the spaces of countable functionals in detail in the next section.

4. Enumerated bases and the decidability theorem

Since C_0, the space of continuous functionals of type 0, clearly has a countable basis, it follows from Propositions 3.1 and 3.2 that all spaces C_σ of continuous functionals have countable bases. In order to be able to talk in recursion theoretic terms about the canonical filter bases introduced by these bases, we need an explicit enumeration of the bases. An enumerated basis is a map from the natural numbers to a basis. We let $\mathbf{N}$ denote the set of natural numbers, and adapting the notation for ordinary bases, write $\mathcal{U} = \{U_i \mid i \in \mathbf{N}\}$ for an enumerated basis.

Before turning to the main business of this section, we give specific enumerated bases for the spaces of continuous functionals. Of course, we never use the details of this coding but it seems best that the readers have something specific in mind. Let $\{e_n \mid n \in \mathbf{N}\}$ be the sequence of finite sets of natural numbers, where e_n is determined by the 1's in the binary notation for n; let $(\ ,\ )$ be a standard (1-1), onto pairing function from $\mathbf{N} \times \mathbf{N}$ to $\mathbf{N}$, with projections $p(\ )$ and $q(\ )$. Then we can define inductively,

$$U_0^0 = C_0, \qquad U_1^0 = \emptyset, \qquad U_{n+2}^0 = \{n\};$$

$$U_k^{\sigma\times\tau} = U_{p(k)}^\sigma \times U_{q(k)}^\tau,$$

$$U_n^{\sigma\to\tau} = \bigcap \{[U_i^\sigma, U_j^\tau] \mid (i, j) \in e_n\}.$$

Then $\mathcal{U}^\sigma = \{U_n^\sigma \mid n \in \mathbf{N}\}$ is an enumerated basis for C_σ. It is easy to see that it is regular by Proposition 3.4.

The final result of this section is a decidability theorem for the bases $\mathcal{U}^\sigma$ for the spaces C_σ. This is a consequence of a number of "structural properties" of the bases $\mathcal{U}^\sigma$. This section is mainly taken up with establishing these in great generality. In Appendix B, we use the decidability theorem to show the equivalence of a number of different formulations of the notions of "recursive" and "recursive in" for the countable functionals; in particular we can show that it makes no difference whether we use "partial" or "total" codes for functionals. Further significant results about the continuous functionals use the decidability theorem in conjunction with results obtained by effectivizing part of its proof. We consider this effectivization in Appendix C.

Definition (provisional). The basis $\mathcal{U} = \{U_i \mid i \in \mathbf{N}\}$ for the filter space (X, F) is *decidable* iff

(i) the set $\{i \mid U_i = \emptyset\}$ is recursive, and

(ii) all sets of the form

$$\{(i_1, \ldots, i_m, k_1, \ldots, k_n) \mid U_{i_1} \cap \cdots \cap U_{i_m} \subseteq U_{k_1} \cup \cdots \cup U_{k_n}\},$$

are recursive (uniformly in m and n).

Remark. As a consequence of (ii), the inclusion between any two positive Boolean combinations of elements of $\mathcal{U}$ can be decided. This is a very strong property, and may not be easy to establish for many bases. But for want of evidence on this point, we let the definition stand for the moment.

Assumption. In order to simplify the proofs which we shall give, we make the assumption that each basis is closed under finite intersection and contains the empty set. These conditions do not in any way affect the theorems we prove.

The reason why we can prove something as strong as condition (ii) above, for the bases $\mathcal{U}^\sigma$, is that they satisfy a *disjunction property* for a basis:

(DP if $U_{i_1} \cap \cdots \cap U_{i_m} \subseteq U_{k_1} \cup \cdots \cup U_{k_n}$, then for some r,

$$1 \leq r \leq n, \quad U_{i_1} \cap \cdots \cap U_{i_m} \subseteq U_{k_r}.$$

(This condition reads more simply when a basis is closed under finite intersection.) Bearing (DP) in mind, we proceed to discuss the problem of showing the $\mathcal{U}^\sigma$'s to be decidable.

Clearly the basis $\mathcal{U}^0$ for C_0 is decidable. We wish to establish decidability of all the $\mathcal{U}^\sigma$'s by induction over the types. Products give no trouble (once we have (DP)), so we concentrate on function spaces. First we introduce some notation for the general situation, which we will use in both this and the next section.

Let $\mathcal{U} = \{U_i \mid i \in \mathbf{N}\}$ and $\mathcal{V} = \{V_i \mid i \in \mathbf{N}\}$ be the bases for the filter spaces (X, F) and (Y, G) respectively. Let $\mathcal{W} = \{W_A \mid A$ is a finite subset of $\mathbf{N} \times \mathbf{N}\}$ be the canonical basis for $\langle [X, Y], [F, G] \rangle$; W_A is the intersection of the $[U_i, V_j]$ such that (i, j) is in A. We use $A, B, C, D, \ldots$ to range over finite subsets of $\mathbf{N} \times \mathbf{N}$. For A a finite subset of $\mathbf{N} \times \mathbf{N}$, we define,

$$U_A = \bigcap \{U_i \mid \text{for some } j, (i, j) \in A\};$$
$$V_A = \bigcap \{V_j \mid \text{for some } i, (i, j) \in A\}.$$

$\mathcal{W}$ has a recursive function coding intersection; for $W_A \cap W_B = W_{A \cup B}$. Thus if we have (DP) for $\mathcal{W}$, it suffices in order to establish condition (ii) for $\mathcal{W}$ to show that inclusion is decidable. To do this, we consider a finitary condition on $\mathcal{U}$ and $\mathcal{V}$ equivalent to "$W_A \subseteq W_B$". The simplest likely condition is,

$$\begin{aligned} W_A \subseteq W_B \text{ iff either for some } & C \subseteq A,\ U_C \neq \emptyset \text{ and } V_C = \emptyset, \text{ or} \\ & \text{whenever } D \subseteq B \text{ and } U_D \neq \emptyset, \text{ then there is} \\ & C \subseteq A \text{ with } U_D \subseteq U_C \text{ and } V_D \supseteq V_C. \end{aligned} \tag{$**$}$$

Of course this condition only makes good sense if (DP) holds for $\mathcal{U}$. Related to $(**)$ (in fact a consequence of it if $\mathcal{V}$ contains the empty set) is,

$$W_A \neq \emptyset \text{ iff whenever } B \subseteq A \text{ and } U_B \neq \emptyset, \text{ then } V_B \neq \emptyset. \tag{$*$}$$

Proposition 4.1. *If $\mathcal{U}$ and $\mathcal{V}$ are decidable and $\mathcal{W}$ satisfies $(*)$, $(**)$ and (DP), then $\mathcal{W}$ is decidable.*

Proof. Trivial by the above discussion.

Next we use (DP) to help us to reduce condition $(**)$ to condition $(*)$. First however let us notice a useful consequence of regularity.

Lemma 4.2. *Let $\mathcal{U}$ be a regular basis for (X, F). Suppose the finite collection $\{U_i \mid i \in I\}$ of elements of $\mathcal{U}$ does not cover $U \in \mathcal{U}$. Then there is $U' \in \mathcal{U}$, $U' \cap U \neq \emptyset$, but for all $i \in I$, $U' \cap U_i = \emptyset$. (In application we will be assuming $\mathcal{U}$ closed under finite intersection and then we can take $U' \neq \emptyset$, $U' \subseteq U$).*

Proof. If $\{U_i \mid i \in I\}$ does not cover U, then $\bigcap\{X \setminus U_i \mid i \in I\}$ is an open set (by regularity) which intersects U. But by the definitions of induced topology and basis, any open set is a union of sets from a basis, whence the result.

Proposition 4.3. *If $\mathcal{U}$ and $\mathcal{V}$ are regular, $\mathcal{U}$ satisfies (DP) and $\mathcal{W}$ satisfies $(*)$, then $\mathcal{W}$ satisfies $(**)$.*

Proof. The non-trivial part of $(**)$ is to derive the right-hand side from the left. Suppose then that the right-hand side does not hold. Then if $C \subseteq A$ and $U_C \neq \emptyset$, then $V_C \neq \emptyset$, and there is $D \subseteq B$ with $U_D \neq \emptyset$ such that for all $C \subseteq A$, if $U_D \subseteq U_C$ then $V_D \nsupseteq V_C$. Pick $C \subseteq A$ maximal such that $U_D \subseteq U_C$. Let $I = \{i \mid U_i \nsubseteq U_D$ for some $j, (i, j) \in A\}$. By (DP) for $\mathcal{U}$, $\{U_i \mid i \in I\}$ does not cover U_D. So by Lemma 4.2 we can pick a non-empty U in $\mathcal{U}$ with $U \subseteq U_D$ and $U \cap U_i = \emptyset$ for all i in I. Similarly since $V_C \nsubseteq V_D$, by Lemma 4.2 we can pick a non-empty V in $\mathcal{V}$ with $V \subseteq V_D$ $V \cap V_D = \emptyset$. Let $W_{A'} = W_A \cap [U, V]$. $W_{A'}$ has been constructed so that by applying $(*)$ we can conclude $W_{A'} \neq \emptyset$. But clearly $W_{A'} \cap W_B = \emptyset$ and $W_{A'} \subseteq W_B$. Thus W_A is not included in $W_{B'}$ and the left-hand side of $(**)$ fails.

We have made considerable use of (DP) above, so it is convenient that we can now show that it can be established by induction through the types. Of course the basis $\mathcal{U}^0$ satisfies (DP). Also it is easy to check that if bases $\mathcal{U}$ and $\mathcal{V}$ satisfy (DP) then so does their canonical product. Finally for function spaces we have the following corollary to Proposition 4.3.

Proposition 4.4. *If $\mathcal{U}$ and $\mathcal{V}$ are regular and satisfy (DP), and $\mathcal{W}$ satisfies $(*)$, then $\mathcal{W}$ satisfies (DP).*

Proof. If $W_A \nsubseteq W_B$, the proof of Proposition 4.3 constructs for us a non-empty $W_{A'} \subseteq W_A$ with $W_{A'} \cap W_B = \emptyset$. Of course we know this must be possible by the regularity of $\mathcal{W}$ (see Proposition 3.4 and Lemma 4.2.) If $\mathcal{V}$ satisfies (DP), we can take the V defined in the proof of Proposition 4.3 so that for any $E \subseteq B$ either $V_E \supseteq V_C$ or $V \cap V_E = \emptyset$. But then the $W_{A'}$ defined has the further property that if $W_A \nsubseteq W_B$, for some B', then $W_{A'} \nsubseteq W_{B'}$. By this device we can easily prove by induction that if W_A is not included in any of $W_{B_1}, \ldots, W_{B_n}$, then there is a non-empty $W_{A'} \subseteq W_A$ with $W_{A'} \cap W_{B_i} = \emptyset$ for all $1 \leq i \leq n$. This is (DP) for $\mathcal{W}$.

In view of Propositions 4.1, 4.3 and 4.4 we have reduced the problem of the decidability of the bases for the countable functionals to that of establishing condition (*). Our proof will be a generalization of the trivial proof that (*) holds when $\mathcal{U}$ and $\mathcal{V}$ are bases of clopen sets for zero-dimensional Hausdorff spaces.

The basis $\mathcal{U}$ for the filter space (X, F) is *separated* iff for any finite collection $\{U_i \mid i \in I\}$ of members of $\mathcal{U}$, there is a collection $\{O_i \mid i \in I\}$ of clopen sets (in the induced topology) with $U_i \subseteq O_i$ such that for all $J \subseteq I$, if $\bigcap\{U_i \mid i \in J\} = \emptyset$, then $\bigcap\{O_i \mid i \in J\} = \emptyset$.

Remark. The above condition can be derived from the fact that any two disjoint members of $\mathcal{U}$ can be separated by a set clopen in the induced topology. Rather than give the niggling argument to show this, we work directly with the more general formulation.

Proposition 4.5. If the basis $\mathcal{U}$ for (X, F) is separated, then $\mathcal{W}$ satisfies condition (*).

Proof. We consider the non-trivial implication. Suppose the right-hand side of (*) holds. Let $I = \{i \mid \text{for some } j, (i, j) \in A\}$ and let $\{O_i \mid i \in I\}$ be the collection of clopen sets guaranteed by $\mathcal{U}$'s being separated. For each $B \subseteq A$ such that $V_B \neq \emptyset$, we pick a canonical element $y_B \in V_B$. Now we define a function $f: X \to Y$ as follows:

For $x \in X$ let $J = \{i \mid x \in O_i\}$, and let $B = \{(i, j) \mid (i, j) \in A \text{ and } i \in J\}$, then $f(x) = y_B$.

f is continuous from X with the induced topology to Y with the discrete topology, and hence using adjointness [i.e. Proposition 2.1] from (X, F) to (Y, G). By construction f is in W_A, so $W_A \neq \emptyset$.

It remains to consider when the canonical bases for product and function spaces are separated.

Lemma 4.6. *If O and O' are clopen in (X, F) and (Y, G) respectively, then $O \times O'$ is clopen in $(X \times Y, F \times G)$.*

Proof. $O \times O'$ is open by Proposition 2.2(a) and closed since $(X \times Y \setminus O \times O')$ is the union of the open sets [by proposition 2.2(a)] $X \times (Y \setminus O')$ and $(X \setminus O) \times Y$.

Corollary 4.7. *The canonical product of separated bases is separated.*

Lemma 4.8. *If x is a point of (X, F) and O is clopen in (Y, G), then $[\{x\}, O]$ is clopen in $([X, Y], [F, G])$.*

Proof. $[\{x\}, O]$ is open by Corollary 2.3 and closed as $[X, Y] \setminus [\{x\}, O]$ is $[\{x\}, Y \setminus O]$, again by Corollary 2.3.

Proposition 4.9. *If $\mathcal{V}$ is separated and $\mathcal{W}$ satisfiies $(*)$ then $\mathcal{W}$ is separated.*

Proof. Let $\{W_{A_i} \mid i \in I\}$ be a finite collection of elements of $\mathcal{W}$. Consider $J \subseteq I$ such that $\bigcap \{W_{A_i} \mid i \in J\} = \emptyset$; i.e. such that $W_{A(J)} = \emptyset$ where $A(J) = \bigcup \{A_i \mid i \in J\}$. By $(*)$ for $\mathcal{W}$, there is $C \subseteq A(J)$ such that $U_C \neq \emptyset$ while $V_C = \emptyset$; pick C maximal with this property. Let $C_i = C \cap A_i$. $\bigcap \{V_{C_i} \mid i \in J\} = \emptyset$, so pick (as $\mathcal{V}$ is separated), a collection $\{O_i \mid i \in J\}$ of clopen sets with $V_{C_i} \subseteq O_i$ and $\bigcap \{O_i \mid i \in J\} = \emptyset$. Pick $x \in U$. For $i \in J$, W_{A_i} is included in the clopen set (by Lemma 4.8) $[\{x\}, O_i]$. If we take all such sets for a given i as J varies, their intersection forms a suitable clopen set $P_i \supseteq W_{A_i}$. $\{P_i \mid i \in I\}$ satisfies the conditions to show $\mathcal{W}$ is separated.

The work of this section can be summarized in terms of the cartesian closedness of various categories of filter spaces with bases or enumerated bases. (These are covering categories of various categories of filter spaces as the morphisms are just the filter space ones.)

Theorem 4.10. *The following categories are cartesian closed:*

(i) *the category of filter spaces with separated bases (in this the bases for function spaces satisfy $(*)$);*

(ii) *the category of filter spaces with bases which are separated, regular and satisfy DP (in this the bases for function spaces satisfy $(**)$);*

(iii) *the category of filter spaces with enumerated bases which are decidable, separated, regular and satisfy DP.*

Corollary 4.11 (Decidability Theorem). *The natural bases (defined at the beginning of this section) for the continuous functionals are decidable.*

Remark. Results of much the same force as Corollary 4.11 were proved for their notions of the countable/continuous functionals by Kleene [15] and Kreisel [17]; there is a more general treatment in Tait [27]. Their proofs are limited to special types (i.e. to maps into the natural numbers); our use of the induced topology shows why this restriction is unnecessary. Yet another line of proof for the special case of the continuous functionals is indicated in Gandy, Hyland [10]. Whether the generality of the results we have proved can be put to good use, has yet to emerge. Early treatments were effective from the start. The effective results seem more transparent when obtained by effectivizing the simple and purely mathematical arguments for Propositions 4.5 and 4.9. We treat this in Appendix C.

5. Closure properties of the continuous functionals

$\mathscr{C}$ is a c.c.c., that is the continuous functionals are closed under explicit definition and λ-abstraction. This section presents further closure properties: it is simply a survey of well-known results.

We first present a proof that the continuous functionals are closed under *computation* via Kleene's schemes S1–S9. We suppose the reader familiar with this notion (see Kleene [16]); Gandy, Hyland [10] contains a survey of computation on the continuous functionals, and further developments are in Norman [22] and Wainer [29].

Proposition 5.1. (a) *Let e be the index for a Kleene computation, $f_1, \ldots, f_k$ continuous functionals of appropriate type such that $\{e\}(f_1, \ldots, f_k)$ converges. Let $\Phi_1, \ldots, \Phi_k$ be filters converging to $f_1, \ldots, f_k$ respectively. Then there exist basis sets $V_1, \ldots, V_k$ with $V_i \in \Phi_i$ such that if $g_1 \in V_1, \ldots, g_k \in V_k$ and $\{e\}(g_1, \ldots, g_k)$ converges, then $\{e\}(f_1, \ldots, f_k) = \{e\}(g_1, \ldots, g_k)$.*

(b) *A total functional defined by S1–S9 (possibly with continuous functional parameters) is a continuous functional, i.e. the continuous functionals are closed under computation.*

Proof. (a) and (b) are proved by simultaneous induction on the indices e. We sketch the only tricky induction step, that of application. We say that the basis sets $V_1, \ldots, V_k$ *determine* e (with value n) whenever if $g_1 \in V_1, \ldots, g_k \in V_k$ and $\{e\}(g_1, \ldots, g_k)$ converges then $\{e\}(g_1, \ldots, g_k) = n$. For (a), suppose that

$$\{e\}(f_1, \ldots, f_k) = f_1(\lambda x.\{e'\}(x, f_1, \ldots, f_k))$$

converges. By induction hypothesis (a) we easily see that the finite intersections of elements of

$$S = \{[U, \{n\}] \mid \text{there are } V_1 \in \Phi_1, \ldots, V_k \in \Phi_k \text{ such that } U, V_1, \ldots, V_k \text{ determines } e' \text{ with value } n\}$$

form a filter base converging to the (by induction hypothesis (b)) continuous functional $\lambda x.\{e'\}(x, f_1, \ldots, f_k)$. Thus there is $W \in \Phi_1$ and $[U_1, \{n_1\}], \ldots, [U_r, \{n_r\}] \in S$ such that

$$W([U_1, \{n_1\}] \cap \cdots \cap [U_r, \{n_r\}]) = \{\{e\}(f_1, \ldots, f_k)\}.$$

Now (taking intersections) we can find $V_1', V_2, \ldots, V_k$ from $\Phi_1, \ldots, \Phi_k$ such that

$$U_i, V_1', V_2 \cdots V_k \text{ determines } e' \quad (1 \leq i \leq r).$$

Set $V_1 = W \cap V_1'$ and we have $V_1, \ldots, V_k$ determining e. The deduction of (b) from (a) is trivial.

Two further results follow from the proof of Proposition 5.1, though for their understanding one requires the recursion theoretic ideas presented in the Appendices.

Proposition 5.2. (Kleene [15]). *The continuous functionals computed by S1–S9 are all recursive (in the sense of Appendix B).*

Proposition 5.3. *The Kreisel generalizations of the continuous functionals as described at the close of Section 1, are closed under computation.*

Proofs. Both Propositions 5.2 and 5.3 rest on a obvious effectivization of Proposition 5.1: by the recursion theorem, there is a partial recursive function ϕ such that if e is a Kleene index then $\phi(e)$ is an index for an r.e. set P_e with the following properties:

(1) P_e consists of $k+1$-tuples $(V_1, \ldots, V_k, n)$ such that $V_1, \ldots, V_k$ determines e with value n,

(2) If $\{e\}(f_1, \ldots, f_k) = n$ and $\Phi_1, \ldots, \Phi_k$ are filters converging to $f_1, \ldots, f_k$ respectively then there are $V_1 \in \Phi_1, \ldots, V_k \in \Phi_k$ with $(V_1, \ldots, V_k, n) \in P_e$.

This result gives Proposition 5.2 directly in view of the work in Appendices A and B. Proposition 5.3 follows by observing that *exactly the same proof* goes through for the generalized type structures: we just need the effectivization to show that the set S in the proof of Proposition 5.1 can be replaced by an r.e. set.

Our next result does *not* extend from the continuous functionals to its generalizations (as Proposition 5.1 did). The bar recursive functionals were introduced in Spector [26].

Proposition 5.4. *The continuous functionals are closed under the schemas which define the bar recursive functionals (i.e. Gödel's T and bar recursion of all finite types).*

Proof. (Kreisel, see footnote 6 to Spector [26].) The continuous functionals computed by S1–S9 are closed under the schemes (so we rely on Proposition 5.1). For the non-trivial schemas, primitive recursion and bar recursion, we can by the recursion theorem find indices for partial computable functionals satisfying the schemas. It remains to show that these are total functionals. This is straightforward for primitive recursion. For bar recursion of type σ, the basis of the proof is an induction over a well-founded C_σ-branching tree.

Remarks. (1) This result fails to generalize because one loses well-foundedness.

(2) The continuous functionals computed by S1–S9 form a type structure which is closed under bar recursion but *does not* satisfy the principle of bar induction (footnote 6 to [26]).

(3) When the theory of the continuous functionals is formalized in analysis (essentially via Section 6), the proof of Proposition 5.4 uses the principles of extended bar induction and intuitionistic logic (cf. Ershov [6]); of course one can use dependent choices and classical logic.

(4) The proof-theoretic significance of Proposition 5.4 is indicated in Kreisel [18].

For completeness, we next present a closure property of the continuous functionals which is discussed in detail in Gandy, Hyland [10]. It is related to a special feature of C_2 namely that it can be inductively defined:

C_2 is the least class C of maps: $C_1 \to C_0$ such that,

(i) all constant maps are in C, and

(ii) if $f_0, f_1, \ldots$ are in C, then so is f defined for $\alpha \in C_1$ by $f(\alpha) = f_{\alpha(0)}(\lambda n.\alpha(n+1))$.

Proposition 5.5. *($\mathscr{C}$ contains a functional giving a modulus of continuity on compact subsets of C_1.) Define the compact subsets K_α of C_1 by $K_\alpha = \{\beta \mid (\forall n)\beta(n) \leq \alpha(n)\}$. There is a continuous functional Φ (not a filter!) of type $(2 \times 1) \to 0$ defined by*

$$\Phi(f, \alpha) = (\text{least } n)(\forall \beta, \gamma \in K_\alpha)[(\forall k < n)\beta(k) = \gamma(k) \to f(\beta) = f(\gamma)].$$

Proof. See Gandy, Hyland [10].

Remarks. (1) Both Φ and another important functional Γ from Gandy, Hyland [10] can be defined by recursion over the inductive definition of C_2 in the sense discussed in Hyland [13]. The continuous functionals are closed under definition by recursion over the inductive definition of C_2, but the significance of this is not entirely clear.

(2) In contrast to Proposition 5.5, if we drop the restriction to compact subsets, $\mathscr{C}$ does *not* contain modulus of continuity functionals (see Proposition 7.7). It appears to be impossible to obtain such functionals without dropping extensionality.

Finally we present a useful modification of Proposition 5.5.

Proposition 5.6. *There is a continuous functional Φ^* such that $\Phi^*(f, \alpha) = n$ iff the finite set $e_n = f(K_\alpha)$.*

6. Kleene's definition of the countable functionals

Our aim in this section is to show that the definition of the continuous functionals which we gave in Section 1, is equivalent to that of the countable functionals in Kleene [15]. Kleene defined the countable functionals at a particular type as a subclass of the collection of all functionals at that type. I have modified his definition so that the domain of a countable functional is the set of countable functionals of the appropriate type. A countable functional in this sense corresponds to an equivalence class of countable functionals in Kleene's original sense: two of Kleene's countable functionals of type $n+1$ are equivalent just when they have the same restriction to (Kleene's) countable functionals of type n. Unfortunately, a computation may terminate on one member of an equivalent pair but not on the other. So on Kleene's procedure *partial* computations via S1–S9 become a mess. On the other hand, partial computations on the restricted type structure of the continuous functionals as we have defined them do have a significant theory (Norman [22]). Thus Kleene's proposal to regard the countable (i.e. continuous) functionals as a subset of the maximal type structure seems to have little value.

[N.B. In Kreisel's original paper [17], the continuous functionals were understood as equivalence classes corresponding precisely to the continuous functionals as defined in this paper.]

We give Kleene's definition of the countable functionals of pure type. As usual the pure types will be denoted by numerals. We write K_n for the collection of countable functionals of type n (in Kleene's sense).

Definition (Kleene). (1) K_0 is the set $\mathbf{N}$ of natural numbers, and $n \in K_0$ has the function $\lambda x \cdot n \in \mathbf{N}^{\mathbf{N}}$ as its only associate.

(2) $f: K_n \to K_0$ in K_{n+1} iff it has an associate α_f; $\alpha_f \in \mathbf{N}^{\mathbf{N}}$ is an associate for f iff

(a) if $g \in K_n$, α_g an associate for g, then for some $k \in \mathbf{N}$, $\alpha_f(\overline{\alpha_g(k)})>0$, and

(b) if $g \in K_n$, α_g an associate for g, and k is such that $\alpha_f(\overline{\alpha_g(k)})>0$, then $\alpha_f(\overline{\alpha_g(k)})=f(g)+1$.

This is a simultaneous definition of the spaces K_n and the associates for members of K_n. (As usual when $f\in \mathbf{N}^{\mathbf{N}}$ and $k \in \mathbf{N}$, $\overline{f(k)}$ is the standard code for the sequence $\langle f(0), \ldots, f(k-1)\rangle$.)

We use the variable α_g to range over associates of g, and define for each n,

$$V_u^n = \{g \mid g \in K \text{ and for some } \alpha_g,\ \overline{\alpha_g(\mathrm{lh}(u))} = u\},$$

where u is a sequence number and $\mathrm{lh}(u)$ is its length. For a given α_g, the collection

$$\phi(\alpha_g) = \{V_u^n \mid u = \overline{\alpha_g(\mathrm{lh}(u))}\},$$

is clearly a filter base.

Theorem 6.1. *For all n, K_n is C_n (i.e. the underlying sets are the same); if $f\in K_n$, α_f an associate for f, then $\phi(\alpha_f)$ converges to f in the sense of the filter structure on C_n: finally, the filter bases $\phi(\alpha_f)$ determines the filter structure on C_n, in the sense that a filter converges to f iff it includes some $\phi(\alpha_f)$.*

Proof. By induction on n; the theorem is clear for $n=0$: we suppose it is true for n and show it true for $n+1$. Suppose we are given $F \in K_{n+1}$ with associates α_F. The filter base $\phi(\alpha_F)$ will be considered for the moment in the full space of functions from K_n to K_0. Clearly it is included in the principle filter on F. We show that if $\Phi \downarrow f$ in $C_n(=K_n)$, then $\phi(\alpha_F)(\Phi)\downarrow F(f)$ in $C_0(=K_0)$. From this and the preceding observation, if follows that $F \in C_{n+1}$. It is sufficient to consider the case when Φ is a $\phi(\alpha_f)$. But given α_f, there is a k such that $\alpha_F(\overline{\alpha_f(k)}) = F(f)+1$. Let $u=\overline{\alpha_f(k)}$. Then $V_v^{n+1}(V_u^n)=\{F(f)\}$. Thus $\phi(\alpha_F)(\phi(\alpha_f))$ tends to $F(f)$. This shows not only that $F\in C_{n+1}$, but that $\phi(\alpha_F)\downarrow F$, (where now $\phi(\alpha_F)$ is considered in C_{n+1}).

Suppose now we are given $F \in C_{n+1}$ with $\Theta \downarrow F$. We define α_F by,

$$\alpha_F(u)=\begin{cases} k+1 & \text{if for some } W\in\Theta,\ W(V_u^n)=\{k\},\\ 0 & \textit{otherwise.}\end{cases}$$

We show that α_F is indeed an associate for F. Take α_g any associate for $g \in K_n$; $\phi(\alpha_g)\downarrow g$ so $\Theta(\phi(\alpha_g))\downarrow F(g)$; thus there is $W\in\Theta$ and $V_u^n \in \phi(\alpha_g)$ with $W(V_u^n)=\{F(g)\}$. Hence $\alpha_F(\overline{\alpha_g(\mathrm{lh}(u))}) = F(g)+1$. Since α_g was arbitrary, this shows that α_F is an associate for F. Thus F is in K_{n+1}. Hence C_{n+1} and K_{n+1} are the same (underlying set). What is more, it is clear from the above definition of α_F that $\phi(\alpha_F)\subseteq[\Theta]$. Hence a filter converges to F iff it includes some $\phi(\alpha_F)$. This completes the proof of the induction step.

Corollary 6.2. *For all types σ considered by Kleene, the countable functionals in his sense K_σ, is the same (underlying set) as C_σ.*

Proof. We have this for Kleene's numerical types by essentially the same proof as for Theorem 6.1. Kleene extends the definition to all his types by use of the isomorphisms in a cartesian closed category. Since $\mathscr{C}$ is a c.c.c., the equivalence is immediate.

7. Some simple counterexamples

The continuous functionals of types 0 and 1 are the natural numbers (with the discrete topology) and the usual Baire space; both are topological. For our counterexamples we need a non-topological filter space, it turns out that the continuous functionals of type 2 form such a space. Before we show this however, we introduce a reformulation of some of Section 6 which is more convenient for discussing the structure of C_2.

For a (sequence) number u, we define $V_u \subseteq C_1$ by,

$$V_u = \{f \mid f\in C_1 \quad \text{and} \quad \text{if } i<\mathrm{lh}(u), \quad \text{then} \quad f(i)=(u)_i\}.$$

Each V_u is a clopen neighbourhood in the topology on C_1 and the collection of such V_u's is both a basis (in the usual sense) for the topology, and (hence) a basis for the filter structure on C_1.

Let A be a finite set of pairs of the form (u, p) where u is a sequence number and $p \in \omega$. Define $W_A \subseteq C_2$ by,

$$W_A = \{F \mid F \in C_2 \quad \text{and if} \quad (u,p) \in A, \quad \text{then} \quad F(V_u) = \{p\}\}.$$

Then the collection of all such W_A's is a basis for the filter structure on C_2.

A (new-style) *type 2 associate* α is a function from sequence numbers to N, satisfying,

(i) for all $f \in C_1$, there is an $n \in \omega$, such that $\alpha(\overline{f(n)}) > 0$;

(ii) if $\alpha(u) > 0$, then for any $n \in \omega$, $\alpha(u^*\langle n \rangle) = \alpha(u)$.

(Here * is the usual concatenation operator.)

Every type 2 associate α determines a functional $F \in C_2$ and a filter Φ converging to F by the stipulations,

(i) if $\alpha(u) = k+1$ then F has constant value k on V_u;

(ii) Φ is generated by a filter base consisting of all W_A such that if $(u,p) \in A$, then $\alpha(u) = p + 1$.

That the collection of such Φ determines the filter structure C_2 was shown (in a trivially different formulation) in Theorem 6.1.

Proposition 7.1. *The filter structure on C_2 is not topological.*

Proof. If a filter space is topological, then for each $x \in X$, $F(x)$ the collection of filters tending to x, has a least member, namely the neighbourhood filter at x. But given an associate α_1 for $F \in C_2$, we can define another associate α_2 by stipulating that,

(i) if $\operatorname{lh}(u) = k+1$ and w is such that $u = w^*\langle (u)_k \rangle$, then $\alpha_2(u) = \alpha_1(w)$;

(ii) $\alpha_2(\langle\,\rangle) = 0$.

Then clearly the filter determined by α_2 is strictly included in that determined by α_1. But the filter determined by α_1 is an arbitrary filter converging to F, so there is no least such filter.

Remark. Since the injection of TOP in FIL preserves all function spaces, Proposition 7.1 shows that TOP is not a c.c.c.

Next we prove a simple lemma about the relation between the basis sets and the open sets in C_2.

Lemma 7.2. *Let $W_A \subseteq C_2$ be a basis set with A non-empty (so that W_A is strictly included in C_2). Then the only open set included in W_A is the empty set.*

Proof. It is sufficient to consider the case when A is the singleton $\{(u, p)\}$. Suppose that O is non-empty and $O \subseteq W_A$. Pick $F \in O$. There exists an associate

α for F such that if $\alpha(v)>0$ then $\mathrm{lh}(v)>\mathrm{lh}(u)$. But clearly W_A (and hence O) cannot be a member of the filter determined by such an α. Thus O is not open.

We now give our counterexamples. For $f\in C_1$ define the compact set $K_f\subseteq C_1$ by,

$$K_f=\{g\mid g\in C_1 \quad \text{and for all} \quad n,\ g(n)\leqslant f(n)\}.$$

Now define $O\subseteq C_2\times C_1$ by,

$$O=\{(F,f)\mid F\in C_2, f\in C_1 \quad \text{and} \quad F(K_f)=\{0\}\}.$$

Proposition 7.3. *O is open (in fact clopen) in the induced topology on $C_2\times C_1$; however O includes no non-empty set of the form $O_2\times O_1$ with O_2 open in C_2 and O_1 open in C_1.*

Proof. Proposition 5.6 shows that O is clopen.

Suppose there is non-empty $O_2\times O_1\subseteq O$ with O_2 open in C_2 and O_1 open in C_1.

Pick $(F,f)\in O_2\times O_1$. For some n, $V_{\overline{f(n)}}\subseteq O_1$. Let U be the finite union of basis sets defined by,

$$U=\bigcup\{V_u\mid \mathrm{lh}(u)=n \quad \text{and if} \quad i<n \quad \text{then,} \quad (u)_i\leqslant f(i)\}.$$

Then clearly, $\{G\}\times V_{\overline{f(n)}}\subseteq O$ iff $G(U)=\{o\}$.

Thus $O_2\subseteq\{G\mid G(U)=\{O\}\}$, which is a basis of the form W_A with A non-empty (in fact $A=\{(u,0)\mid \mathrm{lh}(u)=n$ and if $i<n$ then, $(u)_i\leqslant f(i)\}$). But O_2 is open which contradicts Lemma 7.2.

Corollary 7.4 (justifying a remark at the end of Section 2). *The induced topology on a product is not necessarily the product of the induced topologies.*

For $n\in\omega$ and $f\in\omega^\omega$ let $n*f$ be the function whose value for argument 0 is n, and for the argument $k+1$ is $f(k)$. Let z denote the always zero function. for $F\in C_2$, define $h_F\in\omega^\omega$, by,

$$h_F(n)=F(n+1)*z.$$

Now define $O'\subseteq C_2$ by,

$$O'=\{F\mid F(K_{(0)*h_F})=\{0\}\}.$$

Proposition 7.5. *O' is open (in fact clopen) in the induced topology on C_2; however no non-empty finite intersection of sets of the form $[A,U]$ with A compact in C_1 and U open in C_0, is included in O'.*

Proof. O' is clearly clopen as one only needs a finite amount of information about any type 2 associate to determine whether the corresponding functional is in O' or not. For the rest of the proof, we give the argument to show that no $[A, U]$ as above is included in O'; the reader can easily extend this to finite intersections of such sets. If $[A, U]$ is to be included in O', we may safely assume that U is of the form $\{O\}$. Since any compact set in C_1 is included in some compact set of the form K_f, we may assume that A is K_f, for a suitable f. Now define $F \in C_2$ as follows: F takes the value zero except

(i) on $V_{\langle f(0)+1 \rangle}$ where it takes the value $f(f(0)+1)+1$, and

(ii) on $V_{u*\langle f(f(0)+1)+1 \rangle}$ (where u is the sequence of $f(0)+1$ zeros), where it takes the value 1.

Then one easily checks that $F \in [K_f, \{0\}]$ but $F \notin O'$.

Corollary 7.6 (justifying a remark at the end of Section 2). *The induced topology on a function space is not necessarily the compact-open topology.*

At the close of Section 5, we referred to the fact that $\mathscr{C}$ does not contain any (unrestricted) modulus of continuity functionals. We now prove this.

Proposition 7.7 (Kreisel). *There exist no functionals ϕ in $\mathscr{C}$ such that for $F \in C_{(0 \to \sigma) \to 0}$ and $\alpha \in C_{0 \to \sigma}$,*

$$(\forall \beta \in C_{0 \to \sigma})[(\forall k < \phi(F, \alpha))\beta(k) = \alpha(k) \to F(\beta) = F(\alpha)].$$

Proof. To avoid introducing further notation, we give this for the case $\sigma = 0$: the general case is essentially the same. Let ${}^2 0$ and ${}^1 0$ denote the everywhere zero functionals of types 2 and 1 respectively. Suppose there is a functional ϕ as above and let $\phi({}^2 0, {}^1 0) = n$. By considering an associate for ${}^2 0$ which gives value 0 for all u with $\mathrm{lh}(u) \leq n$, we can find a W_A such that

$${}^2 0 \in W_A, \qquad (u, 0) \in A \to \mathrm{lh}(u) > n, \qquad \phi(W_A \times \{{}^1 0\}) = \{n\}.$$

Construct an α such that $\alpha(k) = 0$ for $k < n$, but α extends no u with $(u, 0) \in A$. Let $m > \mathrm{lh}(u)$ for all u with $(u, 0) \in A$ and $m > n$. Define $F \in C_2$ by

$$F(\beta) = \begin{cases} i & \text{if } \beta \text{ extends } \overline{\alpha(m)}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $F \in W_A$ so $\phi(F, {}^1 0) = n$. However for all $k < n$ $\alpha(k) = 0$ while $F(\alpha) \neq F({}^1 0)$. This contradiction proves the theorem.

Remark. For $\sigma = 0$ or 1, for each $F \in C_{(0 \to \sigma) \to 0}$, it is possible to find a continuous functional ϕ_F such that

$$(\forall \alpha \in C_{0 \to \sigma})(\forall \beta \in C_{0 \to \sigma})[(\forall k < \phi_F(\alpha))\alpha(k) = \beta(k) \to F(\alpha) = F(\beta)]. \qquad (*)$$

Thus Proposition 7.7 simply shows that ϕ_F cannot be chosen uniformly in F. However for levels of the type structure greater than 1, it is impossible in general to find a ϕ_{F} such that $(*)$ holds. Essentially this is because for the higher levels $\alpha \in C_{0\to\sigma}$ can be regarded as a code for a pair of elements of types 2 and 1: so we let F decode and apply one to the other: $(*)$ would now provide a modulus of continuity contradicting Proposition 7.7 for $\sigma = 0$.

8. The L-space approach to the continuous functionals

This section gives a further approach to the continuous functionals based on sequence convergence. The topos-theoretic environment in which this material is embedded has been considered in detail in Johnstone [14].

An L-space $(X, \downarrow)$ is a set X together with a relation $\downarrow$, of sequential convergence, between countable sequences $\langle x_i \rangle \in X^{\mathbf{N}}$ and elements $x \in X$, written $x_i \downarrow x$ ("x_i tends to x"), and satisfying the following:

(1) if all but finitely many x_i are x, then $x_i \downarrow x$;

(2) if $x_i \downarrow x$ and $k(0) < k(1) < \cdots < k(n) < \cdots$, then $x_{k(i)} \downarrow x$;

(3) if not $x_i \downarrow x$, then there is $k(0) < k(1) < \cdots < k(n) < \cdots$, such that for no subsequence $l(0) < l(i) < \cdots < l(n) < \cdots$, do we have $x_{l(i)} \downarrow x$.

Remark. When introducing filter convergence, we put very weak restrictions on the notion of a filter space. But for sequential convergence we will need the strong conditions which we have given to establish the topological approach in Section 10, and again in Section 11.

In what follows, we shall never consider more than one L-structure on a given set; so we shall use "$\downarrow$" for the relation of sequential convergence at all times. No confusion should result

If $(X, \downarrow)$ and $(Y, \downarrow)$ are L-spaces, a map $f : (X, \downarrow) \to (Y, \downarrow)$ (i.e. a map between the underlying sets) is continuous iff whenever $x_i \downarrow x$ in X, then $f(x_i) \downarrow f(x)$ in Y. The L-spaces with the continuous maps as morphisms form the category LSP of L-spaces.

If $(X, \downarrow)$ and $(Y, \downarrow)$ are L-spaces, define in $X \times Y$, $(x_i, y_i) \downarrow (x, y)$ iff $x_i \downarrow x$ in X and $y_i \downarrow y$ in Y. This gives an L-structure on $X \times Y$, which is the categorical product of $(X, \downarrow)$ and $(Y, \downarrow)$ in LSP.

We let $[X, Y]$ denote the set of continuous maps from $(X, \downarrow)$ to $(Y, \downarrow)$. In $[X, Y]$, define $f_i \downarrow f$ iff whenever $x_i \downarrow x$ in X, then $f_i(x_i) \downarrow f(x)$ in Y. Kuratowski [20] showed that this gives an L-structure on $[X, Y]$, and his results amount to a proof of the following proposition.

Proposition 8.1. *LSP is cartesian closed; the sequential convergence we have defined on $[X, Y]$ gives the right adjoint to the product.*

Proof. See Kuratowski [20].

Remark. Kuratowski calls our "L-spaces", "L^*-spaces".

In Section 1, we defined the category of the continuous functionals as a certain sub-c.c.c. of FIL, the category of filter spaces. We can consider the same process applied to LSP, the category of L-spaces. $\mathscr{L}$ is the sub-c.c.c. of the c.c.c. LSP, generated by the space of natural numbers (where a sequence $\langle x_i \rangle$ converges to a natural number n iff all but finitely many x_i's are n). Thus, for each type symbol σ, we have an object L_σ of $\mathscr{L}$, where L_0 is the natural numbers as above, $L_{\sigma\times\tau}$ is the product of L_σ and L_τ, and $L_{\sigma\to\tau}$ is the space of functions from L_σ to L_τ. The objects of $\mathscr{L}$, consist of the L_σ's together with the terminal object of LSP (the one point space).

In the next section, we will establish a natural isomorphism between $\mathscr{L}$ and the category $\mathscr{C}$ of the continuous functionals: thus the underlying sets L_σ and $\mathscr{C}_\sigma$ are the same. The category $\mathscr{L}$ (or rather a full subcategory corresponding to a limited collection of types), was considered by Scarpellini [24] as a model for bar recursion of finite types. Scarpellini's results follow very easily from the equivalence of $\mathscr{L}$ and $\mathscr{C}$. Scarpellini has a notion of *constructive* elements of $\mathscr{L}$—these are determined by indices in the manner of the effective operations (other characterizations may be obtained by means of the Kreisel–Lacombe–Schoenfield theorem, cf. Gandy, Hyland [10]). Now an easy application of the recursion theorem shows that the constructive continuous functionals are included in those computed by Kleene's schemes S1–S9 (see Proposition 5.1) and these include the continuous functionals defined by bar recursion: so we have Scarpellini's result that the bar recursive continuous functionals are constructive. Scarpellini also presents variants of $\mathscr{L}$ which are models for bar recursion: what this amounts to is that (hereditarily) the constructive elements are required to be dense in the spaces (i.e. one only considers limits of sequences of such elements). But from Appendix C and Section 9, we see that there are (even) elementary sequences of elementary functions dense in the continuous functionals. So Scarpellini's variants give nothing new.

The L-space approach to the continuous functionals has also been considered by Vogel [28]. His work is entirely in terms of sequence convergence, while the main interest from our point of view is in the applications to topological questions which we discuss in Section 10 and Section 11. (The L-space approach is also useful for applications to the sheaf models for intuitionism; indications are in Hyland [12].)

9. L-spaces and filter spaces

If $\langle x_i \rangle$ is a sequence in X, we write $[x_i]$ for the usual Fréchet filter on X generated by $\langle x_i \rangle$. $[x_i]$ is generated by the filter base consisting of all $\{x_i \mid i \geq m\}$ as $m \in \mathbf{N}$.

A sequence $\langle x_i \rangle$ in X is *eventually in* $U \subseteq X$ iff for some n, if $m \geqslant n$, $x_m \in U$. If $\langle x_i \rangle$ is not eventually in U, it is continually in $X \setminus U$. If $\langle x_i \rangle$ is a sequence in X and Φ a filter base on X, $\langle x_i \rangle$ is *eventually in* Φ iff $\langle x_i \rangle$ is eventually in every member of Φ. Thus $\langle x_i \rangle$ is eventually in Φ iff $[x_i] \supseteq [\Phi]$, in other words, iff $\langle x_i \rangle$ converges more strongly than Φ.

We can now set up a connection between L-spaces and filter spaces.

(1) Let $(X, \downarrow)$ be an L-space. We define a filter structure F on X by,

$$\Phi \in F(x) \text{ iff for some countable filter base } \Psi, \Phi \supseteq \Psi$$
$$\text{and} \quad \text{if } [x_n] \supseteq [\Psi], x_n \downarrow x.$$

(2) Let (X, F) be a filter space. We define a notion of convergence of sequences by,

$$x_i \downarrow x \quad \text{iff} \quad [x_i] \in F(X).$$

Remark. Instead of (1), the reader may expect to see,

$$\Phi \in F(x) \text{ iff for some } x_n \downarrow x, \Phi \supseteq [x_n]$$

Indeed if we dropped the strong axiom (3) in the definition of L-space, we would have a pair of adjoint functors, presenting LSP (without axiom (3)) as the sequential coreflection of FIL. But this would be no use to us here. It would not give us the way to derive the filter structures on the C_σ's from the L-structure on the L_σ's.

Despite the above remark, (1) and (2) interact quite pleasantly.

Theorem 9.1. *Let $(X, \downarrow)$ be an L-space, and (X, F) the corresponding filter space defined by (1). In terms of F, (2) defines the original L-structure on X. Furthermore if $(X, \downarrow)$ and $(Y, \downarrow)$ are L-spaces, $f: X \to Y$ is continuous with respect to the L-structures iff it is continuous with respect to the corresponding filter structures.*

Thus (1) and (2) provide an embedding of LSP as a (full) subcategory of FIL.

Proof. It is easy to see that (2) defines the original L-structure. Now suppose that $f: X \to Y$ is continuous with respect to the L-structures. Let Ψ be a countable filter base on X such that if $[x_n] \supseteq [\Psi]$, $x_n \downarrow x$. Then $f(\Psi)$ is countable. Also if $[y_n] \supseteq [f(\Psi)]$, we can use countability to find a sequence $\langle x_n \rangle$ such that for all n, $f(x_n) = y_n$ and $[x_n] \supseteq [\Psi]$. $x_n \downarrow x$ so $f(x_n) \downarrow f(x)$ i.e, $y_n \downarrow f(x)$. We can clearly deduce that f is continuous with respect to the figter structures.

Conversely if $f: X \to Y$ is continuous with respect to the filter structures and $x_i \downarrow x$ in X, then as $[x_i]$ is a convergent filter, so is $[f(x_i)]$ and we can deduce $f(x_i) \downarrow f(x)$. Thus f is continuous with respect to the L-structures.

For our purposes, we need more than a simple embedding of LSP in FIL. For we need to preserve the cartesian closed structure; and to do this, we must consider a subcategory of LSP. We only really have to worry about the function spaces as our embedding of LSP in FIL clearly preserves products.

Suppose that $(X, \downarrow)$ and $(Y, \downarrow)$ are L-spaces, with (X, F) and (Y, G) the corresponding filter spaces defined by (1) above. $[X, Y]$ unambiguously denotes the set of continuous maps from X to Y, as by Theorem 9.1, this set is the same whether one works in LSP or in FIL. On the other hand, the natural L-structure and filter structures on $[X, Y]$ in these two categories need not be connected by (1) and (2) above. However, there is an important case where this can be guaranteed.

Theorem 9.2. *In the above situation, suppose that (X, F) and (Y, G) have countable bases. Then the natural L-structure and filter structures on $[X, Y]$ are related by* (1) *and* (2) *above.*

Proof. It is sufficient to show that the filter structure on $[X, Y]$ can be obtained from the L-structure by (1) above. Let Θ be a filter converging to $f \in [X, Y]$ in the natural filter structure $[F, G]$. By assumption and Proposition 3.2, this filter structure has a countable basis, hence there is a countable filter base Θ' included in Θ, converging to f. Suppose $[f_n] \supseteq [\Theta']$ then $[f_n]$ converges to f in $[F, G]$. For any $x_n \downarrow x$ in X, $[x_n] \downarrow x$ in the filter structure F; hence $[f_n]([x_n]) \downarrow f(x)$ in G; clearly $[f_n(x_n)] \supseteq [f_n]([x_n])$ so $[f_n(x_n)] \downarrow f(x)$ in G, i.e. $f_n(x_n) \downarrow f(x)$ in the L-structure on Y. Thus if $[f_n] \supseteq [\Theta']$, $f_n \downarrow f$ in the L-structure on $[X, Y]$. So Θ satisfies the condition (1) above.

Conversely, let Θ be a countable filter base such that if $[f_n] \supseteq [\Theta]$, then $f_n \downarrow f$. We wish to show that $\Theta \downarrow f$ in $[F, G]$. We can assume that $\Theta = \{W_i \mid i \in \mathbf{N}\}$ where for all i, $W_i \supseteq W_{i+1}$. It is sufficient to consider $\Phi \downarrow x$ in the filter structure F, where $\Phi = \{U_i \mid i \in \mathbf{N}\}$ and for all i, $U_i \supseteq U_{i+1}$. Suppose that $[y_n] \supseteq [\Theta(\Phi)]$. We can find sequences $\langle f_n \rangle$ and $\langle x_n \rangle$ such that $f_n(x_n) = y_n$ for each n, and $[f_n] \supset \Theta$ and $[x_n] \supseteq \Phi$. Then $f_n \downarrow f$ and $x_n \downarrow x$, so $y_n \downarrow f(x)$. This shows that $\Theta(\Phi) \downarrow f(x)$ in the filter structure G, so $\Theta \downarrow f$ in the filter structure $[F, G]$.

Theorem 9.3. *For all type symbols σ, C_σ is* (*the same underlying set as*) *L_σ and the connection between the filter structure and L-structure is given by* (1) *and* (2) *above.*

Proof. By induction on the types. The result is trivial at type 0, products give no trouble and Theorem 9.2 takes care of function spaces.

Theorem 9.3, which establishes the equivalence of the filter space and L-space approaches to the continuous functionals, clears up the worries on pp. 139–140 of Scarpellini [19]. In particular we can answer a question of his,

Proposition 9.4. *The L_σ's are separable in the sense that there exists a countable collection of elements such that any element is the limit of a sequence chosen from this collection.*

Proof. Simply choose an element from each non-empty U_k^τ in the basis $\mathcal{U}^\tau$. (By Appendix C, we can even do this effectively.)

Remark. Vogel [28] has shown how to prove Proposition 9.4 purely from the point of view of sequence convergence.

We close this section with an example of the conceptual value of the L-space approach to the continuous functionals. A map $F: C_1 \to C_0$ is continuous iff ${}^2E$ is *not* (S1–S9) computable from F and any element of C_1. Many people must be (at least vaguely) aware that this result will extend in some sense through the continuous functionals. Bergstra has given a very refined version of such a result. Leaving aside extreme refinement, we now give the essential content of this at higher types.

Proposition 9.5. *For $n \geq 2$, C_{n+1} is the maximal collection of $F: C_n \to C_0$ such that ${}^2E$ is not elementary (in the sense of Gandy [9]) in F and elements of C_n.*

Proof. The only problem is to show that if $F: C_n \to C_0$ is discontinuous then ${}^2E$ is elementary in F and some elements of C_n.

But if $F: C_n \to C_0$ is discontinuous, then there is $x_k \downarrow x$ in C_n such that $F(x_k) \not\downarrow F(x)$: we may as well assume all $F(x_k)$'s distinct from $F(x)$, but then there is a continuous map $g: C_1 \to C_n$ where

$$g(\alpha) = \begin{cases} x & \text{if } \alpha = \lambda n \cdot 0 \\ x_k & \text{where } k \text{ is least } i \text{ such that } \alpha(i) \neq 0 \text{ otherwise.} \end{cases}$$

We can explicitly define ${}^2E$ in terms of F and g, and elementary typechanging maps now complete the proof.

Remark. We have not used axiom (3) in the definition of L-spaces in this section.

10. The L-topology and sequential spaces

Let $(X, \downarrow)$ be an L-space. A subset O of X is L-open iff whenever $x_n \downarrow x$ and $x \in O$, then $\langle x_n \rangle$ is eventually in O. It is clear that the collection of L-open sets is a topology on X, which we call the *L-topology*.

In a topological space we define a notion of sequential convergence by $x_n \downarrow x$ iff $\langle x_n \rangle$ is eventually inside each open set containing x. This does define an L-structure on the underlying set of the topological space. Indeed it induces an obvious functor from TOP to LSP which is clearly right adjoint to the functor induced by taking the L-topology. It is more to our purpose however to consider just those topological spaces which can be obtained from L-spaces by taking the L-topology. They satisfy the following equivalent conditions on a topological space:

(a) C is closed iff whenever $x_n \downarrow x$ and each $x_n \in C$, then $x \in C$;

(b) O is open iff whenever $x_n \downarrow x$ and $x \in O$, then $\langle x_n \rangle$ is eventually in O.

The spaces satisfying these conditions were first studied by Franklin [8] in a purely topological context, and are known as the sequential spaces. It is easy to see that we can identify SEQ the category of sequential spaces and continuous maps with a subcategory of LSP (by the functor described at the start of this paragraph). The following proposition is then immediate.

Proposition 10.1. *SEQ is a reflective subcategory of LSP.*

Proof. Trivial.

Corollary 10.2. *SEQ is cartesian closed.*

Proof. It suffices to check that the left adjoint functor "taking the L-topology" preserves products (which is obvious) and then apply a simple argument in category theory. (A simple "reflection theorem".)

Remark. SEQ is of course a "coreflective hull" of TOP, and the proof of Corollary 10.2 is in fact by a quite general method for showing that certain such coreflective hulls are cartesian closed. a more complicated general method is in Day [4], who proves a very strong reflection theorem for an arbitrary (i.e. not necessarily cartesian) closed category.

We say that an L-space $\langle X, \downarrow \rangle$ is *sequentially Hausdorff* iff no sequence converges to two distinct points; a topological space is *sequentially Hausdorff* iff the corresponding notion of sequence convergence is. A Hausdorff space is sequentially Hausdorff, but not vice-versa.

Our next result is that if the L-topology on an L-space $\langle X, \downarrow \rangle$ is sequentially Hausdorff, then $x_n \downarrow x$ in the L-space sense iff $x_n \downarrow x$ with respect to the L-topology. So that there should be no confusion, we use "$\downarrow$" only in the L-space sense in the statement and proof of the proposition.

Proposition 10.3. *Let $(X, \downarrow)$ be an L-space such that the L-topology is sequentially Hausdorff. Then $x_n \downarrow x$ iff whenever O is L-open, $x \in O$, then $\langle x_n \rangle$ is eventually inside O.*

Proof. The only problem is the implication from right to left.

Suppose then that $\langle x_n \rangle$ does not tend to x. By axiom (3) of the definition of L-space, we can take a subsequence $\langle y_n \rangle$ of $\langle x_n \rangle$, none of whose subsequences converge to x. We may assume that x does not appear amongst the y_n's (by omitting a finite initial segment of $\langle y_n \rangle$ if necessary). If the *set* $\{y_n \mid n \in \mathbf{N}\}$ is L-closed, its complement is an L-open set O containing x, such that $\langle x_n \rangle$ is not eventually in O. On the other hand, if $\{y_n \mid n \in \mathbf{N}\}$ is not L-closed, there is a sequence $\langle z_n \rangle$ of elements of $\{y_n \mid n \in \mathbf{N}\}$ such that $z_n \downarrow z$ but z is not among the y_n's. Since the L-topology is sequentially Hausdorff, no sequence converges to two distinct points, so $\langle z_n \rangle$ does not converge to any element of $\{y_n \mid n \in \mathbf{N}\}$, and so no y_m can appear infinitely often in the sequence $\langle z_n \rangle$. Thus we can pick a subsequence $\langle w_n \rangle$ of $\langle z_n \rangle$ which is also a subsequence of $\langle y_n \rangle$. Then $w_n \downarrow z$ but $\langle w_n \rangle$ does not converge to x (by our choice of $\langle y_n \rangle$. Thus z and x are distinct. Now we can see that $\{w_n | n \in \mathbf{N}\} \cup \{z\}$ is closed: for otherwise there is a sequence $\langle s_n \rangle$ of elements from the *set* $\{w_n \mid n \in \mathbf{N}\}$ such that $s_n \downarrow t$ and t is distinct from z; but as above $\langle s_n \rangle$ may be chosen to be a subsequence of $\langle w_n \rangle$, contradicting the fact that $(X, \downarrow)$ is sequentially Hausdorff. Thus in the case $\{y_n \mid n \in \mathbf{N}\}$ not closed, the complement of $\{w_n \mid n \in \mathbf{N}\} \cup \{z\}$ is an L-open set containing x with $\langle x_n \rangle$ not eventually inside it. This completes the proof.

Theorem 10.4. *The category of sequentially Hausdorff sequential spaces is isomorphic to the category of sequentially Hausdorff L-spaces.*

Proof. Proposition 10.3 shows that if you start with a sequentially Hausdorff L-space, reflect into SEQ and inject back in LSP, you get back where you started. The isomorphism of categories now follows from adjointness, Proposition 10.1.

Theorem 10.4 indicates that there is a purely topological approach to the continuous functionals. We have but to check that the L-topology on the countable functionals is sequentially Hausdorff; it is in fact Hausdorff as one can either prove directly as we did Corollary 2.5, or by Corollary 2.5 together with Proposition 11.1. Then we simply consider the sub-c.c.c. of SEQ generated by the natural members with the discrete topology. This clearly gives a category isomorphic to $\mathscr{L}$ (as defined in Section 10). This approach has however a limitation which we discuss in Section 11: the sequential topology is not 1st countable for types 2 and above.

Remarks. (1) It is easy to describe the topology on the categorical product and function space in SEQ. The *sequential coreflection* of a space is obtained by taking the L-topology corresponding to the notion of sequential convergence naturally defined on the space. The product in SEQ of two sequential spaces is then the sequential coreflection of the usual product. (That it is not just the usual product follows from the work of Section 7 in view of Proposition 11.1.) To obtain the function space in SEQ, one can take the sequential coreflection of any of the following topologies:

(a) that with subbasis $\{[A, O] \mid A$ sequentially compact and O open$\}$;

(b) that with subbasis

$$\{[A, O] \mid A \text{ the image of a compact Hausdorff, and } O \text{ open}\};$$

(c) that with subbasis

$$\{[A,O] \mid A \text{ of form } \{x_n \mid n \in N\} \cup \{x\} \text{ where } x_n \downarrow x \text{ and } O \text{ open}\}.$$

(2) A careful analysis of the compact subsets of the spaces of continuous functionals shows that SEQ is by no means the unique cartesian closed category of topological spaces from which one can define the continuous functionals. For example they could equally well be obtained from the more familiar category of compactly generated Hausdorff spaces.

11. Failure of first countability

First we prove an equivalence which we will use tacitly throughout the proof of our main result.

Proposition 11.1. *Let $(X, \downarrow)$ be an L-space and let (X, F) be the corresponding filter space as defined by (1) of Section 9. Then O is L-open iff O is open in the induced topology on (X, F).*

Proof. First suppose O is open in the induced topology. For any $x_n \downarrow x$ with $x \in O$, we have $[x_n] \downarrow x$, so $O \in [x_n]$, that is $\langle x_n \rangle$ eventually in O. Thus O is L-open.

Conversely suppose O is L-open. If $\Phi \downarrow x$, $x \in O$ then $\Phi \supseteq \Psi$ where Ψ is countable and $\Psi \downarrow x$. If O is not in $[\Psi]$, there is $\langle x_n \rangle$ such that $[x_n] \supseteq [\Psi]$ and $\langle x_n \rangle$ is not eventually in O; this contradicts the fact that O is L-open, as by (1) of Section 9, $x_n \downarrow x$. Thus O is in $[\Psi]$ and so in $[\Phi]$. Thus O is open in the induced topology.

Corollary 11.2. *Suppose (X, F) and (Y, G) are the corresponding filter spaces of L-spaces, whose induced topology is sequentially Hausdorff. Then a map $f: X \to Y$ is continuous with respect to the filter structures iff it is continuous with respect to the induced topologies. (In particular this conclusion holds for the spaces of countable functionals.)*

Proof. By Theorems 9.1 and 10.4 and Proposition 11.1.

We now establish a limitation of any purely topological approach to the continuous functionals (such as that via SEQ in Section 10). Any such approach involves topologies which are not 1st countable.

Proposition 11.3. *Suppose τ is a topology on C_2 such that the maps from C_1 to C_0 and from C_1 to C_2 continuous with respect to τ, are just those continuous with respect to the filter structure. Then τ is not 1st countable.*

Proof. In the proof, we use $f_n \downarrow f$ in C_2 to mean $\langle f_n\rangle$ tends to f with respect to the standard L-structure on C_2. First we note that if τ is to give the right maps from C_1 to C_2, then $f_n \downarrow f$ will imply that $\langle f_n\rangle$ is eventually in all members of τ containing f.

We proceed to show the converse, that if $\langle f_n\rangle$ does not converge to f in C_2, then there is $O \in \tau$ such that $f \in O$ but $\langle f_n\rangle$ is not eventually in O. Suppose then that $\langle f_n\rangle$ does not tend to f, and without loss of generality that none of its subsequences do so either. Pick $x_n \downarrow x$ in C_1 such that $\langle f_n(x_n)\rangle$ does not converge to $f(x)$. Now since f is continuous, there is $m \in M$ such that for all $n > m$, $f(x_n) = f(x)$. Consider $A = \{x_n \mid n > m\} \cup \{x\}$, which is clearly compact in C_1. By Proposition 2.7, $O = [A, \{f(x)\}]$ is open in the induced topology on C_2. But $C_2 \setminus O = \bigcup \{[A, \{k\}] \mid f(x) \neq k\}$ is also open by Proposition 2.7, so O is clopen. But now if τ is to give the right maps from C_2 to C_0, O must be in τ. However while $f \in O$, clearly $\langle f_n\rangle$ is not eventually in O.

We have now shown that τ must define the standard notion of sequential convergence on C_2. Thus τ is included in the induced topology, (which must be the sequential coreflection of τ). Now suppose that $f \in C_2$ has a countable basis $\{O_i \mid i \in \mathbf{N}\}$ for its neighbourhood system in τ. By what we have shown, if $[f_n] \supseteq [\{O_i \mid i \in \mathbf{N}\}]$, then $f_n \downarrow f$. But by (1) of Section 11, this means that $[\{O_i \mid i \in \mathbf{N}\}] \downarrow f$ in the usual filter structure on C_2. Then there will be non-trivial basis sets U_i^2 in $[\{O_i \mid i \in \mathbf{N}\}]$, contradicting Lemma 7.2.

Corollary 11.4. *The induced topology on C_2 is not 1st countable.*

Proof. By Theorem 10.4 and Propositions 11.1 and 11.3.

Remark. One can prove Corollary 11.4 more directly by observing that the L-topology on C_2 is not Fréchet (cf. Franklin [8]). Indeed C_2 has such a pleasant structure that one can readily find for each countable ordinal α, a set $A_\alpha \subseteq C_2$ such that one must apply the operation "add all sequential limits" exactly α times to A_α to obtain the closure of A_α in the L-topology.

Appendix A. The recursion theory (general discussion)

The aim of this section is to motivate briefly the definition of the recursion theory which I believe is the natural one to consider on the continuous functionals. The familiar definitions of a recursion theory are given by schemes, that is, by an inductive definition of the computation relation "$\{e\}(x) = y$". The definition discussed here does not arise in this way, however. Rather, the fundamental notion turns out to be the degree-theoretic one "$x \leqslant_e y$", and no notion of computation *on the continuous functionals* turns out to be involved in this. Of course there is a corresponding notion of partial recursive continuous functional (Feferman [7], Hyland [13]) but it has no priority.

This is not the place to discuss (the range or applicability of) the notion of constructivity from the classical point of view, but the reader will be familiar with its typical feature, the interpretation of

$$(\forall x)(\exists y)R(x, y)$$

where the variables range over real numbers. (In the usual case $R(\text{-}, \text{-})$ becomes recursive as a relation on the real number generators.) The natural way to regard the situation is to suppose that x is given by a countable sequence of successively better finite approximations and that we are to use these effectively to determine approximations to an appropriate y. To make such an interpretation precise, one must answer the following questions:

(1) What finite approximations should we consider, and what collections of finite approximations (i.e. codes) should determine elements?; that is (adopting the terminology of Kreisel [19]) what choice of data should we make?

(2) what is the appropriate notion of effective operator on the data?

(3) should the interpretation be extensional or intensional?

Question (3) is of great general interest: experience shows that there are great advantages in dropping extensionability (for example there are modulus of continuity functionals for the intensional continuous functionals in contrast to Proposition 7.7). Though in order to develop a theory appropriate for the continuous functionals, we consider an extensional interpretation (i.e. we will demand operators respecting the obvious equivalence relation on codes), this has no bearing on the appropriate answer to question (3) in the general context of constructivity. (Incidentally, an abstract categorical approach to intensional functionals would be of great value; category theory is perfectly adapted to the considerations required. Of course such an approach could not be based *directly* on categories such as FIL.)

The above considerations suggest a basic form for a definition of "recursive in":

x is recursive in y with index e iff the operator with index e, applied to data determining y gives data determining x.

When we first introduced the notion of a filter space, we said that one should think of filters converging to points as ways of approximating to them. We now propose to make some real use of this idea. Typically, of course, a convergent filter Φ is too big for the application of the ideas of effectivity provided by ordinary recursion theory. What is more, Φ may converge to x but contain elements U of which x is not a member; such a U would seem inappropriate as a finite approximation to (or bit of information about) x.

We dispose of the first difficulty by considering as our basic objects of study, filter spaces (X, F) with explicitly enumerated bases $\mathcal{U} = \{U_i \mid i \in \mathbf{N}\}$. As data for determining an element x of X, we consider codes for filter bases included in $\mathcal{U}$ and converging to x.

As for the second difficulty the approach we shall adopt is to legislate it out of existence. We shall only consider codes for filter bases converging to x all of whose members contain x. (In the special case when the basis is regular there is no need for this legislation: regular bases are further considered in Appendix B.)

Remark. It is worth noting at this point that the use of non-topological filter spaces forces one to consider many codes for an element. Of course we are already used to this: the representations of reals by Cauchy sequences of rationals (with given rate of convergence) and by *oscillating* decimals both involve non-uniqueness of representation. (Decimals are not worth considering as addition is not continuous on the product topology—a good example of the importance of question (1) above.) Since spaces may admit a natural filter structure with countable basis, which is well-related (via the induced topology) to a more usual non-1st-countable topology, the theory we are about to describe has wide application.

Non-uniqueness of representation means that while our definition of relative recursion, will be parasitic upon ordinary recursion theory, it will be a non-trivial extension of that theory.

We have said that as data for determining an element of a filter space with basis, we will be using codes for filter bases. Before we have decided on the precise form of these codes, we can agree upon the appropriate notion of effective operator on the data. For suppose that Φ is a filter base converging to x; if U_i is not in Φ, it may still be that U_i is information about x (i.e. that $x \in U_i$). But this makes it ridiculous to use negative information about the data for x. Similarly, it would be silly to attempt to compute such information about data. it is only the positive information that matters. Now the standard effective operators taking positive information to positive information are the *enumeration operators* (or monotonic Σ_1^0 operators) as described in Rodgers [23]. We adopt these as our effective operators on data.

There remains the question what codes for the filter bases should be considered. Suppose we have a filter space (X, F) with basis $\mathscr{U} = \{U_i \mid i \in \mathbf{N}\}$. We could

(i) code a canonical filter base Φ, uniquely as $\{i \mid U_i \in \Phi\}$, or

(ii) allow any set $I \subseteq \mathbf{N}$ such that $\{U_i \mid i \in I\}$ is Φ, as a code for Φ, or

(iii) allow any set $I \subseteq \mathbf{N}$ such that $\{U_i \mid i \in I\}$ is a filter base generating the same filter as Φ, to be a code for Φ.

It turns out, however, that we do not simply need to make a choice here; we have in fact to change our point of view slightly.

How this comes about can be best shown by looking more closely at our choice of enumeration operators as our effective operators on data. An enumeration operator is determined by an r.e. set W_e (which we identify with the operator), with application (as in Scott [25]) defined by,

$$W_e(I) = \{m \mid \text{for some } n, e_n \subseteq I \text{ and } (n, m) \in W_e\} \quad \text{for} \quad I \subseteq N,$$

(where the enumeration of finite sets $\langle e_n \mid n \in \mathbf{N}\rangle$ and pairing functions (,) are as described at the beginning of Section 4). Suppose now that we have filter spaces (X, F) and (Y, G) with bases $\mathcal{U}$ and $\mathcal{V}$, and suppose we have an enumeration operator W_e, taking data for elements of X to data for elements of Y in such a way that it determines a total map from X to Y. Then we should like this map to be a recursive element (i.e. one recursive in something trivial) of $([X, Y], [F, G])$ with the canonical basis for a function space. Let us suppose that for $\mathcal{U} = \{U_i \mid i \in \mathbf{N}\}$, there is an effective map M such that,

$$U_{M(i,j)} = U_i \cap U_j,$$

(in other words $\mathcal{U}$ is *effectively closed under finite intersection*). (Note that this always holds for the canonical basis for a function space.) Then it is easy to convert our enumeration operator W_e to an r.e. code for a filter base converging to the corresponding map from X to Y, in the sense of possibility (iii) mentioned above. But however we may have coded the data for elements of X and Y, we could not hope to provide more restrictive codes in the sense of (ii) or (i) above. Thus we see that we must allow more or less arbitrary codes. But now a problem arises. For we cannot turn an arbitrary r.e. code for an element of $[X, Y]$ into an enumeration operator W_e of the sort we started with. There is no need for such a code to give information relevant to *all* codes for canonical filter bases on X; it may give information for just *some* codes for each filter base. What this shows is that *we must make the choice of data part of the structure we consider.*

The most appropriate situation seems to be this. We consider a filter space (X, F) with enumerated basis $\mathcal{U} = \{U_i \mid i \in \mathbf{N}\}$, effective map $M_{\mathcal{U}}$ giving the intersection of elements of $\mathcal{U}$ as above, and a set $C_F \subseteq P(\mathbf{N})$ such that

(i) each I in C_F is closed under $M_{\mathcal{U}}$;

(ii) if I is in C_F, then $\{U_i \mid i \in I\}$ is a filter base converging to some member of X, and if $\{U_i \mid i \in I\} \downarrow x$, then $x \in U_i$ for all $i \in I$;

(iii) the collection of filter bases $\{U_i \mid i \in I\}$ for I in C_F, generates the filter structure F (i.e. a filter is in F iff it includes $\{U_i \mid i \in I\}$ for some $I \in C_F$).

(iv) if $I \in C_F$, $J \supseteq I$, J closed under M and $\{U_i \mid j \in I\} \downarrow x$ and $x \in U_j$ for all $j \in J$, then $J \in C_F$. We call such a structure $((X, F), \mathcal{U}, M_{\mathcal{U}}, C_F)$ a *coded filter space.* In a coded filter space, we write "$I \downarrow x$" for "$I \in C_F$ and $\{U_i \mid i \in I\} \downarrow x$".

The morphisms between coded filter spaces are just those between the underlying filter spaces. Thus we have a category of coded filter spaces. Given two coded filter spaces $((X, F), \mathcal{U}, M_{\mathcal{U}}, C_F)$ and $((Y, G), \mathcal{V}, M_{\mathcal{V}}, C_G)$ it is easy to see how to construct their product in the category. To form the function space we take $(\langle [X, Y], [F, G]\rangle, \mathcal{W}, M_{\mathcal{W}}, C_{[F,G]}]$ where

(a) $\mathcal{W}$ is the canonical basis for the function space enumerated by setting $W_k = \bigcap \{[U_i, V_j] \mid (i, j) \in e_k\}$,

(b) if $M_{\mathcal{W}}(k, l) = n$, then $e_k \cup e_l = e_n$, and

(c) $K \in C_{[F,G]}$ iff for some $f \in [F, G]$ whenever $I \in C_F$ and $I \downarrow x$, then $K(I) \downarrow f(x)$, (here $K(I)$ is defined to be $\{j \mid (\exists i)(\exists k)(i \in I \ \& \ k \in K \ \& \ (i, j) \in e_k)\}$), and K is closed under $M_{\mathcal{W}}$.

It is easy to check that the function space as defined is a coded filter space, and we obtain the following proposition.

Proposition. *The category of coded filter spaces is cartesian closed; the function space as defined provides the right adjoint to the product.*

In the context of coded filter spaces, we give a precise definition of "recursive in". Let $((X, F), \mathcal{U}, M_{\mathcal{U}}, C_F)$ and $((Y, G), \mathcal{V}, M_{\mathcal{V}}, C_G)$ be coded filter spaces. Let x be a member of X and y a member of Y.

Definition. x *is recursive in* y *with index* e (tacitly with respect to the coded filter spaces) iff whenever $I \downarrow y$ then $W_e(I) \downarrow x$.

Definition. x *is recursive* (tacitly with respect to the coded filter space) iff there is an r.e. I such that $I \downarrow x$.

Remarks. (1) The above framework enables one to consider the constructivity of many operations in mathematics in a uniform way. I believe that the very general view taken above is the correct one. The reader who finds the definitions somewhat ad hoc should be reassured that they also appear naturally in the lattice theoretic approach to the continuous functionals. A paper on this is in preparation. Of course for the continuous functionals, the definitions can be greatly simplified. We discuss the simplified definitions in Appendix B.

(2) It is clear how to give a definition of *recursive equivalence* of coded filter spaces. We have a typical situation in modern mathematics: a collection of structures (here filter spaces) on some of which we can put some additional structure (here giving rise to coded filter spaces); the additional structures determine special maps between the original structures (here the continuous maps which are recursive elements of the canonical function space). The natural notion of equivalence of the additional structures is that the identities (in both directions) be such special maps. (As a typical example of this consider the procedure for putting a differential structure on a manifold). We let the reader formulate for himself the precise definition of recursive equivalence of coded filter spaces, and content ourselves with making a cautionary remark about the concept. In Section 4 and later in Appendix C, we establish various properties of our specific enumerated bases for the continuous functionals; they are decidable and are effective dense bases. These properties of a basis are *not* invariant under recursive equivalence of coded filter spaces. The technical importance of the properties are that they can be used in the induction through the types. It follows from the existence of effective dense bases that we can effectively enumerate a (countable) dense subset of any space of continuous functionals. This latter fact is invariant under recursive equivalence of coded filter spaces and suffices (for example) to

establish quantifier-free axiom of choice for the continuous functionals. But it appears too weak to be of much use for the induction.

(3) The considerations in Hyland [11] about the recursion theory on general spaces are not sufficiently detailed. As a result the importance of taking the coding into the structure did not emerge there.

Appendix B. The intrinsic recursion theory (on the continuous functionals)

In this section we present some simple characterizations of the notions "recursive in" and "recursive" for the continuous functionals. We start with a quite general result which shows that for filter spaces with decidable bases, we may disregard the rest of the structure of a coded filter space.

Proposition 1. *Let (X, F) and (Y, G) be filter spaces with enumerated bases $\mathcal{U}$ and $\mathcal{V}$.*

(a) *Suppose that $\mathcal{U}$ and $\mathcal{V}$ are decidable and closed under intersection. Let $M_{\mathcal{U}}, M_{\mathcal{V}}, C_F, C_g$ be chosen so that $((X, F), \mathcal{U}, M_{\mathcal{U}}, C_F)$ and $((Y, G), \mathcal{V}, M_{\mathcal{V}}, C_G)$ are coded filter spaces. (Note that there will always exist such choices.) Let x be in X and y in Y.*

Then x is recursive in y (in the sense of Appendix A) iff for some e, whenever $\Phi \in G(y)$ and $\Phi \subseteq [y]$, then there is $\Psi \subseteq [x]$ with $\Psi \in F(x)$ such that,

$$W_e(\{j \mid V_j \in \Phi\}) = \{i \mid U_i \in \Psi\}.$$

(b) *Further, if $\mathcal{U}, \mathcal{V}$ above are regular, we can drop the stipulations "$\Phi \subseteq [y]$" and "$\Psi \subseteq [x]$" from* (a).

Proof. (a) is immediate on the existence of enumeration operators which acting on I in C_F (say) produce $\{i \mid U_i \in [\{U_j \mid j \in I\}]\}$ (and similarly for C_G). (b) is just the characterization of Proposition 3.3(a).

Remark. This proposition requires less than the full notion of decidability as defined in Section 4.

Proposition 1 shows that for decidable and regular bases, recursion can be defined in terms of the canonical filter bases. By Proposition 3.3(a), the elements U of a regular basis may be regarded as pieces of information which are *about* any x such that $U \in \Phi$ and $\Phi \downarrow x$, in the obvious sense that *x is in U.*

We have not introduced the continuous functionals as coded filter spaces, and since the bases we gave for them were decidable, we have no need to do so (though there is of course a natural way to do so).

Definition. Consider the continuous functionals with the enumerated bases as defined in Section 4. Let x be in C_σ and y in C_τ. x is *recursive in* y iff for some e, whenever Φ is a canonical filter basis converging to y, there is a canonical filter base Ψ converging to x such that

$$w_e(\{j \mid U_j^\tau \in \Phi\}) = \{i \mid U_i^\sigma \in \Psi\}.$$

x is *recursive* iff for some canonical filter base Φ converging to x, $\{i \mid U^\sigma \in \Phi\}$ is r.e..

Corollary 2. *However we may extend the continuous functionals as filter spaces with enumerated bases (defined in Section 4) to coded filter spaces, the above definitions agree with those given in Appendix A.*

The above definitions are the core of what I have called the *intrinsic recursion theory* on the continuous functionals: the corresponding notion of a *partial recursive functional* is that of a partial map

$$\{e\} : C_\sigma \to C_\tau,$$

where for x in C_σ and y in C_τ,

$$\{e\}(x) = y \text{ iff } y \text{ is recursive in } x \text{ with index } e.$$

A survey of our present (rather slight) knowledge of this subject is in Hyland [13].

The whole emphasis of the recursion theory we are discussing is on partial objects as codes (that is we only use positive information). Our next result shows that for the continuous functionals we could just as well consider codes as total objects.

Theorem 3. *Let Θ be a filter converging to $f \in C_\sigma$; let $\{i \mid U_i^\sigma \in \Theta\}$ be r.e. in some set A; then there is $\Theta^* \subseteq \Theta$ with Θ^* a filter converging to $f \in C_\sigma$ and with $\{i \mid U_i^\sigma \in \Theta^*\}$ recursive in A. Moreover, from an r.e. index (relative to A) for $\{i \mid U_i^\sigma \in \Theta\}$ we can effectively find an index (relative to A) for the characteristic function of $\{i \mid U_i^\sigma \in \Theta^*\}$.*

Proof. For simplicity we consider only the unrelativized version. We may assume that σ is of the form $(\tau \to 0)$ since (see Section 0) every type may be regarded as a product of such types and of type 0. Let $U_i^\sigma \in \Theta$ iff $(\exists y)T(e, i, y)$ where $T(-, -, -)$ is the usual T-predicate. Define a recursive set R by,

$$(i, n) \in R \quad \text{iff} \quad (\exists y \leq i)(\exists x \leq y)[T(e, x, y) \;\&\; U_x^\sigma(U_i^\tau) = \{n\} \;\&\; (\forall k < i)(U_i^\tau \neq U_k^\tau)].$$

(That R is recursive follows from the decidability of the bases for the continuous functionals.)

R contains enough information to determine the behaviour of Θ on any canonical filter base in C_τ, in the sense of the following trivial Lemma.

Lemma A. *For any filter Φ converging to $g \in C_\tau$, there is $U_i^\tau \in \Phi$ such that $(i, f(g)) \in R$.*

We are about to define a (as we shall show) filter Θ^*. For ease of presentation, we adopt the notations and conventions of Section 4 to our situation: we take $\mathcal{U}^\tau$ as the basis $\mathcal{U}$, $\mathcal{U}^0$ as the basis $\mathcal{V}$, and $\mathcal{U}^\sigma$ as the basis $\mathcal{W}$, and shall use both notations in the same formula! Let $\{U_{a_i}^\tau \mid i \in \mathbf{N}\}$ be a recursive enumeration of the non-empty basis sets in $\mathcal{U}^\tau$ (we mean that the set $\{a_i \mid i \in \mathbf{N}\}$ is recursive). We find such an enumeration by the decidability of the basis $\mathcal{U}^\tau$.

We define Θ^* by defining the set $K = \{D \mid W_D \in \Theta^*\}$. K is defined as follows:

$D \in K$ iff whenever $E \subseteq D$, then one of the following three alternatives holds:
(i) $U_E = \emptyset$,
(ii) $V_E = \mathbf{N}$,
(iii) $V_E = \{n\}$
and for the unique k such that $U_{a_k}^\tau = U_E$, we have
(a) $(\exists i \leqslant a_k)[U_i^\tau \supseteq U_E \;\&\; (i, n) \in R]$ and
(b) $(\forall j < k)$ [if $U_{a_j}^\tau \subseteq U_E$, then $(\exists i \leqslant a_j)[U_i^\tau \supseteq U_{a_j}^\tau \;\&\; (j, n) \in R]]$.

We prove some Lemmas about this rather complicated definition.

Lemma B. *If $W_{D'} \subseteq W_D$ and $D \in K$, then $D' \in K$.*

Proof. Let $E' \subseteq D'$. Take the corresponding $E \subseteq D$ guaranteed by condition (**). By (**) if $U_{E'} \neq \emptyset$ and $V_{E'} \neq \mathbf{N}$, then $U_E \neq \emptyset$ and $V_E \neq \mathbf{N}$. So in that case E satisfies (iii) of the above definition. There are then two possibilities.

(1) The unique j such that $U_{a_j}^\tau = U_{E'}$ is less than the k such that $U_{a_k}^\tau = U_E$. Then (b) above for E ensures both (a) and (b) above for E'.

(2) Otherwise. Then (a) for E ensures (a) for E' while (b) for E' follows from (a) and (b) for E.

This completes the proof of Lemma B.

Lemma C. *If D and D' are in k, so is $D \cup D'$.*

Proof. Let $E \subseteq D \cup D'$ be such that $U_E \neq \emptyset$ and $V_E \neq N$.

Then we may assume (say) that $U_{D \cap E} \neq \emptyset$ and that $V_{D \cap E}$ is a singleton $\{n\}$. $D \cap E$ satisfies (iii) above. $U_E \subseteq U_{D \cap E}$ so as in the proof of Lemma B, we can split into two cases from each of which we deduce (iii) for E.

From the above Lemmas it follows that K is the set $\{D \mid W_D \in \Theta^*\}$ for some filter $\Theta^* \subseteq \Theta$.

Lemma D. *Θ^* converges to f.*

Proof. Let the filter Φ converge to $g \in C_\tau$. Pick by Lemma A, $U_i^\tau \in \Phi$ such that $(i, f(g)) \in R$.

Consider the set

$$B = \{a_k \mid U_i^\tau \supseteq U_{a_k}^\tau \text{ and } a_k \geq i \text{ and not}$$
$$(\forall j < k)[\text{if } U_{a_j}^\tau \subseteq U_{a_k}^\tau, \text{ then } (\exists r \leq a_j)(U_r^\tau \supseteq U_{a_j}^\tau \;\&\; (r, f(g) \in R)]\}.$$

If $a_k \in B$, then there is $a_j < i$ such that

$$U_{a_j}^\tau \subseteq U_{a_k}^\tau.$$

Now choose $U_{a_k}^\tau$ in Φ such that $U_{a_k}^\tau \subseteq U_i^\tau$ and $a_k \geq i$ and for all $a_j < i$ $U_{a_j}^\tau \subseteq U_{a_k}^\tau$. (When τ is trivial this is easy; otherwise make use of the regularity of the bases to show that the intersection of any canonical base is a singleton, while no set in the basis is a singleton.) According to the definition of $K, [U_{a_k}^\tau, \{f(g)\}] \in \Theta^*$. Hence $\Theta^*(\Phi)$ converges to $f(g)$. This completes the proof of Lemma D.

The rest of the proof is now obvious: K is clearly recursive with index effectively obtainable from e.

Corollary 4. *Let x be in C_σ and y in C_τ.*

(a) *x is recursive in y iff for some recursive index e, whenever Φ is a canonical filter base converging to y, there is a canonical filter base Ψ converging to x such that*

$$\{e\}(\{j \mid U_j^\tau \in \Phi\}) = \{i \mid U_i^\sigma \in \Psi\}.$$

(b) *x is recursive iff for some canonical filter base Φ converging to x, $\{i \mid U_i^\sigma \in \Phi\}$ is recursive.*

We can also show the equivalence of these definitions for the continuous functionals to ones in terms of associates (see Section 6).

Corollary 5. *Let x be in C_σ and y in C_τ.*

(a) *x is recursive in y iff for some recursive index e, whenever α is an associate for y, then $\lambda n.\{e\}(\alpha, n)$ is an associate for x.*

(b) *x is recursive iff it has a recursive associate.*

Proof. Both results follow easily from Theorem 3 in view of the constructions in the proof of Theorem 6.1. Details are in Hyland [11].

Remarks. (1) (b) is Kleene's own definition. (a) is a definition suggested to me by Gandy; it initiated my research on the continuous functionals.

(2) The above results show that there is an embedding of the degrees of the continuous functionals (with respect to our notion of "recursive in") in the Mass Problems of Medvedev (see Rogers [23]).

(3) The results of this section apply to the Kreisel generalizations of the continuous functionals (see Section 1).

Appendix C. The recursive density theorem

The aim of this appendix is to prove a result (Theorem 5) which is vital both for the study of recursion theory on the continuous functionals, and for applications to constructive mathematics. It is also needed to characterize the (total) effective operations in filter space terms (as indicated at the end of Section 1), and to characterize Scarpellini's notion of "constructive" (see Section 8). These results of this appendix hold equally well for the Kreisel generalizations of the continuous functionals (see close of Section 1).

The proof involves effectivizing material from Section 4 in particular Propositions 4.4 and 4.9. The reader will readily see how to give effective versions of other parts of Section 4 should he ever feel the need.

In what follows, "effective" may be taken to mean "recursive in the sense of Appendix A". In what follows, *we may assume that all bases are decidable*; thus by Proposition 1 of Appendix B, the recursion theory is independent of choice of coding and depends only on the bases. So there is no mention of coding in the definitions and results below. However only very weak conditions are needed on the notion of "effective": in particular for cases where it makes sense (e.g. any generalization of the continuous functionals in the sense of the discussion of Section 1), "effective" could be taken to mean "primitive recursive" in the sense of Kleene [16], or even "elementary" in the sense of Gandy [9].

In discussing effective enumerations, we identify $\mathbf{N}$ with the filter space C_0 with basis $\mathcal{U}^0$. An *effective map* from $\mathbf{N}$ to (X, F) with decidable basis $\mathcal{U}$ is an effective (recursive) element of the space of (continuous) functions from $\mathbf{N}$ to (X, F). We denote by 2 the two element discrete space (with obvious decidable basis).

Let $\mathcal{U} = \{U_i \mid i \in \mathbf{N}\}$ be a decidable basis for (X, F). $\mathcal{U}$ *is an effective base* iff there are effective maps a from $\mathbf{N}$ to (X, F) (with basis $\mathcal{U}$) and b from $\mathbf{N}$ to 2, such that if $U_i \neq \emptyset$, then $a(i) \in U_i$, and b is the characteristic function of $\{i \mid U_i \neq \emptyset\}$. The set $\{a(i) \mid i \in U_i\}$ is then an *effective dense subset* of (X, F). (It is clear that the closure of $\{a(i) \mid i \in U_i\}$ in the induced topology is X.)

An *effective clopen set* in (X, F) with decidable basis $\mathcal{U}$, is an effective map from (X, F) to 2. There is an obvious filter structure and basis for $[X, 2]$, the set of effective clopen sets. $\mathcal{U}$ is *effectively separated* iff there are effective maps (uniformly in k) $a_k : \mathbf{N}^k \to [X, 2]^k$ such that if$(i_1, \ldots, i_k) \in \mathbf{N}^k$ and $a_k(i_1, \ldots, i_k) = (f_1, \ldots, f_k)$ then for all j, $1 \leq j \leq k$, $f_j(U_{i_j}) = \{1\}$ (i.e. the effective clopen set f_i "includes" U_{i_j}) and if $S \subseteq \{1, \ldots, k\}$ is such that $\bigcap \{U_{i_j} \mid j \in S\} = \emptyset$, then $\prod_{i \in S} f_i$ is everywhere zero.

With the above definitions we can effectivize the latter part of Section 4. We use the notation of that section.

Proposition 1. [Effective version of Proposition 4.5]. *If $\mathcal{U}$ is effectively separated and $\mathcal{V}$ is an effective dense base, then $\mathcal{W}$ is an effective dense base.*

Proof. In the proof of Proposition 4.5, we can define f from the $\{O_i \mid i \in I\}$ and the y_B's explicitly (using definition by cases). But by assumption we can get the $\{O_i \mid i \in I\}$ and by y_B's effectively, so f is effective and is defined uniformly from A. (If $W_A = \emptyset$, it doesn't matter what we associate with W_A.)

Proposition 2 [Effective version of (4.7)]. *The canonical product of effectively separated bases is effectively separated.*

Proof. Obvious given Lemma 4.6.

Proposition 3 [Effective version of Proposition 4.9]. *If $\mathscr{U}$ and $\mathscr{W}$ are effective dense bases and $\mathscr{V}$ is effectively separated, then $\mathscr{W}$ is effectively separated.*

Proof. Lemma 4.8 gives a uniform effective method for finding the effective clopen sets needed to effectivize the proof of Proposition 4.9. One obtains the sets needed by a complicated explicit definition using definitions by cases (which are effective by our assumptions).

As we did in Section 4, we can sum up these results using a suitable category.

Theorem 4. *The category of filter spaces with enumerated bases which have an effective dense base and are effectively separated, is cartesian closed.* [*cf.* *Theorem* 4.10(i)].

Theorem 5 (Recursive density theorem). *The natural bases for the continuous functionals (defined at the beginning of Section 4) are effective dense bases. (Here effective can mean elementary (cf. Gandy [7]) and so certainly recursive in the sense of Appendix B.)*

Acknowledgment

I wish to thank my supervisor Dr. Gandy for the advice and encouragement he has given me, and Professor Scott for his interest in and criticism of my work. I am also grateful to the referee for some very penetrating comments both on the presentation and the content of an earlier draft of this paper.

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