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A characterization of compact-friendly multiplication operators

by Y.A. Abramovich[1], C.D. Aliprantis[2], O. Burkinshaw[1] and A.W. Wickstead[3]

[1] *Department of Mathematical Sciences, IUPUI, Indianapolis, IN 46202-3216, USA*
[2] *Departments of Economics and Mathematics, Purdue University, West Lafayette, IN 47907, USA*
[3] *Department of Pure Mathematics, The Queen's University of Belfast, Belfast BT7 1NN, UK*

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ABSTRACT

Answering in the affirmative a question posed in [3], we prove that a positive multiplication operator on any L_p-space (resp. on a $C(\Omega)$-space) is compact-friendly if and only if the multiplier is constant on a set of positive measure (resp. on a non-empty open set).

In the process of establishing this result, we also prove that any multiplication operator has a family of hyperinvariant bands – a fact that does not seem to have appeared in the literature before. This provides useful information about the commutant of a multiplication operator.

1. PRELIMINARIES

This work will employ techniques and terminology from Banach lattice theory. For terminology which is not explained below, we refer the reader to [4].

In this work the word 'operator' will be synonymous with 'linear operator.' An operator $T: X \to Y$ between two Banach lattices is *positive* if $x \geq 0$ in X implies $Tx \geq 0$ in Y.

A positive operator $S: X \to X$ on a Banach lattice X is said to **dominate** another operator $T: X \to X$ (in symbols, $S \succ T$) if

$$|Tx| \leq S|x|$$

for each $x \in X$. If S dominates T, we shall also say that T *is dominated* by S. Every operator dominated by a positive operator is automatically continuous.

We recall next the notion of a compact-friendly operator that was introduced in [1] and that will play an important role in this work.

Definition 1.1. A positive operator $B : X \to X$ on a Banach lattice is said to be **compact-friendly** if there exist three non-zero operators $R, K, A : X \to X$ with R and K positive and K compact satisfying

$$RB = BR, \quad R \succ A \quad and \quad K \succ A.$$

Regarding the invariant subspace problem for operators on Banach lattices the compact-friendly operators seem to be the analogues of Lomonosov operators. Recall that an operator $T : X \to X$ on a Banach space is a **Lomonosov operator** if there exist non-zero operators $S, K : X \to X$ such that *S is not a multiple of the identity, K is compact, $ST = TS$, and $SK = KS$.*

The invariant subspace theorems for positive operators obtained in [1] (see also [2]) can be viewed as the Banach lattice analogues of the following famous invariant subspace theorem of V.I. Lomonosov.

Theorem 1.2 (Lomonosov [5]). *Every Lomonosov operator T has a non-trivial closed invariant subspace. Moreover, if T itself commutes with a non-zero compact operator, then there exists a non-trivial closed hyperinvariant subspace.*

Besides compact-friendly operators, we shall work here also with multiplication operators on spaces of continuous and measurable functions. If Ω is a compact Hausdorff space and $\varphi \in C(\Omega)$, then a **multiplication operator** M_φ on $C(\Omega)$ is defined by $M_\varphi f = \varphi f$ for each $f \in C(\Omega)$. The function φ is called the *multiplier*.

Similarly, if X is a Banach function space on a measure space (Ω, Σ, μ) and φ is a measurable function, then a multiplication operator M_φ on X is defined by $M_\varphi f = \varphi f$ for each $f \in X$. Observe that a multiplication operator M_φ maps X into itself if and only if the multiplier φ is an (essentially) bounded function. So, for the rest of this paper, whenever we deal with a multiplication operator M_φ on a Banach function space we assume that the multiplier $\varphi \in L_\infty(\mu)$.

It should be noticed that a multiplication operator is positive if and only if its multiplier is a non-negative function.

Obviously each multiplication operator M_φ has non-trivial invariant subspaces and, as was observed in [3], each multiplication operator is a Lomonosov operator. Moreover, as we will prove in the next section (see Theorem 2.2 and Corollary 2.3) each multiplication operator M_φ has hyperinvariant subspaces of a very simple geometrical form, namely, the disjoint bands.

Our next definition describes the kind of multipliers that will be important in our work.

Definition 1.3. A continuous function $\varphi : \Omega \to \mathbb{R}$ on a topological space has a **flat** if there exists a non-empty open set V such that φ is constant on V.

Similarly, a measurable function $\varphi : \Omega \to \mathbb{R}$ on a measure space (Ω, Σ, μ) is said to have a **flat** if φ is constant on some $A \in \Sigma$ with $\mu(A) > 0$.

It was shown in [3] that a positive multiplication operator commutes with a non-zero finite rank operator if and only if the multiplier has a flat. It was then asked whether the flatness condition characterizes also the compact-friendly multiplication operators. The objective of this work is to answer this question affirmatively. Namely, the main result of this paper can be stated as follows.

Theorem 1.4. *A positive multiplication operator M_φ on a $C(\Omega)$-space or on a $L_p(\Omega, \Sigma, \mu)$-space $(1 \leq p \leq \infty)$ is compact-friendly if and only if the multiplier φ has a flat.*

2. THE COMMUTANT OF A MULTIPLICATION OPERATOR

In this section X will denote a Banach function space on a fixed measure space (Ω, Σ, μ). Let $M_\varphi : X \to X$ be the multiplication operator with a multiplier $\varphi \in L_\infty(\mu)$.

Not much is known about the commutant of M_φ. The following discussion will provide some important insights into the structure of the commutant. We precede this discussion by fixing some notation. If $f : \Omega \to \mathbb{R}$ is a function, then its support, Supp(f), is defined by

$$\text{Supp}(f) = \{\omega \in \Omega : f(\omega) \neq 0\}.$$

If $A, B \in \Sigma$, then relations $A \subseteq B$ a.e. and $A = B$ a.e. are understood as usual μ-a.e. For example, $A \subseteq B$ a.e. means that $\mu(\{\omega \in A : \omega \notin B\}) = 0$.

Definition 2.1. Let $T : X \to X$ be a continuous operator and let $E \subseteq \Omega$ be a measurable subset of positive measure. We shall say that T **leaves** E **invariant**, if

$$x \in X \quad \textit{and} \quad \text{Supp}(x) \subseteq E \quad \textit{implies} \quad \text{Supp}(Tx) \subseteq E \textit{ a.e.}$$

This definition is, of course, motivated by a simple observation that an operator T leaves a (measurable) set E invariant if and only if T leaves invariant the band

$$B_E = \{f \in X : f = 0 \text{ on } \Omega \setminus E\}$$

generated by E in X. It is obvious that *if an operator $T : X \to X$ leaves invariant the sets E and F, then it also leaves invariant the sets $E \cap F$ and $E \cup F$.*
Now let us introduce some more notation. For each $\alpha \in \mathbb{R}$, let

$$E_\alpha = \{\omega \in \Omega : \varphi(\omega) \geq \alpha\} \quad \text{and} \quad E^\alpha = \{\omega \in \Omega : \varphi(\omega) \leq \alpha\}.$$

If we need to emphasize that the level set E_α is produced by the function φ, then we shall write $E_\alpha(\varphi)$ instead of E_α. For $\alpha \leq \beta$, we also write

$$E_\alpha^\beta = E_\alpha \cap E^\beta = \{\omega \in \Omega : \alpha \leq \varphi(\omega) \leq \beta\}.$$

And now we come to a simple but important result asserting that all the bands in X generated by the level sets introduced above are left invariant by each operator commuting with M_φ.

Theorem 2.2. *Every operator in the commutant of M_φ leaves invariant all the sets E_α, E^α and E_α^β.*

Proof. Let $R : X \to X$ be a bounded operator commuting with M_φ. We begin by considering the sets E^α. Assume that $\varphi \geq 0$. First we will verify that R leaves invariant the set E^α with $\alpha = 1$, i.e., the set

$$E^1 = E^1(\varphi) = \{\omega \in \Omega : \varphi(\omega) \leq 1\}.$$

To do this, assume by way of contradiction that R does not leave E^1 invariant. This means that there exists some function $x \in X$ with $\mathrm{Supp}(x) \subseteq E^1$ and such that the measurable set $A = \{\omega \in \Omega : Rx(\omega) \neq 0 \ \& \ \varphi(\omega) > 1\}$ has positive measure. Pick some $\gamma > 1$ such that $B = \{\omega \in \Omega : Rx(\omega) \neq 0 \ \& \ \varphi(\omega) > \gamma\}$ has positive measure.

The commutativity property $RM_\varphi = M_\varphi R$ easily implies

$$(*) \qquad R(\varphi^n x) = \varphi^n Rx$$

for each n. Let $\|\cdot\|$ denote the norm on X. We shall reach a contradiction by computing the norm of the function in $(*)$ in two different ways. On one hand, the hypothesis $\mathrm{Supp}(x) \subseteq E^1$ and the fact that $0 \leq \varphi(\omega) \leq 1$ on E^1 imply that $|\varphi^n x| \leq |x|$, and so

$$\|R(\varphi^n x)\| \leq \|R\| \cdot \|\varphi^n x\| \leq \|R\| \cdot \|x\| < \infty.$$

On the other hand, for the element $y = |(Rx)\chi_B| \in X$ we have

$$0 < \gamma^n y \leq |\varphi^n (Rx)\chi_B| \leq |\varphi^n Rx|,$$

whence

$$0 < \gamma^n \|y\| \leq \|\varphi^n Rx\| = \|R(\varphi^n x)\| \leq \|R\| \cdot \|x\| < \infty$$

for each n, contradicting the fact that $\gamma > 1$. Hence, R leaves $E^1(\varphi)$ invariant.

Let us verify now that R leaves invariant each E^α with $\alpha > 0$. Consider $\psi = \alpha^{-1}\varphi$. Obviously the multiplication operator M_ψ also commutes with R and $E^1(\psi) = E^\alpha(\varphi)$. By the previous part R leaves E^α invariant.

Since $E^0 = \cap_{\alpha > 0} E^\alpha$ we see that R leaves E^0 invariant as well. Since $\varphi \geq 0$ the set $E^\alpha = \emptyset$ whenever $\alpha < 0$. Thus, for $\varphi \geq 0$ we have proved that R leaves any set E^α invariant. The assumption made at the beginning of the proof that the multiplier φ is nonnegative can be easily disposed of. Indeed, pick any $t > 0$ such that the function $\psi = \varphi + t\mathbf{1}$ is positive. Obviously M_ψ commutes with R (since M_φ does) and $E^\alpha(\varphi) = E^{\alpha+t}(\psi)$. By the preceding part R leaves $E^{\alpha+t}(\psi)$, that is $E^\alpha(\varphi)$, invariant.

Finally notice that $E_\alpha(\varphi) = E^{-\alpha}(-\varphi)$. This shows that the case of the sets E_α follows immediately from the case of the sets E^α considered above. □

Corollary 2.3. *If $\varphi \in L_\infty(\mu)$ is a non-constant function, then the multiplication operator M_φ has a non-trivial hyperinvariant band. If the (essential) range of the multiplier φ is an infinite set, then M_φ has infinite many disjoint hyperinvariant bands.*

Consider also the following three additional types of the level sets associated with the multiplier φ:

$$\{\omega \in \Omega : \alpha \leq \varphi(\omega) < \beta\}, \{\omega \in \Omega : \alpha < \varphi(\omega) < \beta\}$$

$$\text{and } \{\omega \in \Omega : \alpha < \varphi(\omega) \leq \beta\}.$$

It is easy to see that if R is order continuous (and commutes with M_φ) then R leaves also each of these sets invariant. In particular this is so if the norm on X is order continuous. However, quite surprisingly, it may happen that without this extra assumption the operator R may fail to leave these latter sets invariant.

Even when X has order continuous norm (and so R leaves invariant so many mutually disjoint bands) it is not true in general that R leaves invariant any band. Furthermore, as we shall see in the next example R may even fail to be a disjointness preserving operator. (Recall that an operator R on a vector lattice is said to preserve disjointness if R carries disjoint vectors to disjoint vectors.)

- A positive operator $R : L_\infty \to L_\infty$ commuting with M_φ need not be disjointness preserving even if φ has no flat.

To see this take μ to be the usual 2-dimensional Lebesgue measure on $[0,1] \times [0,1]$ and $\varphi(x,y) = y$. Let $Rf(x,y) = \int_0^1 f(t,y)dt$, then it is easy to see that R commutes with M_φ, the multiplier φ has no flat but R is not disjointness preserving. [If φ has a flat, then the existence of R as required is obvious].

3. MULTIPLICATION OPERATORS ON C(Ω)-SPACES

We start with a useful general criterion for distinguishing between compact-friendly and non-compact-friendly operators on a Banach lattice with order continuous norm.

Proposition 3.1. *Let $A : Y \to Y$ be an operator on a Banach lattice dominated by a positive compact operator. Then for any norm bounded sequence $\{e_n\}$ the following two statements are true.*

1. *The sequence $\{Ae_n\}$ has an order bounded subsequence.*
2. *If Y has order continuous norm and $\{Ae_n\}$ is disjoint, then $\|Ae_n\| \to 0$.*

Proof. (1) Let $K : Y \to Y$ be a compact positive operator dominating A, i.e., $|Ax| \leq K|x|$ holds for each $x \in Y$. Since K is a compact operator and $\{e_n\}$ is a norm bounded sequence, we can extract from $\{K(|e_n|)\}$ a convergent subsequence. Without loss of generality we can assume that the sequence $\{K(|e_n|)\}$ itself converges in Y, that is, there exists $y \in Y$ such that $K|e_n| \to y$. By passing to another subsequence if necessary, we can also assume without loss of generality that $\|K|e_n| - y\| < 2^{-n}$ holds for each n. Letting $e = \sum_{n=1}^\infty |K|e_n| - y|$ we see that $e \in Y^+$ and clearly $|K|e_n| - y| \leq e$, whence $K|e_n| \leq e + |y|$ for each n. It remains to note that

$$|Ae_n| \leq K|e_n| \leq e + |y|$$

for each n.

(2) Assume that $\{Ae_n\}$ is a disjoint sequence and let $\{f_n\}$ be a subsequence of $\{e_n\}$. By part (1), there exists a subsequence $\{g_n\}$ of $\{f_n\}$ (and hence of $\{e_n\}$) such that the pairwise disjoint sequence $\{Ag_n\}$ is order bounded. Since Y has order continuous norm, it follows that $Ag_n \to 0$ in Y; see [4, Theorem 12.13, p. 183]. Thus, we have shown that every subsequence of $\{Ae_n\}$ has a subsequence convergent to zero, and consequently $Ae_n \to 0$ in Y. □

The next theorem is a characterization of the compact-friendly multiplication operators on $C(\Omega)$-spaces.

Theorem 3.2. *A positive multiplication operator M_φ on a $C(\Omega)$-space is compact-friendly if and only the multiplier φ has a flat.*

Proof. Let $0 \leq \varphi \in C(\Omega)$. If φ is constant on a non-empty open subset of Ω, then M_φ commutes with a non-zero positive rank-one operator (see [3, Theorem 2.6]), and so M_φ is compact-friendly.

For the converse, assume that M_φ is compact-friendly, and consequently there exist non-zero bounded operators R, K, $A : C(\Omega) \to C(\Omega)$ with R, K positive, K compact and such that

$$M_\varphi R = RM_\varphi, \quad R \succ A \quad \text{and} \quad K \succ A.$$

Taking adjoints, we see that

$$M^*_\varphi R^* = R^* M^*_\varphi, \quad R^* \succ A^* \quad \text{and} \quad K^* \succ A^*.$$

The following three properties follow in a rather straightforward way.

(1) For each $\omega \in \Omega$ the support of the measure $R^*\delta_\omega$ is contained in the set $W_\omega = \varphi^{-1}(\varphi(\omega))$, where δ_ω denotes the unit mass at ω. This claim is immediate from consideration of the identity

$$M^*_\varphi R^*\delta_\omega = R^* M^*_\varphi \delta_\omega = \varphi(\omega) R^*\delta_\omega.$$

(2) Since $R \succ A$, it follows immediately from (1) that for each $\omega \in \Omega$ the measure $A^*\delta_\omega$ is also supported by W_ω.

(3) Pick $h \in C(\Omega)$ with $\|h\| = 1$ and $Ah \neq 0$. Next, choose a non-empty open set U on which $|Ah(\omega)| \geq \varepsilon > 0$ for some $\varepsilon > 0$. Then for each $\omega \in U$ we have $\|A^*\delta_\omega\| \geq \varepsilon$. Indeed, to see this, notice that

$$\|A^*\delta_\omega\| \geq |\langle A^*\delta_\omega, h\rangle| = |\langle \delta_\omega, Ah\rangle| = |Ah(\omega)| \geq \varepsilon.$$

To complete the proof, assume by way of contradiction that the set W_ω has an empty interior for each $\omega \in \Omega$. Then the non-empty open set U, chosen in (3) must meet infinitely many sets W_ω. Pick a sequence $\{\omega_n\}$ in U with

$\varphi(\omega_m) \neq \varphi(\omega_n)$ if $m \neq n$, and let $e_n = |A^*\delta_{\omega_n}|$ for each n. Then $\|e_n\| \geq \varepsilon$ for each n. Furthermore, since each e_n is supported by the set W_{ω_n} and the sequence $\{W_{\omega_n}\}$ is pairwise disjoint, the sequence $\{A^*\delta_{\omega_n}\}$ is also disjoint. However, by Proposition 3.1 (which is applicable since the norm in $C(\Omega)^*$ is order continuous) we should have $\|A^*\delta_{\omega_n}\| \to 0$, a contradiction. This completes the proof of the theorem. □

Since each $L_\infty(\mu)$ space can be represented as $C(\Omega)$ space on its Stone space, the previous theorem implies immediately the following result.

Theorem 3.3. *A multiplication operator M_φ on L_∞, where $\varphi \in L_\infty(\mu)$, is compact-friendly if and only if its multiplier φ has a flat.*

4. COMPACT-FRIENDLY MULTIPLICATION OPERATORS ON L_p-SPACES

For the rest of our discussion, (Ω, Σ, μ) will denote a fixed measure space, and $\|\cdot\|$ will denote the standard norm on $L_p(\mu)$. The main result in this section is the following L_p-version of Theorems 3.2 and 3.3.

Theorem 4.1. *A multiplication operator M_φ on an arbitrary $L_p(\mu)$-space, where $0 \leq \varphi \in L_\infty(\mu)$ and $1 \leq p < \infty$, is compact-friendly if and only if φ has a flat.*[1]

Proof. It was shown in [3] that if φ has a flat, then M_φ commutes with a positive rank-one operator – and hence M_φ is compact-friendly.

In the converse direction, assume that M_φ is compact-friendly and that, contrary to our claim, φ is not constant on any set of positive measure. Pick three non-zero bounded operators R, A, $K : L_p(\mu) \to L_p(\mu)$ such that R and K are positive, K is compact and

$$RM_\varphi = M_\varphi R, \quad R \succ A \quad \text{and} \quad K \succ A.$$

To obtain a contradiction, it will suffice (in view of Proposition 3.1) to construct a sequence $\{e_n\}$ in $L_p(\mu)$ satisfying the following properties:

(i) $\|e_n\| = 1$ for each n,
(ii) $\{Ae_n\}$ is a disjoint sequence, and
(iii) $\|Ae_n\| \geq \delta$ for each n and for some $\delta > 0$.

The construction of such a sequence is quite involved and will be presented in a series of lemmas below. □

The rest of this section will be devoted to construction of a sequence $\{e_n\}$ that satisfies the properties (i), (ii) and (iii) stated at the end of the proof of our Theorem 4.1. We begin with some preliminary comments.

[1] We do not know if this theorem is true for arbitrary Banach function spaces.

1. The assumption that φ does not have a flat means that for each $\gamma \geq 0$ the set $E_\gamma^\gamma = \{\omega \in \Omega : \varphi(\omega) = \gamma\} = \varphi^{-1}(\{\gamma\})$ has measure zero. In particular, this implies that for any $\gamma \in (\alpha, \beta)$ the level sets E_α^γ and E_β^γ are essentially disjoint (in the sense that $E_\alpha^\gamma \cap E_\gamma^\beta = E_\gamma^\gamma$ is a set of measure zero).

2. By Theorem 2.2 the operator R leaves all the level sets of φ invariant, and so does the operator A since it is dominated by R.

3. Since $A \neq 0$ there exists some $x \in L_p(\mu)$ with $y = Ax \neq 0$. The functions x and y will be fixed throughout the discussion in this section. If we let $\alpha_0 = 0$ and $\beta_0 = \|\varphi\|_\infty$, then obviously $E_{\alpha_0}^{\beta_0} = \Omega$ and so

$$\operatorname{Supp}(x) \subseteq E_{\alpha_0}^{\beta_0}.$$

Lemma 4.2. *There exists some $\gamma_0 \in (\alpha_0, \beta_0)$ such that*

$$\|y\chi_{E_{\alpha_0}^{\gamma_0}}\| = \|y\chi_{E_{\gamma_0}^{\beta_0}}\| = c\|y\|,$$

where $c = 1/\sqrt[p]{2}$.

Proof. Consider the function $N : [\alpha_0, \beta_0] \to \mathbb{R}$ defined by

$$N(\gamma) = \|y\chi_{E_{\alpha_0}^{\gamma}}\|.$$

Clearly, $N(\alpha_0) = 0, N(\beta_0) = \|y\|$, and the function N is continuous by virtue of the 'no flats' assumption about φ. Therefore, there exists some $\gamma_0 \in (\alpha_0, \beta_0)$ such that $N(\gamma_0) = c\|y\|$.

Since $y\chi_{E_{\alpha_0}^{\gamma_0}} + y\chi_{E_{\gamma_0}^{\beta_0}} = y$, and since the sets $E_{\alpha_0}^{\gamma_0}, E_{\gamma_0}^{\beta_0}$ are essentially disjoint, the p-additivity of the norm in $L_p(\mu)$ implies that

$$\|y\chi_{E_{\alpha_0}^{\gamma_0}}\|^p + \|y\chi_{E_{\gamma_0}^{\beta_0}}\|^p = \|y\|^p.$$

Consequently,

$$\|y\chi_{E_{\gamma_0}^{\beta_0}}\|^p = \|y\|^p - \|y\chi_{E_{\alpha_0}^{\gamma_0}}\|^p = \|y\|^p - c^p\|y\| = \frac{1}{2}\|y\|^p = c^p\|y\|^p,$$

that is, $\|y\chi_{E_{\gamma_0}^{\beta_0}}\| = c\|y\|$, as required. □

Using the sets $E_{\alpha_0}^{\gamma_0}$ and $E_{\gamma_0}^{\beta_0}$ we can represent x as

$$x = x\chi_{E_{\alpha_0}^{\gamma_0}} \oplus x\chi_{E_{\gamma_0}^{\beta_0}},$$

and denote by a_1 the summand with smaller (or equal) norm. The other summand will be denoted by b_1. So, if $\|x\chi_{E_{\alpha_0}^{\gamma_0}}\| \leq \|x\chi_{E_{\gamma_0}^{\beta_0}}\|$, then we let $a_1 = x\chi_{E_{\alpha_0}^{\gamma_0}}$ and $b_1 = x\chi_{E_{\gamma_0}^{\beta_0}}$, and thus

$$x = a_1 \oplus b_1.$$

Having chosen a_1 and b_1, we let

$$u_1 = y\chi_{E_{\alpha_0}^{\gamma_0}} \quad \text{and} \quad v_1 = y\chi_{E_{\gamma_0}^{\beta_0}}$$

and also $\alpha_1 = \alpha_0$ and $\beta_1 = \gamma_0$. (However, if $\|x\chi_{E_{\gamma_0}^{\beta_0}}\| < \|x\chi_{E_{\alpha_0}^{\gamma_0}}\|$, then $a_1 = x\chi_{E_{\gamma_0}^{\beta_0}}$, $b_1 = x\chi_{E_{\alpha_0}^{\gamma_0}}$, and we let $u_1 = y\chi_{E_{\gamma_0}^{\beta_0}}$ and $v_1 = y\chi_{E_{\alpha_0}^{\gamma_0}}$, so that the functions u_1 and a_1

are supported by the same set. In this case we accordingly choose $\alpha_1 = \gamma_1$ and $\beta_1 = \beta_0$.)

In accordance with our construction the support sets of u_1 and v_1 are the disjoint sets $E_{\alpha_0}^{\gamma_0}$ and $E_{\gamma_0}^{\beta_0}$ respectively, which are left invariant by A. The same disjoint sets are the support sets of the elements a_1 and b_1. This implies (in view of the equality $x = a_1 \oplus b_1$) that $y = Ax = Aa_1 \oplus Ab_1$, and therefore

$$Aa_1 = u_1 \quad \textit{and} \quad Ab_1 = v_1.$$

In the next lemma, we present some simple estimates on the norms of a_1 and b_1.

Lemma 4.3. *For the functions a_1 and b_1 introduced above, we have:*

$$c\frac{\|y\|}{\|A\|} \leq \|a_1\| \leq c\|x\| \quad \textit{and} \quad \|b_1\| \leq c_1\|x\|,$$

where $c = 1/\sqrt[p]{2}$ and $c_1 = [1 - (\frac{c\|y\|}{\|A\| \cdot \|x\|})^p]^{1/p} > 0$.

Proof. Since $a_1 \oplus b_1 = x$ and $\|a_1\| \leq \|b_1\|$, the p-additivity of the norm yields

$$2\|a_1\|^p \leq \|a_1\|^p + \|b_1\|^p = \|x\|^p,$$

whence $\|a_1\| \leq c\|x\|$.

From $u_1 = Aa_1$ we have $\|u_1\| \leq \|A\|\|a_1\|$. So, taking into account that (in view of Lemma 4.2) $\|u_1\| = \|u_2\| = c\|y\|$, we see that

$$c\frac{\|y\|}{\|A\|} = \frac{\|u_1\|}{\|A\|} \leq \|a_1\|.$$

For the last inequality, note that

$$\begin{aligned}
\|b_1\|^p &= \|x\|^p - \|a_1\|^p \\
&\leq \|x\|^p - \frac{c^p\|y\|^p}{\|A\|^p} \\
&= \|x\|^p[1 - \frac{c^p\|y\|^p}{\|A\|^p\|x\|^p}] = c_1^p\|x\|^p,
\end{aligned}$$

and the proof of the lemma is finished. □

The rest of the construction must be done inductively. For instance, at the next step we will apply the above described procedure to the functions u_1, a_1 satisfying $u_1 = Aa_1$ and to the interval $[\alpha_1, \beta_1]$. That is, we take u_1 for y and a_1 for x and we repeat the same procedure, keeping in mind that the support set of either of these two functions lies in $E_{\alpha_1}^{\beta_1}$.

Afterwards, we will have $u_1 = u_2 \oplus v_2$ with $\|u_2\| = \|v_2\| = c\|u_1\|$ and with the support sets of these new functions also invariant under A. Next we will have $a_1 = a_2 \oplus b_2$ with $\|a_2\| \leq \|b_2\|$ and $Aa_2 = u_2, Ab_2 = v_2$ and with the corresponding estimates on the norms of a_2, b_2. The precise details of this inductive construction can be formulated as follows.

Assume that we have already constructed the functions u_k, v_k, a_k and b_k and scalars $\alpha_{k-1} < \gamma_{k-1} < \beta_{k-1}$ satisfying the following conditions:

$$\|u_k\| = \|v_k\| = c\|u_{k-1}\| = c^k\|y\|$$
$$u_{k-1} = u_k \oplus v_k$$
$$\mathrm{Supp}(u_k) \subseteq E_{\alpha_{k-1}}^{\gamma_{k-1}}$$
$$\mathrm{Supp}(v_k) \subseteq E_{\gamma_{k-1}}^{\beta_{k-1}}$$
$$a_k = a_{k-1}\chi_{E_{\alpha_{k-1}}^{\gamma_{k-1}}}$$
$$b_k = a_{k-1}\chi_{E_{\gamma_{k-1}}^{\beta_{k-1}}}$$
$$(1) \qquad c^k\frac{\|y\|}{\|A\|} \le \|a_k\| \le c^k\|x\|$$
$$(2) \qquad \|a_k\| \le \|b_k\| \le c_1c^{k-1}\|x\|.$$

For this choice of a_k and b_k we let $\alpha_k = \alpha_{k-1}$ and $\beta_k = \gamma_{k-1}$.

Now we are ready to describe the induction step to produce u_{k+1}, v_{k+1}, a_{k+1} and b_{k+1}, and the scalars α_{k+1} and β_{k+1}. Namely, to the elements u_k, a_k, satisfying $u_k = Aa_k$, we apply the very first step described in detail above. As a consequence, we find first the scalar $\gamma_k \in (\alpha_k, \beta_k)$ such that the functions $u_k\chi_{E_{\alpha_k}^{\gamma_k}}$ and $u_k\chi_{E_{\gamma_k}^{\beta_k}}$ have the same norm

$$\|u_k\chi_{E_{\alpha_k}^{\gamma_k}}\| = \|u_k\chi_{E_{\gamma_k}^{\beta_k}}\| = c\|u_k\|.$$

Next we consider the functions $a_k\chi_{E_{\alpha_k}^{\gamma_k}}$ and $a_k\chi_{E_{\gamma_k}^{\beta_k}}$ and denote by a_{k+1} the one with the smaller norm – if both have the same norm, a_{k+1} can be either one. The other function is denoted by b_{k+1}. Without loss of generality, we can assume that $a_{k+1} = a_k\chi_{E_{\alpha_k}^{\gamma_k}}$. Subsequently, we let $\alpha_{k+1} = \alpha_k$ and $\beta_{k+1} = \gamma_k$. (Recall however, that if $\|a_k\chi_{E_{\gamma_k}^{\beta_k}}\| < \|a_k\chi_{E_{\alpha_k}^{\gamma_k}}\|$, then $a_{k+1} = a_k\chi_{E_{\gamma_k}^{\beta_k}}$, and accordingly $\alpha_{k+1} = \gamma_k$ and $\beta_{k+1} = \beta_k$.)

We are ready to verify now that the functions a_{k+1} and b_{k+1} satisfy the desired estimates.

Lemma 4.4. *The functions a_{k+1}, b_{k+1} constructed above satisfy the following inequalities:*

$$c^{k+1}\frac{\|y\|}{\|A\|} \le \|a_{k+1}\| \le c^{k+1}\|x\| \quad \textit{and} \quad \|b_{k+1}\| \le c_1c^k\|x\|.$$

Proof. By Lemma 4.3 we have $\|a_{k+1}\| \le c\|a_k\|$. This and the right inequality in (1) imply that $\|a_{k+1}\| \le c^{k+1}\|x\|$. The equalities $Aa_{k+1} = u_{k+1}$ and $\|u_k\| = c^k\|y\|$ imply

$$\|a_{k+1}\| \ge \frac{\|u_{k+1}\|}{\|A\|} = c\frac{\|u_k\|}{\|A\|} = c^{k+1}\frac{\|y\|}{\|A\|}.$$

Finally, we use the identity $a_{k+1} \oplus b_{k+1} = a_k$ and again the above estimate $\|a_k\| \le c^k\|x\|$ to get:

$$\|b_{k+1}\|^p = \|a_k\|^p - \|a_{k+1}\|^p \le (c^k\|x\|)^p - \|a_{k+1}\|^p$$
$$\le c^{kp}\|x\|^p - (c^{k+1}\frac{\|y\|}{\|A\|})^p$$
$$= c^{kp}\|x\|^p[1 - (\frac{c\|y\|}{\|A\|\|x\|})^p] = c_1^p c^{kp}\|x\|^p.$$

This implies $\|b_{k+1}\| \le c_1 c^k \|x\|$, as desired. □

Using the sequence $\{b_k\}$ and the estimates obtained so far, we can finally produce a sequence $\{e_k\}$ satisfying the properties required in the proof of Theorem 4.1.

Lemma 4.5. *If* $e_n = \frac{b_n}{\|b_n\|}$, *then the sequence* $\{e_n\}$ *satisfies the following properties:*

(i) $\|e_n\| = 1$ *for each* n,
(ii) $\{Ae_n\}$ *is a disjoint sequence, and*
(iii) $\|Ae_n\| \ge \delta$ *for each* n *and for some* $\delta > 0$.

Proof. Since, by their definition, the vectors b_n are pairwise disjoint and have the sets $E_{\gamma_n}^{\beta_{n-1}}$ (which are disjoint and invariant under our operator A) as their support sets, we see that the vectors Ab_n, $n = 1, 2, \ldots$, are also pairwise disjoint. Now recalling that $Ab_n = v_n$ and using the right inequality in (2) we can easily estimate $\|Ae_n\|$:

$$\|Ae_n\| = \frac{\|Ab_n\|}{\|b_n\|} = \frac{\|v_n\|}{\|b_n\|} = c^n\frac{\|y\|}{\|b_n\|}$$
$$\ge \frac{c^n}{c_1 c^{n-1}}\frac{\|y\|}{\|x\|} = \frac{c}{c_1}\cdot\frac{\|y\|}{\|x\|}.$$

This completes the proof. □

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