Existence of positive solutions for a variational inequality of Kirchhoff type

OUIDAD Frites, TOUFIK MOUSSAOUI*

Department of Mathematics, Laboratory of Fixed Point Theory and Applications, E.N.S. Kouba, Algiers, Algeria

Received 9 October 2014; received in revised form 18 February 2015; accepted 19 February 2015
Available online 2 March 2015

Abstract. In the present paper, we study existence of nontrivial positive solutions for a Kirchhoff type variational inequality. Our approach is based on the non-smooth critical point theory for Szulkin-type functionals.

Keywords: Variational inequality; Critical point; Mountain pass theorem; Minimization; Szulkin-type functionals

1. INTRODUCTION

Variational inequalities describe phenomena from mathematical physics. They have applications in physics, mechanics, engineering, optimization, and elliptic inequalities, see, for example, [1–4] and [5].

The aim of this work is to study a Kirchhoff type variational inequality which is defined on a bounded interval $(0, 1)$ by using a non-smooth critical point theory due to Szulkin. In [7], the author has proved a number of existence theorems for critical points of functionals which are not smooth. He has generalized some minimization and minimax methods in critical point theory to a class of functionals which are not necessarily continuous and has introduced a new concept of compactness which is suitable to study these kinds of problems.

In the present paper, by using a minimization principle and the Mountain pass theorem of Szulkin-type, we prove existence of positive solutions to a variational inequality of Kirchhoff-type in a closed convex set.

Let $K = \{ u \in H^1_0(0, 1) : u \geq 0 \}$ be the closed convex set in the Sobolev space $H^1_0(0, 1)$ and we consider the problem, denoted by $(P)$:

* Corresponding author.
E-mail addresses: ofrites@yahoo.fr (O. Frites), moussaoui@ens-kouba.dz (T. Moussaoui).
Peer review under responsibility of King Saud University.
Given \( f : [0, 1] \times \mathbb{R} \to \mathbb{R} \) a continuous function and \( a, b > 0 \), find \( u \in K \) such that

\[
(a + b \int_0^1 |u'(x)|^2 \, dx) \int_0^1 u'(x)(v'(x) - u'(x)) \, dx - \int_0^1 f(x, u(x))(v(x) - u(x)) \, dx \geq 0, \quad \forall v \in K.
\]

Such kind of problems are called obstacle problems and they have been largely studied due to its physical applications. See, for example, the classical books Kinderlehrer and Stampacchia [4], Rodrigues [6] and Troianiello [8] and the references therein.

## 2. Szulkin-type functionals

Let \( X \) be a real Banach space and \( X^* \) its dual. Let \( E \) be a functional which is of class \( C^1 \) and let \( \psi : X \to \mathbb{R} \cup \{+\infty\} \) be a proper (i.e. \( \psi \neq +\infty \)), convex, lower semicontinuous functional. We say that \( I = E + \psi \) is a Szulkin-type functional, see [7]. An element \( u \in X \) is called a critical point of \( I = E + \psi \) if

\[
E'(u)(v - u) + \psi(v) - \psi(u) \geq 0 \quad \text{for all } v \in X,
\]

which is equivalent to

\[
0 \in E'(u) + \partial \psi(u) \quad \text{in } X^*,
\]

where \( \partial \psi(u) \) is the subdifferential of the convex functional \( \psi \) at \( u \in X \).

**Definition 2.1.** The functional \( I = E + \psi \) satisfies the Palais–Smale condition at level \( c \in \mathbb{R} \), denoted by \( (PSZ)_c \) if every sequence \( \{u_n\} \subset X \) such that \( \lim_{n \to \infty} I(u_n) = c \) and

\[
(E'(u_n), v - u_n)_X + \psi(v) - \psi(u_n) \geq -\varepsilon_n\|v - u_n\| \quad \text{for all } v \in X,
\]

where \( \varepsilon_n \to 0 \), possesses a convergent subsequence.

**Theorem 2.1 ([7]).** Let \( X \) be a Banach space, \( I = E + \psi : X \to \mathbb{R} \cup \{+\infty\} \) a Szulkin-type functional which is bounded below. If \( I \) satisfies the \( (PSZ)_c \)-condition for \( c = \inf_{u \in X} I(u) \), then \( c \) is a critical value.

Szulkin has proved the following version of the Mountain Pass theorem.

**Theorem 2.2 ([7]).** Let \( X \) be a Banach space, \( I = E + \psi : X \to \mathbb{R} \cup \{+\infty\} \) a Szulkin-type functional and assume that

(i) \( I(u) \geq \alpha \) for all \( \|u\| = \rho \) for some \( \alpha, \rho > 0 \), and \( I(0) = 0 \);

(ii) there is \( e \in X \) with \( \|e\| > \rho \) and \( I(e) \leq 0 \).

If \( I \) satisfies the \( (PSZ)_c \)-condition for

\[
c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I(\gamma(t)),
\]

then \( c \) is a critical value.
with
\[ Γ = \{ γ ∈ C([0, 1], X) : γ(0) = 0, γ(1) = e\}, \]
then \( c \) is a critical value of \( I \) and \( c ≥ α \), i.e., there exists \( u^* \) in \( X \) such that \( I'(u^*) = 0 \) and \( I(u^*) = c ≥ α \).

### 3. Main results

We now formulate the main results of this paper. We denote by \( F \) the function defined by
\[ F(x, s) = \int_0^s f(x, t)dt. \]

**Theorem 3.1.** Let \( f : [0, 1] × \mathbb{R} → \mathbb{R} \) be a continuous function which satisfies the following condition:
\[ (f_1) \text{ there exists } β > 0 \text{ such that } \limsup_{|t| → ∞} \frac{F(x, t)}{t^2} ≤ β, \text{ uniformly with respect to } x ∈ [0, 1]. \]

Then the problem \( (P) \) has at least one solution \( u ∈ K \).

**Theorem 3.2.** Let \( f : [0, 1] × \mathbb{R} → \mathbb{R} \) be a continuous function which satisfies the following conditions:
\[ (h_1) \text{ There exists } ν > 4 \text{ and } M > 0 \text{ such that } 0 < ν F(x, t) ≤ t f(x, t) \text{ for } |t| ≥ M, \forall x ∈ [0, 1]. \]
\[ (h_2) \limsup_{|t| → 0} \frac{F(x, t)}{|t|^2} < \frac{a}{2}, \text{ uniformly with respect to } x ∈ [0, 1]. \]
Then the problem \( (P) \) has at least one nontrivial solution \( u ∈ K \).

**Remark 3.1.** The hypotheses in Theorems 3.1 and 3.2 are respectively of sublinear and superlinear types, so they are natural conditions.

We define the functional \( E : H^1_0(0, 1) → \mathbb{R} \) by
\[ E(u) = \frac{1}{2} a\|u\|^2 + \frac{1}{4} b\|u\|^4 - \int_0^1 F(x, u(x))dx. \]
Because \( f : [0, 1] × \mathbb{R} → \mathbb{R} \) is continuous, by using the Lebesgue theorem on dominated convergence and the compact embedding of \( H^1_0(0, 1) \) in \( C([0, 1]) \), we can prove easily that \( E ∈ C^1(H^1_0(0, 1), \mathbb{R}) \).

We define the indicator functional of the set \( K \) by
\[ ψ_K(u) = \begin{cases} 0, & \text{if } u ∈ K \\ +∞, & \text{if } u ∉ K. \end{cases} \]
We remark that the functional \( ψ_K \) is convex, proper, and lower semicontinuous. So, \( I = E + ψ_K \) is a Szulkin-type functional.

**Proposition 3.1.** Every critical point \( u ∈ H^1_0(0, 1) \) of \( I = E + ψ_K \) is a solution of \( (P) \).
Proof. Since \( u \in H^1_0(0,1) \) is a critical point of \( I = E + \psi_K \), we have

\[
E'(u)(v - u) + \psi_K(v) - \psi_K(u) \geq 0, \quad \forall v \in H^1_0(0,1).
\]

Note that \( u \) belongs to \( K \). For if this were not true we had \( \psi_K(u) = +\infty \) and taking \( v = 0 \in K \) in the above inequality, we obtain a contradiction. We fix \( v \in K \). Since

\[
0 \leq E'(u)(v - u) = (a + b\|u\|^2) \int_0^1 u'(x)(v'(x) - u'(x))dx - \int_0^1 f(x, u(x))(v(x) - u(x))dx,
\]

the inequality is proved. \( \square \)

4. Proof of Theorem 3.1

We assume that the hypothesis of Theorem 3.1 is satisfied and prove the existence of a solution for the problem \((P)\) by using Theorem 2.1.

Proposition 4.1. If the function \( f \) satisfies the hypothesis \((f_1)\), then \( I = E + \psi_K \) is coercive and bounded from below in \( H^1_0(0,1) \).

Proof. We have

\[
I(u) = E(u) = \frac{1}{2} \left( a\|u\|^2 + \frac{1}{2} b\|u\|^4 \right) - \int_0^1 F(x, u(x))dx
\]

for every \( u \in K \). By the hypothesis \((f_1)\), there exists \( A > 0 \) such that \( F(x, t) \leq \beta t^2 \) for every \( |t| > A \) and \( x \in [0,1] \). By using the compactness embedding of \( H^1_0(0,1) \) in \( L^2[0,1] \), we obtain that \( \|u\|_{L^2(0,1)} \leq \|u\|_{H^1_0(0,1)} \). Hence

\[
I(u) \geq \frac{1}{2} a\|u\|^2 + \frac{1}{4} b\|u\|^4 - \beta \int_0^1 u^2(x)dx
\]

\[
= \frac{1}{2} a\|u\|^2 + \frac{1}{4} b\|u\|^4 - \beta \|u\|_{L^2}^2
\]

\[
\geq \frac{1}{2} a\|u\|^2 + \frac{1}{4} b\|u\|^4 - \beta \|u\|^2
\]

\[
= \left( \frac{1}{2} a - \beta \right) \|u\|^2 + \frac{1}{4} \|u\|^4,
\]

which implies that the functional \( I = E + \psi_K \) is coercive. Therefore \( I \) is bounded from below in \( H^1_0(0,1) \). If this is not true, there exists a sequence \( \{u_n\} \) in \( H^1_0(0,1) \) such that \( \|u_n\| \to +\infty \) and \( I(u_n) \to -\infty \), which is a contradiction with the coerciveness of \( I \). \( \square \)

Proposition 4.2. If the function \( f \) satisfies \((f_1)\), then \( I = E + \psi_K \) satisfies \((PSZ)_c\) for every \( c \in \mathbb{R} \).

Proof. Let \( c \in \mathbb{R} \) be fixed. Let \( \{u_n\} \) be a sequence in \( H^1_0(0,1) \) such that

\[
I(u_n) = E(u_n) + \psi_K(u_n) \to c;
\]

(2)
and

\[ E'(u_n)(v - u_n) + \psi_K(v) - \psi_K(u_n) \geq -\varepsilon_n\|v - u_n\|, \quad \forall v \in H^1_0(0, 1), \]  

Equation (3)

\{\varepsilon_n\} a sequence in \([0, \infty)\) with \(\varepsilon_n \to 0\). By (2), we obtain that the sequence \(\{u_n\}\) is in \(K\). By Proposition 4.1, since \(I\) is coercive on \(H^1_0(0, 1)\), the sequence \(\{u_n\}\) is bounded in \(K\). Because the sequence \(\{u_n\}\) is bounded in \(H^1_0(0, 1)\). Hence there exists a subsequence still denoted by \(\{u_n\}\) which converges weakly in \(H^1_0(0, 1)\). So there exists \(u \in H^1_0(0, 1)\) such that

\[ u_n \rightharpoonup u \quad \text{in} \quad H^1_0(0, 1); \]  

Equation (4)

\[ u_n \to u \quad \text{in} \quad L^2(0, 1), \]  

Equation (5)

\[ u_n \to u \quad \text{in} \quad C([0, 1]). \]  

Equation (6)

As \(K\) is weakly closed, \(u \in K\). Setting \(v = u\) in (3), we obtain that

\[ (a + b\|u_n\|^2) \int_0^1 u'_n(x)(u'(x) - u'_n(x))\,dx + \int_0^1 f(x, u_n(x))(u_n(x) - u(x))\,dx \geq -\varepsilon_n\|u - u_n\|. \]

Therefore, for large \(n \in \mathbb{N}\), we have

\[ (a + b\|u_n\|^2)\|u - u_n\|^2 \leq (a + b\|u_n\|^2) \int_0^1 u'(x)(u'(x) - u'_n(x))\,dx + \int_0^1 f(x, u_n(x))(u_n(x) - u(x))\,dx + \varepsilon_n\|u - u_n\| \]

\[ \leq (a + b\|u_n\|^2)(u, u - u_n)_{H^1_0} + \|u - u_n\|_{L^2}
\]

\[ \left( \int_0^1 |f(x, u_n(x))|^2\,dx \right)^\frac{1}{2} + \varepsilon_n\|u - u_n\|. \]

Since \(\{u_n\}\) is bounded in \(H^1_0(0, 1)\), it is also bounded in \(C([0, 1])\). Therefore, there exists a constant \(M > 0\) such that \(\|u_n\|_{\infty} \leq M\), which together with the continuity of \(f\) implies that \(|f(x, u_n(x))| \leq M_1\) for some \(M_1 > 0\). We obtain that

\[ (a + b\|u_n\|^2)\|u - u_n\|^2 \leq (a + b\|u_n\|^2)(u, u - u_n)_{H^1_0} + M_1\|u - u_n\|_{L^2} + \varepsilon_n\|u - u_n\|. \]  

Equation (7)

By (4) and the fact that \(\{u_n\}\) is bounded in \(H^1_0(0, 1)\), we have

\[ \lim_{n \to \infty} (a + b\|u_n\|^2)(u, u - u_n)_{H^1_0} = 0. \]

We conclude by (5) that the second term in (7) also converges to 0. Since \(\varepsilon_n \to 0^+\), \(\{u_n\}\) converges strongly to \(u\) in \(H^1_0(0, 1)\). This completes the proof. \(\Box\)
By Proposition 4.2, the functional $I$ satisfies the $(PSZ)_c$ condition, and by Proposition 4.1, the functional $I$ is bounded from below. Therefore, the number $c_1 = \inf_{u \in H^1_0(0,1)} I(u)$ is a critical value of $I$ by Theorem 2.1. It remains to apply Proposition 3.1 which concludes that the critical point $u_1 \in H^1_0(0,1)$ which corresponds to $c_1$, is actually an element of $K$ and a solution of the problem $(P)$.

Example 4.1. Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be defined by $f(x, t) = x(|t|^{\frac{3}{2}} + 1)$. It satisfies $(f_1)$. Indeed, we have $F(x, t) = x\left(\frac{2}{3}|t|^{\frac{3}{2}} + t\right)$ and

$$
\frac{F(x, t)}{t^2} = x\left(\frac{2}{3}|t|^{\frac{3}{2}} + 1\right),
$$

so

$$
\limsup_{|t| \to \infty} \frac{F(x, t)}{t^2} = 0.
$$

5. Proof of Theorem 3.2

We assume that all the hypotheses of Theorem 3.2 are satisfied. Now we prove the existence of a nontrivial solution for the problem $(P)$ by using the Mountain Pass theorem of Szulkin type (see Theorem 2.2).

Proposition 5.1. If the function $f$ satisfies $(h_1)$, then the functional $I = E + \psi_K$ satisfies $(PSZ)_c$ for every $c \in \mathbb{R}$.

Proof. Let $c \in \mathbb{R}$ be a fixed number. Let $\{u_n\}$ be a sequence in $H^1_0(0,1)$ such that

$$
I(u_n) = E(u_n) + \psi_K(u_n) \to c; \quad (8)
$$

and

$$
E'(u_n)(v - u_n) + \psi_K(v) - \psi_K(u_n) \geq -\varepsilon_n \|v - u_n\|, \quad \forall v \in H^1_0(0,1), \quad (9)
$$

where $\{\varepsilon_n\}$ is a sequence in $[0,\infty)$ with $\varepsilon_n \to 0$. By (8), we see that the sequence $\{u_n\}$ belongs to $K$. We put $v = 2u_n$ in (9) and obtain

$$
E'(u_n)(u_n) \geq -\varepsilon_n \|u_n\|.
$$

Therefore, we obtain that

$$
a\|u_n\|^2 + b\|u_n\|^4 - \int_0^1 f(x, u_n(x))u_n(x)dx \geq -\varepsilon_n \|u_n\|. \quad (10)
$$

Because (8) is satisfied for large $n \in \mathbb{N}$

$$
c + 1 \geq \frac{1}{2}a\|u_n\|^2 + \frac{b}{4}\|u_n\|^4 - \int_0^1 F(x, u_n(x))dx. \quad (11)
$$
By $(h_1)$, we have

$$
u F(x, t) - tf(x, t) \leq c_1 \quad \text{for } x \in [0, 1], t \in \mathbb{R}.$$ 

Multiplying (10) by $\nu^{-1}$, and by adding this to (11) and by using $(h_1)$, for large $n \in \mathbb{N}$, we obtain that

$$c + 1 + \frac{1}{\nu} \|u_n\| \geq a \left( \frac{1}{2} - \frac{1}{\nu} \right) \|u_n\|^2 + b \left( \frac{1}{4} - \frac{1}{\nu} \right) \|u_n\|^4$$

$$- \int_0^1 F(x, u_n(x)) - \frac{1}{\nu} f(x, u_n(x))u_n(x)dx$$

$$\geq a \left( \frac{1}{2} - \frac{1}{\nu} \right) \|u_n\|^2 + b \left( \frac{1}{4} - \frac{1}{\nu} \right) \|u_n\|^4$$

$$- \frac{1}{\nu} \int_0^1 \nu F(x, u_n(x)) - f(x, u_n(x))u_n(x)dx$$

$$\geq a \left( \frac{1}{2} - \frac{1}{\nu} \right) \|u_n\|^2 + b \left( \frac{1}{4} - \frac{1}{\nu} \right) \|u_n\|^4 - \frac{c_1}{\nu}. $$

Since $\nu > 4$ we deduce that the sequence $\{u_n\}$ is bounded in $K$. So there exists a subsequence which converges weakly in $H_0^1(0, 1)$. We can assume that there exists $u \in H_0^1(0, 1)$ such that

$$u_n \rightharpoonup u \quad \text{in } H_0^1(0, 1); \quad u_n \to u \quad \text{in } C([0, 1]).$$

(12) (13)

As $K$ is weakly closed, $u \in K$. When we put $v = u$ in (9), we obtain that

$$(a + b\|u_n\|^2) \int_0^1 u'_n(x)(u'(x) - u'_n(x))dx$$

$$+ \int_0^1 f(x, u_n(x))(u_n(x) - u(x))dx \geq -\varepsilon_n\|u - u_n\|. $$

Hence, for large $n \in \mathbb{N}$, we have

$$(a + b\|u_n\|^2)\|u - u_n\|^2 \leq (a + b\|u_n\|^2) \int_0^1 u'(x)(u'(x) - u'_n(x))dx$$

$$+ \int_0^1 f(x, u_n(x))(u_n(x) - u(x))dx + \varepsilon_n\|u - u_n\|$$

$$\leq (a + b\|u_n\|^2)(u, u - u_n)_{H_0^1} + \|u - u_n\|_{C([0, 1])}$$

$$\times \int_0^1 \max_{s \in [-R, R]} |f(x, s)|dx + \varepsilon_n\|u - u_n\|,$$

where $R = \|u\|_{C([0, 1])} + 1$. By (12) and the fact that $\{u_n\}$ is bounded in $H_0^1(0, 1)$, we have

$$\lim_{n} (a + b\|u_n\|^2)(u, u - u_n)_{H_0^1} = 0.$$
By using (13), the second term in the last expression also tends to 0. Since $\varepsilon_n \to 0^+$, $\{u_n\}$ converges strongly to $u$ in $H^1_0(0, 1)$. This completes the proof. \(\square\)

**Proposition 5.2.** If the function $f$ satisfies $(h_1)$ and $(h_2)$, then the following assertions are true:

(i) there exist constants $\alpha > 0$ and $\rho > 0$ such that $I(u) \geq \alpha$ for all $\|u\| = \rho$;

(ii) there exists $e \in H^1_0(0, 1)$ with $\|e\| > \rho$ and $I(e) \leq 0$.

**Proof.** (i) By condition $(h_2)$, there exist $\varepsilon > 0$ and $\rho > 0$ such that

$$\frac{F(x, t)}{|t|^2} \leq \frac{a}{2} - \varepsilon \quad \text{for } |t| \leq \rho.$$ 

Therefore, by using the compactness embedding of $H^1_0(0, 1)$ in $L^2(0, 1)$ with $\|u\|_{L^2(0, 1)} \leq \|u\|_{H^1(0,1)}$, we have

$$I(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_0^1 F(x, u(x))dx$$

$$\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_0^1 \left(\frac{a}{2} - \varepsilon\right)|u(x)|^2dx$$

$$= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \left(\frac{a}{2} - \varepsilon\right)\|u\|^2_{L^2}$$

$$\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \left(\frac{a}{2} - \varepsilon\right)\|u\|^2$$

$$= \varepsilon\|u\|^2 + \frac{b}{4}\|u\|^4.$$ 

For $\|u\| = \rho$ we have $\alpha = \varepsilon\rho^2 + \frac{b}{4}\rho^4 > 0$, and the assertion of (i) holds true.

(ii) The condition $(h_1)$ implies that the function $t \to \frac{F(x, t)}{|t|^\nu}$ is increasing for $t \geq M$ and decreasing for $t \leq -M$ as one can see by differentiation, so there exists $r_1 > 0$ such that $F(x, t) \geq r_1|t|^{\nu}$, for $x \in [0, 1], |t| \geq M$. Also the function $t \to F(x, t)$ is continuous on the compact $[0, 1] \times [-M, M]$, then there exists $r_2 > 0$ such that $F(x, t) \geq -r_2$, for $x \in [0, 1], |t| \leq M$, so

$$F(x, t) \geq r_1|t|^{\nu} - r_2, \quad \text{for } x \in [0, 1], t \in \mathbb{R}.$$ 

Fix $u_0 \in K \setminus \{0\}$. Letting $u = su_0$ ($s > 0$), we have that

$$I(su_0) = \frac{a}{2}s^2\|u_0\|^2 + \frac{b}{4}s^4\|u_0\|^4 - \int_0^1 F(x, su_0(x))dx$$

$$\leq \frac{a}{2}s^2\|u_0\|^2 + \frac{b}{4}s^4\|u_0\|^4 - \int_0^1 (r_1s^{\nu}|u_0|^{\nu} - r_2)dx$$

$$= \frac{a}{2}s^2\|u_0\|^2 + \frac{b}{4}s^4\|u_0\|^4 - r_1s^{\nu}\|u_0\|_{L^{\nu}}^{\nu} + r_2.$$
Since \( \nu > 4 \) we obtain that \( I(su_0) \to -\infty \) as \( s \to +\infty \). Thus, it is possible to take \( s \) so large such that for \( e = su_0 \), we have \( \|e\| > \rho \) and \( I(e) \leq 0 \). The proof of the proposition is achieved.

By Proposition 5.1, the functional \( I \) satisfies the \((PSZ)_c\)-condition for every \( c \in \mathbb{R} \), and \( I(0) = 0 \). By Proposition 5.2 it follows that there exist constants \( \alpha, \rho > 0 \) and \( e \in H^1_0(0,1) \) such that \( I \) satisfies all the conditions of Theorem 2.2. Therefore,

\[
c_2 = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),
\]

is a critical value of \( I \) with \( c_2 \geq \alpha > 0 \), where

\[
\Gamma = \{ \gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e \}.
\]

We remark that the critical point \( u_2 \in H^1_0(0,1) \) associated to the critical value \( c_2 \) cannot be trivial because \( I(u_2) = c_2 > 0 = I(0) \). By Proposition 3.1, we conclude that \( u_2 \) is a solution of \((P)\).

**Example 5.1.** Let \( f : [0,1] \times \mathbb{R} \to \mathbb{R} \) be defined by \( f(x,t) = \frac{1}{1+x^2} \frac{a}{2} t(1 + t^2)e^{t^2} \). As we will show, it satisfies \((h_1)\) and \((h_2)\). We have \( F(x,t) = \frac{1}{1+x^2} \frac{a}{4} t^2 e^{t^2} \), and

\[
6F(x,t) - tf(x,t) = \frac{1}{1+x^2} \frac{a}{2} t^2(2 - t^2)e^{t^2} \leq 0,
\]

for all \( |t| \geq \sqrt{2} \). So there exist \( \nu = 6 > 4 \) and \( M = \sqrt{2} > 0 \) such that

\[
0 < \nu F(x,t) \leq tf(x,t).
\]

Moreover

\[
\limsup_{|t| \to 0} \frac{F(x,t)}{|t|^2} = \limsup_{|t| \to 0} \frac{1}{1+x^2} \frac{a}{4} e^{t^2} = \frac{a}{4} < \frac{a}{2}.
\]

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