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Differentiably Simple Lie Superalgebras and Representations of Semisimple Lie Superalgebras

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The purpose of this paper is two-fold. First it supplies the proof of Kač's classification theorem of finite dimensional differentiably simple Lie superalgebras of characteristic 0. Next, we use this theorem to obtain a classification of finite dimensional representations of semisimple Lie superalgebras whose simple components are simple Lie superalgebras which have only inner derivations.

1. INTRODUCTION

Kač stated in [K1] the following theorem about differentiably simple superalgebras: *Every finite dimensional differentiably simple superalgebra over an algebraically closed field of characteristic* 0 *is isomorphic to the tensor product of a simple superalgebra and a Grassmann superalgebra*. He then used this fact to describe semisimple Lie superalgebras. The purpose of Chapter I is to provide a proof of this theorem using as a guidance a paper by Block [B], as was suggested in [K1]. In order to make it as self-contained as possible, we have chosen not to just "prove statements by referring to analogous results in [B]"; rather, we have decided to include all necessary arguments, even when some of them are just obvious generalization of results in [B]. Chapter I is organized as follows: In Sections 2–6, the main result is shown. In Section 7, we use this result to describe semisimple Lie superalgebras in terms of its simple components, as was suggested by Kač [K1].

In Chapter II, we will use the classification of semisimple Lie superalgebras obtained in Chapter I to study their finite dimensional representations. We first prove a result describing irreducible representations of a

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direct sum of superalgebras. It turns out that they are not necessarily just tensor products, as it is in the case of algebras. Next, we study irreducible representations of differentiably simple Lie superalgebras. This is necessary, as they form the "simple components" of a semisimple Lie superalgebra. Finally, in Section 10, we show that irreducible representations of certain semisimple Lie superalgebras (those with "nice" simple components) are all obtained by inducing from a subalgebra, whose irreducible representations can be determined using results of previous sections.

1. Differentiably Simple Superalgebras

Some notations and definitions: All our fields are assumed to have characteristic 0. k will always stand for such a field. A *superalgebra* over k is a $\mathbb{Z}_2$-graded vector space $A = A_{\bar{0}} \oplus A_{\bar{1}}$ such that $A_\varepsilon A_\delta \subseteq A_{\varepsilon+\delta}$, where $\varepsilon, \delta \in \mathbb{Z}_2$. An element $a \in A_\varepsilon$, $\varepsilon \in \mathbb{Z}_2$, is said to be *homogeneous* and ε is called its *degree*. It is convenient to call elements of degree $\bar{0}$ *even*, and elements of degree $\bar{1}$ *odd*. Furthermore we will assume that all our superalgebras are finite dimensional over k unless otherwise stated. If R is a ring and M is an R-module, then $\mathrm{End}_R(M)$ will denote the endomorphism ring of "R-linear" maps from M into itself. We will talk about them in more detail in Section 2. A *derivation* of degree deg d, where deg $d \in \mathbb{Z}_2$ is k-linear map of A into itself satisfying $d(ab) = d(a)b + (-1)^{(\deg d)(\deg a)}ad(b)$, where $a, b \in A$ and deg a denotes the degree of a. A ($\mathbb{Z}_2$-graded) ring will be called *unitary* if it contains a unit element.

2. Chain Condition for A

Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a superalgebra over a field k of characteristic 0. Define left and right multiplication operators by

$$L_x(a) = xa \qquad \text{and} \qquad R_x(a) = (-1)^{(\deg a)(\deg x)}ax,$$

where a and x are homogeneous elements of A. Denote by $T(A)$ the associative multiplication superalgebra generated by those elements. (Note that for a Lie superalgebra A $T(A)$ is the associative superalgebra generated by left multiplication of elements of A.) Let $\mathrm{End}_k(A)$ denote the $\mathbb{Z}_2$-graded ring of k-endomorphisms of A. Let

$$C(A) = \{c \in \mathrm{End}_k(A) | [c, y] = 0, \forall y \in T(A)\}.$$

That is, $C(A)$ is the *centroid* of A. Note that $C(A)$ itself is an associative superalgebra over k. Let M and N be *left A-modules*, i.e., $M = M_{\bar{0}} \oplus M_{\bar{1}}$

and $A_i M_k \subseteq M_{i+k}$, for $i, k \in \mathbb{Z}_2$, and similarly for N. A homogeneous element f in the $\mathbb{Z}_2$-graded $\mathrm{Hom}_k(M, N)$ will be called a homogeneous *left A-homomorphism* if $f(ax) = (-1)^{(\deg a)(\deg f)} af(x)$ for all homogeneous $a \in A$ and $x \in M$. Similarly we define right A-modules. In contrast to left A-homomorphisms we require a *right A-homomorphism* to satisfy the rule $f(xa) = f(x)a$. If M and N are *A-modules*, i.e., left and right A-modules, then an *A-homomorphism* will be a k-homomorphism which is both a left and right A-homomorphism. We denote this $\mathbb{Z}_2$-graded ring by $\mathrm{Hom}_A(M, N)$. The reader might find this definition of A-homomorphisms a bit strange. We certainly could have defined A-homomorphisms differently. For example we could require an A-anti-homomorphism to satisfy the conditions $f(ax) = af(x)$ and $f(xa) = (-1)^{(\deg f)(\deg a)} f(x)a$. These conditions are essentially equivalent to the ones we have given earlier. It is not hard to see that there is a 1–1 correspondence S between A-homomorphisms and A-anti-homomorphisms given by $S(f)(x) := (-1)^{(\deg f)(\deg x)} f(x)$. This map S will change an A-homomorphism into an A-anti-homomorphism and vice versa. Now, one might suggest that we drop the sign in the definition of an A-homomorphism altogether. But we will not, for we feel that this is a good way to describe the centroid.

Suppose that A is a differentiably simple superalgebra over a field k of characteristic 0. *Differentiably simple* means that A has no ideal invariant under the set of k-linear derivations of A and $A^2 \neq 0$. The Lie superalgebra of k-linear derivations of A will be denoted by $\mathrm{der}_k A$. The set $Z(A) = \{a \in A \mid t(a) = 0, \forall t \in T(A)\}$ is a differential ideal, hence either $Z(A) = 0$ or $A^2 = 0$. Furthermore A^2 is also a differential ideal, so either $A^2 = 0$ or $A^2 = A$. So we have $Z(A) = 0$.

Suppose that A has a minimal ideal I_1. Starting from this minimal ideal I_1 we can construct a chain of ideals using derivations in the following way:

(1) The derivations of A form a Lie superalgebra, so we have the degree decomposition $\mathrm{der}_k A = (\mathrm{der}_k A)_{\bar{0}} \oplus (\mathrm{der}_k A)_{\bar{1}}$. If A is differentiably simple, then A is $\{(\mathrm{der}_k A)_{\bar{0}}\} \cup \{(\mathrm{der}_k A)_{\bar{1}}\}$*-simple*.

(2) Let $d_1 \in \mathrm{der}_k A$, homogeneous of degree $\deg d_1$, such that $d_1(I_1) \subseteq I_1$. Define $I_2 = I_1 + d_1(I_1)$. This is an ideal, since d_1 is a homogeneous derivation. d_1 induces a map $\delta_1\colon I_1 \mapsto I_2/I_1$ defined by $\delta_1(x) := d_1(x) + I_1$. δ_1 is an A-homomorphism, hence $\ker \delta_1$ is an ideal contained in I_1. Therefore δ_1 *is an A-isomorphism*.

(3) Suppose we have $0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k$ such that $I_i/I_{i-1} \cong I_1$, $1 \leq i \leq k$. Pick a derivative d_k, homogeneous of degree $\deg d_k$, such that $d_k(I_k) \not\subseteq I_k$. Let $i \in \mathbb{N}$ be such that $d(I_i) \not\subseteq I_k$ and $d(I_{i-1}) \subseteq I_k$. Then d_k, similarly as d_1 before, induces an *A-isomorphism* $\delta_k\colon I_i/I_{i-1} \mapsto I_{k+1}/I_k$, where $I_{k+1} = d_k(I_i) + I_k$.

(4) So we get a *chain of ideals*

$$0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_{L-1} \subseteq I_L = A,$$

if we know that A is finite dimensional. This is also true if A is not assumed to be finite dimensional. The interested reader is referred to [B]. (The key point here is that $Z(A) = 0$ and the δ_i's are A-homomorphisms.)

(5) Let $N = I_{L-1}$. Since $A/N \cong I_i/I_{i-1}$, as A-modules, for $1 \leq i \leq L$, we have $NI_i \subseteq I_{i-1}$; hence N is nilpotent. Also N is a maximal ideal of A, since $A/N \cong I_1$. N indeed is the *unique maximal ideal* of A. For if N' is another maximal ideal, then $N' + N = A$. So $N/(N \cap N') \cong A/N'$. But then the left-hand side is nilpotent while the right-hand side is simple, which is not possible. Summarizing, we have the following.

THEOREM 2.1. *Let A be a finite dimensional differentiably simple superalgebra. Then there exists a chain of ideals of A*

$$0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_{L-1} \subseteq I_L = A$$

such that each factor is A-isomorphic to I_1. Furthermore the nilpotent maximal ideal I_{L-1} is the unique maximal ideal of A.

Remark. Theorem 2.1 is just the superalgebraic analogue of Block's chain condition for algebras. But in order to get a chain of ($\mathbb{Z}_2$-graded) ideals, we cannot use arbitrary derivations to go from one ideal to another. As we have seen, this is most clearly done using homogeneous derivations. This is in general too restrictive. A more general way to obtain a chain of ideals, using not necessarily homogeneous derivations, will be discussed later.

PROPOSITION 2.1. *If A is a superalgebra such that $A = A^2$, then $C(A)$ is a unitary commutative associative superalgebra.*

Proof. It is easy to see from the definition that $C(A) = \mathrm{End}_A(A)$. Hence we have $T \in C(A)$ if and only if

$$T(xy) = (-1)^{(\deg T)(\deg x)} xT(y) = T(x)y, \qquad \text{for all homogeneous } x, y \in A.$$

Hence

$$T_1T_2(xy) = (-1)^{(\deg T_2)(\deg x)} T_1(x)T_2(y) \qquad \text{and}$$

$$T_2T_1(xy) = (-1)^{(\deg T_2)(\deg x) + (\deg T_1)(\deg T_2)} T_1(x)T_2(y).$$

Therefore

$$T_1T_2 - (-1)^{(\deg T_1)(\deg T_2)}T_2T_1 = [T_1, T_2] = 0,$$

i.e., A is commutative. ∎

Remark. We have the degree decomposition $C(A) = C(A)_{\bar{0}} \oplus C(A)_{\bar{1}}$. So if $A = A^2$, then $C(A)_{\bar{0}}$ is a commutative ring and $C(A)_{\bar{1}}$ is nilpotent. Now if A is simple, then every homogeneous element in $C(A)$ is invertible. This implies that $C(A) = C(A)_{\bar{0}}$ is a field.

3. Differential Simplicity of the Centroid

Let A be a differentiably simple superalgebra with a minimal ideal I_1. Let $d \in \operatorname{der}_k A$, $x \in A$ and $T \in C(A)$, all assumed to be homogeneous. Since $[d, L_x] = L_{d(x)}$, we have

$$[[d,T], L_x] = [d,[T,L_x]] - (-1)^{(\deg T)(\deg d)}[T,[d,L_x]] = 0.$$

Similarly $[[d,T], R_x] = 0$. Hence d induces a map $\phi(d)$: $C(A) \mapsto C(A)$. That is, we have a map ϕ: $\operatorname{der}_k A \mapsto \operatorname{End}_k(C(A))$ given by $\phi(d)(T) := [d,T]$. Clearly ϕ is degree-preserving. Consider $C(A)$ as an associative superalgebra. Then

$$\phi(d)(T_1T_2) = [d, T_1T_2] = dT_1\,T_2 - (-1)^{\deg d(\deg T_1 + \deg T_2)}T_1T_2d.$$

Also

$$\begin{aligned}\phi(d)(T_1)T_2 &+ (-1)^{(\deg d)(\deg T_1)}T_1\phi(d)(T_2)\\ &= [d,T_1]T_2 + (-1)^{(\deg d)(\deg T_1)}T_1[d,T_2]\\ &= dT_1\,T_2 - (-1)^{(\deg d)(\deg T_1 + \deg T_2)}T_1T_2d.\end{aligned}$$

Thus $\phi(d)$ is a derivation of $C(A)$ as an associative superalgebra. We will discuss the map ϕ in more detail later.

Suppose we have a chain as in the previous section:

$$0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_{L-1} = N \subseteq I_L = A.$$

We know that there exists an A-isomorphism θ: $A/N \mapsto I_1$. Let ι: $I_1 \mapsto A$ be the natural injection and π: $A \mapsto A/N$ be the natural projection. Both maps are A-homomorphisms of degree $\bar{0}$. Let $\gamma \in C(A/N) = \operatorname{End}_A(A/N)$. Define σ_1: $\operatorname{End}_A(A/N) \mapsto \operatorname{End}_A(A)$ by

$$\sigma_1(\gamma) := \iota\theta\gamma\pi.$$

Since $\sigma_1(\gamma)A = I_1$ and $\sigma_1(\gamma)N = 0$ for $\gamma \neq 0$, $\sigma_1(\gamma)$ induces an A-isomorphism

$$\overline{\sigma_1(\gamma)}: A/N \mapsto I_1.$$

Conversely, if ψ is a (homogeneous) A-isomorphism of A/N onto I_1, then

$$\overline{\sigma_1(\theta^{-1}\psi)} = \psi;$$

so the map $\overline{\sigma_1}$: $\mathrm{End}_A(A/N) \mapsto \mathrm{Hom}_A(A/N, I_1)$, defined by $\overline{\sigma_1}(\gamma) = \overline{\sigma_1(\gamma)}$, is a k-isomorphism.

We will define σ_i: $C(A/N) \mapsto C(A)$ recursively: Suppose that σ_i has been defined for $i \leq q$ satisfying the following properties:

(a) $\sigma_i(\gamma)A + I_{i-1} = I_i$.

(b) $\sigma_i(\gamma)N \subseteq I_{i-1}$.

(c) $\overline{\sigma_i}$: $C(A/N) \mapsto \mathrm{Hom}_A(A/N, I_i/I_{i-1})$ is a k-isomorphism.

Define σ_{q+1} as follows: Let d_q be the homogeneous derivation used to go from I_q to I_{q+1}. Let us say that d_q induces the A-isomorphism

$$\delta_q: I_i/I_{i-1} \mapsto I_{q+1}/I_q.$$

So we have $d_q(I_{i-1}) \subseteq I_q$ and $I_{q+1} = I_q + d_q(I_i)$. Now given $\gamma \in C(A/N)$. Define

$$\sigma_{q+1} := \phi(d_q)(\sigma(\gamma)).$$

This element is in $C(A)$, since $\sigma_i(\gamma) \in C(A)$. We have the following commutative diagram:

$$\begin{array}{ccc} A/N & \xrightarrow{\overline{\phi(d_q)(\sigma_i(\gamma))}} & I_{q+1}/I_q \\ \Big\downarrow \mathrm{id} & & \Big\downarrow \delta_q^{-1} \\ A/N & \xrightarrow{\overline{\sigma_i(\gamma)}} & I_i/I_{i-1} \end{array}$$

So $\overline{\sigma_{q+1}(\gamma)} = \delta_q \overline{\sigma_i(\gamma)}$, i.e., $\overline{\sigma_{q+1}}(\gamma) = \delta_q \overline{\sigma_i}(\gamma)$. Now δ_q is invertible, hence $\overline{\sigma_{q+1}}$ is a k-isomorphism (since $\overline{\sigma_i}$ is a k-isomorphism). It follows that conditions (a) and (b) are also satisfied with $q + 1$ replacing i. Also each σ_i is a k-monomorphism. Since $\sigma_i(\gamma) = 0$ implies that $\overline{\sigma_i(\gamma)} = \overline{\sigma_i}(\gamma) = 0$ and $\overline{\sigma_i}$ is a k-monomorphism. This discussion leads to the following.

PROPOSITION 3.1. *The map*

$$\sigma\colon \underbrace{C(A/N) \oplus C(A/N) \oplus \cdots \oplus C(A/N)}_{L \text{ times}} \mapsto C(A)$$

defined by

$$\sigma(\gamma_1, \gamma_2, \ldots, \gamma_L) := \sum_{i=1}^{L} \sigma_i(\gamma_i)$$

is a k-isomorphism.

Proof. If $\sigma(\gamma_1, \gamma_2, \ldots, \gamma_L) = 0$, then

$$\sum_{i=1}^{L} \sigma_i(\gamma_i) A = 0.$$

So

$$\sigma_L(\gamma_L) A = \sum_{i=1}^{L-1} \sigma_i(\gamma_i) A \subseteq N.$$

Therefore $\bar{\sigma}(\gamma_L) = 0$, which implies that $\gamma_L = 0$. Proceeding this way we can show that σ is 1–1. Now let $c \in C(A)$ be homogeneous. Then there exists an i such that $cA \subseteq I_j$ and $cA \not\subseteq I_{j-1}$. But then there is a γ_j in $C(A/N)$ such that $(c - \sigma_j(\gamma_j))A \subseteq I_{j-1}$. Now consider $c - \sigma_j(\gamma_j) \in C(A)$. We can now apply the same argument, that we applied to c above, to this new element. Proceeding this way we get

$$c = \sum_{i=1}^{j} \sigma_i(\gamma_i) = \sigma(\gamma_1, \gamma_2, \ldots, \gamma_j, 0, \ldots, 0).$$

So σ is onto. ∎

PROPOSITION 3.2. $m = \{c \in C(A) | cA \subseteq I_1\}$ *is a minimal ideal of* $C(A)$.

Proof. First m is an ideal of A and it is homogeneous of either degree $\bar{0}$ or $\bar{1}$. Let c and c' be nonzero elements of m. Then

$$\bar{c}\colon A/N \mapsto I_1 \quad \text{and} \quad \bar{c'}\colon A/N \mapsto I_1$$

are both A-isomorphisms. Hence

$$\bar{c}^{-1}\bar{c'}\colon A/N \mapsto A/N$$

is an A-isomorphism. Since $\overline{\sigma_L}$ is an automorphism of $C(A/N)$, there exists a $\beta \in C(A/N)$ such that $\overline{\sigma_L}(\beta) = \bar{c}^{-1}\bar{c'}$. Now $\overline{\sigma_L}(\beta) = \overline{\sigma_L(\beta)}$, hence $\bar{c}\overline{\sigma_L(\beta)} = \bar{c'}$. Now both sides annihilate N. Thus $c\sigma_L(\beta) = c'$. ∎

It is not hard to see that $m = \sigma_1(C(A/N))$. Proposition 3.1 tells us that given a chain of ideals of A, we can construct a corresponding chain of ideals of $C(A)$. Furthermore ϕ gives a correspondence between derivations used in the chain of A and derivations used in the chain of $C(A)$. So differential simplicity of the centroid $C(A)$ can be expected.

THEOREM 3.1. *$C(A)$ is $\phi(\mathrm{der}_k A)$-simple.*

Proof. Given a chain of ideals of A as before. We let $J_1 = m$ and define

$$J_2 = \sigma_2(C(A/N)) + m = \phi(d_1)(m) + m = \{c \in C(A) | cA \subseteq I_2\}.$$

We define J_i using σ_i as in the definition of J_2. In this fashion we get a chain of ideals of $C(A)$. $C(A)$ is $\phi(\mathrm{der}_k A)$-simple; for if H is an ideal invariant under $\phi(\mathrm{der}_k A)$, then for $h \in H$, $d \in \mathrm{der}_k A$ and $a \in A$, we have

$$d(ha) = \phi(d)(h)(a) + (-1)^{(\deg d + \deg h)} hd(a) \in HA.$$

So HA is a differential ideal of A, hence $HA = A$. So there exists a homogeneous $h \in H$ such that $hA + N = A$. Now given $0 \neq c \in m$, $chA \neq 0$. Therefore $ch \neq 0$. So $0 \neq ch \in H \cap m$. Since m is minimal, $m \subseteq H$. Now by assumption H is a $\phi(\mathrm{der}_k A)$-ideal. But $C(A)$ is obtained by letting $\phi(\mathrm{der}_k A)$ act on m. Therefore $H = C(A)$. ∎

4. THE STRUCTURE OF THE CENTROID

Let A be a finite dimensional differentiably simple associative unitary commutative superalgebra over a field k. Furthermore let m be a minimal ideal of A. By Theorem 2.1 there exists a unique maximal ideal N, which we know is nilpotent. We like to show that $A = k + N$. For this we need Wedderburn's Theorem for $\mathbb{Z}_2$-graded rings.

PROPOSITION 4.1. *If R be an associative simple unitary $\mathbb{Z}_2$-graded ring and I is a minimal left ideal of R, then $R \cong \mathrm{End}_D(I)$. Furthermore $D = D_{\bar{0}} \otimes C(1)$ or $D = D_{\bar{0}}$, where $D_{\bar{0}}$ is a division ring and $C(1)$ denotes the Clifford superalgebra in one indeterminate.*

Proof. One can use standard arguments from simple ring theory to deduce that $R \cong \mathrm{End}_D(I)$. To prove the last statement we can assume that $D_{\bar{1}} \neq 0$. In this case we fix an element $0 \neq x_0 \in D_{\bar{1}}$ and define a left

$D_{\bar{0}}$-homomorphism

$$f: D_{\bar{1}} \mapsto D_{\bar{0}} \text{ by } f(x) := xx_0, \qquad x \in D_{\bar{1}}.$$

f gives an isomorphism of the $D_{\bar{0}}$-spaces $D_{\bar{0}}$ and $D_{\bar{1}}$. Hence D is the Clifford superalgebra in one indeterminate over $D_{\bar{0}}$. ∎

Now assume that k is algebraically closed. We will apply Proposition 4.1 to A/N above. We have $A/N \cong \mathrm{End}_D(I)$. By assumption A/N is commutative. The only way this can happen is when $\dim_D I = 1$, $D_{\bar{1}} = 0$ and $D_{\bar{0}} = k$. Hence $A/N \cong k$ and so $A = k + N$, as desired.

Let $\{1, \bar{\xi}_1, \bar{\xi}_2, \dots, \bar{\xi}_n\}$ denote the generators of the Grassmann superalgebra $\Lambda(n)$.

THEOREM 4.1. *If A is a differentiably simple unitary associative commutative superalgebra over an algebraically closed field k, then $A \cong \Lambda(n)$.*

Proof. Pick a homogeneous basis for N/N^2 over k. Let's say that it consists of $\{\xi_1, \xi_2, \dots, \xi_n\} \cup \{x_1, x_2, \dots, x_m\}$, where $\deg \xi_i = \bar{1}$ and $\deg x_i = \bar{0}$. Since $\xi_i \xi_j = -\xi_j \xi_i$ and $x_i x_j = x_j x_i$, we obtain a homomorphism of k-superalgebras

$$\tau : \Lambda(n) \otimes k[\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_m] \mapsto A$$

defined by $\tau(\tilde{\xi}_i) := \xi_i$, $1 \le i \le n$, $\tau(\tilde{x}_j) := x_j$, $1 \le j \le m$ and $\tau(1) = 1$. τ is clearly onto. Now a derivation d of A can be lifted to a derivation d' of $\Lambda(n) \otimes k[\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_m]$ in an obvious way such that $d\tau = \tau d'$. Let D' denote the set of derivations in $\Lambda(n) \otimes k[\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_m]$ that come from derivations of A. Then $\ker \tau$ is a D'-ideal, since $\tau d' = d\tau$. Indeed it is a maximal D'-ideal, since A is differentiably simple. Let $\Delta = \mathrm{der}_k(\Lambda(n) \otimes k[\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_m])$ and consider $I = \ker \tau + \Delta(\ker \tau)$. We will show that $I = \ker \tau$, which implies that $\ker \tau$ is invariant under Δ. It follows that $\ker \tau$ is a differential ideal of $\Lambda(n) \otimes k[\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_m]$, and since $\Lambda(n) \otimes k[\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_m]$ is differentiably simple, τ must be an isomorphism. Since A is assumed to be finite dimensional over k, $m = 0$, and the theorem follows. So it suffices to show that $I = \ker \tau$.

Let $a \in \Lambda(n) \otimes k[\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_m]$, $b \in \ker \tau$ and $\delta \in \Delta$, all assumed to be homogeneous. We have

$$a\delta(b) = (-1)^{(\deg a)(\deg \delta)}(\delta(ab) - \delta(a)b) \in I.$$

Furthermore if $d' \in D'$, then

$$d'\delta(b) = (-1)^{(\deg d')(\deg \delta)}\delta d'(b) - [d', \delta](b)) \in I,$$

since $[d', \delta] \in \Delta$ and $d'(b) \in \ker \tau$. Therefore I is a D'-ideal containing $\ker \tau$. But we know that $\ker \tau$ is a maximal D'-ideal. Hence either $I = \ker \tau$ or $I = \Lambda(n) \otimes k[\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m]$. But if $I = \Lambda(n) \otimes k[\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m]$, then $\ker \tau$ contains an element with a nonzero monomial term of degree 1. The fact that

$$\sum_{i=1}^{n} \lambda_i \tilde{\xi}_i + \sum_{i=1}^{m} \mu_i \tilde{x}_i + o(2) \xrightarrow{\tau} 0$$

for some $\lambda_i, \mu_i \in$ k means that

$$\sum_{i=1}^{n} \lambda_i \xi_i + \sum_{i=1}^{m} \mu_i x_i \equiv 0 \bmod N^2.$$

(Here $o(2)$ is an element of $\tilde{N}^2$, where $\tilde{N}$ denotes the unique maximal ideal of $\Lambda(n) \otimes k[\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m]$.) But the set $\{\xi_1, \xi_2, \ldots, \xi_n\} \cup \{x_1, x_2, \ldots, x_m\}$ is chosen to be linearly independent. Hence $\sum_{i=1}^{n} \lambda_i \tilde{\xi}_i + \sum_{i=1}^{m} \mu_i \tilde{x}_i = 0$, which is a contradiction. Thus $I = \ker \tau$, as required. ∎

5. The d-Simplicity of A

At this point Block in [B] proves that a differentiably simple algebra with a minimal ideal is d-simple for a fixed derivation d, which enables him to create a chain of ideals using just one derivation d. One might suggest that the natural generalization would be that the same is true for superalgebras with d replaced by some homogeneous derivation. But this is certainly not true, as it does not hold for the Grassmann superalgebra. One way to get the superalgebraic analogue is to consider derivations, which act "homogeneously at each step." We will discuss this now.

Let $d \in \operatorname{der}_k A$, not necessarily homogeneous, and let I be an ideal of A. We say that *d acts homogeneously* on I, if $\forall x \in I$

$$d(x) \equiv d_h(x) \bmod I,$$

where d_h is some homogeneous derivation of A. Note that if $d = d_{\bar{0}} + d_{\bar{1}}$ is the degree decomposition of d, then $d(x) \equiv d_h(x) \bmod I$ implies that

$$d_h(x) \equiv d_\varepsilon(x) \bmod I, \qquad \varepsilon \in \mathbb{Z}_2.$$

This follows from the fact that I is an ideal, and hence graded. Therefore we may assume that $d(x) \equiv d_\varepsilon(x) \bmod I$, where $\varepsilon \in \mathbb{Z}_2$. Suppose now that no proper ideal of A is stable under the action of d, i.e., A is d-simple, for a single d. Let I_1 be a minimal ideal of A. If d acts homogeneously on I_1, then $d(I_1) + I_1 = I_2$ is an ideal of A. If d continues to act homogeneously on I_2, then on I_3 and so on, then we will obtain a chain of ideals of A such that only one single d is needed to go from one ideal to another. In this case we will say that *d acts homogeneously at each step*. Note that ε may vary as i varies.

Let $c \in C(A)$ and $d \in \operatorname{der}_k A$, all assumed to be homogeneous. Then it is straightforward to check that cd is a derivation of A homogeneous of degree $\deg c + \deg d$. Thus $\operatorname{der}_k A$ is a left $C(A)$-module. It is clear that $\operatorname{der}_k C(A)$ is also a left $C(A)$-module. We have a map

$$\phi\colon \operatorname{der}_k A \mapsto \operatorname{der}_k C(A) \qquad \text{given by } \phi(d)(c) = [d, c].$$

It is straightforward to check that ϕ is a left $C(A)$-homomorphism of degree $\bar{0}$. So the image of ϕ is a Lie superalgebra which is also a left $C(A)$-module.

We know that $C(A)$ is $\operatorname{Im}(\phi)$-simple. Now suppose that $C(A)$ is $\phi(d)$-simple for some fixed $d \in \operatorname{der}_k A$, such that $\phi(d)$ acts homogeneously at each step starting from the unique minimal ideal J_1. (The unique minimal ideal of $\Lambda(n)$ is of course $k(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n)$, where $\xi_1, \xi_2, \ldots, \xi_n$ generate the unique maximal ideal of $\Lambda(n)$.) As before write $d = d_{\bar{0}} + d_{\bar{1}}$. Let d_ε denote either of the degree component of d. Let I_1 be a minimal ideal of A. Since the minimal ideal of $C(A)$ is unique, $J_1 = \{c \in C(A) | cA \subseteq I_1\}$. Because $\phi(d)$ acts homogeneously on J_1 and ϕ is degree-preserving,

$$\phi(d)(x) \equiv \phi(d_\varepsilon)(x) \bmod J_1, \qquad \forall x \in J_1.$$

But then we have $\phi(d - d_\varepsilon)J_1 \subseteq J_1$. Hence $[d - d_\varepsilon, J_1] \subseteq J_1$. Now $d - d_\varepsilon = d'$ is a homogeneous element, so $[d', J_1] \subseteq J_1$ implies that $d'c \pm cd' = c'$, where c' and c are in J_1. Choose c to be nonzero. Then $cA = I_1$. So we have $(d'c \pm cd')A = c'A$. This gives us $d'I_1 \subseteq I_1$, hence $d(x) + I_1 = d_\varepsilon(x) + I_1, \forall x \in I_1$. Therefore d acts homogeneously on I_1. Repeating the same argument for I_2, I_3 and so on, we conclude that d acts homogeneously at each step starting with the minimal ideal I_1. Thus we get a chain of ideals of A. Indeed we claim that this chain goes all the way up to A, i.e., A is d-simple. For if not, then we can use a minimal ideal going up to a d-invariant ideal $I_q \neq A$. Let $H = \{c \in C(A) | cA \subseteq I_q\}$. We know that H is a proper ideal of $C(A)$. But then

$$\phi(d)(h) + H = \phi(d_\alpha)(h) + H,$$

where d_α is a homogeneous derivation and $h \in H$. Therefore for every $h \in H$ there is some $h' \in H$ such that

$$\phi(d)(h)A = [d_\alpha, h]A + h'A \subseteq I_q.$$

But then H is invariant under $\phi(d)$, which contradicts the fact that $C(A)$ is $\phi(d)$-simple. Therefore we have reduced the problem of d-simplicity of A to the problem of $\phi(d)$-simplicity of the centroid $C(A)$. So in order to show that A is d-simple, where d is some (fixed) derivation acting homogeneously at each step, it suffices to prove the following.

THEOREM 5.1. *Let A denote the Grassmann superalgebra over an algebraically closed field k of characteristic 0. Suppose that A is D-simple, where $D \subseteq \operatorname{der}_k A$ is a subalgebra and a left A-module. Then there exists a $d \in D$ such that A is d-simple and d acts homogeneously at each step.*

Proof. We know that $A = k + N$, where N is the unique maximal ideal of A. By D-simplicity there exists an element $n \in N$ and $d_1 \in D$ such that $d_1(n)$ is a unit in A. It is not hard to see that we can assume that d_1 and n are both homogeneous, and $d_1(n) = 1$. Let $D_0 = \{d - d(n)d_1 | d \in D\}$. Then $D_0 = \{d \in D | d(n) = 0\}$. Hence D_0 is an A-module and $[d, d'] \in D_0$, $\forall d,\ d' \in D_0$. A is not D_0-simple, since An is a proper D_0-ideal of A. Let $H = \langle D_0, T(A)\rangle$. Then A as an H-module is d_1-simple. Starting with a minimal H-ideal M_1 and using d_1 at each step, we get a chain of H-ideals of A all way up to A, i.e.

$$0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_{L-1} \subseteq M_L = A.$$

Note that $[d_1, d] \in D_0$, $\forall d \in D_0$. Therefore d_1 induces for each i an H-isomorphism

$$\delta_i \colon M_i / M_{i-1} \mapsto M_{i+1} / M_i .$$

In particular we have $A / M_{L-1} \cong M_1$ as H-modules. Since M_1 is D_0-simple, so is A / M_{L-1}. By previous section $A / M_{L-1} \cong \Lambda(m)$, for some $m \geq 0$. If $m = 0$, then A / M_{L-1} is isomorphic to k. In this case A must be d_1-simple and we are done. Hence we can assume that $m > 0$. By induction there exists a $d_0 \in D_0$ such that A / M_{L-1} is d_0-simple and d_0 acts homogeneously at each step. Pick homogeneous elements $\xi_1, \xi_2, \ldots, \xi_m$ in A that correspond to the nilpotent generators of A / M_{L-1}. Let $N_L = \langle \xi_1, \xi_2, \ldots, \xi_m \rangle + M_{L-1}$ and $m_L = \langle \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_m \rangle + M_{L-1}$, i.e., N_L and m_L are the unique maximal and minimal ideals of A / M_{L-1}, respectively. Suppose that

$$M_1 \xrightarrow{\delta_1} M_2 / M_1 \xrightarrow{\delta_2} M_3 / M_2 \xrightarrow{\delta_3} \cdots \xrightarrow{\delta_{L-2}} M_{L-1} / M_{L-2} \xrightarrow{\delta_{L-1}} A / M_{L-1},$$

where δ_i, for $1 \leq i \leq L-1$, is an H-isomorphism. Define

$$N_i := (\delta_{L-1} \delta_{L-2} \ldots \delta_i)^{-1} N_L + M_{i-1}$$

and

$$m_i := (\delta_{L-1}\delta_{L-2}\dots\delta_i)^{-1}m_L + M_{i-1}.$$

Set $d = d_0 + (\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_m)d_1$. Let m_1 be the minimal ideal in M_1 constructed above. Then

$$d(m_1) = d_0(m_1) + (\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_m)d_1(m_1).$$

Since δ_i is H-linear for all i and $(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_m)A/M_{L-1} = 0$, we have

$$(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_m)M_1 = 0.$$

Now $d_1(m_1) \subseteq m_2$ by definition. Set $w = \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_m$. It follows that

$$\begin{aligned} wd_1(m_1) \subseteq wm_2 &= w(d_1(m_1) + M_1) \\ &= wd_1(m_1) + wM_1 \subseteq d_1(wm_1) + d_1(w)m_1 \subseteq m_1, \end{aligned}$$

since $[d_1, L_w] = L_{d_1(w)}$. So $d(m_1) + m_1 = d_0(m_1) + m_1$, hence d acts homogeneously on m_1. After applying d to m_1 $2^m - 1$ times (in which the action of d is the same as the action of d_0), we get M_1. Then

$$d(M_1) = d_0(M_1) + wd_1(M_1) = M_1 + wM_2 = M_1 + m_2,$$

since M_1 is D_0-stable. We see that the action here is entirely the action of wd_1, and hence d acts homogeneously at this step as well. Now we can start all over again and use

$$wd_1(m_2 + M_1) \subseteq wd_1(m_2) + wM_2 \subseteq d_1(wm_2) + d_1(w)(m_2) + m_2 \subseteq m_2.$$

(This is because $wM_2 \subseteq m_2$ and $wm_2 \subseteq m_1$.) Again we see that d acts homogeneously at each step. ∎

6. The Isomorphism of $A/N \otimes C(A)$ and A

Let A be a finite dimensional differentiably simple superalgebra over a field k of characteristic 0. Suppose that $I_1 \subseteq A$ is a minimal ideal. We have shown that there exists a derivation d such that A is d-simple and d acts homogeneously at each step starting with I_1. This gives us a chain of ideals going all the way up to A. Let N denote the unique maximal ideal of A and let I_i be as before. Let $S = \{a \in A | d(a) \in I_1\}$. By d-simplicity we have $d(N) + I_1 = A$. So given $a \in A$, there exist elements $n \in N$ and

$a_1 \in I_1$ such that $d(a) = d(n) + a_1$. Thus $d(a - n) = a_1 \in I_1$, i.e., $A = S + N$. If there exists $0 \neq a \in S \cap N$, then there is an $i \geq 1$ such that $a \in I_i$, but $a \notin I_{i-1}$. But then $d(a) \notin I_i$, which is a contradiction. Hence we have $A = S \oplus N$ as vector spaces. It follows that S is also a superalgebra, which implies that S is a simple superalgebra isomorphic to A/N.

A being differentiably simple implies that the *differential constants* $R = \{c \in C(A) | [c, d] = 0, d \in \mathrm{der}_k A\}$ is a field containing k. A result in [P] and [AN] (they actually proved it for associative rings, but it also applies to superalgebras) tells us that a differentiably simple algebra over the field of its differential constants stays differentiably simple, when we extend the ground field to its algebraic closure. Using this result we find that $C(A) \otimes_R \bar{k}$ is differentiably simple over $\bar{k}$, where $\bar{k}$ denotes the algebraic closure of k. But we know that $C(A) \otimes_R \bar{k}$ is isomorphic to $\Lambda(n, \bar{k})$, i.e., the Grassmann superalgebra in n indeterminates over the field $\bar{k}$. Assume for now that it follows that $C(A) \cong \Lambda(n, R)$. We will use this to derive our main result.

Let

$$0 \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_{L-1} = N' \subseteq A' = C(A)$$

be the corresponding chain of ideals of the centroid. By definition $J_1 A = I_1$, hence $C(A) I_1 = I_1$, hence I_1 is an R-submodule of A. Similarly I_i is an R-submodule of A, for $1 \leq i \leq L$. So S is an R-subalgebra. Therefore $A = S \oplus N$ is a splitting of R-superalgebras. Now J_1 is 1-dimensional over R. So $C(A/N)$ is 1-dimensional over R by our discussion from Section 2. Since R acts unitarily on A/N, we must have $C(A/N) = R$. We are going to construct an isomorphism of superalgebras $F\colon S \otimes_R C(A) \mapsto A$.

First let's denote by $f\colon S \mapsto A$ the natural embedding of R-superalgebras. Clearly f preserves the product. Now define

$$F(s \otimes c) := (-1)^{(\deg s)(\deg c)} c(f(s)).$$

Pick a homogeneous R-basis $\{c_1, c_2, \ldots, c_L\}$ of $C(A)$. Then we have $c_i \in J_i$ and $c_i \notin J_{i-1}$. The map $\bar{F}\colon S \times C(A) \mapsto A$ defined by

$$\bar{F}(s \times c) := (-1)^{(\deg s)(\deg c)} c(f(s))$$

is clearly R-bilinear. Thus F is well-defined. Since f preserves the product, it is straightforward to check that F preserves the product. Hence F is an R-homomorphism of superalgebras. Suppose now that

$$x = \sum_{i=1}^{L} s_i \otimes c_i$$

is in ker F. Write $s_i = s_{i\bar{0}} + s_{i\bar{1}}$ according to its degree decomposition. Now if $s_L \neq 0$, then $f(s_L) \notin N$. So $c_L(f(s_L)) \notin N$, since c_L induces an A-automorphism of A/N. But then $F(x) = 0$ implies that

$$c_L f(s_{L\bar{0}}) + (-1)^{\deg c_L} c_L f(s_{L\bar{1}}) = -\sum_{i=1}^{L-1} c_i f(s_{i\bar{0}}) + (-1)^{\deg c_i} c_i f(s_{i\bar{1}}) \in N.$$

Hence $c_L f(s_{L\bar{0}}) + (-1)^{\deg c_L} c_L f(s_{L\bar{1}}) \in N$. Because f preserves degrees and f is a monomorphism, we have $s_{L\bar{0}} = s_{L\bar{1}} = 0$, i.e., $s_L = 0$. Proceeding this way, we can show that $s_i = 0$, $\forall i$. Hence $\ker F = 0$ and F is an isomorphism.

Combining all the results we have

THEOREM 6.1. *Let A be a finite dimensional differentiably simple superalgebra over a field of characteristic* 0. *Then $A \cong S \otimes_R \Lambda(n, R)$, where R is the field of differential constants, S some simple superalgebra and $\Lambda(n, R)$ the Grassmann superalgebra over R.*

It remains to prove

PROPOSITION 6.1. *Let B be a commutative associative unitary superalgebra over a field R. Suppose E is an algebraic field extension over R such that $B \otimes_R E \cong \Lambda(n, E)$. Then $B \cong \Lambda(n, R)$.*

Proof. Let N be the unique maximal ideal of $\Lambda(n, E)$. Pick homogeneous generators $z_1, z_2, \dots, z_n$ of N over E. For each $i = 1, 2, \dots, n$ we have

$$z_i = \sum_s \lambda_{is} \otimes b_{is}, \qquad \lambda_{is} \in E \quad \text{and} \quad b_{is} \in B.$$

Let F be the field extension of R obtained by adjoining all λ_{is}'s. Let us say that F has dimension m over R. Pick a basis $f_1, f_2, \dots, f_m$ of F over R. We have then

$$\lambda_{is} = \sum_{k=1}^{m} \mu_{isk} f_k, \qquad \mu_{isk} \in R.$$

Therefore

$$z_i = \sum_k f_k \otimes \left(\sum_s \mu_{isk} b_{is} \right) = \sum_k f_k \otimes x_{ik},$$

where $x_{ik} \in B$. Each x_{ik} must be homogeneous of degree 1. Since $z_1 \wedge z_2 \wedge \cdots \wedge z_n \neq 0$, we must have $x_{1k_1} \wedge x_{2k_2} \wedge \cdots \wedge x_{nk_n} \neq 0$, for some $k_1, k_2, \ldots, k_n$. For convenience of notation we will denote x_{ik_i} by x_i. Now

$$x_i = \sum_{s=1}^{n} a_{is} z_s + n^2, \qquad a_{is} \in E, n^2 \in N^2.$$

$x_1 \wedge x_2 \wedge \cdots \wedge x_n \neq 0$ means that $\det(a_{is}) \neq 0$. Hence

$$z_i = \sum_{s=1}^{n} c_{is} x_s + n^2, \qquad \text{where } c_{is} \in E.$$

So $x_1, x_2 \ldots, x_n$ generate N over E. From this it follows immediately that the R-superalgebra generated by $\{1, x_1, x_2, \ldots, x_n\}$ is contained in B. So we have $\Lambda(n, R) \subseteq B$, hence $\Lambda(n, R) = B$. ∎

As an immediate consequence of Theorem 6.1, we have

COROLLARY 6.1. *Let A be a finite dimensional differentiably simple superalgebra over an algebraically closed field k of characteristic* 0. *Then $A \cong S \otimes_k \Lambda(n)$, where S is some simple superalgebra and $\Lambda(n)$ is the Grassmann superalgebra in n indeterminates over k.*

Remark. The determination of the centroid in the case of superalgebras differs more than just slightly from [B]. If we are only interested in the algebraically closed case, then most of Block's ideas carry through. That is, one can, without digressing too far from [B], show that the centroid in this case is a Grassmann superalgebra, since we have $C(A) = k + N$ and the proof of the Theorem 5.1 shows that $C(A)$ indeed is isomorphic to $\Lambda(n)$. However, if the ground field is not algebraically closed, we have chosen a different approach. We first determine differentiably simple commutative superalgebras over an algebraically closed field and then use this to establish the structure of the centroid when the field is not necessarily algebraically closed.

7. SEMISIMPLE LIE SUPERALGEBRAS

As was suggested in [K1], one can describe semisimple Lie superalgebras using Theorem 6.1. This is because the minimal ideals of a semisimple Lie superalgebra L are ad L-simple, and hence differentiably simple. In this section L will denote a finite dimensional semisimple Lie superalgebra over an algebraically closed field k of characteristic 0. We need a preliminary result about derivations of a superalgebra of the form $S \otimes_k \Lambda$, where S is a superalgebra and Λ is a commutative superalgebra with 1. Since we are only interested in finite dimensional Lie superalgebras, we will assume that S and Λ are both finite dimensional, although this requirement can be weakened somewhat. It should be mentioned that similar arguments were used by Block in [B] to describe semisimple Lie algebras of characteristic p.

PROPOSITION 7.1. $\operatorname{der}_k(S \otimes_k \Lambda) \cong \operatorname{der}_k S \otimes_k \Lambda + C(S) \otimes_k \operatorname{der}_k \Lambda$, *where the actions are given by*

$$(d_S \otimes \lambda)(s \otimes \lambda') = (-1)^{(\deg \lambda)(\deg s)} d_S(s) \otimes \lambda\lambda' \quad \text{and}$$
$$(\gamma \otimes d_\Lambda)(s \otimes \lambda') = (-1)^{(\deg d_\Lambda)(\deg s)} \gamma(s) \otimes d_\Lambda(\lambda'),$$

where $d_S \in \operatorname{der}_k S$, $d_\Lambda \in \operatorname{der}_k \Lambda$, $\gamma \in C(S)$(= *the centroid of* S), $s \in S$ *and* $\lambda, \lambda' \in \Lambda$.

Proof. This result is the superalgebraic analogue of a result of Block. One can prove it following Block with only slight changes. We will only give the beginning of the proof for the case when S has a unit element and trust that the interested reader will have no problem finishing it on his or her own.

Let $Z(S)$ denote the center of S. We can identify the centroid $C(S)$ with $Z(S)$. Pick a homogeneous basis $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ of Λ and a homogeneous basis $\{s_1, s_2, \ldots, s_k\}$ of $Z(S)$. Extend it to $\{s_1, s_2, \ldots, s_k, s_{k+1}, \ldots, s_n\}$, a homogeneous basis of S. Given a homogeneous derivation d of $S \otimes \Lambda$, define homogeneous derivations d_{S_j} and d_{Λ_i} as follows:

$$d(s \otimes 1) = \sum_{j=1}^{m} (-1)^{(\deg s)(\deg \lambda_j)} d_{S_j}(s) \otimes \lambda_j \quad \text{and}$$
$$d(1 \otimes \lambda) = \sum_{i=1}^{n} s_i \otimes d_{\Lambda_i}(\lambda), \quad \text{where } s \in S \text{ and } \lambda \in \Lambda.$$

It is not hard to see that $d_{S_j} \in \operatorname{der}_k S$ and $d_{\Lambda_i} \in \operatorname{der}_k \Lambda$. They are of degrees $\deg d + \deg \lambda_j$ and $\deg d + \deg s_i$, respectively. From here on, it is just straight computation and a matter of keeping track of signs. ∎

Let us return to semisimple Lie superalgebras. From the theory of Lie algebras over an algebraically closed field of characteristic 0, we know that any finite dimensional semisimple Lie algebra is a direct sum of simple ones. This is certainly not true for Lie superalgebras. However, similar to the theory of Lie algebras of characteristic p, we can describe semisimple Lie superalgebras in terms of simple. This we will do now.

Let M denote the *socle* of L, i.e., M is the maximal sum of minimal ideals of L. We have $M = M_1 \oplus M_2 \oplus \cdots \oplus M_r$, where the M_i's are minimal ideals of L. Consider the map

$$\kappa\colon L \mapsto \operatorname{der}_k M \text{ given by } \kappa(x) = \operatorname{ad}_M(x), \qquad x \in L.$$

κ is 1–1, since the centralizer of M in L is 0, i.e., $C_L(M) = 0$. For it not, then $C_L(M)$ contains a minimal ideal, which must be abelian by definition of M. But this would contradict the assumption that L is semisimple. So if we identify L with $\kappa(L)$, we have

$$\operatorname{inder}_k(M) \subseteq L \subseteq \operatorname{der}_k M,$$

where $\operatorname{inder}_k(M)$ denotes the set of inner derivations of M. By Corollary 6.1 $M_i = S_i \otimes_k \Lambda(n_i)$, where S_i is simple and n_i is a nonnegative integer $1 \leq i \leq r$. By Proposition 7.1,

$$\bigoplus_{i=1}^{r} \operatorname{inder}_k S_i \otimes \Lambda(n_i) \subseteq L \subseteq \bigoplus_{i=1}^{r} \left(\operatorname{der}_k S_i \otimes \Lambda(n_i) + 1_{S_i} \otimes \operatorname{der}_k \Lambda(n_i)\right).$$

This discussion quickly leads to

PROPOSITION 7.2. *Every finite dimensional semisimple Lie superalgebra L arises in this manner, i.e.,*

$$\bigoplus_{i=1}^{r} \operatorname{inder}_k S_i \otimes \Lambda(n_i) \subseteq L \subseteq \bigoplus_{i=1}^{r} \left(\operatorname{der}_k S_i \otimes \Lambda(n_i) + 1_{S_i} \otimes \operatorname{der}_k \Lambda(n_i)\right),$$

where the S_i's are simple Lie superalgebras.

PROPOSITION 7.3. *Let L be a Lie superalgebra such that* $\operatorname{inder}_k(M) \subseteq L \subseteq \operatorname{der}_k M$ *with M as above. Then L is semisimple if and only if each M_i is L_i-simple, where L_i is the ith simple component of L in* $\operatorname{der}_k M$.

Proof. If M_i is not L_i-simple, then M_i contains a proper L_i-ideal (and hence an M_i-ideal). Differential simplicity of M_i implies that this M_i-ideal (which of course is an ideal of L) is nilpotent, contradicting the semisimplicity of L. Conversely let r_L be the radical of L. Since M_i is L-simple, $[r_L, M_i] = 0\ \forall i$, which implies that $[r_L, M] = 0$, which gives $r_L = 0$. ∎

We will now describe the derivations of L, when L is semisimple. Let us denote $\operatorname{der}_k L$ by D. Let M_D be the D-socle of L, i.e., the maximal sum of minimal D-ideals of L. First every minimal D-ideal I_D is a minimal ideal of L. For otherwise I_D would properly contain an ideal of L, which must be nilpotent, since I_D is differentiably simple. So if we denote the socle of L by M, we have $M_D \subseteq M$. Now if M contains M_D properly, then the centralizer of M_D in L is a nonzero D-ideal of L. So it contains a minimal D-ideal, which must be abelian. But L being semisimple implies that L is D-semisimple. Therefore $M = M_D$.

For $d \in D$, we have $d(M) = d(M_D) \subseteq M_D = M$. Hence the restriction of d to M is a derivation of M. Now if $d(M) = 0$, then $[d(L), M] = 0$, thus $d(L) = 0$. This enables us to embed D into $\operatorname{der}_k M$. We have the following description of D:

PROPOSITION 7.4. $D = N_{\operatorname{der}_k M}(L)$, *the normalizer of* L *in* $\operatorname{der}_k M$.

Proof. It is clear that $N_{\operatorname{der}_k M}(L) \subseteq D$. On the other hand let $d \in D$ and $x \in \operatorname{der}_k M$ such that $d(m) = x(m) = [x, m]$, $\forall m \in M$. Given $y \in L$ we have for $s \in M$

$$\begin{aligned} d(y)(s) = [d(y), s] &= -(-1)^{(\deg d)(\deg y)}[y, d(s)] + d[y, s] \\ &= -(-1)^{(\deg d)(\deg y)}[y, [x, s]] + [x, [y, s]] \\ &= [x, y](s). \end{aligned}$$

Hence $[x, y] = d(y) \in L$. ∎

II. Representations of Semisimple Lie Superalgebras

In this chapter let k be an algebraically closed field of characteristic 0. $A = A_{\bar{0}} \oplus A_{\bar{1}}$ and $B = B_{\bar{0}} \oplus B_{\bar{1}}$ will always denote the degree decomposition of the k-superalgebras A and B, respectively. By modules and ideals we will mean left modules and left ideals, respectively. All modules are assumed to be finite dimensional over k. V_A will denote an irreducible A-module and π_A will be its corresponding representation. Similarly we define π_B and V_B. In general if M and N are A-modules, then an A-homomorphism of degree $\deg f \in \mathbb{Z}_2$ is a k-linear map $f\colon M \mapsto N$ satisfying $f(am) = (-1)^{(\deg f)(\deg a)} af(m)$, where $a \in A$, homogeneous of degree $\deg a$, and m is a homogeneous element of M. Corresponding to f we can define a map $f^*\colon M \mapsto N$ by setting $f^*(m) = (-1)^{(\det f)(\deg m)} f(m)$. f^* satisfies the condition $f^*(am) = af^*(m)$. We will call such a map an A-anti-homomorphism. We note that there is a 1–1 correspondence between A-homomorphisms and A-antihomomorphisms.

8. A Tensor Product Theorem

In this section we will prove a tensor product theorem which will be quite useful in our later study of representations of certain Lie superalgebras, which arise naturally as components of semisimple Lie superalgebras. One basic result in the representation theory of finite dimensional associative unitary algebras is the following theorem: *If A and B are associative unitary algebras, then all irreducible representations of $A \otimes B$ are of the form $V_A \otimes V_B$. Furthermore all such modules are irreducible.* Our goal in this section is to prove a similar result when A and B are allowed to be superalgebras. However, in this case, $V_A \otimes V_B$ is not necessarily irreducible, as we will see in the following example:

Let $C(n)$ denote the Clifford superalgebra in n indeterminates with nondegenerate bilinear form. It is clear that $C(n)$ has a unique nontrivial irreducible representation V_n of dimension $2^{[n/2]}$, where $[n/2]$ is the smallest integer bigger than or equal to $n/2$. Now $C(1) \otimes C(1) = C(2)$. But

$$\dim_k(V_1 \otimes V_1) = 2 \cdot 2 = 4 \neq 2^{[2/2]} = 2 = \dim_k(V_2).$$

Before we state our main result we like to give a quick proof of the theorem in the case when A and B are both k-algebras. Indeed we can prove a little bit more. We first need a lemma.

LEMMA 8.1. *Let B be a k-algebra and let A be a k-superalgebra. Suppose V_B is an irreducible B-module. Then all $A \otimes B$-submodules of $A \otimes V_B$ are of the form $I_A \otimes V_B$, where I_A is an ideal of A.*

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of V_B over k and let I be an $A \otimes B$-submodules of $A \otimes V_B$. Let $x \in I$. Then

$$x = \sum_{i=1}^{n} a_i \otimes v_i, \qquad a_i \in A.$$

Let p_j be the projection of I into its v_j-th coordinate, i.e., $p_j\colon I \mapsto A$ is given by $p_j(x) = a_j$. It is clear that $p_j(I)$ is an ideal of A.

Now B acts on V_B as the full $n \times n$ matrix ring over k by Burnside's theorem. Thus there exists, for every j, an element $b_j \in B$ such that

$$b_j v_j = v_j \qquad \text{and} \qquad b_j v_i = 0,\ i \neq j.$$

Hence $(1 \otimes b_j)I = p_j(I) \otimes v_j$, for arbitrary j. Therefore we have

$$\bigoplus_{i=1}^{n} p_i(I) \otimes v_i = I.$$

Now, using Burnside's theorem again, there exists for every pair of vectors $\{v_j, v_k\}$ an element $b_{jk} \in B$ such that

$$b_{jk} v_j = v_k \qquad \text{and} \qquad b_{jk} v_k = v_j.$$

Then

$$(1 \otimes b_{jk})(p_j(I) \otimes v_j) = p_j(I) \otimes v_k,$$

and hence

$$p_j(I) \subseteq p_k(I).$$

Similarly, we have $p_k(I) \subseteq p_j(I)$, and so

$$p_j(I) = p_k(I), \qquad \forall j, k.$$

Now set $I_A = p_1(I)$ and our lemma follows. ∎

PROPOSITION 8.1. *Let A and B be as before. Then all irreducible representations of $A \otimes B$ are of the form $V_A \otimes V_B$. Furthermore all such modules are irreducible.*

Proof. Let V be an irreducible $A \otimes B$-module. Let $V_B \subseteq V$ be an irreducible B-module. We have $(A \otimes V_B)/m \cong V$, where m is some maximal $A \otimes B$-submodule of $A \otimes V_B$. By Lemma 8.1 $m = m_A \otimes V_B$, where m_A is some maximal ideal of A. Therefore

$$V \cong (A \otimes V_B)/(m_A \otimes V_B) \cong (A/m_A) \otimes V_B.$$

Clearly $A/m_A = V_A$ is an irreducible A-module.

Conversely, let $0 \neq x \in V_A \otimes V_B$. Then we can write, using the notation from Lemma 8.1,

$$x = \sum_{i=1}^{n} w_i \otimes v_i, \qquad w_i \in V_A.$$

We can assume that $w_1 \neq 0$. Then $(1 \otimes b_1)x = w_1 \otimes v_1$. Clearly this element generates $V_A \otimes V_B$. ∎

The proof of Proposition 8.1 essentially contains all the ideas behind the proof of our main theorem. For this we will need to generalize Lemma 8.1, which in turn relies heavily on Burnside's theorem. So our first step will be to generalize Burnside's theorem. Note that the proofs of Proposition 8.1 and Lemma 8.1 only require that B acts as the full $n \times n$ matrix ring over k. That is,

$$\pi_B(B) \cong \mathrm{End}_k(V_B). \tag{1}$$

In fact B can be a k-superalgebra as long as (1) holds.

PROPOSITION 8.2 (Burnside's Theorem). *Let B be an associative superalgebra over k and let (π_B, V_B) be a simple B-module. Then $\pi_B(B) \cong \mathrm{End}_D(V_B)$, where $D = \mathrm{End}_B(V_B)$ is either k or $C(1)$.*

Proof. As in the case where B is a k-algebra, one uses the density theorem and Schur's lemma. While the density theorem still holds if one considers only homogeneous elements, Schur's lemma states that $\mathrm{End}_B(V_B)$ is either k or $C(1)$. ∎

So we see that Proposition 8.1 holds if either $\mathrm{End}_B(V_B) \cong k$ or $\mathrm{End}_A(V_A) \cong k$. Hence from now on we only need to consider the case when $\mathrm{End}_A(V_A) \cong C(1)$ and $\mathrm{End}_B(V_B) \cong C(1)$. Since $\mathrm{End}_B(V_B) \cong C(1)$, there exists an element $d_B \in \mathrm{End}_B(V_B)$ such that d_B is of degree $\bar{1}$ and $(d_B)^2 = -1$. Let $V_B = (V_B)_{\bar{0}} \oplus (V_B)_{\bar{1}}$ be the degree decomposition of V_B. Suppose that $\{v_1, v_2, \ldots, v_n\}$ is a k-basis of $(V_B)_{\bar{0}}$. Then $\{v_1, v_2, \ldots, v_n, d_B(v_1), d_B(v_2), \ldots, d_B(v_n)\}$ is a k-basis of V_B. With respect to this basis d_B take the form

$$d_B = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

where I_n denotes the $n \times n$ identity matrix. We have by Proposition 8.2

$$\pi_B(B) \cong \mathrm{End}_{C(1)}(V_B).$$

So $\pi_B(B)$ consists of all $2n \times 2n$ matrices supercommuting with d_B. It is easy to see that

$$\pi_B(B) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \middle| a, b \text{ are arbitrary } n \times n \text{ matrices over } k \right\}. \quad (2)$$

Remark. If d_B is a B-homomorphism of degree $\bar{1}$ with $(d_B)^2 = 1$, then the corresponding anti-homomorphism d_B^* will satisfy $(d_B^*)^2 = -1$.

PROPOSITION 8.3. *Every $A \otimes B$-submodule M of $A \otimes V_B$ contains a submodule of the form $I \otimes V_B$, where I is an ideal of A. Furthermore every element $\bar{x} \in \overline{M} = M/(I \otimes V_B) \subseteq (A/I) \otimes V_B$ has the form*

$$\bar{x} = \sum_{i=1}^{n} q_i \otimes v_i + d^*(q_i) \otimes d_B(v_i),$$

where $q_i \in J/I$, J an ideal of A. Furthermore if $I \subsetneq J$, then $d^*\colon J/I \mapsto J/I$ is an A-anti-homomorphism of degree $\bar{1}$ with $(d^*)^2 = -1$.

Proof. We will continue to use the notation defined above. Let x be a homogeneous element of M. Then

$$x = \sum_{i=1}^{n} a_i \otimes v_i + a'_i \otimes d_B(v_i), \qquad a_i, a'_i \in A.$$

For every j, $1 \leq j \leq n$, define $p_j\colon M \mapsto A$ and $p'_j\colon M \mapsto A$ by

$$p_j(x) = a_i \qquad \text{and} \qquad p'_j(x) = a'_i.$$

Clearly $p_j(m)$ and $p'_j(M)$ are ideals of A. For every j we can find an element $b_j \in B_{\bar{1}}$ such that

$$b_j v_j = d_B(v_j) \qquad \text{and} \qquad b_j v_i = 0,\ i \neq j.$$

This is because of (2). Then $(1 \otimes b_j)M \subseteq M$ implies that

$$p_j(M) = p'_j(M), \qquad \forall j.$$

Now let $1 \leq k, j \leq n$. Again by (2) we can choose an element $b_{jk} \in B_{\bar{0}}$ such that

$$b_{jk} v_j = v_k \qquad \text{and} \qquad b_{jk} v_k = v_j.$$

As in the proof of Lemma 8.1, we conclude that

$$p_j(M) = p_k(M), \qquad \forall j, k.$$

Now for every j, $1 \leq j \leq n$, we define

$$X_j = \{x \in M \mid p_j(x) \neq 0 \text{ and } p'_j(x) = 0\};$$
$$X'_j = \{x \in M \mid p_j(x) = 0 \text{ and } p'_j(x) \neq 0\}.$$

Clearly X'_j and X_j are ideals of A. Furthermore since $(1 \otimes b_j)X_j \subset X'_j$, we have $p_j(X_j) \subseteq p'_j(X'_j)$. Similarly $p'_j(X'_j) \subseteq p_j(X_j)$. Thus

$$p_j(X_j) = p'_j(X'_j).$$

For arbitrary j, k we have $(1 \otimes b_{jk})X_j \subseteq X_k$ and $(1 \otimes b_{jk})X_k \subseteq X_j$, which implies that

$$p_j(X_j) = p_k(X_k).$$

By (2) there exists $\bar{b}_j \in B_{\bar{0}}$ such that $\bar{b}_j v_j = v_j$ and $\bar{b}_j v_i = 0$, for $i \neq j$. Then

$$p_j(X_j) \otimes v_j = \left(1 \otimes \bar{b}_j\right) X_j \subseteq X_j \subseteq M, \forall j.$$

Set $I = p_1(X_1)$, then it follows that

$$I \otimes V_B \subseteq M.$$

This proves the first statement of Proposition 8.3.

To prove the second statement consider $\overline{M} = M/(I \otimes V_B) \subseteq (A/I) \otimes V_B$. So every element $\bar{x} \in \overline{M}$ can be written as

$$\bar{x} = \sum_{i=1}^{n} q_i \otimes v_i + q_i' \otimes d_B(v_i),$$

where $q_i, q_i' \in A/I$. We will denote the projection maps of $\overline{M}$ to its v_jth and $d_B(v_j)$th coordinates by $\bar{p}_j$ and $\bar{p}_j'$, respectively. Clearly $\bar{p}_j(\overline{M}) = p_j(M)/I$ and $\bar{p}_j'(\overline{M}) = p_j'(M)/I = p_j(M)/I$. Define for each j a map ϕ_j: $p_j(M)/I \mapsto p_j'(M)/I$ as follows:
Let $q \in p_j(M)/I$. Pick $a \in p_j(M)$ such that $a + I = q$. Let $x \in M$ such that

$$x = \sum_{i=1}^{n} a_i \otimes v_i + a_i' \otimes d_B(v_i) \qquad \text{with } a_j = a.$$

Set $\phi_j(q) = a_i' + I$. ϕ_j is well-defined and indeed it is an A-anti-homomorphism of $p_j(M)/I$ with $\phi_j^2 = -1$. As in the previous arguments we can use the existence of elements b_{jk} for arbitrary $1 \leq j, k \leq n$ to show that

$$\phi_j = \phi_k, \qquad \forall j, k.$$

The existence of $\bar{b}_j$ enables us to prove that

$$p_j(M)/I \otimes v_j + \phi_j\big(p_j(M)/I\big) \otimes d_B(v_j) \subseteq \overline{M}, \qquad \forall j.$$

Now set $J = p_1(M)$ and we are done. ∎

Remark. One assumption of Proposition 8.3 can be weakened. Indeed, in the proof we never require our module to have the form $A \otimes V_B$. If we take our module to be $M \otimes V_B$, where M is an arbitrary A-module, then the proof of Proposition 8.3 allows us to obtain a similar result describing the structure of $A \otimes B$-submodules of $M \otimes V_B$.

Before stating our main result, we need one more lemma, which will be used implicitly throughout the proof of Proposition 8.4.

LEMMA 8.2. *Let V be an irreducible $A \otimes B$-module and suppose $V_A \subseteq V$ is an irreducible A-module. Then every irreducible A-module contained in V is isomorphic to V_A.*

Proof. We can use arguments similar to the ones in Section 2 to construct a composition series of V, where every factor is isomorphic to V_A.

Set $V_0 = V_A$. If $V = V_0$, then the statement of the lemma is obvious. So we may assume that $V \neq V_0$. In this case we can find a homogeneous $b_0 \in B$ such that $(1 \otimes b_0)V_0 \not\subseteq V_0$. Define $V_1 = V_0 + (1 \otimes b_0)V_0$. Then the map $\rho_0\colon V_0 \mapsto V_1/V_0$ given by $\rho_0(v) = (1 \otimes b_0)v + V_0$ is an A-isomorphism of the same degree as b_0. Suppose that $\{\rho_{i-1}, V_i\}$ has been defined for $1 \leq i \leq k$. Define ρ_k and V_{k+1} as follows: If $V_k = V$, then we are done. If not, then there exists a homogeneous element $b_k \in B$ such that $(1 \otimes b_k)V_k \not\subseteq V_k$. Find the smallest j such that $(1 \otimes b_k)V_j \not\subseteq V_k$. Set $V_{k+1} = V_k + (1 \otimes b_k)V_j$. The map $\rho_k\colon V_j/V_{j-1} \mapsto V_{k+1}/V_k$ given by $\rho_k(v) = (1 \otimes b_k)v + V_k$ is an A-isomorphism. Now V is assumed to be finite dimensional, hence this chain must stop eventually. Thus we get the composition series described above.

Now if V_A' is another irreducible A-submodule of V, then we can construct a composition series using V_A' as our V_0 above. Since a composition series is unique up to renumbering of its factors, we must have $V_A' \cong V_A$. ∎

PROPOSITION 8.4. *Every irreducible $A \otimes B$-module V is either isomorphic to $V_A \otimes V_B$, or it is isomorphic to a proper subspace V of $V_A \otimes V_B$ such that every homogeneous element $v \in V$ has the form*

$$v = \sum_{i=1}^{n} w_i \otimes v_i + d_A^*(w_i) \otimes d_B(v_i), \tag{3}$$

where d_A is an A-homomorphism of V_A of degree $\bar{1}$ such that $d_A^2 = 1$ and the w_i's are arbitrary homogeneous elements of V_A.

Proof. Let V be an irreducible $A \otimes B$-module. Let $V_A \subseteq V$ and $V_B \subseteq V$ be irreducible A- and B-modules, respectively. If either $\mathrm{End}_A(V_A) \cong k$ or $\mathrm{End}_B(V_B) \cong k$, then the proof of Proposition 8.1 shows that $V \cong V_A \otimes V_B$. Hence we can assume that we have $\mathrm{End}_A(V_A) \cong C(1)$ and $\mathrm{End}_B(V_B) \cong C(1)$.

Consider $V_B \subseteq V$. Then we have $(A \otimes V_B)/m \cong V$, where m is some maximal $A \otimes B$-submodule of $A \otimes V_B$. By Proposition 8.3 m contains a submodule of the form $I \otimes V_B$ such that $m/(I \otimes V_B)$ is generated by elements of the form $q \otimes v_i + d^*(q) \otimes d_B(v_i)$, where $1 \leq i \leq n$, $q \in J/I$ and J is some ideal of A. If $m = I \otimes V_B$, then $V \cong A/I \otimes V_B$. But in this case V is not irreducible, since it contains the submodule described in (3) with $d_A \in \mathrm{End}_A(V_A)$ and $(d_A)^2 = 1$. Hence $I \otimes V_B \subsetneq m$. Now m is maximal. Thus $J = A$, for otherwise $m \subsetneq J \otimes V_B$. Now if I is maximal, then

$V \subseteq V_A \otimes V_B$. By Proposition 8.3 V must be of the form in (3). So we can assume that I is not maximal.

Let N be a maximal ideal of A properly containing I. Then $d^*(N/I) \not\subseteq N/I$, since m is maximal. Denote by $d^*(N)$ the inverse image of $d^*(N/I)$ under the natural projection $A \mapsto A/I$. Then we have $d^*(N) + N = A$. On the other hand, $d^*(N/I) \cap N/I$ is invariant under d^*. Since I is "maximal" with respect to this property, we have $d^*(N) \cap N = I$. Therefore

$$A/I \cong A/N \oplus A/d^*(N) \cong A/N \oplus A/N.$$

Thus if we let $V_A = A/N$, then $A/I = V_A \oplus V_A$. Hence

$$V \subseteq (V_A \oplus V_A) \otimes V_B.$$

By Proposition 8.3 and the irreducibility of V, we know that

$$V = \left\{ \sum_{i=1}^{n} w_i \otimes v_i + d^*(w_i) \otimes d_B(v_i) \,\middle|\, w_i \in K,\ K \text{ some submodule of } V_A \oplus V_A \right\}.$$

If $K \cong V_A$, then we are done. So suppose that $K = V_A \oplus V_A$. We will write $K = (V_A)^1 \otimes (V_A)^2$ to distinguish these two components.

If $d^*((V_A)^1) \subseteq (V_A)^1$, then our module V is not irreducible. Hence we can assume that $d^*((V_A)^1) \subseteq (V_A)^2$. But then

$$V = \sum_{i=1}^{n} \{V_A \otimes v_i + d^*(V_A) \otimes d_B(v_i)\} \oplus \sum_{i=1}^{n} \{d^*(V_A) \otimes v_i - V_A \otimes d_B(v_i)\},$$

where $\oplus$ here denotes a direct sum of $A \otimes B_{\bar{0}}$-modules. We define a map $f\colon V \mapsto V_A \otimes V_B$ by

$$f(w_i \otimes v_i + d^*(w_i) \otimes d_B(v_i)) = w_i \otimes v_i,$$
$$f(d^*(w_i) \otimes v_i - w_i \otimes d_B(v_i)) = -w_i \otimes d_B(v_i).$$

f is clearly an $A \otimes B_{\bar{0}}$-homomorphism. It is easily checked that f is indeed an $A \otimes B$-homomorphism. Now f is necessarily 1–1. But from dimensional considerations f must be onto. This cannot be, since $V_A \otimes V_B$ is not irreducible in this case, as it contains (3).

Conversely, the irreducibility of every such module is an immediate consequence of the remark following Proposition 8.3. ∎

Remark. The module V in (3) can be obtained in the following different way: Let d_A be the degree $\bar{1}$ A-homomorphism in (3), so that $d_A \otimes d_B$ is an $A \otimes B$-homomorphism. Then

$$V = \{x \in V_A \otimes V_B | (d_A \otimes d_B)(x) = x\}.$$

Now $(d_A)^2 = 1$, so $(-d_A)^2 = 1$. Hence we could replace d_A in (3) by $-d_A$ and obtain another irreducible $A \otimes B$-module V' inside $V_A \otimes V_B$. It is not hard to see that

$$V' = \{x \in V_A \otimes V_B | (d_A \otimes d_B)(x) = -x\}.$$

Furthermore the $A \otimes B$-automorphism

$$(d_A \otimes 1) : V_A \otimes V_B \mapsto V_A \otimes V_B$$

sends V into V'. Thus we have

$$V_A \otimes V_B \cong V \oplus V.$$

Proposition 8.4 and the remark following it allow us to define

$$V_A \hat{\otimes} V_B = \begin{cases} V_A \otimes V_B, & \text{if } V_A \otimes V_B \text{ is irreducible} \\ V \subsetneq V_A \otimes V_B, & \text{if } V_A \otimes V_B \text{ is not irreducible.} \end{cases}$$

Restating Proposition 8.4 we can say that *every irreducible* $A \otimes B$*-module is of the form* $V_A \hat{\otimes} V_B$*. Furthermore every such module is irreducible.*

9. Representations of $S \otimes \Lambda(n)$

In the remaining sections of this chapter we will study representations of certain types of Lie superalgebras that arise naturally in the description of semisimple Lie superalgebras obtained in Section 7. We will keep the notation of previous sections. By representations and modules we will mean finite dimensional representations and modules, respectively. If $A \subseteq B$ are superalgebras and M is an A-module, then the induced module will be denoted by $\mathrm{Ind}_A^B M$. If A and B are Lie superalgebras, then the vector space structure of $\mathrm{Ind}_A^B M$ is explicitly given by the Poincaré–Birkhoff–Witt theorem for Lie superalgebras (see [MM] for details). Also if M is an A-module, then we will write π_M for the corresponding representation of A. As before we will restrict ourselves to finite dimensional superalgebras over an algebraically closed field k. $\Lambda(n)$ stands for the Grassmann superalgebra in n indeterminates, and finally its derivation superalgebra will be denoted by $W(n)$.

Recall that if L is a semisimple Lie superalgebra, then

$$\bigoplus_{i=1}^{r} S_i \otimes \Lambda(n_i) \subseteq L \subseteq \bigoplus_{i=1}^{r} (\mathrm{der}_k S_i \otimes \Lambda(n_i) + W(n_i)),$$

where the n_i's are nonnegative integers and the S_i's are simple Lie superalgebras.

Thus to understand representations of semisimple Lie superalgebras, one is led naturally to consider irreducible representations of differentiably simple Lie superalgebras. In this section we will study irreducible representations of the Lie superalgebra $S \otimes \Lambda(n)$, where S is an arbitrary Lie superalgebra. It turns out that they are "more or less" just representations of S.

Let $S = S_{\bar{0}} \oplus S_{\bar{1}}$ be a Lie superalgebra. Since $S_{\bar{0}}$ is a Lie algebra, we have by Levi's theorem

$$S_{\bar{0}} = s_{\bar{0}} \ltimes r_{\bar{0}},$$

where $s_{\bar{0}}$ is a semisimple Lie subalgebra of $S_{\bar{0}}$, and $r_{\bar{0}}$ is a solvable ideal of $S_{\bar{0}}$. Let N be the unique maximal ideal of $\Lambda(n)$. Set $S \otimes N = \{s \otimes \lambda | s \in S, \lambda \in N\}$. Choose $\bar{\mathfrak{b}}$ a Borel subalgebra of $s_{\bar{0}}$ and let $\mathfrak{b}' = \bar{\mathfrak{b}} + r_{\bar{0}}$. Then $\mathfrak{b}'$ is a solvable subalgebra of $S_{\bar{0}}$. Hence $\mathfrak{b} = \mathfrak{b}' + S \otimes N$ is a solvable subalgebra of $S \otimes \Lambda(n)$. A very useful result for constructing irreducible representations of Lie algebras in Lie's theorem. For Lie superalgebras it is the following result due to Kač [K1]:

PROPOSITION 9.1. *Let $\mathfrak{b} = \mathfrak{b}_{\bar{0}} \oplus \mathfrak{b}_{\bar{1}}$ be a finite dimensional solvable Lie superalgebra. Then every finite dimensional irreducible representation of $\mathfrak{b}$ is one-dimensional if and only if $[\mathfrak{b}_{\bar{1}}, \mathfrak{b}_{\bar{1}}] \subseteq [\mathfrak{b}_{\bar{0}}, \mathfrak{b}_{\bar{0}}]$.*

LEMMA 9.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{b}$ a Borel subalgebra of $\mathfrak{g}$. If V is a finite dimensional representation of $\mathfrak{g}$, then*

$$\mathfrak{g} \cdot V = \mathfrak{b} \cdot V.$$

(Here $\mathfrak{g} \cdot V = \{\Sigma_i g_i v_i | g_i \in \mathfrak{g}$ and $v_i \in V\}$, and similarly for $\mathfrak{b} \cdot V$.)

Proof. By complete reducibility of V we can assume that V is irreducible. First suppose that $\mathfrak{g}$ is simple. Write $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^+$, where $\mathfrak{h}$ is a Cartan subalgebra and $\mathfrak{n}^+$ is the corresponding nilpotent subalgebra. Let $\mathfrak{U}(\mathfrak{n}^+)$ denote the universal enveloping algebra corresponding to $\mathfrak{n}^+$. Suppose that V has lowest weight Λ. If $\Lambda = 0$, then V is the trivial representation and hence

$$\mathfrak{g} \cdot V = 0 = \mathfrak{b} \cdot V.$$

Hence we can assume that $\Lambda \neq 0$. Consider the Verma module $M(\Lambda)$ with lowest weight Λ. Since $M(\Lambda) \cong \mathfrak{U}(\mathfrak{n}^+)$ and $\Lambda \neq 0$, we have

$$\mathfrak{b} \cdot M(\Lambda) = M(\Lambda).$$

Now $V \cong M(\Lambda)/m$, where m is some maximal submodule of $M(\Lambda)$. Thus $\mathfrak{b} \cdot V \cong V$. Therefore $\mathfrak{b} \cdot V = \mathfrak{g} \cdot V$, as required.

Now suppose that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k$, where the $\mathfrak{g}_j$'s are simple Lie algebras. It follows that $V = V_1 \otimes V_2 \otimes \cdots \otimes V_k$, where V_j is an irreducible representation of $\mathfrak{g}_j$ for $1 \leq j \leq k$. Clearly we also have $\mathfrak{b} = \mathfrak{b}_1 \oplus \mathfrak{b}_2 \oplus \cdots \oplus \mathfrak{b}_k$, where $\mathfrak{b}_j$ is some Borel subalgebra of $\mathfrak{g}_j$. Now

$$\begin{aligned} \mathfrak{b} \cdot V = \mathfrak{b}_1 \cdot V_1 \otimes V_2 \otimes \cdots \otimes V_k + V_1 \otimes \mathfrak{b}_2 \cdot V_2 \otimes \cdots \otimes V_k \\ + \cdots V_1 \otimes V_2 \otimes \cdots \otimes \mathfrak{b}_k \cdot V_k \end{aligned}$$

and

$$\begin{aligned} \mathfrak{g} \cdot V = \mathfrak{g}_1 \cdot V_1 \otimes V_2 \otimes \cdots \otimes V_k + V_1 \otimes \mathfrak{g}_2 \cdot V_2 \otimes \cdots \otimes V_k \\ + \cdots V_1 \otimes V_2 \otimes \cdots \otimes \mathfrak{g}_k \cdot V_k \end{aligned}$$

Since $\mathfrak{g}_j V_j = \mathfrak{b}_j V_j$ for every j, our lemma follows. ∎

From this lemma we obtain two simple identities that will be used later.

LEMMA 9.2. *Using the notation defined above we have*

$$\text{(i) } [\mathfrak{b}', S_{\bar{0}}] = [S_{\bar{0}}, S_{\bar{0}}];$$
$$\text{(ii) } [\mathfrak{b}', S_{\bar{1}}] = [S_{\bar{0}}, S_{\bar{1}}].$$

Proof.

$$\begin{aligned} \text{(i) } [\mathfrak{b}', S_{\bar{0}}] &= [\bar{\mathfrak{b}}, S_{\bar{0}}] + [r_{\bar{0}}, S_{\bar{0}}] \\ &= [s_{\bar{0}}, S_{\bar{0}}] + [r_{\bar{0}}, S_{\bar{0}}] \qquad \text{by Lemma 9.1} \\ &= [S_{\bar{0}}, S_{\bar{0}}]. \\ \text{(ii) } [\mathfrak{b}', S_{\bar{1}}] &= [\bar{\mathfrak{b}}, S_{\bar{1}}] + [r_{\bar{0}}, S_{\bar{1}}] \\ &= [s_{\bar{0}}, S_{\bar{1}}] + [r_{\bar{0}}, S_{\bar{1}}] \qquad \text{by Lemma 9.1} \\ &= [S_{\bar{0}}, S_{\bar{1}}]. \end{aligned}$$ ∎

LEMMA 9.3. *Let $s \otimes \lambda \in S \otimes N$ and $d \in S$. Then $[d \otimes 1, s \otimes \lambda] = [d, s] \otimes \lambda$ is either of degree $\bar{1}$ or else is contained in $[\mathfrak{b}, \mathfrak{b}]$.*

Proof. Suppose that $[d, s] \otimes \lambda$ is not of degree $\bar{1}$. We have two cases.

(1) $\lambda \in N^2$, where N is the maximal ideal of $\Lambda(n)$. We can write $\lambda = \sum_{i=1}^m \lambda_i$, where $\lambda_i = \lambda_{i1} \wedge \lambda_{i2}$ with $\lambda_{ij} \in N$, $j = 1, 2$, and $i = 1, \dots, m$. Now for each summand of $[d, s] \otimes \lambda = \sum_{i=1}^m [d, s] \otimes \lambda_i$ we have

$$[d, s] \otimes \lambda_i = (-1)^{(\deg \lambda_{i1})(\deg s)}[d \otimes \lambda_{i1}, s \otimes \lambda_{i2}] \in [\mathfrak{b}, \mathfrak{b}].$$

Thus $[d, s] \otimes \lambda \in [\mathfrak{b}, \mathfrak{b}]$.

(2) $\lambda \notin N^2$. Then λ necessarily has degree $\bar{1}$. We can assume that $\deg s = \bar{0}$ and $\deg d = \bar{1}$. But then

$$\begin{aligned}[s, d] \otimes \lambda \in [S_{\bar{0}}, S_{\bar{1}}] \otimes \lambda &= [\mathfrak{b}', S_{\bar{1}}] \otimes \lambda \qquad \text{by Lemma 9.2(ii)}\\ &= [\mathfrak{b}', S_{\bar{1}} \otimes \lambda] \subseteq [\mathfrak{b}, \mathfrak{b}]. \quad \blacksquare\end{aligned}$$

As an immediate consequence of Lemma 9.3, we have

COROLLARY 9.1. *If $[S, S] = S$, then $(S \otimes N)_{\bar{0}} \subseteq [\mathfrak{b}, \mathfrak{b}]$.*

PROPOSITION 9.2. *Let S be a Lie superalgebra and let V be an irreducible representation of $S \otimes \Lambda(n)$. Then $S \otimes N$ acts as scalars on V, and hence V is an irreducible S-module.*

Proof. We will continue to use the notation defined above. First we claim that the Lie superalgebra $\mathfrak{b} = \mathfrak{b}_{\bar{0}} \oplus \mathfrak{b}_{\bar{1}}$ satisfies the condition $[\mathfrak{b}_{\bar{1}}, \mathfrak{b}_{\bar{1}}] \subseteq [\mathfrak{b}_{\bar{0}}, \mathfrak{b}_{\bar{0}}]$. To see this consider $s_1 \otimes \lambda_1$ and $s_2 \otimes \lambda_2$ both homogeneous of degree $\bar{1}$ in $S \otimes N$. Suppose that $s_1 \in S_{\bar{1}}$. Then λ_1 has to be of degree $\bar{0}$. We can assume that there exist $\mu_1, \mu_2 \in N$ both of degree $\bar{1}$ such that $\mu_1 \wedge \mu_2 = \lambda_1$. Write

$$[s_1 \otimes \lambda_1, s_2 \otimes \lambda_2] = \pm[s_1 \otimes \mu_1, s_2 \otimes \mu_2 \wedge \lambda_2] \in [\mathfrak{b}_{\bar{0}}, \mathfrak{b}_{\bar{0}}].$$

Hence we can assume that s_1 and s_2 are both of degree $\bar{0}$. So $[s_1, s_2] \in [S_{\bar{0}}, S_{\bar{0}}]$. By Lemma 9.2(i) there exist $b \in \mathfrak{b}'$ and $s \in S_0$ such that $[s_1, s_2] = [b, s]$. Thus

$$[s_1 \otimes \lambda_1, s_2 \otimes \lambda_2] = \pm[b \otimes 1, s \otimes \lambda_1 \wedge \lambda_2] \in [\mathfrak{b}_{\bar{0}}, \mathfrak{b}_{\bar{0}}].$$

Now $\mathfrak{b}$ is solvable. Hence by Proposition 9.1 all irreducible representations of $\mathfrak{b}$ are 1-dimensional.

Suppose that V is an irreducible $S \otimes \Lambda(n)$-module. Let $\{\pi_v, v\}$ be an irreducible $\mathfrak{b}$-submodule of V. Then we have an $S \otimes \Lambda(n)$-homomorphism

$$\mathrm{Ind}_{\mathfrak{b}}^{S \otimes \Lambda(n)} v \mapsto V.$$

This map is onto, since V is irreducible. To complete the proof of the proposition it suffices to show that $S \otimes N$ acts as scalars on $\mathrm{Ind}_{\mathfrak{b}}^{S \otimes \Lambda(n)} v$. Let $\{b_1, b_2, \dots, b_l\}$ be a homogeneous basis of $\mathfrak{b}'$. Extend this to a homogeneous basis of S, say $\{b_1, \dots, b_l, d_1, \dots, d_m\}$. Then $\mathrm{Ind}_{\mathfrak{b}}^{S \otimes \Lambda(n)} v$ is generated (over k) by elements of the form

$$x = d_1^{k_1} d_2^{k_2} \dots d_m^{k_m} v, \qquad \text{where the } k_i\text{'s are nonnegative integers.}$$

It suffices to prove that $s \otimes \lambda \in S \otimes N$ acts as $\pi_v(s \otimes \lambda)$ on all such x's. We will show this by induction on $t = \sum_{i=1}^m k_i$. For $t = 0$ the assertion is obvious. Suppose that $t > 0$. We have

$$\begin{aligned}
&(s \otimes \lambda) d_1 d_1^{k_1 - 1} d_2^{k_2} \dots d_m^{k_m} v \\
&\quad = (-1)^{((\deg s) + (\deg \lambda))(\deg d_1)} d_1 (s \otimes \lambda) d_1^{k_1 - 1} d_2^{k_2} \dots d_m^{k_m} v \\
&\qquad + [s \otimes \lambda, d_1] d_1^{k_1 - 1} d_2^{k_2} \dots d_m^{k_m} v. \\
&\quad = (-1)^{((\deg s) + (\deg \lambda))(\deg d_1)} d_1 \pi_v (s \otimes \lambda) d_1^{k_1 - 1} d_2^{k_2} \dots d_m^{k_m} v \\
&\qquad + \pi_v([s \otimes \lambda, d_1]) d_1^{k_1 - 1} d_2^{k_2} \dots d_m^{k_m} v.
\end{aligned}$$

Now $\pi_v([s \otimes \lambda, d_1]) = 0$ by Lemma 9.3. Since $\pi_v(s \otimes \lambda) = 0$, if $s \otimes \lambda$ is odd, it remains to consider the case when $s \otimes \lambda$ is even. In this case we have

$$\begin{aligned}
(s \otimes \lambda) d_1 d_1^{k_1 - 1} d_2^{k_2} \dots d_m^{k_m} v &= d_1 \pi_v (s \otimes \lambda) d_1^{k_1 - 1} d_2^{k_2} \dots d_m^{k_m} v \\
&= \pi_v (s \otimes \lambda) d_1^{k_1} d_2^{k_2} \dots d_m^{k_m} v,
\end{aligned}$$

which is exactly what we have claimed. ∎

If we assume that S is simple in the hypothesis of Proposition 9.2, then we can obtain a slightly stronger result. We will make this precise:

COROLLARY 9.2. *Let S be a Lie superalgebra satisfying $[S, S] = S$ and let V be an irreducible representation of $S \otimes \Lambda(n)$. Then V is an irreducible S-module with $S \otimes N$ acting trivially on V.*

Proof. From the proof of Proposition 9.2 we only need to show that $\pi_v(s \otimes \lambda) = 0$ for all $s \otimes \lambda \in S \otimes N$. For this it suffices to show that $(S \otimes N)_{\bar{0}}$ is contained in $[\mathfrak{b}, \mathfrak{b}]$. But this is precisely the statement of Corollary 9.1. ∎

While the representation theory of $S \otimes \Lambda(n)$ is relatively simple, the theory becomes a bit more involved if one is to study representations of its central extensions. There are many inequivalent such extensions, but nevertheless they can be classified quite nicely [C1, C2].

10. Representations of $S \otimes \Lambda(n) \rtimes \mathfrak{D}$

In Section 7 we have seen that every semisimple Lie superalgebra L can be written as

$$\bigoplus_{i=1}^{r} S_i \otimes \Lambda(n_i) \subseteq L \subseteq \bigoplus_{i=1}^{r} (\mathrm{der}_k S_i \otimes \Lambda(n_i) + W(n_i)),$$

where the S_i's are simple Lie superalgebras and the $W(n_i)$'s are the derivation superalgebras of the corresponding $\Lambda(n_i)$'s. In particular if all derivations of S_i are inner for all i, $1 \leq i \leq r$, then we are left with

$$L = \bigoplus_{i=1}^{r} S_i \otimes \Lambda(n_i) \rtimes \mathfrak{D},$$

where $\mathfrak{D}$ is some subalgebra of $\oplus_{i=1}^{r} W(n_i)$. We will study representations of this type of Lie superalgebras in this section. However, we will only need that $S_i = [S_i, S_i]$ for all i. From now on we will assume that all our Lie superalgebras satisfy this condition unless otherwise stated.

Let $\Lambda(n)$ be the Grassmann superalgebra in the n indeterminates $\xi_1, \xi_2, \ldots, \xi_n$; and let N be its maximal ideal. Since $\Lambda(n)$ has a natural $\mathbb{Z}$-grading, it induces a $\mathbb{Z}$-grading on its derivation superalgebra $W(n)$. Let $\partial/\partial\xi_1, \partial/\partial\xi_2, \ldots, \partial/\partial\xi_n$ be elements of $W(n)_{-1}$ (the subspace consisting of elements of degree -1) such that $(\partial/\partial\xi_i)(\xi_j) = \delta_{ij}$. Then

$$W(n) = \left\{ \sum_{i=1}^{n} \lambda_i \frac{\partial}{\partial \xi_i} \,\middle|\, \lambda_i \in \Lambda(n) \right\}.$$

Let $W(n)^+$ denote the subalgebra of $W(n)$ consisting of elements of nonnegative degrees. It is clear that

$$W(n)^+ = \left\{ \sum_{i=1}^{n} \lambda_i \frac{\partial}{\partial \xi_i} \,\middle|\, \lambda_i \in N \right\}.$$

Thus $W(n)/W(n)^+$ is generated over k by $\partial/\partial\xi_1, \partial/\partial\xi_2, \ldots, \partial/\partial\xi_n$.

Now suppose that $\mathfrak{D}^+$ is a subalgebra of $\oplus_{i=1}^{r} W(n_i)^+$. Let N_i denote the maximal ideal of $\Lambda(n_i)$ for $1 \leq i \leq r$. We will construct irreducible representations of $\oplus_{i=1}^{r} S_i \otimes \Lambda(n_i) \rtimes \mathfrak{D}^+$ as follows:

Let V_S be an irreducible representation of $\oplus_{i=1}^r S_i \otimes \Lambda(n_i)$. By Proposition 8.4, we know that $V_S \cong V_1 \hat{\otimes} V_2 \hat{\otimes} \cdots \hat{\otimes} V_r$, where V_i is an irreducible representation of S_i for $1 \le i \le r$. By Corollary 9.2 $S_i \otimes N_i$ acts trivially on V_S. Since $[S_i \otimes N_i, \mathfrak{D}^+] \subseteq S_i \otimes N_i$, V_S extends to an irreducible representation of $\oplus_{i=1}^r S_i \otimes \Lambda(n_i) \rtimes \mathfrak{D}^+$ by letting $\mathfrak{D}^+$ act trivially. Now if $V_{\mathfrak{D}^+}$ is an irreducible representation of $\mathfrak{D}^+$, then we can extend $V_{\mathfrak{D}^+}$ to an irreducible representation of $\oplus_{i=1}^r S_i \otimes \Lambda(n_i) \rtimes \mathfrak{D}^+$ by letting $\oplus_{i=1}^r S_i \otimes \Lambda(n_i)$ act trivially. We can form the irreducible $\oplus_{i=1}^r S_i \otimes \Lambda(n_i) \rtimes \mathfrak{D}^+$-module $V_S \hat{\otimes} V_{\mathfrak{D}^+}$. Moreover, we have the following proposition:

PROPOSITION 10.1. *Every irreducible representation of* $\oplus_{i=1}^r S_i \otimes \Lambda(n_i) \rtimes \mathfrak{D}^+$*-module is obtained in this way.*

Proof. Let V be an irreducible $\oplus_{i=1}^r S_i \otimes \Lambda(n_i) \rtimes \mathfrak{D}^+$-module. By Proposition 8.4 it suffices to prove that $\oplus_{i=1}^r S_i \otimes N_i$ acts trivially on V. But this is an easy consequence of Corollary 9.2 and the next lemma. ∎

LEMMA 10.1. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be Lie superalgebras. Let* $\mathfrak{a}^+ \subseteq \mathfrak{a}$ *be a subalgebra such that for every irreducible representation* $V_\mathfrak{a}$ *of* $\mathfrak{a}$ *we have* $\mathfrak{a}^+ V_\mathfrak{a} = 0$. *Suppose that* $\mathfrak{g} = \mathfrak{a} \rtimes \mathfrak{b}$ *and* $[\mathfrak{a}^+, \mathfrak{b}] \subseteq \mathfrak{a}^+$. *Then* $\mathfrak{a}^+$ *acts trivially on every irreducible representation of* $\mathfrak{g}$.

Proof. Suppose that V is an irreducible $\mathfrak{g}$-module. Let $V_\mathfrak{a}$ be an irreducible $\mathfrak{a}$-module inside V. Pick a homogeneous basis $\{b_1, b_2, \ldots, b_k\}$ of $\mathfrak{b}$. Then we have a $\mathfrak{g}$-homomorphism

$$\operatorname{Ind}_\mathfrak{a}^\mathfrak{g} V_\mathfrak{a} \mapsto V.$$

Clearly this map is onto. So it suffices to show that $\mathfrak{a}^+$ acts trivially on the induced representation. Now every element x of $\operatorname{Ind}_\mathfrak{a}^\mathfrak{g} V_\mathfrak{a}$ is of the form

$$x = b_1^{m_1} b_2^{m_2} \ldots b_k^{m_k} v_\mathfrak{a},$$

where the m_i's are nonnegative integers and $v_\mathfrak{a} \in V_\mathfrak{a}$. Let $a \in \mathfrak{a}^+$ be homogeneous of degree deg a. We will show that $a \cdot x = 0$ using induction on $n = \sum_{i=1}^k m_i$. If $n = 0$, then the statement is clear. Now

$$\begin{aligned} a \cdot x &= a b_1^{m_1} b_2^{m_2} \ldots b_k^{m_k} v_\mathfrak{a} \\ &= a b_1 b_1^{m_1 - 1} b_2^{m_2} \ldots b_k^{m_k} v_\mathfrak{a} \\ &= (-1)^{(\deg a)(\deg b_1)} b_1 a b_1^{m_1 - 1} b_2^{m_2} \ldots b_k^{m_k} v_\mathfrak{a} \\ &\quad + [a, b_1] b_1^{m_1 - 1} b_2^{m_2} \ldots b_k^{m_k} v_\mathfrak{a} \\ &= 0 \quad \text{by induction.} \end{aligned}$$ ∎

It is an interesting question to ask what the irreducible representations of $\oplus_{i=1}^r S_i' \otimes \Lambda(n_i) \rtimes \mathfrak{D}^+$ look like when the S_i''s are arbitrary Lie superalgebras. The answer is a little bit more complicated than Proposition 10.1 as we will show below. Since we will not use this result anywhere else (and especially to keep our notation simple), we will only describe representations of Lie superalgebras of the form $L' = S' \otimes \Lambda(n) \rtimes \mathfrak{D}^+$. However the same proof can be applied to obtain a similar result in the general case.

Let S' be an arbitrary Lie superalgebra and let $(\pi, V_{S'})$ be an irreducible representation of $S' \otimes \Lambda(n)$. As before let N denote the unique maximal ideal of $\Lambda(n)$. Pick the standard basis $\Xi = \{\xi_1, \dots, \xi_n, \xi_1 \wedge \xi_2, \xi_1 \wedge \xi_3, \dots, \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n\}$ of N. Finally let $\Sigma = \{s_1, s_2, \dots, s_k\}$ be a homogeneous basis of S'. For each $s \in \Sigma$ and $\lambda \in \Xi$ we can define a linear map $F_{s\otimes\lambda}\colon \mathfrak{D}^+ \mapsto k$ by setting

$$F_{s\otimes\lambda}(\partial) = \pi([\partial, s \otimes \lambda]), \partial \in \mathfrak{D}^+.$$

($\pi([\partial, s \otimes \lambda]) \in k$ follows from Proposition 9.2.) Let

$$K = \bigcap_{\substack{s\in\Sigma \\ \lambda\in\Xi}} \ker F_{s\otimes\lambda}.$$

Then K is a subalgebra of $\mathfrak{D}^+$. We can choose a subset $\{s^1 \otimes \lambda^1, s^2 \otimes \lambda^2, \dots, s^m \otimes \lambda^m\}$ of $\{s \otimes \lambda | s \in \Sigma, \lambda \in \Xi\}$ such that K is a proper intersection of $\ker F_{s^i\otimes\lambda^i}$, $i = 1, \dots, m$. This enables us to find $\{\partial_1, \partial_2, \dots, \partial_m\} \in \mathfrak{D}^+$ such that

$$\partial_j \in \bigcap_{\substack{i=1 \\ i\neq j}}^{m} \ker F_{s^i\otimes\lambda^i} \qquad \text{and} \qquad \partial_j \notin \ker F_{s^j\otimes\lambda^j}.$$

By construction $\{\partial_1, \partial_2, \dots, \partial_m\}$ is a basis of $\mathfrak{D}^+/K$. Set $(L')^+ = S' \otimes \Lambda(n) \rtimes K$. We can extend $(\pi, V_{S'})$ to a representation of $(L')^+$ by letting K act trivially on $V_{S'}$. Let V_K be an irreducible representation of K. Extend V_K to a representation of $(L')^+$ by letting $S' \otimes \Lambda(n)$ act trivially on V_K. Now by Proposition 8.4 $V_{(L')^+} = V_{S'} \hat{\otimes} V_K$ is an irreducible $(L')^+$-module. We have the follow description of irreducible L'-modules.

PROPOSITION 10.2. *$\mathrm{Ind}_{(L')^+}^{L'}$ is irreducible and every irreducible L'-module is of this form.*

Proof. Let $\{v_1, v_2, \dots, v_l\}$ be a homogeneous basis of $V_{(L')^+}$ and let $0 \neq y \in \mathrm{Ind}_{(L')^+}^{L'} V_{(L')^+}$. Then

$$y = \sum_{i=1}^{l} p_i(\partial_1, \partial_2, \dots, \partial_m) v_i,$$

where $p_i(\partial_1, \partial_2, \ldots, \partial_m)$ are "polynomial" in the variables $\partial_1, \partial_2, \ldots, \partial_m$ written in a fixed order. (They are not actually polynomials, since the ∂_i's do not necessarily commute.) Define the *degree* of y to be the highest degree of the p_i's, $i = 1, \ldots, m$. It will suffice to show that if the degree of y is D, then we can find a nonzero element of degree $D - 1$ in the L'-submodule generated by y, provided $D \neq 0$. We may assume that $\{\partial_1, \partial_2, \ldots, \partial_m\}$ are ordered in such a way that the odd elements have the first indices. There are two cases:

(i) ∂_1 is odd and it appears in one of the p_i's that have degree D. We can write

$$y = \partial_1 \sum_{i=1}^{l} q_i(\partial_2, \ldots, \partial_m) v_i + \sum_{i=1}^{l} h_i(\partial_2, \ldots, \partial_m) v_i,$$

where the q_i's and h_i's are "polynomials" in $\{\partial_2, \ldots, \partial_m\}$. Let q_1 have degree $D - 1$. By applying elements of S' to y if necessary, we can assume that $[\partial_1, s^1 \otimes \lambda^1] v_1 \neq 0$. Consider

$$\begin{aligned}(s^1 \otimes \lambda^1) y = (s^1 \otimes \lambda^1) \partial_1 \sum_{i=1}^{l} q_i(\partial_2, \ldots, \partial_m) v_i \\ + (s^1 \otimes \lambda^1) \sum_{i=1}^{l} h_i(\partial_2, \ldots, \partial_m) v_i.\end{aligned}$$

Its term of degree D is 0, since $(s^1 \otimes \lambda^1)$ necessarily is odd. The first sum has a nonzero term of degree D_1 coming from

$$\pm q_1(\partial_2, \ldots, \partial_m) [\partial_1, s^1 \otimes \lambda^1] v_i \neq 0.$$

The terms of degree $D - 1$ in the second sum is 0.

(ii) Only even ∂_i's appear in p_j's that have degree D. Let p_1 have degree D and say ∂_k appears in p_1 with k as small as possible. Let $\pi(s^1 \otimes \lambda^1) = c \in k$. Then

$$(s^1 \otimes \lambda^1) y - cy$$

is of degree $D - 1$ and it has a nonzero term of degree $D - 1$ coming from $(\partial / \partial(\partial_k)) y$.
This shows irreducibility.

Now suppose that M is an irreducible L'-module. In M we can find an irreducible $S' \otimes \Lambda(n)$-module $V_{S'}$. Corresponding to this representation we can find K as above. Consider the $(L')^+$-module $\mathfrak{U}(K) \cdot V_{S'}$. The fact that $S' \otimes \Lambda(n)$ acts as scalars on this module combined with Proposition 8.4

enables us to find an irreducible $(L')^+$-module of the form $V_{S'} \hat{\otimes} V_K$, where V_K is some irreducible K-module, inside $\mathfrak{U}(K) \cdot V_{S'}$. Now induce from this module and use irreducibility of such induced modules. ∎

Remarks. (i) The irreducible representations constructed in Proposition 10.2 are not necessarily finite dimensional. It is clear that it is finite dimensional if and only if D^+/K has a homogeneous basis consisting of odd elements.

(ii) The subalgebra K constructed above coincides with the "stabilizer" we will discuss in more detail later.

The next two lemmas will be useful later.

LEMMA 10.2. *Let V_1 and V_2 be two vector spaces over k. Suppose that $D \subseteq V_1 \oplus V_2$ is a subspace. Let p_1: $D \mapsto V_1$ and p_2: $D \mapsto V_2$ be the natural projections of D into V_1 and V_2, respectively. Then there exists a basis $\{d_1, d_2, \dots, d_k\}$ of D such that the nonzero elements of the set $\{p_i(d_1), p_i(d_2), \dots, p_i(d_k)\}$ are linearly independent for $i = 1, 2$.*

Proof. Write $D = (D \cap V_1) \oplus (D \cap V_2) \oplus W$, where $W \subseteq D$, not contained in either V_1 or V_2. Pick a basis of V_1, a basis of V_2 and a basis for W. It is not hard to see that this basis will have the desired property. ∎

LEMMA 10.3. *Let $b, a_1, a_2, \dots, a_n$ be homogeneous elements of an associative superalgebra. Then we have*

$$\begin{aligned} ba_1a_2 \dots a_n &= (-1)^{(\deg b)(\deg a_1 + \cdots + \deg a_n)} a_1 a_2 \dots a_n b \\ &\quad + \sum_{i=1}^{n} (-1)^{(\deg b)(\deg a_1 + \cdots + \deg a_{i-1})} a_1 \dots a_{i-1} [b, a_i] a_{i+1} \dots a_n. \end{aligned}$$

Proof. This is easily proved using induction on n. ∎

We are now ready to describe irreducible representations of L. We will first consider a special case. The general case is very similar and will be discussed in the remark following the proof of Theorem 10.1.

THEOREM 10.1. *Let $L = \oplus_{i=1}^r S_i \otimes \Lambda(n_i) \rtimes \mathfrak{D}$, where $\mathfrak{D}$ is a subalgebra of $\oplus_{i=1}^r W(n_i)$. Set $\mathfrak{D}^+ = \mathfrak{D} \cap \oplus_{i=1}^r W(n_i)^+$ and $L^+ = \oplus_{i=1}^r S_i \otimes \Lambda(n_i) \rtimes \mathfrak{D}^+$. Let V_{L^+} be an irreducible L^+-module on which all the S_i's act nontrivially. Then $\operatorname{Ind}_{L^+}^L V_{L^+}$ is an irreducible L-module. Furthermore all irreducible L-modules on which all the S_i's act nontrivially are obtained this way.*

Proof. We only need to show that $V = \operatorname{Ind}_{L^+}^{L} V_{L^+}$ is irreducible, since this will imply that all irreducible L-modules are necessarily of this form. To simplify notation we will write W_n for $\oplus_{i=1}^{r} W(n_i)$ and W_n^+ for $\oplus_{i=1}^{r} W(n_i)^+$.

Let $\{d_1, d_2, \ldots, d_k\}$ be a linearly independent set in $\mathfrak{D}$ such that its image under the natural projection map $\mathfrak{D} \mapsto \mathfrak{D}/\mathfrak{D}^+$ form a basis of $\mathfrak{D}/\mathfrak{D}^+$. Since $\mathfrak{D}/\mathfrak{D}^+ \cong (\mathfrak{D} + W_n^+)/W_n^+$ as vector spaces, $\{d_1, d_2, \ldots, d_k\}$ is linearly independent modulo W_n^+. We can write $d_i = \partial_i + \partial_i^+$, where $\partial_i \in W_n$ and $\partial_i^+ \in W_n^+$. Consider the vector space $\sum_{i=1}^{k} k\partial_i$ as a subspace of $W(n_1)_{-1} \oplus (\oplus_{i=2}^{r} W(n_i)_{-1})$. By Lemma 10.2 and after a suitable change of coordinates in $\Lambda(n_1)$ (which of course induces a change of coordinates in $W(n_1)$) if necessary, we may assume that

$$\partial_1 = \frac{\partial}{\partial \xi_{11}} + \delta_1$$

$$\partial_2 = \frac{\partial}{\partial \xi_{12}} + \delta_2$$

$$\vdots$$

$$\partial_{s_1} = \frac{\partial}{\partial \xi_{1 s_1}} + \delta_{s_1}$$

$\partial_{s_1+1}, \partial_{s_1+2}, \ldots, \partial_k \in \oplus_{i=2}^{r} W(n_i)_{-1}$ and $\delta_1, \delta_2, \ldots, \delta_{s_1} \in \oplus_{i=2}^{r} W(n_i)_{-1}$.

Now we consider $\sum_{i=s_1+1}^{k} k\partial_i \in W(n_2)_{-1} \oplus (\oplus_{i=3}^{r} W(n_i)_{-1})$ and proceed as above. Hence, by a similar argument as before, we may assume that

$$\partial_{s_1+1} = \frac{\partial}{\partial \xi_{21}} + \delta_{s_1+1}$$

$$\partial_{s_1+2} = \frac{\partial}{\partial \xi_{22}} + \delta_{s_1+2}$$

$$\vdots$$

$$\partial_{s_1+s_2} = \frac{\partial}{\partial \xi_{2 s_2}} + \delta_{s_1+s_2},$$

and

$$\partial_{s_1+s_2+1}, \partial_{s_1+s_2+2}, \ldots, \partial_k \in \oplus_{i=3}^{r} W(n_i)_{-1}; \; \delta_{s_1+1}, \delta_{s_1+2}, \ldots, \delta_{s_1+s_2} \in \oplus_{i=3}^{r} W(n_i)_{-1}.$$

Repeating this procedure if necessary, we may assume that

$$\partial_1 = \frac{\partial}{\partial \xi_{11}} + \delta_1$$

$$\vdots$$

$$\partial_{s_1} = \frac{\partial}{\partial \xi_{1s_1}} + \delta_{s_1}$$

$$\partial_{s_1+1} = \frac{\partial}{\partial \xi_{21}} + \delta_{s_1+1}$$

$$\vdots$$

$$\partial_{s_1+s_2} = \frac{\partial}{\partial \xi_{2s_2}} + \delta_{s_1+s_2}$$

$$\partial_{s_1+s_2+1} = \frac{\partial}{\partial \xi_{31}} + \delta_{s_1+s_2+1}$$

$$\vdots$$

$$\partial_{s_1+s_2+s_3} = \frac{\partial}{\partial \xi_{3s_3}} + \delta_{s_1+s_2+s_3}$$

$$\vdots$$

$$\partial_k = \frac{\partial}{\partial \xi_{rs_r}}$$

$\delta_{s_1+s_2+1}, \dots, \delta_{s_1+s_2+s_3} \in \oplus_{i=4}^r W(n_i)_{-1}$ and so on.

Now every element in V is a linear combination of elements of the form

$$y = (\partial_{i_1} + \partial_{i_1}^+)(\partial_{i_2} + \partial_{i_2}^+) \dots (\partial_{i_t} + \partial_{i_t}^+)v, \tag{1}$$

where $v \in V_{L^+}$ and $k \geq i_1 > i_2 > \cdots > i_t \geq 1$.

By Lemma 10.1 $\oplus_{i=1}^r S \otimes N_i$ acts trivially on V_{L^+}. To show that V is irreducible, we need to know how $\oplus_{i=1}^r S \otimes N_i$ acts on y in (1). The basis $\{d_1, d_2, \dots, d_k\}$ we have chosen above will make this task easier.

Let

$$0 \neq x = \sum_{i_1 > i_2 \cdots > i_t} d_{i_1} d_{i_2} \dots d_{i_t} v_{i_1 i_2 \dots i_t},$$

where $v_{i_1 i_2 \dots i_t} \in V_{L^+}$. For each summand $d_{i_1} d_{i_2} \dots d_{i_t} v_{i_1 i_2 \dots i_t}$ define the *length* to be the integer t. Consider the unique summand of x, say

$d_{\alpha_1} d_{\alpha_2} \dots d_{\alpha_m} v_{\alpha_1\alpha_2\dots\alpha_m}$, that has the maximal length among all summand of x and such that among those summands with equal length the m-tuple $(\alpha_1, \alpha_2, \dots, \alpha_m)$ is minimal in the lexicographical order. Let

$$d_j = \partial_j + \partial_j^+ = \frac{\partial}{\partial \xi_{lp}} + \delta_j + \partial_j^+ .$$

Set $\xi_j = \xi_{lp}$. Let $s_j \in \bigcup_{i=1}^r S_i$ such that $s_j \otimes \xi_j \in \bigcup_{i=1}^r S_i \otimes N_i$. We can choose $s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_m}$ such that

$$s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_m} v_{\alpha_1\alpha_2\dots\alpha_m} \neq 0.$$

We claim that

$$(s_{\alpha_m} \otimes \xi_{\alpha_m}) \dots (s_{\alpha_2} \otimes \xi_{\alpha_2})(s_{\alpha_1} \otimes \xi_{\alpha_1})x = \pm s_{\alpha_m} \dots s_{\alpha_2} s_{\alpha_m} v_{\alpha_1\alpha_2\dots\alpha_m} \neq 0. \tag{2}$$

From this it follows that V is irreducible. (2) will follow from the next three lemmas. It is convenient to introduce one more definition.

Let $\lambda \in \Lambda(n_j)$. Since $\Lambda(n_j)$ is $\mathbb{Z}$-graded, we can write $\lambda = \sum_{i=0}^{n_j} \lambda_i$, where λ_i is homogeneous of degree i. Define the *height* of λ to be the smallest integer i for which $\lambda_i \neq 0$. We will write $\mathrm{ht}(\lambda)$ for the height of λ. Now for $s \otimes \lambda \in S_j \otimes \Lambda(n_j)$ we define the *height* of $s \otimes \lambda$ to be the height of λ. Similarly we shall write $\mathrm{ht}(s \otimes \lambda)$ for this integer.

LEMMA 10.4. *Let $s_1 \otimes \lambda_1, s_2 \otimes \lambda_2, \dots, s_m \otimes \lambda_m \in \bigcup_{i=1}^r S_i \otimes \Lambda(n_i)$ and $d_{i_1} d_{i_2} \dots d_{i_t} v \in V$. If $\sum_{i=1}^m \mathrm{ht}(s_i \otimes \lambda_i) > t$, then*

$$(s_1 \otimes \lambda_1)(s_2 \otimes \lambda_2) \dots (s_m \otimes \lambda_m) d_{i_1} d_{i_2} \dots d_{i_t} v = 0.$$

Proof. We will use induction on t. If $t = 0$, then the result is clear from earlier discussions. Now assume $t > 0$. We have by Lemma 10.3

$$\begin{aligned}(s_1 \otimes \lambda_1) \dots (s_m \otimes \lambda_m) d_{i_1} \dots d_{i_t} v &= \pm d_{i_1}(s_1 \otimes \lambda_1) \dots (s_m \otimes \lambda_m) d_{i_2} \dots d_{i_t} v \\ &\quad + \sum_{i=1}^{m} \pm (s_1 \otimes \lambda_1) \dots [s_j \otimes \lambda_j, d_{i_1}] \dots (s_m \otimes \lambda_m) d_{i_2} \dots d_{i_t} v.\end{aligned}$$

The first summand is 0 by induction. Now if $[s_j \otimes \lambda_j, d_{i_1}] \neq 0$, then $\mathrm{ht}(\lambda_j) - 1 \leq \mathrm{ht}([s_j \otimes \lambda_j, d_{i_1}])$. Hence, applying induction again, we obtain

$$(s_1 \otimes \lambda_1) \dots [s_j \otimes \lambda_j, d_{i_1}] \dots (s_m \otimes \lambda_m) d_{i_2} \dots d_{i_t} v = 0. \quad ∎$$

LEMMA 10.5. $(s_t \otimes \xi_{i_t}) \dots (s_2 \otimes \xi_{i_2})(s_1 \otimes \xi_{i_1}) d_{i_1} d_{i_2} \dots d_{i_t} v = \pm s_t \dots s_2 s_1 v$

Proof. This is easy to see for $t = 1$. Suppose now that $t > 1$. Using Lemma 10.3 we have

$$\begin{aligned}
&(s_t \otimes \xi_{i_t}) \dots (s_1 \otimes \xi_{i_1}) d_{i_1} \dots d_{i_t} v \\
&\quad = \pm d_{i_1}(s_t \otimes \xi_{i_t}) \dots (s_2 \otimes \xi_{i_2})(s_1 \otimes \xi_{i_1}) d_{i_2} \dots d_{i_t} v \\
&\qquad + \sum_{j=2}^{t} \pm (s_t \otimes \xi_{i_t}) \dots \left[s_j \otimes \xi_{i_j}, d_{i_1}\right] \dots (s_1 \otimes \xi_{i_1}) d_{i_2} \dots d_{i_t} v \\
&\qquad \pm (s_t \otimes \xi_{i_t}) \dots (s_2 \otimes \xi_{i_2})(s_1 \otimes (1 + \mu)) d_{i_2} \dots d_{i_t} v,
\end{aligned}$$

where $\mu \in \bigcup_{i=1}^{r} N_i$. The first summand is 0 by Lemma 10.4. The last summand equals to $\pm s_t \dots s_1 s_1 v$ by Lemma 10.4 and induction. Since $i_1 > i_j$ for $j \geq 2$, $\partial_{i_1}(\xi_{i_j}) = 0$. Hence if $[s_j \otimes \xi_{i_j}, d_{i_1}] \neq 0$, then $\mathrm{ht}([s_j \otimes \xi_{i_j}, d_{i_1}]) \geq 1$. This implies that each summand in the sum above equals to 0 by Lemma 10.4. ∎

LEMMA 10.6. *Let $(j_1, j_2, \dots, j_t) > (i_1, i_2, \dots, i_t)$ in the lexicographical order. Then*

$$(s_t \otimes \xi_{i_t}) \dots (s_2 \otimes \xi_{i_2})(s_1 \otimes \xi_{i_1}) d_{j_1} d_{j_2} \dots d_{j_t} v = 0.$$

Proof. Suppose that $i_1 = j_1, i_2 = j_2, \dots, i_\beta = j_\beta$ and $i_{\beta+1} < j_{\beta+1}$. By the proof of the previous lemma

$$\begin{aligned}
&(s_t \otimes \xi_{i_t}) \dots (s_2 \otimes \xi_{i_2})(s_1 \otimes \xi_{i_1}) d_{j_1} d_{j_2} \dots d_{j_t} v \\
&\quad = \pm (s_t \otimes \xi_{i_t}) \dots (s_{\beta+1} \otimes \xi_{i_{\beta+1}}) d_{j_{\beta+1}} d_{j_{\beta+2}} \dots d_{j_t} v \\
&\quad = \pm d_{j_{\beta+1}}(s_t \otimes \xi_{i_t}) \dots (s_{\beta+1} \otimes \xi_{i_\beta}) d_{j_{\beta+2}} \dots d_{j_t} v \\
&\qquad + \sum_{c=\beta+1}^{t} \pm (s_t \otimes \xi_{i_t}) \dots \left[s_c \otimes \xi_{i_c}, d_{j_{\beta+1}}\right] \dots (s_{\beta+1} \otimes \xi_{i_{\beta+1}}) d_{j_{\beta+2}} \dots d_{j_t} v.
\end{aligned}$$

The first summand is 0 by Lemma 10.4. Since $i_{\beta+1} < j_c$, for $c \geq \beta + 1$, all the summands in the sum above is 0 as in the proof of the previous lemma. ∎

Using Lemma 10.4, 10.5, and 10.6, Eq. (2) follows. This completes the proof of Theorem 10.1. ∎

Remark. If V_L is an irreducible L-module containing only trivial irreducible S_i-modules for $i = 1, \dots, k$, $k \leq r$, then we have to modify $\mathscr{D}^+$ above for Theorem 10.1 to be true. This can be done by taking $\mathscr{D}^+ = \mathscr{D} \cap (\oplus_{i=1}^{k} W(n_i) + \oplus_{i=k+1}^{r} W(n_i)^+)$. Subsequently we let L^+ be as before using our new definition of $\mathscr{D}^+$. Using similar arguments as in the proof of Theorem 10.1, suitably modified, one shows that $\mathrm{Ind}_{(L)^+}^{L}$ is irreducible. Now irreducible representations of L^+ on which S_i acts trivially for $i = 1, \dots, k$ can be shown to have the same tensor product form as the ones in Proposition 10.1 using almost exactly the same arguments. Note that this is a well-defined tensor product, since V_{S_i} for $i = 1, \dots, k$ is trivial.

It should be mentioned that Theorem 10.1 can be obtained in a different way. In [Bl], Blattner proved an irreducibility theorem for Lie algebras. A similar result for Lie superalgebras can be used to prove Theorem 10.1 with less effort. Before we can state this result, we need to introduce the notion of a stabilizer associated to a representation.

Let L be a Lie superalgebra and $I \subseteq L$ be an ideal of L. Let (π, V_I) be an irreducible representation of I. Associated to π, we have a subalgebra K_π defined as follows:

$$K_\pi = \{k \in L | \exists A_k \in \mathrm{End}(V_I) \text{ with } \pi([k, i]) = [A_k, \pi(i)], \forall i \in I\}.$$

It is clear that $I \subseteq K_\pi$. We will call K_π the *stabilizer* of the representation π.

THEOREM 10.2. *Let V_K be an irreducible representation of K_π such that as an I-module V_K is a direct sum of copies of π. If the $\mathbb{Z}_2$-graded vector space L/K_π is spanned by elements of the same parity, then $\mathrm{Ind}_{K_\pi}^{L} V_K$ is an irreducible L-module.*

Before we give a proof of Theorem 10.2 we like to show how Theorem 10.1 follows. We let $L = \oplus_{i=1}^{r} S_i \otimes \Lambda(n_i) \rtimes \mathscr{D}$, $I = \oplus_{i=1}^{r} S_i \otimes \Lambda(n_i)$. For any irreducible representation of I, the stabilizer can be easily computed using Corollary 9.2. It turns out that the stabilizer is independent of the representation in the settings of Theorem 10.1. Furthermore it is equal to $L^+ = \oplus_{i=1}^{r} S_i \otimes \Lambda(n_i) \rtimes \mathscr{D}^+$. The main difficulty in the proof of Theorem 10.1 is to show that $\mathrm{Ind}_{L^+}^{L} V_{L^+}$ is irreducible. This follows from Theorem 10.2, since by Proposition 10.1 and Proposition 8.4 V_{L^+} is contained in the tensor product of an irreducible $\mathscr{D}^+$-module and an irreducible $\oplus_{i=1}^{r} S_i \otimes \Lambda(n_i)$-module, which is a direct sum of copies of this irreducible $\oplus_{i=1}^{r} S_i \otimes \Lambda(n_i)$-module. Furthermore, L/L^+ is an "odd" vector space. Therefore Theorem 10.2 applies and so the induced representation is irreducible. We note that if one of the tensor factors of a representation π of I is trivial, then π generates a stabilizer distinct from the one mentioned above. In this case, however, one can easily see that the stabilizer of π is the algebra L^+ defined in the remark following Theorem 10.1.

Proof of Theorem 10.2. Let $D = \{d_1, d_2, \dots, d_k\}$ be a fixed ordered (homogeneous) basis for L/K_π and assume that all the d_i's have the same parity. Let x be a homogeneous element of $\mathrm{Ind}_{K_\pi}^L V_K$. By assumption we have $V_k = \oplus_j V_j$, where $V_j \cong V_l$, $\forall j$. Let $\{m_i(D)\}$ be a basis for $\mathfrak{U}(L)/\mathfrak{U}(K_\pi)$ consisting of ordered monomials in D. Then

$$x = \sum_i m_i(D) \otimes v_i, \qquad \text{where the } v_i\text{'s are homogeneous elements of } V_K.$$

Define the *degree* of x to be the highest degree of the monomials $m_i(D)$. We will denote this integer by $|x|$. Let M be an L-submodule of $\mathrm{Ind}_{K_\pi}^L V_K$. Suppose that $0 \neq x \in M$ has the lowest degree among all elements in M. If its degree is 0, then clearly the L-module generated by x is necessarily the entire induced module. Hence we may assume that x has positive degree, say p. We can assume that $m_1(D)$ has degree p and that $v_1 \in V_1$. Now if $m_j(D)$ has degree p, then v_j must have the same parity as v_1. Hence by Burnside's theorem we can assume that $p_1(v_j) = \lambda_j v_1$, where p_1: $V_K \mapsto V_1$ is the natural projection map and $\lambda_j \in k$ is some scalar for all j. (Here we use the parity assumption.)

Let a be a homogeneous element in $\mathfrak{U}(I)$ and suppose that d_1 appears in $m_1(D)$ as a nonzero power. Let $m_l(D)$ be the monomial in D identical to $m_1(D)$ with the exception that its d_1th power is one less than that of $m_1(D)$. Consider the $m_l(D)$-th coefficient of $p_1(ax)$. One shows in the case when all the d_i's are even that this equals to

$$av_l - [a, y]v_1, \qquad \text{where } y \notin K_\pi.$$

Two quick remarks:

1. The reason that $y \notin K_\pi$ is because the $m_1(D)$-th coefficient of x is nonzero.

2. In the case when all the d_i's are odd, one obtain the same sum, up to the sign $(-1)^{(\deg a)(|m_l(D)|)}$, where $|m_l(D)|$ is the degree of the monomial $m_l(D)$.

However, $av = 0$ implies that $|x| > |ax|$. This fact assures us that the map T_y: $V_l \mapsto V_l$ given by

$$T_y(av_1) = (-1)^{(\deg a)(\deg y)}(av_l - [a, y]v_1)$$

is well-defined. A simple computation shows that $\pi([b, y]) = [\pi(b), T_y]$, $\forall b \in I$, and hence $y \in K_\pi$. This gives us the desired contradiction. ∎

The proof above, with the exception of some minor changes, is based on [Bl].

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