Physics Letters B 749 (2015) 167–171

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Quasi-bound states of massive scalar fields in the Kerr black-hole spacetime: Beyond the hydrogenic approximation

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ARTICLE INFO

Article history:
Received 8 July 2015
Received in revised form 25 July 2015
Accepted 28 July 2015
Available online 31 July 2015
Editor: M. Cvetič

ABSTRACT

Rotating black holes can support quasi-stationary (unstable) bound-state resonances of massive scalar fields in their exterior regions. These spatially regular scalar configurations are characterized by instability timescales which are much longer than the timescale M set by the geometric size (mass) of the central black hole. It is well-known that, in the small-mass limit $\alpha \equiv M\mu \ll 1$ (here μ is the mass of the scalar field), these quasi-stationary scalar resonances are characterized by the familiar *hydrogenic* oscillation spectrum: $\omega_R/\mu = 1 - \alpha^2/2\bar{n}_0^2$, where the integer $\bar{n}_0(l,n;\alpha\to 0) = l+n+1$ is the principal quantum number of the bound-state resonance (here the integers $l = 1,2,3,\dots$ and $n = 0,1,2,\dots$ are the spheroidal harmonic index and the resonance parameter of the field mode, respectively). As it depends only on the principal resonance parameter $\bar{n}_0$, this *small*-mass ($\alpha \ll 1$) hydrogenic spectrum is obviously *degenerate*. In this paper we go beyond the small-mass approximation and analyze the quasi-stationary bound-state resonances of massive scalar fields in rapidly-spinning Kerr black-hole spacetimes in the regime $\alpha = O(1)$. In particular, we derive the non-hydrogenic (and, in general, *non*-degenerate) resonance oscillation spectrum $\omega_R/\mu = \sqrt{1-(\alpha/\bar{n})^2}$, where $\bar{n}(l,n;\alpha) = \sqrt{(l+1/2)^2 - 2m\alpha + 2\alpha^2} + 1/2 + n$ is the *generalized* principal quantum number of the quasi-stationary resonances. This analytically derived formula for the characteristic oscillation frequencies of the composed black-hole-massive-scalar-field system is shown to agree with direct numerical computations of the quasi-stationary bound-state resonances.

1. Introduction

Recent analytical [1] and numerical [2] studies of the coupled Einstein-scalar equations have revealed that rotating black holes can support stationary spatially regular configurations of *massive* scalar fields in their exterior regions. These bound-state resonances of the composed black-hole-scalar-field system owe their existence to the well-known phenomenon of superradiant scattering [3,4] of integer-spin (bosonic) fields in rotating black-hole spacetimes.

The stationary black-hole-scalar-field configurations [1,2] mark the boundary between stable and unstable bound-state resonances of the composed system. In particular, these stationary scalar field configurations are characterized by azimuthal frequencies ω_{field} which are in resonance with the black-hole angular velocity Ω_H [5]:

$$\omega_{\text{field}} = m\Omega_H\ , \tag{1}$$

where $m = 1,2,3,\dots$ is the azimuthal quantum number of the field mode. Bound-state field configurations in the superradiant regime $\omega_{\text{field}} < m\Omega_H$ are known to be unstable (that is, grow in time), whereas bound-state field configurations in the regime $\omega_{\text{field}} > m\Omega_H$ are known to be stable (that is, decay in time) [3,4].

The bound-state scalar resonances of the rotating Kerr black-hole spacetime are characterized by at least two different time scales: (1) the typical oscillation period $\tau_{\text{oscillation}} \equiv 2\pi/\omega_R \sim 1/\mu$ of the bound-state massive scalar configuration (here μ is the mass of the scalar field [6]), and (2) the instability growth time scale $\tau_{\text{instability}} \equiv 1/\omega_I$ associated with the superradiance phenomenon. Former studies [7–11] of the Einstein-massive-scalar-field system have revealed that these two time scales are well separated. In particular, it was shown [7–11] that the composed system is characterized by the relation

$$\tau_{\text{instability}} \gg \tau_{\text{oscillation}}\ , \tag{2}$$

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http://dx.doi.org/10.1016/j.physletb.2015.07.071

or equivalently

$$\omega_I \ll \omega_R\ . \tag{3}$$

The strong inequality (2) implies that the bound-state massive scalar configurations may be regarded as the *quasi-stationary* resonances of the composed system.

As shown in [12,13], the physical significance of the characteristic black-hole-scalar-field oscillation frequencies $\{\omega_R(n)\}_{n=0}^{n=\infty}$ [14] lies in the fact that the corresponding quasi-stationary scalar resonances dominate the dynamics of massive scalar fields in curved black-hole spacetimes. In particular, recent numerical simulations [12,13] of the dynamics of massive scalar fields in the Kerr black-hole spacetime have demonstrated explicitly that these quasi-stationary bound-state resonances dominate the characteristic Fourier power spectra $P(\omega)$ of the composed black-hole-massive-scalar-field system [15].

2. The small-mass hydrogenic spectrum

As shown by Detweiler [7], the massive scalar resonances can be calculated analytically in the small-mass regime $M\mu \ll 1$. In particular, one finds [7] that the quasi-stationary bound-state scalar resonances are characterized by the familiar *hydrogenic* spectrum

$$\frac{\omega_R(\bar{n}_0)}{\mu} = 1 - \frac{\alpha^2}{2\bar{n}_0^2} \quad \text{for} \quad \alpha \equiv M\mu \ll 1\ , \tag{4}$$

where the integer

$$\bar{n}_0(l, n; \alpha \to 0) = l + 1 + n \tag{5}$$

is the principal quantum number of the quasi-bound-state resonances. Here the integer $l \geq |m|$ is the spheroidal harmonic index of the field mode and $n = 0, 1, 2, \ldots$ is the resonance parameter.

It is worth emphasizing that the hydrogenic spectrum (4) depends only on the principal resonance parameter (quantum number) $\bar{n}_0 = l + 1 + n$. This small-mass oscillation spectrum is therefore *degenerate*. That is, two different modes, (l, n) and (l', n') with $l + n = l' + n'$, are characterized by the *same* resonant frequency: $\omega_R(l, n) = \omega_R(l', n')$ for $l + n = l' + n'$.

To the best of our knowledge, the oscillation frequency spectrum which characterizes the quasi-stationary bound-state resonances of massive scalar fields in the rotating Kerr black-hole spacetime has not been studied analytically beyond the hydrogenic regime (4) of small ($\alpha \ll 1$) field masses. The main goal of the present paper is to analyze the oscillation spectrum of the composed black-hole-massive-scalar-field system in the $\alpha = O(1)$ regime. To that end, we shall use the resonance equation [see Eq. (11) below] derived in [10] for the bound-state resonances of massive scalar fields in rapidly-rotating (near-extremal) Kerr black-hole spacetimes. As we shall show below, this resonance equation can be solved *analytically* to yield the characteristic oscillation spectrum $\{\omega_R(n)\}_{n=0}^{n=\infty}$ of the quasi-stationary bound-state scalar resonances in the regime $\alpha \lesssim 1$.

3. Description of the system

We consider a scalar field Ψ of mass μ linearly coupled to a rapidly-rotating (near-extremal) Kerr black hole of mass M and dimensionless angular momentum $a/M \to 1^-$. The dynamics of the scalar field in the black-hole spacetime is governed by the Klein–Gordon (Teukolsky) wave equation

$$(\nabla^a\nabla_a - \mu^2)\Psi = 0\ . \tag{6}$$

Substituting the field decomposition [16–18]

$$\Psi(t, r, \theta, \phi) = \int \sum_{l,m} e^{im\phi} S_{lm}(\theta) R_{lm}(r) e^{-i\omega t} d\omega \tag{7}$$

into the wave equation (6), one finds [19,20] that the radial function R and the angular function S obey two ordinary differential equations of the confluent Heun type [21,22].

The angular eigenfunctions, known as the spheroidal harmonics, are determined by the angular Teukolsky equation [19–23]

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial S_{lm}}{\partial\theta}\Big) + \Big[a^2(\omega^2 - \mu^2)\cos^2\theta - \frac{m^2}{\sin^2\theta} + A_{lm}\Big]S_{lm} = 0\ . \tag{8}$$

The regularity requirements of these functions at the two boundaries $\theta = 0$ and $\theta = \pi$ single out a discrete set of angular eigenvalues $\{A_{lm}\}$ [see Eq. (14) below] labeled by the integers l and m.

The radial Teukolsky equation is given by [19,20,24]

$$\Delta\frac{d}{dr}\Big(\Delta\frac{dR_{lm}}{dr}\Big) + \Big\{[(r^2 + a^2)\omega - am]^2 - \Delta(a^2\omega^2 - 2ma\omega + \mu^2 r^2 + A_{lm})\Big\}R_{lm} = 0\ , \tag{9}$$

where $\Delta \equiv (r - r_+)(r - r_-)$ [25]. Note that the angular Teukolsky equation (8) and the radial Teukolsky equation (9) are coupled by the angular eigenvalues $\{A_{lm}\}$.

The quasi-stationary bound-state resonances of the massive scalar fields in the black-hole spacetime are characterized by the boundary conditions of purely ingoing waves at the black-hole horizon (as measured by a comoving observer) and a spatially decaying (bounded) radial eigenfunction at asymptotic infinity [7–9, 26]:

$$R_{lm} \sim \begin{cases} \frac{1}{r}e^{-\sqrt{\mu^2-\omega^2}\,y} & \text{as } r \to \infty \ \ (y \to \infty)\ ; \\ e^{-i(\omega - m\Omega_H)y} & \text{as } r \to r_+ \ \ (y \to -\infty)\ , \end{cases} \tag{10}$$

where Ω_H is the angular velocity of the black-hole horizon [see Eq. (12) below]. The boundary conditions (10) imposed on the radial eigenfunctions single out a discrete set of eigenfrequencies $\{\omega_n(a/M, l, m, \alpha)\}_{n=0}^{n=\infty}$ which characterize the quasi-stationary bound-state resonances of the massive scalar fields in the Kerr black-hole spacetime [7–9].

4. The characteristic resonance equation and its regime of validity

Solving analytically the radial Klein–Gordon (Teukolsky) equation (9) in two different asymptotic regions and using a standard matching procedure for these two radial solutions in their common overlap region [see Eq. (15) below], we have derived in [10] the characteristic resonance equation

$$\frac{1}{\Gamma(\frac{1}{2} + \beta - \kappa)} = \Big[\frac{\Gamma(-2\beta)}{\Gamma(2\beta)}\Big]^2 \frac{\Gamma(\frac{1}{2} + \beta - ik)}{\Gamma(\frac{1}{2} - \beta - ik)\Gamma(\frac{1}{2} - \beta - \kappa)} \times \Big[8iMr_+\sqrt{\mu^2 - \omega^2}(m\Omega_H - \omega)\Big]^{2\beta} \tag{11}$$

for the bound-state resonances of the composed Kerr-black-hole-massive-scalar-field system. Here

$$\Omega_H = \frac{a}{r_+^2 + a^2} \tag{12}$$

is the angular velocity of the black-hole horizon, and

$$k \equiv 2\omega r_+ \quad , \quad \kappa \equiv \frac{\omega k - \mu^2 r_+}{\sqrt{\mu^2-\omega^2}} \quad ,$$

$$\beta^2 \equiv a^2\omega^2 - 2ma\omega + \mu^2 r_+^2 + A_{lm} - k^2 + \frac{1}{4} \, , \tag{13}$$

where $\{A_{lm}\}$ are the angular eigenvalues which couple the radial Teukolsky equation (9) to the angular (spheroidal) equation (8). These angular eigenvalues can be expanded in the form [23]

$$A_{lm} = l(l+1) + \sum_{k=1}^{\infty} c_k a^{2k}(\mu^2-\omega^2)^k \, , \tag{14}$$

where the expansion coefficients $\{c_k\}$ are given in [23].

Before proceeding, it should be emphasized that the validity of the resonance equation (11) is restricted to the regime

$$\tau \ll M(m\Omega_{\text{H}} - \omega) \ll x_0 \ll \frac{1}{M\sqrt{\mu^2-\omega^2}} \, , \tag{15}$$

where $\tau \equiv (r_+ - r_-)/r_+ \ll 1$ is the dimensionless temperature of the rapidly-rotating (near-extremal) Kerr black hole, and the dimensionless coordinate $x_0 \equiv (r_0 - r_+)/r_+$ belongs to the *overlap* region in which the two different solutions of the radial Teukolsky equation (hypergeometric and confluent hypergeometric radial wave functions) can be matched together, see [10,27] for details. The inequalities in (15) imply that the resonance condition (11) should be valid in the regime [28]

$$M^2(m\Omega_{\text{H}} - \omega)\sqrt{\mu^2-\omega^2} \ll 1 \, . \tag{16}$$

5. The quasi-stationary bound-state resonances of the composed black-hole-massive-scalar-field system

As we shall now show, the resonance condition (11) can be solved *analytically* in the physical regime (16). In particular, in the present section we shall derive a (remarkably simple) analytical formula for the discrete spectrum of oscillation frequencies, $\{\omega_{\text{R}}(l,m,\alpha;n)\}_{n=0}^{n=\infty}$, which characterize the quasi-stationary bound-state resonances of the composed Kerr-black-hole-massive-scalar-field system.

Our analytical approach is based on the fact that the right-hand-side of the resonance equation (11) is small in the regime (16) with $\beta \in \mathbb{R}$ [29]. The resonance condition can therefore be approximated by the simple zeroth-order equation

$$\frac{1}{\Gamma(\frac{1}{2}+\beta-\kappa)} = 0 \, . \tag{17}$$

As we shall now show, this zeroth-order resonance condition can be solved analytically to yield the real oscillation frequencies which characterize the bound-state scalar resonances. We first use the well-known pole structure of the Gamma functions [23] in order to write the resonance equation (17) in the form [10]

$$\frac{1}{2} + \beta - \kappa = -n, \tag{18}$$

where the integer $n = 0, 1, 2, \ldots$ is the resonance parameter of the field mode.

Defining the dimensionless variable [30]

$$\epsilon \equiv M\sqrt{\mu^2-\omega^2} \, , \tag{19}$$

one finds from Eq. (13)

$$\beta^2 = \beta_0^2 + O(\epsilon^2, \tau) \quad \text{and} \quad \kappa = \frac{\alpha^2}{\epsilon} - 2\epsilon + O(\tau) \, , \tag{20}$$

Table 1
Quasi-stationary resonances of massive scalar fields in the rotating Kerr black-hole spacetime. The data shown is for the fundamental $l = m = 1$ mode with $a/M = 0.99$, $\alpha \equiv M\mu = 0.42$, and $n = 0, 1, 2, 3, 4$. We display the dimensionless ratio between the *analytically* derived oscillation frequencies $\omega_{\text{R}}^{\text{ana}}(n)$ [see Eq. (24)] and the *numerically* computed resonances $\omega_{\text{R}}^{\text{num}}(n)$ of [12]. One finds a good agreement between the analytical formula (24) and the numerical data of [12].

Resonance parameter n	0	1	2	3	4
$\omega_{\text{R}}^{\text{ana}}(n)/\omega_{\text{R}}^{\text{num}}(n)$	0.9999	1.0006	1.0004	1.0002	0.9994

where

$$\beta_0^2 \equiv (l+1/2)^2 - 2m\alpha - 2\alpha^2 \, . \tag{21}$$

Substituting (20) into the resonance condition $\beta^2 = [\kappa - (n+1/2)]^2$ [see Eq. (18)], one obtains the characteristic equation

$$\epsilon^2 \cdot [(2l+1)^2 - 8m\alpha + 8\alpha^2 - (2n+1)^2] + \epsilon \cdot 4(2n+1)\alpha^2 - 4\alpha^4 + O(\tau, \epsilon^3) = 0 \tag{22}$$

for the dimensionless parameter ϵ. This resonance equation can easily be solved to yield

$$\epsilon(l,m;n) = \frac{2\alpha^2}{\sqrt{(2l+1)^2 - 8m\alpha + 8\alpha^2} + 1 + 2n} \, . \tag{23}$$

Finally, taking cognizance of the relation (19), one finds the discrete spectrum of oscillation frequencies

$$\frac{\omega_{\text{R}}(n)}{\mu} = \sqrt{1 - \Big(\frac{\alpha}{\ell+1+n}\Big)^2} \tag{24}$$

which characterize the quasi-stationary bound-state resonances of the composed black-hole-massive-scalar-field system. Here

$$\ell \equiv \frac{1}{2}\Big[\sqrt{(2l+1)^2 - 8m\alpha + 8\alpha^2} - 1\Big] \tag{25}$$

is the generalized (finite-mass) spheroidal harmonic index. Note that $\ell \to l$ in the small mass $\alpha \ll 1$ limit, in which case one recovers from (24) the well-known hydrogenic spectrum (4) of [7].

It is worth noting that, in general, the parameter $\ell(\alpha)$ is not an integer. This implies that, for generic values of the dimensionless mass parameter α, the non-hydrogenic oscillation spectrum (24) is *not* degenerate [31].

6. Numerical confirmation

It is of physical interest to test the accuracy of the analytically derived formula (24) for the characteristic oscillation frequencies $\omega_{\text{R}}^{\text{ana}}(n)/\mu$ of the quasi-stationary massive scalar configurations. The quasi-bound-state resonances can be computed using standard numerical techniques, see [9,12] for details. In Table 1 we present a comparison between the analytically derived oscillation frequencies (24) and the numerically computed resonances [12]. The data presented is for the fundamental $l = m = 1$ mode with $a/M = 0.99$ [32] and $\alpha = 0.42$ [33,34]. One finds a good agreement between the *analytically* calculated oscillation frequencies (24) and the *numerically* computed resonances of [12].

In order to compare the accuracy of the newly derived analytical formula (24) with the accuracy of the familiar hydrogenic (small-mass) spectrum (4), we display in Table 2 the physical quantity $\epsilon(n)$ [see Eq. (19)] which provides a quantitative measure for the deviation of the resonant oscillation frequency $\omega_{\text{R}}(n)$ from the field mass parameter μ. In particular, we present the dimensionless ratios $\epsilon^{\text{ana}}/\epsilon^{\text{num}}$ and $\epsilon^{\text{ana-hydro}}/\epsilon^{\text{num}}$, where $\epsilon^{\text{ana}}(n)$ is given by the analytical formula (24), $\epsilon^{\text{ana-hydro}}(n)$ is defined from

Table 2
Quasi-stationary resonances of massive scalar fields in the rotating Kerr black-hole spacetime. The data shown is for the fundamental $l = m = 1$ mode with $a/M = 0.99$, $\alpha \equiv M\mu = 0.42$, and $n = 0, 1, 2, 3, 4$. We display the dimensionless ratios $\epsilon^{\text{ana}}/\epsilon^{\text{num}}$ and $\epsilon^{\text{ana-hydro}}/\epsilon^{\text{num}}$, where $\epsilon^{\text{ana}}(n)$ is given by the analytical formula (24), $\epsilon^{\text{ana-hydro}}(n)$ is defined from the hydrogenic spectrum (4), and $\epsilon^{\text{num}}(n)$ is obtained from the numerically computed resonances of [12]. One finds that, in general, the newly derived analytical formula (24) performs better than the hydrogenic formula (4) [35].

Resonance parameter n	0	1	2	3	4
$\epsilon^{\text{ana}}(n)/\epsilon^{\text{num}}(n)$	1.001	0.974	0.971	0.974	1.140
$\epsilon^{\text{ana-hydro}}(n)/\epsilon^{\text{num}}(n)$	0.910	0.916	0.928	0.939	1.107

the hydrogenic spectrum (4), and $\epsilon^{\text{num}}(n)$ is obtained from the numerically computed resonances of [12]. One finds that, in general, the newly derived formula (24) performs better than the hydrogenic formula (4) [35].

7. Summary

In summary, we have studied the resonance spectrum of quasi-stationary massive scalar configurations linearly coupled to a near-extremal (rapidly-rotating) Kerr black-hole spacetime. In particular, we have derived a compact analytical expression [see Eq. (24)] for the characteristic oscillation frequencies $\omega^{\text{ana}}_{\text{R}}(n)/\mu$ of the bound-state massive scalar fields. It was shown that the *analytically* derived formula (24) agrees with direct *numerical* computations [12] of the black-hole-scalar-field resonances.

It is well known that the characteristic hydrogenic spectrum (4) in the *small*-mass $\alpha \ll 1$ limit is highly degenerate – it depends only on the principal resonance parameter $\bar{n}_0 \equiv l+1+n$ [36] [Thus, according to (4), two different modes which are characterized by the integer parameters (l, n) and (l', n') with $l + n = l' + n'$ share the *same* resonant frequency $\omega_{\text{R}}(\bar{n}_0)$ in the $\alpha \to 0$ limit]. On the other hand, the newly derived resonance spectrum (24), which is valid in the $\alpha = O(1)$ regime, is no longer degenerate. That is, for generic values of the dimensionless mass parameter α, two quasi-stationary modes with different sets of the integer parameters (l, n) are characterized, according to (24), by *different* oscillation frequencies [37].

Finally, it is worth emphasizing again that the physical significance of the characteristic oscillation frequencies (24) lies in the fact that these quasi-stationary (long-lived) resonances dominate the dynamics [and, in particular, dominate the characteristic Fourier power spectra $P(\omega)$ [12]] of the massive scalar fields in the black-hole spacetime.

Acknowledgements

This research is supported by the Carmel Science Foundation. I would like to thank Yael Oren, Arbel M. Ongo, Ayelet B. Lata, and Alona B. Tea for helpful discussions.

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[28] In particular, since each inequality in (15) roughly corresponds to an order-of-magnitude difference between two quantities, one concludes that the resonance condition (11) should be valid in the regime $M^2(m\Omega_{\text{H}} - \omega)\sqrt{\mu^2 - \omega^2} \lesssim 10^{-2}$.
[29] One finds that $\beta \in \mathbb{R}$ in the regime $\alpha \le [\sqrt{2(l + 1/2)^2 + m^2} - m]/2$, see Eq. (21) below.
[30] This physical parameter provides a quantitative measure for the deviation of the resonant oscillation frequency $\omega_{\text{R}}(n)$ from the field mass parameter μ.
[31] Compare this property of the non-hydrogenic spectrum (24) with the familiar hydrogenic spectrum (4) which is known to be *degenerate*.
[32] This is the largest black-hole spin for which we have exact (numerical) data [12] for the characteristic oscillation frequencies $\omega_{\text{R}}(n)/\mu$ of the quasi-stationary bound-state resonances. Note that the numerical results of [9] indicate that, for $\alpha \lesssim 0.6$ and $a \to 1^-$, the values of the fundamental resonant frequencies ω_{R}/μ depend very weakly on the black-hole rotation parameter a (see, in particular, Fig. 5 of [9]).
[33] As discussed above, our resonance equation (11) is restricted to the regime $M^2(m\Omega_{\text{H}} - \omega)\sqrt{\mu^2 - \omega^2} \lesssim 10^{-2}$ [28]. Substituting our solution (24) into the left-hand-side of (16), one finds that, for the fundamental $l = m = 1$ mode, the requirement $M^2(m\Omega_{\text{H}} - \omega)\sqrt{\mu^2 - \omega^2} \lesssim 10^{-2}$ is respected in the regime $\alpha \lesssim 0.58$. Note that this mass regime is consistent with our previous requirement $\beta \in \mathbb{R}$ [29], which implies the weaker restriction $\alpha \lesssim 0.67$ for the fundamental $l = m = 1$ mode.

[34] Note that Eqs. (21) and (23) yield $\beta_0^2 \simeq 1.057$ and $\epsilon(n=0) \simeq 0.097$ for the fundamental $l = m = 1$ mode with $\alpha = 0.42$. One therefore finds the characteristic small ratio $\epsilon^2/\beta_0^2 \simeq 0.009 \ll 1$. This strong inequality justifies our previous assumption $\epsilon^2/\beta_0^2 \ll 1$ [see Eq. (20)]. Moreover, since $\epsilon(n)$ is a *decreasing* function of the resonance parameter n [see Eq. (23)], one finds that the approximation $\epsilon^2/\beta_0^2 \ll 1$ becomes even better for the excited ($n \geq 1$) resonant modes.

[35] The only exception is the excited $n = 4$ mode, for which the agreement between the numerically computed deviation-parameter $\epsilon^{\text{num}}(n = 4)$ and the analytical formulas (4) and (24) is quite poor.

[36] Note that this parameter is an integer.

[37] Note that, for generic values of the mass parameter α, the generalized (finite-mass) spheroidal harmonic index $\ell(\alpha)$ that appears in the resonance spectrum (24) is not an integer.